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# Fixed Point Theorems and their Applications 

Věty o pevných bodech a jejich aplikace

Bc. Zbyšek Machaczek

Diploma Thesis
Supervisor: doc. Mgr. Petr Vodstrčil, Ph.D.
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# Diploma Thesis Assignment 

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The thesis will be devoted to some important fixed point theorems (e.g. Banach's theorem and its extensions, Brouwer's theorem and Schauder's theorem). We will also review some important applications of these theorems (e.g., the Picard-Lindeloef theorem, the Lax-Milgram lemma, and the so-called Collage Theorem).

References:
Pata Vittorino: Fixed Point Theorems and Applications, 2019

Extent and terms of a thesis are specified in directions for its elaboration that are opened to the public on the web sites of the faculty.

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#### Abstract

Abstrakt

Předpokládejme že máme nějakou množinu $X$ a zobrazení $T: X \rightarrow X$, bod $x \in X$ nazveme pevným bodem zobrazení $T$ pokud $x=T(x)$. V první částí této práce se budeme zabývat podmínkami, které zaručují, že tento pevný bod existuje. V druhé části předvedeme některé aplikace těchto vět o pevných bodech.


## Klíčová slova

diplomová práce; věty o pevných bodech; funkcionální analýza


#### Abstract

Suppose that we have some set $X$ and a mapping $T: X \rightarrow X$, we call $x \in X$ a fixed point of $T$ if $x=T(x)$. In the first part of this thesis we will be discussing conditions under which such a fixed point exists and in the second part we will introduce several applications for these fixed point theorems.


## Keywords

master thesis; fixed point theorems; functional analysis

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## Contents

List of symbols and abbreviations ..... 6
1 Introduction ..... 7
2 Theorems ..... 10
2.1 Banach Fixed Point Theorem and Its Extensions ..... 10
2.2 Brouwer Fixed Point Theorem and Mapping Degree ..... 28
2.3 Schauder Fixed Point Theorem ..... 39
2.4 Kakutani Fixed Point Theorem ..... 43
3 Applications ..... 50
3.1 Picard-Lindelöf Theorem ..... 50
3.2 Peano Theorem ..... 53
3.3 Lax-Milgram Lemma ..... 55
3.4 Fractals ..... 58
3.5 Game Theory Applications ..... 73
4 Conclusion ..... 84
Bibliography ..... 86

## List of symbols and abbreviations

| s.t. | - such that |
| :---: | :---: |
| iff | - if and only if |
| $\bar{\Omega}^{X}$ | - $\left\{x \in X: \exists\left(x_{n}\right) \subset \Omega: x_{n} \rightarrow x\right\}$, if it is clear what $X$ is we will instead write $\bar{\Omega}$ |
| $\operatorname{int}_{X} \Omega$ | - $\Omega \cap \bar{\Omega}^{X}$, we may also write int $\Omega$ if it is clear what $X$ is |
| $\partial_{X} \Omega$ | - $\bar{\Omega}^{X} \backslash \Omega$, again we may also write $\partial \Omega$ |
| $B_{X}(a, b)$ | - $\quad\{x \in X:\\|x-a\\|<b\}$ |
| conv $\Omega$ | - The unique minimal convex set $C$ such that $\Omega \subset C$ |
| $\operatorname{span}\left\{x_{1}, \ldots, x_{n}\right\}$ | - $\left\{\sum_{i=1}^{n} \lambda_{i} x_{i}: \lambda_{i} \in \mathbb{R}\right\}$ |
| $\operatorname{diam}_{d} \Omega$ | $-\sup _{x, y \in \Omega} d(x, y)$, if it is clear what metric $d$ is used we will instead write $\operatorname{diam} \Omega$ |
| $\times_{i=1}^{n} \Omega_{i}$ | - $\Omega_{1} \times \Omega_{2} \times \ldots \times \Omega_{n}$ |

## Chapter 1

## Introduction

Let us analyze a very simple case where $f:[0,1] \rightarrow[0,1]$, suppose that $\bar{x} \in[0,1]$ is a fixed point of $f$, let us define a function $g:[0,1] \rightarrow \mathbb{R}$ as

$$
g(x) \equiv f(x)-x
$$

Observe that $g(\bar{x})=0$, it is also easy to see that

$$
g(0)=f(0) \geq 0,
$$

and

$$
g(1)=f(1)-1 \leq 0 .
$$

We know that if $g$ is continuous, then by the intermediate value theorem there exists $\bar{x} \in[0,1]$ such that $g(\bar{x})=0$, meaning that $f(\bar{x})=\bar{x}$, thus we have found a condition under which $f$ has a fixed point. In fact this example is the 1-dimensional version of the Brouwer fixed point theorem which we will be introducing later on.

Many mathematical problems can be converted into fixed point problems, one simple example is equations, suppose we have a vector space $X$ and a mapping $T: X \rightarrow X$, if we are looking for roots of $T$, i.e. $\bar{x} \in X$ such that

$$
T(\bar{x})=0_{X},
$$

we can easily turn this into a fixed point problem by finding a fixed point of the mapping $x \mapsto$ $T(x)+x$.

Arguably one of the most fundamental results in functional analysis is the Banach fixed point theorem which guarantees the existence of a unique fixed point $\bar{x} \in X$ and requires $X$ to be a complete metric space along with $T$ being a so called contraction which can be understood as $T$ in some sense moving any two points closer together, this theorem can be used to prove many existence theorems for solutions of, e.g., partial differential equations, initial value problems for systems of
ordinary differential equations, variational inequalities and many more. The advantage of using this particular theorem to show the existence of some problem is that it provides a numerical method for finding this fixed point given that the mapping $T$ we apply the Banach fixed point theorem can be reasonably constructed on a computer. In particular if we take any $x \in X$ and apply the mapping $T$ to it again and again we can get arbitrarily close to the fixed point $\bar{x}$.

As an illustration of this property, take the space of all compact sets in $\mathbb{R}^{2}$, on this set we can define a metric $d_{H}$ so that $\left(\mathbb{R}^{2}, d_{H}\right)$ is a complete metric space. A fixed point of some mapping $f$ that satisfies the assumptions of the Banach fixed point theorem is of course a set in $R^{2}$. Let us take two different sets $X, \tilde{X} \subset \mathbb{R}^{2}$ and start applying one such mapping to it and see what happens.


Figure 1.1: (left) $X$, (right) $\tilde{X}$.


Figure 1.2: (left) $f(X),($ right $) f(\tilde{X})$


Figure 1.3: (left) $f(f(X)),($ right $) f(f(\tilde{X}))$


Figure 1.4: (left) $f(f(f(X)))$, (right) $f(f(f(\tilde{X})))$


Figure 1.5: (left) $f(f(f(f(X))))$, (right) $f(f(f(f(X))))$

We can see that after applying $f$ a few times we cannot visually determine which initial set was used. The third iteration also visually looks the same as the fourth iteration, signifying that we are close the fixed point of $f$ (close in the sense of $d_{H}$ ).

## Chapter 2

## Theorems

This chapter will be dedicated to exploring various fixed point theorems.

### 2.1 Banach Fixed Point Theorem and Its Extensions

First we will present the Banach Fixed Point Theorem which is a fundamental and very well known statement and then show some of its generalizations, more can be found in [1]. This theorem has wide application in functional analysis and other fields.

Theorem 2.1 (Banach Fixed Point) Let

- $(X, d)$ be a complete metric space,
- $T: X \rightarrow X$ satisfies $\forall x, y \in X: d(T(x), T(y)) \leq \lambda d(x, y)$ for some $\lambda \in[0,1)$ ( $T$ is a contraction).

Then $\exists!x \in X: T(x)=x$.
Proof First we shall prove the existence. Let $\left(x_{n}\right) \subset X$ be a sequence such that $x_{0} \in X$ is chosen arbitrarily and $x_{n}:=T\left(x_{n-1}\right), n \in \mathbb{N}$. Because $T$ is a contraction, for $n \in \mathbb{N}$ we can write

$$
d\left(x_{n+1}, x_{n}\right)=d\left(T\left(x_{n}\right), T\left(x_{n-1}\right)\right) \leq \lambda d\left(x_{n}, x_{n-1}\right)
$$

and inductively

$$
d\left(x_{n+1}, x_{n}\right) \leq \lambda^{n} d\left(x_{1}, x_{0}\right) . \quad(\forall n \in \mathbb{N})
$$

Now for some natural $n, m$ where $n>m$ consider

$$
\begin{equation*}
d\left(x_{n}, x_{m}\right) \leq \sum_{i=m}^{n-1} d\left(x_{i+1}, x_{i}\right) \leq d\left(x_{1}, x_{0}\right) \sum_{i=m}^{n-1} \lambda^{i} \leq d\left(x_{1}, x_{0}\right) \sum_{i=m}^{\infty} \lambda^{i}=d\left(x_{1}, x_{0}\right) \frac{\lambda^{m}}{1-\lambda} . \tag{2.1}
\end{equation*}
$$

Since $\lambda<1$ we have $d\left(x_{n}, x_{m}\right) \rightarrow 0$ as $m \rightarrow \infty$, hence $\left(x_{n}\right)$ is Cauchy. Since $X$ is a complete metric space, we know that $\left(x_{n}\right)$ has a limit $x \in X$. The relation $T(x)=x$ comes from the fact that

$$
T\left(x_{n}\right)=x_{n+1} \rightarrow x
$$

and

$$
d\left(T\left(x_{n}\right), T(x)\right) \leq \lambda d\left(x_{n}, x\right) \rightarrow 0
$$

this means that

$$
T\left(x_{n}\right) \rightarrow T(x)
$$

As a sequence can only have one limit, we conclude that $T(x)=x$. To prove the uniqueness, take $y_{1}, y_{2} \in X$ and assume that $y_{1}=T\left(y_{1}\right)$ and $y_{2}=T\left(y_{2}\right)$. It holds

$$
d\left(y_{1}, y_{2}\right)=d\left(T\left(y_{1}\right), T\left(y_{2}\right)\right) \leq \lambda d\left(y_{1}, y_{2}\right) .
$$

Due to the fact that $\lambda \in[0,1)$ this can only be satisfied if $y_{1}=y_{2}$.

Remark 2.1 The proof of this theorem also gives us a numerical method to compute the fixed point $x$. If we keep iteratively applying the mapping $T$ to any point within $X$ we can get arbitrarily close to $x$. Furthermore we have an estimate of how close we are to $x$. That is

$$
d\left(x, x_{m}\right) \leq d\left(x, x_{n}\right)+d\left(x_{n}, x_{m}\right) \leq d\left(x, x_{n}\right)+d\left(x_{1}, x_{0}\right) \frac{\lambda^{m}}{1-\lambda} .
$$

Here we utilize (2.1). Sending $n \rightarrow \infty$ and utilizing the fact that $d\left(x, x_{n}\right) \rightarrow 0$ we get

$$
d\left(x, x_{m}\right) \leq d\left(x_{1}, x_{0}\right) \frac{\lambda^{m}}{1-\lambda} .
$$

Next we will look at this simple extension of the Banach fixed point theorem

Theorem 2.2 If

- $(X, d)$ is a complete metric space,
- $T: X \rightarrow X$ and $T^{m}$ is a contraction for m-th iteration of $T$

Then $\exists!\bar{x} \in X: T(\bar{x})=\bar{x}$.
Proof The Banach fixed point theorem guarantees that $T^{m}$ has a unique fixed point $\bar{x}$, we have

$$
T^{m}(T(\bar{x}))=T\left(T^{\bar{x}}(x)\right)=T(\bar{x}),
$$

so $T(\bar{x})$ is a fixed point of $T^{m}(\bar{x})$ as well as $\bar{x}$, but the fixed point of $T^{m}$ is unique, implying that

$$
T(\bar{x})=\bar{x} .
$$

There are no other fixed points of $T$ since being such means it automatically also a fixed point of $T^{m}$ which is subject to the Banach fixed point theorem and has only one fixed point.

Remark 2.2 We should consider whether a situation that a mapping is not a contraction and one of its iterations is can occur. Otherwise this theorem would be redundant. One such function is the Dirichet function $D: \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$
D(x)= \begin{cases}1 & \text { if } x \in \mathbb{Q} \\ 0 & \text { if } x \notin \mathbb{Q} .\end{cases}
$$

Which is not a contraction but $D(D(x))=1$ is constant and thus a contraction.
This next theorem is an extension of the Banach fixed point theorem. It hinges on a condition that is less strict but still similar to constructiveness

Theorem 2.3 (Boyd-Wong) Let $(X, d)$ be a complete metric space, let $T: X \rightarrow X$ and $\varphi$ : $[0, \infty) \rightarrow[0, \infty)$, if $\varphi$ is continuous and satisfies $\varphi(r)<r$ for $r>0$ and

$$
d(T(x), T(y)) \leq \varphi(d(x, y)), \quad \forall x, y \in X
$$

Then $\exists!\bar{x} \in X: T(\bar{x})=\bar{x}$.
Proof Let $\left(x_{n}\right) \subset X$ be a sequence defined by $x_{n}=T\left(x_{n-1}\right)$ where $x_{0} \in X$ is picked arbitrarily and let $\left(a_{n}\right) \subset \mathbb{R}$ be a sequence defined by $a_{n}=d\left(x_{n+1}, x_{n}\right)$. It holds

$$
a_{n+1}=d\left(T\left(x_{n+1}\right), T\left(x_{n}\right)\right) \leq \varphi\left(a_{n}\right) \leq a_{n} .
$$

So $\left(a_{n}\right)$ is non-increasing and is bounded bellow by 0 . Therefore ( $a_{n}$ ) converges to some $a \geq 0$. From the inequality above we also have $\varphi\left(a_{n}\right) \rightarrow a$. Using continuity of $\varphi$ we get $\varphi\left(a_{n}\right) \rightarrow \varphi(a)$, implying $\varphi(a)=a$, thus $a=0$. Now assume that $\left(x_{n}\right)$ is not a Cauchy sequence. Then there exist $\varepsilon>0$ and integers $m_{k}>n_{k} \geq k$ such that

$$
d_{k} \equiv d\left(x_{m_{k}}, x_{n_{k}}\right) \geq \varepsilon \quad(\forall k \in \mathbb{N})
$$

We will choose $m_{k}$ to be as small as possible while still satisfying the condition above, i.e.,

$$
d\left(x_{m_{k}-1}, x_{n_{k}}\right)<\varepsilon .
$$

We have

$$
\varepsilon \leq d_{k} \leq d\left(x_{m_{k}}, x_{m_{k}-1}\right)+d\left(x_{m_{k}-1}, x_{n_{k}}\right)<a_{m_{k}-1}+\varepsilon .
$$

This tells us that $d_{k} \rightarrow \varepsilon$ as $k \rightarrow \infty$ because $a_{k} \rightarrow 0$. It also holds
$d_{k} \leq d\left(x_{m_{k}}, x_{m_{k}+1}\right)+d\left(x_{m_{k}+1}, x_{n_{k}+1}\right)+d\left(x_{n_{k}+1}, x_{n_{k}}\right)=a_{m_{k}}+d\left(T\left(x_{m_{k}}\right), T\left(x_{n_{k}}\right)\right)+a_{n_{k}} \leq a_{m_{k}}+\varphi\left(d_{k}\right)+a_{n_{k}}$.

We know that

- $d_{k} \rightarrow \varepsilon$,
- $a_{k} \rightarrow 0$,
- $\varphi$ is continuous.

By sending $k \rightarrow \infty$ in (2.1) we finally get $\varepsilon \leq \varphi(\varepsilon)$ which is a contradiction to $\varepsilon>0$. Meaning that $\left(x_{n}\right)$ is Cauchy. Same as in the proof of the Banach fixed point theorem $\left(x_{n}\right)$ has a limit $\bar{x}$ because of completeness of $X$ and this limit is the fixed point in question because

$$
T\left(x_{n}\right)=x_{n+1} \rightarrow \bar{x}
$$

and

$$
d\left(T\left(x_{n}\right), T(\bar{x})\right) \leq \varphi\left(d\left(x_{n}, \bar{x}\right)\right) \Rightarrow T\left(x_{n}\right) \rightarrow T(\bar{x}) .
$$

For the proof of uniqueness, for arbitrary $y_{1}, y_{2} \in X$ both of which are fixed points of $T$ it holds

$$
d\left(y_{1}, y_{2}\right)=d\left(T\left(y_{1}\right), T\left(y_{2}\right)\right) \leq \varphi\left(d\left(y_{1}, y_{2}\right)\right) .
$$

Implying that $y_{1}=y_{2}$.
Theorem 2.4 Let $(X, d)$ be a metric space, $T: X \rightarrow X$ and suppose that assumptions of the Boyd-Wong theorem are satisfied, denote the unique fixed point of $T$ as $\bar{x}$ and define

$$
\delta_{n} \equiv d\left(\bar{x}, x_{n}\right) .
$$

Suppose that in addition to assumptions of Theorem 2.3 it further holds for some $a>0$ and $p>0$ that

$$
\varphi(r)=r-a r^{1+p}+\psi(r), \quad(\forall r>0)
$$

where $\psi$ satisfies

$$
\lim _{r \rightarrow 0_{+}} \frac{\psi(r)}{r^{1+p}}=0
$$

Then for some $c>0$ we have

$$
\delta_{n} \leq \frac{c}{n^{1 / p}} .
$$

Proof Using the inequality $d(T(x), T(y)) \leq \varphi(d(x, y))$ with $x=\bar{x}$ and $y=x_{n+1}$ we get

$$
\delta_{n+1} \leq \varphi\left(\delta_{n}\right)=\delta_{n}-a \delta_{n}^{1+p}+\psi\left(\delta_{n}\right)
$$

If $\delta_{n}=0$ for some $n$, the theorem holds, otherwise set

$$
z_{n} \equiv \frac{1}{\delta_{n}^{p}}
$$

Since

$$
\frac{1}{\delta_{n+1}} \geq \frac{1}{\delta_{n}-a \delta_{n}^{1+p}+\psi\left(\delta_{n}\right)}
$$

it holds

$$
z_{n+1} \geq z_{n}\left(1-\frac{a-\tilde{\psi}\left(\delta_{n}\right)}{z_{n}}\right)^{-p}
$$

where $\tilde{\psi} \equiv \frac{\psi(r)}{r^{1+p}}$, i.e.,

$$
\lim _{r \rightarrow 0_{+}} \tilde{\psi}(r)=0
$$

We also have

$$
\begin{equation*}
z_{n}\left(1-\frac{a-\tilde{\psi}\left(\delta_{n}\right)}{z_{n}}\right)^{-p}-z_{n} \rightarrow a p \tag{2.2}
\end{equation*}
$$

This can be shown by applying the mean value theorem to the function $x^{-p}$, we get

$$
\left(1-\frac{a-\tilde{\psi}\left(\delta_{n}\right)}{z_{n}}\right)^{-p}-1=\frac{a-\tilde{\psi}\left(\delta_{n}\right)}{z_{n}} \cdot p \cdot \xi_{n}^{-p-1}
$$

for some $\xi \in\left(1-\frac{a-\tilde{\psi}\left(\delta_{n}\right)}{z_{n}}, 1\right)$. This gives us

$$
\left(a-\tilde{\psi}\left(\delta_{n}\right)\right) p \leq z^{n}\left(\left(1-\frac{a-\tilde{\psi}\left(\delta_{n}\right)}{z_{n}}\right)^{-p}-1\right) \leq\left(a-\tilde{\psi}\left(\delta_{n}\right)\right) p\left(1-\frac{a-\tilde{\psi}\left(\delta_{n}\right)}{z_{n}}\right)^{-p-1}
$$

Implying the claim by squeeze theorem as both bounds go to $a p$. We can also write this limit as

$$
z_{n}\left(1-\frac{a-\tilde{\psi}\left(\delta_{n}\right)}{z_{n}}\right)^{-p}=z_{n}+a p+\varepsilon_{n}
$$

with $\varepsilon_{n} \rightarrow 0$. From (2.2) we inductively get

$$
z_{n} \geq z_{0}+n a p+\sum_{i=0}^{n-1} \varepsilon_{i}=n a p(1+\underbrace{\frac{z_{0}}{n a p}}_{\rightarrow 0}+\underbrace{\frac{\sum_{i=0}^{n-1} \varepsilon_{i}}{n a p}}_{\rightarrow 0})=n a p\left(1+\alpha_{n}\right)
$$

for some $\alpha_{n} \rightarrow 0$. And thus by the definition of $z_{n}$ we have

$$
\delta_{n} \leq \frac{1}{n^{\frac{1}{p}}}\left(\frac{1}{a p\left(1+\alpha_{n}\right)}\right)^{\frac{1}{p}} \leq \frac{c}{n^{\frac{1}{p}}}, \quad(\text { for some } c>0),
$$

this finishes the proof.

Remark 2.3 The Banach fixed point theorem is a specific case of theorem 2.3, if we take

$$
\varphi(r)=\lambda r, \quad(\lambda \in[0,1))
$$

Let us also look at an example that does not satisfy the conditions of the Banach fixed point theorem but satisfies the condition of theorem 2.3. Take $T: \mathbb{R} \rightarrow \mathbb{R}$ as $T(x)=\sin (x)$. This function is not a contraction, if it was, its derivative would be lesser than 1 in all points, but this is not the case. Next we want to show that conditions of theorem 2.3 are satisfied. We are looking for $\varphi:[0, \infty) \rightarrow[0, \infty)$ such that

$$
|\sin (x)-\sin (y)| \leq \varphi(|x-y|)<|x-y|, \quad(\forall x, y \in \mathbb{R}, x \neq y)
$$

and is also continuous. We have

$$
|\sin (x)-\sin (y)|=\left|2 \sin \left(\frac{x-y}{2}\right) \cos \left(\frac{x+y}{2}\right)\right| \leq\left|2 \sin \left(\frac{x-y}{2}\right)\right| .
$$

For $\left|\frac{x-y}{2}\right| \leq \pi$ it holds

$$
\left|2 \sin \left(\frac{x-y}{2}\right)\right|=2 \sin \left(\frac{|x-y|}{2}\right)
$$

Hence we will define $\varphi$ as

$$
\varphi(r)= \begin{cases}2 \sin \left(\frac{r}{2}\right) & (r \in[0, \pi]) \\ 2 \sin \left(\frac{\pi}{2}\right) & (r>\pi)\end{cases}
$$

Observe that $\varphi$ is continuous and it holds

$$
\varphi(r)=2 \sin \left(\frac{r}{2}\right) \leq 2 \sin \left(\frac{r}{2}\right)<r, \quad(\forall r \in(0, \pi])
$$

and

$$
\varphi(r)=2 \sin \left(\frac{\pi}{2}\right)=2<r, \quad(\forall r \in(\pi, \infty))
$$

Such function clearly satisfies all conditions of theorem 2.3.
In regards to theorem 2.4 from the Taylor expansion of $\varphi$ we can see that

$$
\varphi(r)=r-a r^{1+2}+o\left(r^{1+2}\right)
$$

Thus $\delta_{n} \equiv d\left(\bar{x}, x_{n}\right)$, where $\bar{x}$ is the fixed point of $T$ should satisfy $\delta_{n} \leq \frac{c}{\sqrt{n}}$ for some $c>0$. To illustrate this let us look at some numerical data

$$
\begin{aligned}
& \sin ^{10^{1}}(1) \approx 0.46295 \\
& \sin ^{10^{3}}(1) \approx 0.05459 \\
& \sin ^{10^{5}}(1) \approx 0.00547
\end{aligned}
$$

Definition 2.1 (Weak Contraction) Let $(X, d)$ be a metric space. We call a mapping $T: X \rightarrow$ $X$ a weak contraction if $\forall x, y \in X, x \neq y$ it holds

$$
d(T(x), T(y))<d(x, y)
$$

Remark 2.4 Every weak contraction has at most one fixed point. This can be shown exactly as in the proof of the Banach fixed point theorem. Being a weak contraction is however not a sufficient condition for having a fixed point. For example on the complete metric space $X=[1, \infty)$ take function

$$
T(x)=x+\frac{1}{x},
$$

that maps $X$ into itself. We have

$$
|T(x)-T(y)|=\left|x-y+\frac{y-x}{x y}\right|=|x-y| \cdot\left|1-\frac{1}{x y}\right|<|x-y| . \quad(\forall x, y \in X, x \neq y)
$$

Showing that $T$ is a weak contraction, however it has no fixed points as it would imply that a fixed point $\bar{x}$ satisfies

$$
\bar{x}+\frac{1}{\bar{x}}=\bar{x},
$$

which is not possible.

Theorem 2.5 Let $(X, d)$ be a compact metric space and $T: X \rightarrow X$ be a weak contraction. Then there exists a unique fixed point $\bar{x} \in X$ of $T$ and for all $x \in X$ it holds $\left(T^{n}(x)\right) \rightarrow \bar{x}$ as $n \rightarrow \infty$.

Proof Suppose that some $\bar{x} \in X$ satisfies

$$
\bar{x}=\min _{x \in X} d(x, T(x)),
$$

such $\bar{x}$ exists since $X$ is compact and the function $x \mapsto d(x, T(x))$ is continuous due to the fact that it is a composition of two continuous functions $T$ and $d: X \times X \rightarrow \mathbb{R}^{+}$which as a metric is automatically continuous. If $\bar{x} \neq T(\bar{x})$, then

$$
\left.d(\bar{x}, T(\bar{x}))=\min _{x \in X} d(x, T(x))\right) \leq d(T(\bar{x}), T(T(\bar{x})))<d(\bar{x}, T(\bar{x})) .
$$

As this is a contradiction we can conclude that $\bar{x}=T(\bar{x})$. With existence out of the way let us now prove the claim about convergence of the above mentioned sequence. We pick $x \in X$ arbitrarily. If $T^{n}(x)=\bar{x}$ for some $n \in \mathbb{N}$ there is nothing to prove. Otherwise define

$$
\delta_{n}=d\left(\bar{x}, T^{n}(x)\right) .
$$

Since

$$
\delta_{n+1}=d\left(\bar{x}, T\left(T^{n}(x)\right)\right)=d\left(T(\bar{x}), T\left(T^{n}(x)\right)\right)<d\left(\bar{x}, T^{n}(x)\right)=\delta_{n},
$$

we can see that $\left(\delta_{n}\right)$ is decreasing and has a limit $r \geq 0$. Let $\left(T^{n_{k}}(x)\right)$ be a subsequence of $\left(T^{n}(x)\right)$ that converges to some $x_{0} \in X$. If $x_{0} \neq \bar{x}$, then

$$
d\left(\bar{x}, x_{0}\right)=\lim _{k \rightarrow \infty} d\left(\bar{x}, T^{n_{k}}(x)=\lim _{k \rightarrow \infty} \delta_{n_{k}}=r,\right.
$$

but we also have

$$
r=\lim _{k \rightarrow \infty} \delta_{n_{k}+1}=\lim _{k \rightarrow \infty} d\left(\bar{x}, T\left(T^{n_{k}}(x)\right)\right)=d\left(\bar{x}, T\left(x_{0}\right)\right),
$$

showing that

$$
\begin{equation*}
d\left(\bar{x}, x_{0}\right)=d\left(\bar{x}, T\left(x_{0}\right)\right) \tag{2.3}
\end{equation*}
$$

Since $x_{0} \neq \bar{x}$ it holds

$$
d\left(\bar{x}, T\left(x_{0}\right)\right)=d\left(T(\bar{x}), T\left(x_{0}\right)\right)<d\left(\bar{x}, x_{0}\right),
$$

which contradicts with (2.3), showing that any convergent subsequence of $\left(T^{n}(x)\right)$ converges to $\bar{x}$. But $X$ is compact so we can conclude that $T^{n}(x) \rightarrow \bar{x}$.

Now we will prove that if $(X, d)$ is a compact metric space than in fact theorems 2.5 and 2.3 are equivalent. If the conditions of theorem 2.3 are met and $(X, d)$ is compact then the condition of theorem 2.5 are automatically met. Let us show that the opposite implication also holds true.

This following lemma is key to proving a theorem we will showcase next.

Lemma 2.6 Let $\rho:[0,1] \rightarrow[0,1]$, if

- $\rho$ is right continuous.
- $\rho$ is increasing.
- $(\forall r \in(0,1]): \rho(r)<r$.

Then there exists an increasing continuous function $\varphi:[0,1] \rightarrow[0,1]$ such that

$$
\rho(r) \leq \varphi(r)<r, \quad(\forall r \in(0,1])
$$

Proof We will begin by showing that for all $0<a<b \leq 1$

$$
\inf _{r \in[a, b]}(r-\rho(r))>0 .
$$

If this is not the case, then for any fixed interval $[a, b]$ with $0<a<b \leq 1$ there exists a sequence $\left(r_{n}\right) \subset[a, b]$ with

$$
r_{n}-\rho\left(r_{n}\right) \rightarrow 0 .
$$

As $r_{n}$ is a bounded sequence in a compact metric space, there exists $r_{0} \in[a, b]$ such that a subsequence of $\left(r_{n}\right)$ has $r_{0}$ for a limit. For convenience let us denote this subsequence again as $\left(r_{n}\right)$. We will show that there are only finitely many members of $\left(r_{n}\right)$ greater or equal to $r_{0}$. If not, then there exists a subsequence $\left(r_{n_{k}}\right)$ of $\left(r_{n}\right)$ such that $r_{n_{k}} \rightarrow r_{0}$ with $r_{n_{k}} \geq r$ for all $k \in \mathbb{N}$. For this subsequence we have

$$
0=\lim _{k \rightarrow \infty}\left(r_{n_{k}}-\rho\left(r_{n_{k}}\right)\right)=r_{0}-\rho\left(r_{0}\right),
$$

where the second equality results from the right continuity of $\rho$. This contradicts $\rho\left(r_{0}\right)<r_{0}$. Now we can suppose that for each $n \in N$ it holds $r_{n}<r_{0}$, if not we just remove the finitely many members of $\left(r_{n}\right)$ greater or equal to $r_{0}$. Define

$$
\varepsilon \equiv r_{0}-\rho\left(r_{0}\right)
$$

note that $\varepsilon>0$. Using the fact that $r_{n} \rightarrow r_{0}$ and $r_{n}-\rho\left(r_{n}\right) \rightarrow 0$ we pick $m \in \mathbb{N}$ large enough so that $r_{0}-r_{m}<\frac{\varepsilon}{2}$ and $r_{m}-\rho\left(r_{m}\right)<\frac{\varepsilon}{2}$. But then

$$
\rho\left(r_{0}\right)=r_{0}-\varepsilon=r_{0}-r_{m}+r_{m}-\rho\left(r_{m}\right)+\rho\left(r_{m}\right)-\varepsilon<\rho\left(r_{m}\right) .
$$

As $r_{n}<r_{0}$ this is a contradiction with $\rho$ being increasing. Hence we have shown that for all $0<a<b \leq 1$ it holds $\inf _{r \in[a, b]}(r-\rho(r))>0$.

Now let us get to constructing $\varphi$. For $n \in \mathbb{N}$ we define $a_{n}=2^{-(n-1)}$ and

$$
\varepsilon_{n}=\inf _{r \in\left[a_{n+1}, a_{n}\right]}(r-\rho(r))>0 .
$$

We construct $\hat{\varphi}:[0,1] \rightarrow[0,1]$ on intervals $I_{n}=\left(a_{n+1}, a_{n}\right]$ in the following way. Set

$$
\hat{\varphi}(r)=r-\omega_{1}, \quad\left(\forall r \in I_{1}\right),
$$

where

$$
\omega_{1}=\varepsilon_{1}
$$

and for $n \geq 1$ we set

$$
\hat{\varphi}(r)=r-\omega_{n}, \quad\left(\forall r \in I_{n}\right),
$$

with

$$
\begin{equation*}
\omega_{n+1}=\min \left\{\varepsilon_{n+1}, \omega_{n}\right\} \tag{2.4}
\end{equation*}
$$

Finally we set $\hat{\varphi}(0)=0$. We can see that such $\hat{\varphi}$ satisfies $\hat{\varphi}(r)<r$ for all $r \in(0,1]$. But $\hat{\varphi}$ need not be continuous. To deal with continuity define $\varphi:[0,1] \rightarrow[0,1]$ again interval by interval so that

$$
\varphi(r)=q_{n+1}(r), \quad\left(\forall r \in I_{n}\right)
$$

where $q_{n}(r)$ is a straight line connecting the points $\left(a_{n+1}, \hat{\varphi}\left(a_{n+1}\right)\right)$ and $\left(a_{n}, \hat{\varphi}\left(a_{n}\right)\right)$


Figure 2.1: Construction of $\varphi$.

Such $\varphi$ is now continuous and is greater of equal to $\hat{\varphi}$. It also satisfies $\varphi(r)<r$ for all $r \in(0,1]$, this fact translates from $\hat{\varphi}$ having the same property. Moreover $\varphi$ is increasing, since $\varphi$ is defined by connecting point it will suffice to show that the $y$-coordinate of each subsequent point is greater than the previous one, i.e., we have to show that for all $n \in \mathbb{N}$ it holds $\hat{\varphi}\left(a_{n}\right)>\hat{\varphi}\left(a_{n+1}\right)$. Let $n \in \mathbb{N}$ be fixed, by definition of $I_{n}$, the point $a_{n}$ lies in $I_{n}$ and $a_{n+1}$ does not, thus

$$
\varphi\left(\hat{a}_{n}\right)=a_{n}-\omega_{n}
$$

and

$$
\varphi\left(\hat{a_{n+1}}\right)=a_{n+1}-\omega_{n+1}
$$

but from (2.4) it is clear that $\omega_{n} \geq \omega_{n+1}$, since $a_{n}>a_{n+1}$ we conclude that $\hat{\varphi}\left(a_{n}\right)>\hat{\varphi}\left(a_{n+1}\right)$ and we also conclude this proof.

Theorem 2.7 Let $(X, d)$ be a metric space and let $T: X \rightarrow X$, if

- $X$ is non-trivial (contains at least two elements).
- $X$ is compact.
- $T$ is a weak contraction.

Then there exists an increasing continuous function $\varphi:[0, \infty) \rightarrow[0, \infty)$ satisfying

$$
\varphi(r)<r, \quad(\forall r>0)
$$

and such that

$$
d(T(x), T(y)) \leq \varphi(d(x, y)) . \quad(\forall x, y \in X)
$$

Proof We will assume that $\operatorname{diam}(X) \leq 1, X$ is already bounded and we could easily extend lemma 2.6 to a function $\rho:[0, d] \rightarrow[0, d]$ with $d>0$ that satisfies the same requirements but it would make the proof even more technical than it already is. We will define $\rho:[0,1] \rightarrow[0,1]$ (in general it would be $\rho:[0, \operatorname{diam}(X)] \rightarrow[0, \operatorname{diam}(X)])$ as

$$
\begin{equation*}
\rho(r)=\sup _{d(x, y) \leq r} d(T(x), T(y)) . \tag{2.5}
\end{equation*}
$$

Now fix $r>0$, there exist $x, y \in X, x \neq y$ that satisfy the two following conditions

$$
d(x, y) \leq r,
$$

and

$$
\rho(r)=d(T(x), T(y)) .
$$

This can be shown in the following way, choose $\left(x_{n}\right),\left(y_{n}\right) \subset X$ such that $d\left(x_{n}, y_{n}\right)<r$ such that

$$
d\left(T\left(x_{n}\right), T\left(y_{n}\right)\right) \rightarrow \rho(r)
$$

From the compactness of $X$ and continuity of $T$ we get the existence of subsequences of $\left(x_{n}\right)$ and $\left(y_{n}\right)$ converging to some $x$ and $y$ in $X$ with $d(T(x), T(y))=\rho(r)$ satisfying $d(x, y) \leq r$. The fact that $T$ is a weak contraction implies that

$$
\begin{equation*}
\rho(r)=d(T(x), T(y))<d(x, y) \leq r \tag{2.6}
\end{equation*}
$$

To show that $\rho$ is right continuous fix $r \in[0,1)$ and take $\left(\varepsilon_{n}\right) \subset \mathbb{R}^{+}$such that $\varepsilon_{n} \rightarrow 0$. Take $\left(x_{n}\right),\left(y_{n}\right) \subset X$ that satisfy

$$
\begin{equation*}
d\left(x_{n}, y_{n}\right) \leq r+\varepsilon_{n} \tag{2.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\rho\left(r+\varepsilon_{n}\right)=d\left(T\left(x_{n}\right), T\left(y_{n}\right)\right) . \tag{2.8}
\end{equation*}
$$

The existence of such sequences is clear from the definition of $\rho$ (2.5). Using the compactness of $X$ we can find $x, y \in X$ (not necessarily the same as $x, y$ used previously) such that $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$ up to subsequences. The function $\rho$ is clearly non-decreasing, this gives us that

$$
\rho(r) \leq \rho\left(r+\varepsilon_{n}\right) \rightarrow d(T(x), T(y)) .
$$

Furthermore sending $n \rightarrow \infty$ in (2.7) we get

$$
d(x, y) \leq r .
$$

this implies that

$$
\rho(r)=\sup _{d(\tilde{x}, \tilde{y}) \leq r} d(T(\tilde{x}), T(\tilde{y})) \geq d(T(x), T(y)) .
$$

This together with equation (2.6) gives us $\rho(r)=d(T(x), T(y))$, meaning that

$$
\rho\left(r+\varepsilon_{n}\right) \rightarrow \rho(r),
$$

showing that $\rho$ is indeed right continuous. With this we have shown that the conditions of the lemma are satisfied. Giving us the existence of a strictly increasing continuous function $\varphi:[0,1] \rightarrow[0,1]$ s.t.

$$
\begin{equation*}
\rho(r) \leq \varphi(r)<r, \quad(r \in(0,1]) \tag{2.9}
\end{equation*}
$$

This is the function we are looking for, from (2.5), (2.9) we can easily see that

$$
d(T(\tilde{x}), T(\tilde{y})) \leq \rho(d(\tilde{x}, \tilde{y})) \leq \varphi(d(\tilde{x}, \tilde{y})) . \quad(\forall \tilde{x}, \tilde{y} \in X)
$$

The function $\varphi$ can be extended for $r>1$ by setting

$$
\varphi(r)=r-1+\varphi(1) .
$$

Remark 2.5 We will demonstrate that the condition of compactness in the theorem above is necessary. Consider the weak contraction $T(x)=x+\frac{1}{x}$ on $X=[1, \infty)$. We will show that there does
not exist a function $\varphi$ as described in theorem 2.3. We are looking for a function such that

$$
\begin{equation*}
|T(x)-T(y)|=|x-y|\left(1-\frac{1}{x y}\right) \leq \varphi(|x-y|)<|x-y| \quad(\forall x, y \in[1, \infty), x \neq y) \tag{2.10}
\end{equation*}
$$

Substituting $y=x+1$ into the inequality above we get

$$
1\left(1-\frac{1}{x(x+1)}\right) \leq \varphi(1)<1 \quad(\forall x, y \in[1, \infty))
$$

Now sending $x \rightarrow \infty$ we have

$$
1 \leq \varphi(1)<1
$$

As this is impossible we conclude that no $\varphi$ satisfying (2.10) exists.

Definition 2.2 (Non-Expansive Mapping) Let $T: X \rightarrow X$ with $(X, d)$ being a metric space. We call $T$ a non-expansive map if

$$
d(T(x), T(y)) \leq d(x, y) \quad(\forall x, y \in X)
$$

We already know that a weak contraction need not have any fixed points in a complete metric space, this extends to non-expansive maps. The question is what conditions are sufficient for a non-expansive map to have a fixed point.

Definition 2.3 (Uniformly Convex Normed Space) Let $X$ be a normed linear space. If for all $\varepsilon \in(0,2]$ there exists $\delta>0$ such that for $x, y \in X$ with $\|x\|=\|y\|=1$ the condition

$$
\|x-y\| \geq \varepsilon
$$

implies

$$
\left\|\frac{x+y}{2}\right\| \leq 1-\delta .
$$

We call $X$ a uniformly convex normed linear space.

Note that the condition $\|x\|=\|y\|=1$ can be replaced with $\|x\| \leq 1$ and $\|y\| \leq 1$.

Remark 2.6 The condition of uniform convexity tells us that for any two points $x, y$ inside the unit ball separated by distance at least $\varepsilon$ the midpoint of the line between $x$ and $y$ will be at a distance of at most $1-\delta$ from the origin, this guarantees that the midpoint has distance of at least $\delta$ from the boundary of the unit ball.


Figure 2.2: Uniform convexity.
We can see that uniform convexity can be associated with "roundness" of unit balls. The Banach space $\mathbb{R}^{2}$ endowed with the maximum norm $\left\|\left(x_{1}, x_{2}\right)\right\|=\max \left\{\left|x_{1}\right|,\left|x_{2}\right|\right\}$ serves as an example of a space that is not uniformly convex. Following figure shows the boundary of the unit ball in this Banach space.


Figure 2.3: Boundary of the unit ball in $\mathbb{R}^{2}$ with the supremum norm.
We can see that it is a square and indeed for, e.g., $x=(0,1)$ and $y=\left(\frac{1}{2}, 1\right)$ satisfying $\|x\|=$ $\|y\|=1$ we have $\left\|\frac{x+y}{2}\right\|=\left\|\left(\frac{1}{4}, 1\right)\right\|=1$ showing that this space is not uniformly convex. An interesting fact to note is that this space is also an example of a reflexive space that is not uniformly convex (all finite-dimensional normed linear spaces are reflexive because they are isomorphic to $\mathbb{R}^{n}$ for some $n \in \mathbb{N}$ endowed with the Euclid norm).

Theorem 2.8 Every inner product space is uniformly convex.
Proof Every inner product space $H$ satisfies the parallelogram law, i.e., for all $x, y \in H$ it holds $\|x+y\|^{2}=2\|x\|^{2}+2\|y\|^{2}-\|x-y\|^{2}$. Now let $x, y \in H$ with $\|x\|=\|y\|=1$ and $\|x-y\| \geq \varepsilon \in(0,2]$ be given, substituting into the parallelogram law we get

$$
\|x+y\|=\sqrt{4-\|x-y\|^{2}} \leq \sqrt{4-\varepsilon^{2}}
$$

giving us

$$
\left\|\frac{x+y}{2}\right\|=\frac{\|x+y\|}{2} \leq \sqrt{1-\frac{\varepsilon^{2}}{4}}=1-\underbrace{\left(1-\sqrt{1-\frac{\varepsilon^{2}}{4}}\right)}_{\delta},
$$

thus using $\delta=\left(1-\sqrt{1-\frac{\varepsilon^{2}}{4}}\right)$ we have

$$
\left\|\frac{x+y}{2}\right\| \leq 1-\delta,
$$

where $\delta \in(0,1]$, implying the claim.

Theorem 2.9 (Milman 1938-Pettis 1939) Every uniformly convex Banach space is reflexive The proof can be found in [2].

Theorem 2.10 If $M$ is a non-empty closed convex subset of a normed linear space $X$, then $M$ is weakly closed.

We will prove that the result holds for a Hilbert space $H$, the general proof uses the Hahn-Banach theorem and can be found in [3].
Proof Suppose that $\left(x_{n}\right) \subset M$ with $x_{n} \rightharpoonup x \in H$ as $n \rightarrow \infty$, we will show that $x \in M$. Since $M$ is non-empty, closed and convex there exists a projection $P x$ of $x$ into $M$, this projection satisfies

$$
(y-P x, x-P x) \leq 0 \quad(\forall y \in M),
$$

hence

$$
\begin{equation*}
\left(x_{n}-P x, x-P x\right) \leq 0 \quad(\forall n \in \mathbb{N}) \tag{2.11}
\end{equation*}
$$

Now since an inner product is bilinear and continuous with $x_{n}-P x \rightharpoonup x-P x$ it follows that

$$
\begin{equation*}
\left(x_{n}-P x, x-P x\right) \rightarrow(x-P x, x-P x) . \tag{2.12}
\end{equation*}
$$

Finally combining (2.11) and (2.12) gives us

$$
(x-P x, x-P x) \leq 0
$$

implying that $x=P x$ which means that $x \in M$.

Theorem 2.11 (Browder-Kirk) Let $B$ be a uniformly convex Banach space and $T: C \rightarrow C a$ non expansive mapping with $C \subset B$ being closed, bounded, convex and non-empty. Then there exists a fixed point of $T$ in $C$.

We first will demonstrate a proof in the case that instead of a uniformly convex Banach space $B$ we have a Hilbert space $H$, we will later include a proof for the general case which however uses another strong result making it hard to see how exactly the assumptions guarantee us a fixed point.

Proof We will begin by defining the operator $T_{n}$ for $n=1,2, \ldots$ as

$$
T_{n}(x) \equiv\left(1-\frac{1}{n}\right) T(x)+\frac{p}{n},
$$

where $p \in C$ is fixed. Since $C$ is convex, we have $T_{n}: C \rightarrow C$. It also holds

$$
\left\|T_{n}(x)-T_{n}(y)\right\|=\left(1-\frac{1}{n}\right)\|T(x)-T(y)\| \leq\left(1-\frac{1}{n}\right)\|x-y\|,
$$

showing that $T_{n}$ is a contraction, hence by the Banach fixed point theorem for all $n \in \mathbb{N}$ there exists a $x_{n} \in C$ with $x_{n}=T_{n}\left(x_{n}\right)$. As a Hilbert space $H$ is reflexive, this gives us that the bounded sequence $\left(x_{n}\right) \subset C$ has a subsequence which we shall again denote $\left(x_{n}\right)$ that weakly converges to some $\bar{x}$. By theorem $2.10 \bar{x} \in C$. Let us show that $\bar{x}$ is a fixed point of $T$.

$$
\left.\left\|T(\bar{x})-x_{n}\right\|^{2}-\left\|\bar{x}-x_{n}\right\|^{2}=\|T(\bar{x})\|^{2}-2\left(T(\bar{x}), x_{n}\right)+\left\|x_{n}\right\|^{2}-\left(\|\bar{x}\|^{2}-2\left(x_{n}, \bar{x}\right)+\left\|x_{n}\right\|^{2}\right)\right) .
$$

Notice that $x_{n} \rightharpoonup \bar{x}$ implies $\left(T(\bar{x}), x_{n}\right) \rightarrow(T(\bar{x}), \bar{x})$ and $\left(x_{n}, \bar{x}\right) \rightarrow(\bar{x}, \bar{x})$, this means that

$$
\begin{equation*}
\left\|T(\bar{x})-x_{n}\right\|^{2}-\left\|\bar{x}-x_{n}\right\|^{2} \rightarrow\|T(\bar{x})\|^{2}-2(\bar{x}, T(\bar{x}))+\|\bar{x}\|^{2}=\|T(\bar{x})-\bar{x}\|^{2}, \tag{2.13}
\end{equation*}
$$

as $n \rightarrow \infty$. We also have

$$
\begin{aligned}
\left\|T(\bar{x})-x_{n}\right\| & \leq\left\|T(\bar{x})-T\left(x_{n}\right)\right\|+\left\|T\left(x_{n}\right)-x_{n}\right\| \leq\left\|\bar{x}-x_{n}\right\|+\left\|T\left(x_{n}\right)-x_{n}\right\| \\
& =\left\|\bar{x}-x_{n}\right\|+\frac{1}{n}\left\|T\left(x_{n}\right)-p\right\|,
\end{aligned}
$$

where the first inequality is a triangle inequality, the second inequality comes from $T$ being a nonexpansive mapping and we can obtain the equality on the last line by using $x_{n}=T_{n}\left(x_{n}\right)$. We can also write this as

$$
\left\|T(\bar{x})-x_{n}\right\|-\left\|\bar{x}-x_{n}\right\| \leq \frac{1}{n}\left\|T\left(x_{n}\right)-p\right\|,
$$

multiplying both sides of the last inequality by $\left(\left\|T(\bar{x})-x_{n}\right\|+\left\|\bar{x}-x_{n}\right\|\right)$ we get

$$
\begin{aligned}
\left\|T(\bar{x})-x_{n}\right\|^{2}-\left\|\bar{x}-x_{n}\right\|^{2} & \leq \frac{1}{n}\left\|T\left(x_{n}\right)-p\right\|\left(\left\|T(\bar{x})-x_{n}\right\|+\left\|\bar{x}-x_{n}\right\|\right) \\
& \leq \frac{1}{n}\left(\left\|T\left(x_{n}\right)\right\|+\|p\|\right)\left(\|T(\bar{x})\|+\left\|x_{n}\right\|+\|\bar{x}\|+\left\|x_{n}\right\|\right) .
\end{aligned}
$$

Clearly $T\left(x_{n}\right)$ and $x_{n}$ are both members of the bounded set $C$, so sending $n \rightarrow \infty$ while using (2.13) on the left side yields

$$
\|T(\bar{x})-\bar{x}\|^{2} \leq 0
$$

implying that

$$
T(\bar{x})=\bar{x} .
$$

Now we will introduce a new concept and a result regarding it which will be used to prove the general version of theorem ??.

Definition 2.4 Let $B$ be a Banach space. The operator $T: S \subset B \rightarrow B$ is called demiclosed if for every sequence $\left(x_{n}\right) \subset S$ the conditions

$$
x_{n} \rightharpoonup x \in B, \quad T\left(x_{n}\right) \rightarrow y \in B
$$

imply that $x \in S$ and

$$
T(x)=y .
$$

Theorem 2.12 The following three conditions imply that the operator $I-T: S \rightarrow B$, i.e., ( $I-$ $T)(x)=x-T(x)$ is demiclosed

- $T: S \subset B \rightarrow B$ is a non-expansive operator.
- $B$ is a uniformly convex Banach space.
- $S$ is closed, bounded and convex.

The proof of this theorem is rather technical and be found in [4].
Now we are ready to fully prove theorem 2.11.
Proof This proof will take the same direction as the one for Hilbert spaces, theorem 2.12 will in some sense replace the tricks with inner products we used there.

Again we will define the operator $T_{n}: C \rightarrow C$ for $n=1,2, \ldots$ as

$$
T_{n}(x)=\left(1-\frac{1}{n}\right) T x+\frac{p}{n},
$$

where $p \in C$ is fixed. As we have already shown $T_{n}$ is a contraction, hence by the Banach fixed point theorem for all $n \in \mathbb{N}$ there exists a $x_{n} \in C$ with $x_{n}=T_{n}\left(x_{n}\right)$.

From theorem 2.9 we know that $B$ is reflexive, this gives us that the bounded sequence $\left(x_{n}\right) \subset C$ has a weakly convergent subsequence that we shall again denote as $\left(x_{n}\right)$ with

$$
\begin{equation*}
x_{n} \rightharpoonup \bar{x} \in B \quad \text { as } n \rightarrow \infty \tag{2.14}
\end{equation*}
$$

Since $x_{n}$ is a fixed point of $T_{n}$, we also have

$$
x_{n}-\left(1-\frac{1}{n}\right) T\left(x_{n}\right)-\frac{p}{n}=0,
$$

meaning that

$$
\begin{equation*}
x_{n}-T\left(x_{n}\right) \rightarrow 0 \quad \text { as } n \rightarrow \infty \tag{2.15}
\end{equation*}
$$

Using nothing but the assumptions of this theorem we can see that by theorem 2.12, the operator $I-T$ is demiclosed, hence (2.14) and (2.15) imply that $\bar{x} \in C$ and $\bar{x}=T(\bar{x})$.

Remark 2.7 Let us show that none of the assumptions of theorem 2.11 can be omitted

- Starting with the trivial ones, if $C=\mathbb{R}^{n}$ is not bounded, the non-expansive mapping $T(x)=$ $x+a$ with $0 \neq a \in \mathbb{R}^{n}$ will have no fixed points.
- If $C=\overline{B_{\mathbb{R}^{2}}(0,1)} \backslash B_{\mathbb{R}^{2}}\left(0, \frac{1}{2}\right)$ is not convex but is compact, the non-expansive mapping $T$ that rotates every point around the origin say by $\pi / 2$ does not have a fixed point in $C$.
- Next taking the open set $C=B_{1}(0) \subset \mathbb{R}^{n}$ we can use the mapping $T(x)=\frac{x+(1,0, \ldots, 0)}{2}$ that shifts points halfway towards a point on the boundary which has no fixed points in C. Mapping $T$ is even a contraction.
- Take $\ell^{2}$, the space of sequences satisfying $\sum_{i=0}^{\infty}\left|x_{i}\right|^{2}<\infty$ endowed with the norm $\|x\|=$ $\left(\sum_{i=0}^{\infty}\left|x_{i}\right|^{2}\right)^{1 / 2}$. Let $\varepsilon \in(0,1]$ and define $T_{\varepsilon}: \overline{B_{\ell^{2}}(0,1)} \rightarrow \overline{B_{\ell^{2}}(0,1)}$ as

$$
T_{\varepsilon}(x)=\left(\varepsilon(1-\|x\|), x_{0}, x_{1}, \ldots\right)
$$

$T_{\varepsilon}$ maps the closed unit ball to itself because we have

$$
\left\|T_{\varepsilon}(x)\right\|=\left([\varepsilon(1-\|x\|)]^{2}+\|x\|^{2}\right)^{1 / 2} \leq\left((\varepsilon(1-\|x\|)+\|x\|)^{2}\right)^{1 / 2}=\varepsilon(1-\|x\|)+\|x\|=\varepsilon+\|x\|(1-\varepsilon),
$$

and for $\|x\| \leq 1$ this turns to $\left\|T_{\varepsilon}(x)\right\| \leq 1$. We also have

$$
\left\|T_{\varepsilon}(x)-T_{\varepsilon}(y)\right\|^{2}=(\varepsilon(\|y\|-\|x\|))^{2}+\|x-y\|^{2} \leq(\varepsilon(\|x-y\|))^{2}+\|x-y\|^{2},
$$

giving us that $\left\|T_{\varepsilon}(x)-T_{\varepsilon}(y)\right\| \leq \sqrt{1+\varepsilon^{2}}\|x-y\|$. In summary $\bar{B}_{l^{2}}(0,1)$ is a non-empty bounded closed convex subset of a Hilbert space (Hilbert spaces are uniformly convex) and $T_{\varepsilon}$ is Lipschitz continuous with a Lipschitz constant arbitrarly close to 1 but still greater and $T_{\varepsilon}$ has no fixed points in $\bar{B}_{l^{2}}(0,1)$.

- Next we shall investigate if the assumption of uniform convexity is necessary. Let $X$ be the set of sequences of real numbers vanishing at infinity endowed with the supremum norm, such $X$ is a Banach space, define $C=\bar{B}_{X}(0,1)$. Then the mapping $T(x)=\left(1, x_{0}, x_{1}, \ldots\right)$ from $C$ to $C$ where $T$ is clearly non-expansive (it preserves distances in fact) has no fixed points.
- Finally let us break the assumption of completeness. Let $X$ be the vector space of all sequences with a finite number of non-zero elements endowed with the norm $\left\|\left(x_{0}, x_{1}, \ldots\right)\right\|=$ $\left(\sum_{i=0}^{\infty}\left|x_{n}\right|^{2}\right)^{\frac{1}{2}}$ and define the operator $T$ as

$$
T\left(x_{0}, x_{1}, \ldots\right)=\left(\frac{1}{2}, \frac{x_{0}}{2}, \frac{x_{1}}{2}, \ldots\right)
$$

it holds $T: \overline{B_{X}(0,1)} \rightarrow \overline{B_{X}(0,1)}$ since for all $x \in \overline{B_{X}(0,1)}$ it holds

$$
\left\|T\left(x_{0}, x_{1}, \ldots\right)\right\|=\left(\frac{1}{4}+\sum_{i=0}^{\infty}\left|\frac{x_{i}}{2}\right|^{2}\right)^{\frac{1}{2}} \leq\left(\frac{1}{2}\right)^{\frac{1}{2}} \leq 1 .
$$

$X$ is a uniformly convex space since it is an inner product space and the set $\bar{B}_{X}(0,1)$ is non-empty, closed, bounded and convex, however $T$ has no fixed points in $\bar{B}_{X}(0,1)$.

### 2.2 Brouwer Fixed Point Theorem and Mapping Degree

The Brouwer fixed point theorem gives us a fixed point property for continuous mappings from convex compacts in $\mathbb{R}^{n}$ onto themselves. Over the years many proofs of this theorem have been put forward using various fields of mathematics, in this chapter we will demonstrate two such ways one using mathematical analysis and one using graph theory coupled with mathematical analysis. First we will prove this theorem by using the Brouwer mapping degree theory.

Brouwer mapping degree is motivated by counting the number of solutions $f(x)=y$ inside a domain $U$. In complex analysis we can define such a degree for holomorphic functions using the concept of a winding number. We define a winding number of a closed path $\gamma:[0,1] \rightarrow \mathbb{C}$ around a point $z_{0} \in \mathbb{C}$ by

$$
n\left(\gamma, z_{0}\right)=\frac{1}{2 \pi i} \int_{\gamma} \frac{\mathrm{d} z}{z-z_{0}} .
$$

Assuming $U$ is a simply connected domain $n\left(\gamma, z_{0}\right)$ gives us the number of times $\gamma$ encircles $z_{0}$ with respect to orientation of $\gamma$. Further more for holomorphic $f$ we have

$$
n(f(\langle\gamma\rangle), 0)=\frac{1}{2 \pi i} \int_{\gamma} \frac{f^{\prime}(z)}{f(z)} \mathrm{d} z=\sum_{k} n\left(\gamma, z_{k}\right) \alpha_{k},
$$

where $\langle\gamma\rangle$ denotes the image of $\gamma$, i.e., $\langle\gamma\rangle=\{\gamma(z): z$ is in the domain of $\gamma\}$. Each point $z_{k}$ is zero of $f$ in $U$ and $\alpha_{k}$ its multiplicity. If $\gamma$ is a positively oriented Jordan curve, then $n\left(\gamma, z_{k}\right)=1$ if $z_{k} \in \operatorname{int}(\gamma)$. With this we can $\operatorname{define} \operatorname{deg}(f, U, 0)=n(f(\langle\gamma\rangle), 0)$, where $\operatorname{int}(\gamma)=U$ which counts the number of zeros inside $U$. We can set $\operatorname{deg}\left(f, U, z_{0}\right)=n\left(f(\langle\gamma\rangle)-z_{0}, 0\right)$ if $z_{0} \notin\langle\gamma\rangle$.

While this result is interesting it does not give us much in terms of having a good way to compute this degree without knowing the zeros of $f$. This is where homotopy invariance comes in. If we can find a homotopy $H:[0,1] \times\langle\gamma\rangle \rightarrow \mathbb{C} \backslash 0$ between $f:\langle\gamma\rangle \rightarrow \mathbb{C} \backslash\{0\}$ and $g:\langle\gamma\rangle \rightarrow \mathbb{C} \backslash\{0\}$, i.e., continuous $H$ s.t. $H(0, z)=f(z)$ and $H(1, z)=g(z)$ for all $x \in\langle\gamma\rangle$ then $n(f(\gamma), 0)=n(g(\gamma), 0)$. Brouwer mapping degree extends this concept for continuous functions from $\mathbb{R}^{n}$ to $\mathbb{R}^{n}$.

Definition 2.5 Let $D_{y}^{r}\left(\bar{U}, \mathbb{R}^{n}\right)=\left\{f \in C^{r}\left(\bar{U}, \mathbb{R}^{n}\right): y \notin f(\partial U)\right\}$ and let deg be a function that for each triplet $f \in D_{y}=D_{y}^{0}$, a bounded open set $U \subset \mathbb{R}^{n}$ and $y \in \mathbb{R}^{n}$ assigns a real number $\operatorname{deg}(f, U, y)$. We call deg a degree if

$$
C 1 \operatorname{deg}(f, U, y)=\operatorname{deg}(f-y, U, 0) .
$$

C2 $\operatorname{deg}(\operatorname{Id}, U, y)=1$ if $y \in U$.
C3 For two disjoint open sets $U_{1}, U_{2} \subset U$ with $y \notin f\left(\bar{U} \backslash\left(U_{1} \cup U_{2}\right)\right)$ it holds $\operatorname{deg}(f, U, y)=$ $\operatorname{deg}\left(f, U_{1}, y\right)+\operatorname{deg}\left(f, U_{2}, y\right)$.

C4 If $H(t)=(1-t) f+t g \in D_{y}\left(\bar{U}, \mathbb{R}^{n}\right), t \in[0,1]$, then $\operatorname{deg}(f, U, y)=\operatorname{deg}(g, U, y)$

Theorem 2.13 The following properties can be derived from the definition of degree given $f, g \in$ $D_{y}\left(\bar{U}, \mathbb{R}^{n}\right)$

- $\operatorname{deg}(f, \emptyset, y)=0$.
- If $U_{i}, 1 \leq i \leq N$ are disjoint open subsets of $U$ with $y \notin f\left(\bar{U} \backslash \cup_{i=1}^{N} U_{i}\right)$, then $\operatorname{deg}(f, U, y)=$ $\sum_{i=1}^{N} \operatorname{deg}\left(f, U_{i}, y\right)$.
- If $y \notin f(U)$, then $\operatorname{deg}(f, U, y)=0$. Also if $\operatorname{deg}(f, U, y) \neq 0$, then $y \in f(U)$.
- If for all $x \in \partial U$ it holds $\|f(x)-g(x)\|<d(y, f(\partial U))$, then $\operatorname{deg}(f, U, y)=\operatorname{deg}(g, U, y)$.

Proof To prove the fist property take $U_{1}=U$ and $U_{2}=\emptyset$ in C3. The second property follows from C 3 inductively. The third property is a consequence of taking $U_{1}=\emptyset, U_{2}=\emptyset$ in C3. Now let us
prove the last property. For all $x \in \partial U$ it holds

$$
d(y, f(\partial U)) \leq\|y-f(x)\| \leq\|H(t, x)-y\|+\|H(t, x)-f(x)\| \leq\|H(t, x)-y\|+\|g(x)-f(x)\|
$$

giving us

$$
\|H(t, x)-y\| \geq d(y, f(\partial U))-\|f(x)-g(x)\| .
$$

From the assumptions we can see that $\|H(t, x)-y\|>0$ hence by $\mathrm{C} 4 \operatorname{deg}(f, U, y)=\operatorname{deg}(g, U, y)$.

Theorem 2.14 There exists a unique degree that satisfies C1-C4. Moreover, $\operatorname{deg}(., U, y): D_{y}\left(\bar{U}, \mathbb{R}^{n}\right) \rightarrow$ $\mathbb{Z}$ is constant on each component. For $f \in D_{y}\left(\bar{U}, \mathbb{R}^{n}\right)$ we have

$$
\operatorname{deg}(f, U, y)=\sum_{x \in f^{-1}(y)} \operatorname{sgn} \operatorname{det} \tilde{f}^{\prime}(x),
$$

where $\tilde{f} \in D_{y}^{2}\left(\bar{U}, \mathbb{R}^{n}\right)$ with $|f-\tilde{f}|<d(y, f(\partial U))$ so that $\forall x \in \tilde{f}^{-1}(y): \operatorname{det} \tilde{f}^{\prime}(x) \neq 0$.
The proof of this theorem can be found in [5]. The process of developing this explicit formulation of the degree is quite intricate. We need to differentiate between regular values and critical values of $f$, we can define the set of regular values as $\operatorname{RV}(f)=\left\{y \in \mathbb{R}^{n} \mid \forall x \in f^{-1}(y): \operatorname{det} f^{\prime}(x) \neq 0\right\}$ and the set of critical values $\mathrm{CV}(f)$ as its complement in $\mathbb{R}^{n}$. It can be shown that given $f \in D_{y}^{1}\left(\bar{U}, \mathbb{R}^{n}\right)$ and $y \notin \mathrm{CV}(f)$ a degree exists in the form of

$$
\operatorname{deg}(f, U, y)=\sum_{x \in f^{-1}(y)} \operatorname{sgn} J_{f}(x),
$$

where the sum is finite and we set $\sum_{x \in \emptyset}=0$. This equality holds in particular because when we without restriction consider $y=0$ we have $f^{-1}(y)=\left\{x_{1}, \ldots, x_{N}\right\}$, avoiding the trial case where $f^{-1}(y)=\emptyset$. Picking neighborhoods $U\left(x_{i}\right)$ around $x_{i}$ small enough we get

$$
\operatorname{deg}(f, U, 0)=\sum_{i=1}^{N} \operatorname{deg}\left(f, U\left(x_{i}, 0\right)\right)
$$

We can substitute $U\left(x_{i}\right)$ with $B_{\mathbb{R}^{n}}\left(x_{i}, \delta\right)$. Focusing on a single one of these $x_{i}$ and assuming $x_{i}=0$ we have $\operatorname{deg}\left(f, B_{\delta}(0), 0\right)=\operatorname{deg}\left(f^{\prime}(0), B_{\delta}(0), 0\right)$ for $\delta$ small enough (here we understand $f^{\prime}(0)$ as the linear mapping $\left.x \mapsto f^{\prime}(0) \cdot x\right)$, this is due to homotopy invariance. This allows us to work with matrices as opposed to functions. What is interesting about matrices is that two full rank $n \times n$ matrices $M_{1}$ and $M_{2}$ are homotopic iff sgn $\operatorname{det} M_{1}=\operatorname{sgn} \operatorname{det} M_{2}$ and in fact $\operatorname{deg}\left(M, B_{\delta}(0), 0\right)=\operatorname{sgn} \operatorname{det} M$ for an invertable $n \times n$ matrix $M$. The real tricky part of proving the theorem above is extending this formula to all $f \in D_{y}\left(\bar{U}, \mathbb{R}^{n}\right)$ and all $y \in \mathbb{R}^{n}$ (with $y \notin f(\partial U)$ ). Without going into too many details, a big win is that the set of regular values of $f \in C^{1}\left(U, \mathbb{R}^{n}\right)$ is dense in $\mathbb{R}^{n}$. We first run into problems admitting critical values which are resolved by making use of the fact that $f \in C^{2}\left(\bar{U}, \mathbb{R}^{n}\right)$
the map $\operatorname{deg}(f, U,):. \mathbb{R}^{n} \rightarrow \mathbb{Z}$ is locally constant with the exception of $f(\partial U)$, so we define

$$
\operatorname{deg}(f, U, y)=\operatorname{deg}(f, U, \tilde{y}), \quad y \notin f(\partial U), f \in C^{2}\left(\bar{U}, \mathbb{R}^{n}\right)
$$

where $\tilde{y} \in R V(f)$ with $|\tilde{y}-y|<d(y, f(\partial U))$. This gives us more problems because now we have to deal with the condition $f \in C^{2}$. As a matter of fact the way this is resolved can be observed in the formulation of the theorem above.

In the proof of the Brouwer fixed point theorem we will require the following result from topology. Afterwards we will have everything we need.

Theorem 2.15 Let $X$ and $Y$ be Banach spaces and let $K$ be a closed subset of $X$. If $F \in C(K, Y)$ then there exists $\tilde{F} \in C(X, Y)$ with $\tilde{F}(x)=F(x)$ for all $x \in K$. Furthermore $\tilde{F}(X) \subset \operatorname{conv}(F(K))$.

The proof can be found in [5].
Theorem 2.16 (Brouwer fixed point in $\mathbb{R}^{n}$ ) Let $K$ be a non-empty compact convex subset of $\mathbb{R}^{n}$ for some $n \in \mathbb{N}$. Then every continuous mapping $f: K \rightarrow K$ has a fixed point.

Proof Let us first prove the theorem for the case where $K=B_{\mathbb{R}^{n}}(0, r)$ for some $r>0$. If there is a fixed point on $\partial B_{\mathbb{R}^{n}}(0, r)$ the statement is true, otherwise set $H(t, x)=x-t f(x), t \in[0,1]$. We will show that $H(t) \in D_{0}\left(\bar{K}, \mathbb{R}^{n}\right)$ for all $t \in[0,1)$, for all $x \in \partial K$ we have

$$
\|H(t, x)\| \geq\|x\|-t\|f(x)\| \geq(1-t) r>0, \quad(0 \leq t<1)
$$

and the possibility that $H(1, x)=0$ has already been ruled out since that would imply that $f$ has a fixed point on $\partial B_{\mathbb{R}^{n}}(0, r)$. Hence $\operatorname{deg}\left(x-f(x), B_{\mathbb{R}^{n}}(0, r), 0\right)=\operatorname{deg}\left(x, B_{\mathbb{R}^{n}}(0, r), 0\right)=1$. The claim follows from property 3 in theorem 2.13. Let us move on to the general case. Clearly $K \subset \overline{B_{\mathbb{R}^{n}}(0, r)}$ for some $r>0$. Theorem 2.15 gives the existence of a continuous retraction $R: \mathbb{R}^{n} \rightarrow \operatorname{conv}(K)=K$, i.e., $R(x)=x$ for all $x \in K$. Set $\tilde{f}=f \circ R$, then $\tilde{f}: \mathbb{R}^{n} \rightarrow K$, clearly $\tilde{f}$ is a continuous function from $\overline{B_{\mathbb{R}^{n}}(0, r)}$ into itself. Our previously shown claim gives us a fixed point $\bar{x} \in \overline{B_{\mathbb{R}^{n}}(0, r)}$ of $\tilde{f}$. Since range $(\tilde{f}) \subset K$, then $\bar{x}$ must lie in $K$.

Theorem 2.17 (Brouwer fixed point) Let $K$ be a non-empty compact convex subset of a finite dimensional normed linear space $X$. Then every continuous function $f: K \rightarrow K$ has a fixed point.

Proof Let $n=\operatorname{dim} X$, then there exists an isomorphism $L: X \rightarrow \mathbb{R}^{n}$, i.e., $L$ is linear, $L$ is a bijection and there exist $k, K>0$ such that $\forall x \in X$ it holds $k\|L x\| \leq\|x\|_{X} \leq K\|L x\|$, where $\|L x\|$ is the Euclidian norm. We claim that if $L(K)$ is a non-empty compact convex subset of $\mathbb{R}^{n}$ and that if $\bar{y} \in L(K)$ is a fixed point of $g: L(K) \rightarrow L(K)$ defined as $g=L \circ f \circ L^{-1}$ in $L(K)$, then $\bar{x}=L^{-1} \bar{y}$ is a fixed point of $f$ in $K$, moreover $g$ is continuous. Let us go through these one by one

- Since $K$ is non-empty $L(K)$ is also non-empty.
- $L(K)$ is convex: Let $y_{1}, y_{2} \in L(K)$ and let $y=(1-t) y_{1}+t y_{2}$ for some $t \in[0,1]$, we have

$$
y=(1-t) L x_{1}+t L x_{2}
$$

for some $x_{1}, x_{2} \in K$, since $L$ is linear and $K$ is convex it holds

$$
y=L\left((1-t) x_{1}+t x_{2}\right) \in L(K) .
$$

- $L(K)$ is compact: We have $k\|L x\| \leq\|x\|_{X}$ meaning $L$ is bounded and since it is also linear we know that $L$ is continuous. Now let $\left(y_{n}\right) \subset L(K)$, there exists $x_{n} \in K$ for all $n \in \mathbb{N}$ such that $y_{n}=L x_{n}$, because $K$ is compact there exists a subsequence $\left(x_{n_{k}}\right)$ of $\left(x_{n}\right)$ that converges to some $x \in K$. From continuity of $L$ we get

$$
y_{n_{k}}=L x_{n_{k}} \rightarrow L(x) \in L(K),
$$

showing that $\left(y_{n}\right)$ has a subsequence that converges to a point in $L(K)$.

- If $\bar{y}$ is a fixed point of $g: L(K) \rightarrow L(K)$ defined as $g=L \circ f \circ L^{-1}$ and we define $\bar{x}=L^{-1}(\bar{y})$. Then

$$
\bar{y}=g(\bar{y})=L\left(f\left(L^{-1}(\bar{y})\right)\right)=L(f(\bar{x})),
$$

applying $L^{-1}$ to both sides we get

$$
L^{-1} \bar{y}=f(\bar{x})
$$

hence $\bar{x}=f(\bar{x})$.

- Continuity of $g$ : Function $g$ is a composition of functions $L, f$ and $L^{-1}$. We have already shown that $L$ is continuous and continuity of $f$ is one of the assumption of the theorem, all we need to do is show that $L^{-1}$ is continuous. First let us note that since $L$ is one-to-one we have $L\left(L^{-1}(x)\right)=L^{-1}(L(x))=x$. For $\alpha \in \mathbb{R}$ and $y_{1}, y_{2} \in \mathbb{R}^{n}$ it holds

$$
L^{-1}\left(\alpha y_{1}\right)+L^{-1}\left(\alpha y_{2}\right)=L^{-1}\left(L\left(L^{-1}\left(\alpha y_{1}\right)+L^{-1}\left(\alpha y_{2}\right)\right)\right)=L^{-1}\left(\left(\alpha y_{1}\right)+\left(\alpha y_{2}\right)\right)
$$

meaning that $L^{-1}$ is linear. Now let $y \in \mathbb{R}^{n}$, since $L$ is onto we can write $y=L(x)$ for some $x \in X$. We have

$$
k\|L x\| \geq\|x\|_{X}=\left\|L^{-1}(L(x))\right\|
$$

by substituting $y=L x$ we obtain

$$
k\|y\| \geq\left\|L^{-1} y\right\|,
$$

where $y \in \mathbb{R}^{n}$ was chosen arbitrarily, so $L^{-1}$ is bounded and hence continuous.

By the Brouwer fixed point theorem in $\mathbb{R}^{n}$, the function $g$ has a fixed point $\bar{y}$ in $L(K)$, by our earlier considerations $\bar{x}=L^{-1}(\bar{y}) \in K$ is a fixed point of $f$.

Remark 2.8 All the assumptions of this theorem are necessary, all of these assumptions except a few that we will cover shortly were already discussed in the part regarding theorem 2.11. We will not be looking into the condition of finite-dimensionality since it will be explored in the next section and we will see that if $K$ is a non-empty closed bounded convex subset of a normed linear space $X$, then a continuous mapping from $X$ to $X$ need not have fixed points. As an example of a discontinuous function that has no fixed points over a non-empty compact convex set $\bar{B}_{\mathbb{R}^{2}}(0,1)$ we can use mapping $T$ that rotates all points around the origin by say $\pi$ where we define $T(0,0)=(0,1)$. However a question still remains whether this theorem could be extended to infinite-dimensional Banach spaces and the answer will be given in the next section regarding the Schauder fixed point theorem.

While the Brouwer degree is a powerful tool for finding out whether a function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ has a fixed point in a given region, this proof of the Brouwer fixed point theorem is not very telling of how exactly the assumptions guarantee us a fixed point, so we will also include an elementary proof albeit in the case that $f: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$, this proof will make use of the Sperner lemma from graph theory.

Theorem 2.18 (Sperner Lemma) Consider the following situation, we are given a triangle with vertices $A_{1}, A_{2}, A_{3}$ and perform a division (i.e., triangulation) of the initial triangle, splitting it into $n$ triangles, denote the set of these triangles as $D$ and denote the set of vertices of triangles in $D$ as $V$. Now with each vertex in $V$ we associate an integer ranging from 1 to 3 in accordance with these rules:

- If $v \in V$ lies on the side $A_{i} A_{j}$ of the initial triangle then $v$ can only be assigned numbers $i$ or $j$.
- If $v \in V$ does not lie on a side of the initial triangle then any number from the set $\{1,2,3\}$ can be assigned.
(Note that these rules imply that the vertex $A_{i}$ is numbered $i$ ).


Figure 2.4: Vertex numbering.

The Sperner lemma states that regardless of the used division and specific choice of numbering there always exists a triangle in $D$ with vertices numbered $1,2,3$.

Proof We will define a graph $G$ in the following way:

- Each vertex of $G$ will represent a face of the triangulation $D$, this includes a vertex for the exterior face $v$, i.e., one of the vertices represents the outside of the triangle. Because of this association we will be referring to triangles in $D$ and the exterior face as if they were vertices of $G$.
- Two faces of $D$ will be connected by an edge if the side that separates them has end points numbered 1 and 2 , it does not matter in which order. This is also the case with the exterior face $v$, it will be connected to triangles that have a side with end points numbered 1 and 2 provided this side overlaps with a side of the initial triangle.


Figure 2.5: Graph $G$.

Consider the degree of each triangle in $D$ :

- The degree is non-zero if and only if one of its vertices is numbered 1 and another one is numbered 2.
- If the degree is non-zero and if the last vertex is also numbered 1 or 2 , the triangle has degree 2.
- If the degree is non-zero and if the last vertex is numbered 3 , the triangle has degree 1 .

Notice that no other options can occur, so a triangle has odd degree if and only if it has vertices numbered 1, 2 and 3 .

Let us show that the exterior face $v$ has odd degree. Clearly edges with an end point in $v$ can only cross the side $A_{1} A_{2}$ of the initial triangle, the vertex $A_{1}$ must be numbered 1 and similarly $A_{2}$ must be numbered 2 . If there are other triangle vertices lying on the side $A_{1} A_{2}$, they are numbered 1 or 2 , so we are looking to show that given a string of numbers 1,2 beginning with 1 and ending with 2 contains an odd number of times there is a switch from 1 to 2 or from 2 to 1 . The string will begin with a sequence of 1 s until there is a switch to a sequence of 2 s , this gives us one switch which must always be there, if there is another switch to 1 the string cannot end until a switch back to 2 is made giving us a total of 3 switches, now we can see that the string cannot end on an even number of switches since that would mean that the string cannot end with number 1.

The so called first theorem of graph theory states that the sum of degrees of all vertices in a graph is equal to twice the number of edges, so this sum must be an even number, so there must be at least one triangle in $D$ with an odd degree, by our previous considerations there exists a triangle
with vertices numbered 1,2 and 3 , not only that, we are also guaranteed that the number of such triangles must always be odd, as a side note, this observation is key to proving the Sperner lemma for higher-dimensional simplices.

Theorem 2.19 (Brouwer fixed point in $\mathbb{R}^{2}$ ) Let $K$ be a non-empty compact convex subset of $\mathbb{R}^{2}$ and let $f: K \rightarrow K$ be continuous, then there exists a fixed point of $f$ in $K$.

Proof First let us show that this holds if $f$ maps the triangle $\triangle$ given by vertices $(0,1)=A_{1}$, $(1,0)=A_{2}$, and $(0,0)=A_{3}$ into itself. We will define functions $\alpha_{1}, \alpha_{2}, \alpha_{3}: \triangle \rightarrow[0,1]$ to map $x=\left(x_{1}, x_{2}\right) \in \triangle$ as follows

$$
\alpha_{1}(x)=x_{2}, \quad \alpha_{2}(x)=x_{1}, \quad \alpha_{3}(x)=1-x_{2}-x_{1} .
$$

If we were to write $x$ as a convex combination of vertices of $\triangle$, i.e., $x=\sum_{i=1}^{3} \lambda_{i} A_{i}$ where $\lambda_{i}$ satisfy $\sum_{i=1}^{3} \lambda_{i}=1$ and $\lambda_{i} \geq 0$ for $i=1,2,3$, then $\alpha_{i}(x)=\lambda_{i}$, these functions are referred to as barycentric coordinates. Now we will define the sets $M_{1}, M_{2}, M_{3}$ so that

$$
M_{i}=\left\{x \in \triangle: \alpha_{i}(x) \geq \alpha_{i}(f(x))\right\} . \quad(i \in\{1,2,3\})
$$

Notice that if $x \in M_{i}$ for some $i \in\{1,2,3\}$, then $f$ does not shift $x$ any further from the side of $\triangle$ opposite to $A_{i}$. Suppose that we have $\bar{x} \in M_{1} \cap M_{2} \cap M_{3}$, then by definition of $M_{i}$ it holds

$$
\begin{equation*}
\alpha_{i}(f(\bar{x})) \leq \alpha_{i}(\bar{x}), \quad(\forall i \in\{1,2,3\}) \tag{2.16}
\end{equation*}
$$

we also have

$$
\begin{equation*}
\alpha_{1}(x)+\alpha_{2}(x)+\alpha_{3}(x)=1, \quad(\forall x \in \triangle) \tag{2.17}
\end{equation*}
$$

Equations (2.16), (2.17) give us

$$
1=\alpha_{1}(f(\bar{x}))+\alpha_{2}(f(\bar{x}))+\alpha_{3}(f(\bar{x})) \leq \alpha_{1}(\bar{x})+\alpha_{2}(\bar{x})+\alpha_{3}(\bar{x})=1 .
$$

The only way this can be satisfied is if for each $i \in\{1,2,3\}$ we have $\alpha_{i}(f(\bar{x}))=\alpha_{i}(\bar{x})$. Remember that the functions $\alpha_{i}, i \in\{1,2,3\}$ have the property that

$$
x=\sum_{i=1}^{3} \alpha_{i}(x) A_{i}, \quad(\forall i \in\{1,2,3\})(\forall x \in \triangle)
$$

hence the relation $(\forall i \in\{1,2,3\}): \alpha_{i}(f(\bar{x}))=\alpha_{i}(\bar{x})$ implies that $f(\bar{x})=\bar{x}$.
Next we will show that such $\bar{x}$ exists. Let $\left(D_{n}\right)$ be a sequence of divisions of $\triangle$ into smaller triangles such that $\max _{T \in D_{n}} \operatorname{diam}(T) \rightarrow 0$ as $n \rightarrow \infty$, such exists, applying any division that splits the triangle into somewhat equal-sized pieces recursively will do, see for example the barycentric
simplicial divison [6]. We will number the vertices of all triangles in $D_{n}$ according to the rules proposed in the Sperner Lemma and we will also require that if a triangle vertex is numbered $i$, then the vertex is a member of $M_{i}$, let us see that this is indeed possible. Starting with the vertices of $\triangle$ themselves, $A_{1}=(0,1)$, so $\alpha_{1}\left(A_{1}\right)=1$ which is the maximum value $\alpha_{1}$ can attain, so automatically $A_{1} \in M_{1}$, similarly $\alpha_{2}\left(A_{2}\right)=1$, so $A_{2} \in M_{2}$, lastly $\alpha_{3}\left(A_{3}\right)=1$, so $A_{3} \in M_{3}$.

Moving onto sides of $\triangle$, if $x \in \triangle$ lies on the side $A_{i} A_{j}$ where $i, j \in\{1,2,3\}, i \neq j$, let $k \in\{1,2,3\}$ satisfy $k \neq i, k \neq j$. Since $x$ can be written as a convex combination of vertices $A_{i}, A_{j}$ and since $x$ can be uniquely written as

$$
x=\alpha_{i}(x) A_{i}+\alpha_{j}(x) A_{j}+\alpha_{k}(x) A_{k},
$$

we have $\alpha_{k}(x)=0$, this implies that

$$
\begin{equation*}
\alpha_{i}(x)+\alpha_{j}(x)=1 . \tag{2.18}
\end{equation*}
$$

Suppose that $x \notin M_{i}$ and $x \notin M_{j}$, this means that

$$
\begin{aligned}
& \alpha_{i}(x)<\alpha_{i}(f(x)), \\
& \alpha_{j}(x)<\alpha_{j}(f(x)) .
\end{aligned}
$$

This coupled with (2.18) gives us

$$
1=\alpha_{i}(x)+\alpha_{j}(x)<\alpha_{i}(f(x))+\alpha_{j}(f(x)) \leq \alpha_{i}(f(x))+\alpha_{j}(f(x))+\alpha_{k}(f(x))=1,
$$

bringing us to a contradiction.
For triangle vertices lying on the inside of $\triangle$ there are no numbering rules, but we must show that each $x=\left(x_{1}, x_{2}\right) \in \triangle$ must be a member of $M_{i}$ for some $i \in\{1,2,3\}$. Suppose that

$$
\alpha_{i}(x)<\alpha_{i}(f(x)), \quad(\forall i \in\{1,2,3\})
$$

from this we get

$$
\sum_{i=1}^{3} \alpha_{i}(x)<\sum_{i=1}^{3} \alpha_{i}(f(x))
$$

but from the definition of $\alpha_{1}, \alpha_{2}, \alpha_{3}$ we can see that both sides must be equal to 1 giving us a contradiction.

By the Sperner lemma for an arbitrary $n \in \mathbb{N}$ the division $D_{n}$ contains a triangle with vertices numbered 1,2 and 3 , we will denote its vertices as $a_{n, 1}, a_{n, 2}, a_{n, 3} \in \triangle$ ), where vertex $a_{n, i}$ is always numbered $i$, this establishes three sequences. Since all of these sequences are contained inside the compact set $\triangle$, we they all have subsequences that converge with the same index set, to show this let $\left(a_{k^{\prime}, 1}\right)$ be a convergent subsequence of $\left(a_{k, 1}\right)$, next there exists a convergent subsequence
$\left(a_{k^{\prime \prime}, 2}\right)$ of $\left(a_{k^{\prime}, 2}\right)$ and finally there is also a convergent subsequence $\left(a_{k^{\prime \prime \prime}, 3}\right)$ of $\left(a_{k^{\prime \prime}, 3}\right)$, clearly $a_{k^{\prime \prime \prime}, i}$ converges for all $i \in\{1,2,3\}$. We will take the convenience of denoting these subsequences as $\left(a_{k, 1}\right),\left(a_{k, 2}\right),\left(a_{k, 3}\right)$ again. Remember that

$$
\max _{T \in D_{k}} \operatorname{diam}(T) \rightarrow 0, \quad \text { as } k \rightarrow \infty
$$

this implies that

$$
\lim _{k \rightarrow \infty} a_{k, 1}=\lim _{k \rightarrow \infty} a_{k, 2}=\lim _{k \rightarrow \infty} a_{k, 3} \equiv a .
$$

For each $k \in \mathbb{N}, i \in\{1,2,3\}, a_{k, i}$ satisfies

$$
\alpha_{i}\left(a_{k, i}\right) \geq \alpha_{i}\left(f\left(a_{k, i}\right)\right)
$$

sending $k \rightarrow \infty$ and using continuity of $f$ (and of $\alpha_{1}, \alpha_{2}, \alpha_{3}$ ) we get

$$
a \in M_{1} \cap M_{2} \cap M_{3},
$$

meaning that $a$ is a fixed point of $f$.

Now we will show that this result is maintained if we scale $\triangle$ by a constant and shift it. Let $c \in \mathbb{R}^{+}$and $b \in \mathbb{R}^{2}$ be given and define $\sigma: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ as

$$
\sigma(x)=c \cdot x+b .
$$

We will now define the set

$$
\triangle_{c, b}=\sigma(\triangle)=\{\sigma(x): x \in \triangle\} .
$$

it is easy to see that $\sigma$ has an inverse

$$
\sigma^{-1}(x)=\frac{x-b}{c}
$$

note that both the function $\sigma: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ and its inverse are continuous (because of this property and the fact that $\triangle_{c, b}=\sigma(\triangle)$ we can call the sets $\triangle$ and $\triangle_{c, b}$ homeomorphic). Let $f_{\triangle_{c, b}}: \triangle_{c, b} \rightarrow \triangle_{c, b}$ be continuous, clearly $\sigma^{-1} \circ f_{\triangle_{c, b}} \circ \sigma: \triangle \rightarrow \Delta$ is a continuous mapping, thus by our previous considerations there exists $a \in \triangle$ such that

$$
\left(\sigma^{-1} \circ f_{\triangle_{c, b}} \circ \sigma\right)(a)=a
$$

applying the mapping $\sigma$ to both sides we get

$$
f_{\triangle_{c, b}}(\sigma(a))=\sigma(a),
$$

hence $\sigma(a) \in \triangle_{c, b}$ is a fixed point of $f_{\triangle_{c, b}}$. Lastly we will use the same trick as in the original proof using theorem 2.15. It is clear that $K \subset \triangle_{c, b}$ for some $c \in \mathbb{R}^{+}$and $b \in \mathbb{R}^{2}$. Theorem 2.15 gives the existence of a continuous retraction $R: \mathbb{R}^{2} \rightarrow \operatorname{conv}(K)=K$, i.e., $R(x)=x$ for all $x \in K$. Set $\tilde{f}=f_{\triangle_{c, k}} \circ R$, then $\tilde{f}: \mathbb{R}^{2} \rightarrow K$, clearly $\tilde{f}$ is a continuous function from $\triangle_{c, b}$ into itself. Our previously shown claim gives us a fixed point $\bar{x}=\tilde{f}(\bar{x})$, since range $(\tilde{f}) \subset K$, then $\bar{x}$ must lie in $K$.

This proof can also be extended into $\mathbb{R}^{n}$. There is a variation of the Sperner lemma for simplices, using this we would prove that the Brouwer fixed point theorem holds on a unit simplex $S$ with vertices $A_{1}, \ldots, A_{d+1}$ for some $d \in \mathbb{N}$, instead of functions $\alpha_{1}, \alpha_{2}, \alpha_{3}$ we would use $n+1$ functions $\alpha_{1}, \ldots, \alpha_{n+1}: S \rightarrow[0,1]$ satisfying

- For all $x \in S$ it holds $x=\sum_{i=1}^{n+1} \alpha_{i}(x) A_{i}$
- For all $x \in S$ it holds $\sum_{i=1}^{n+1} \alpha_{i}(x)=1$.


### 2.3 Schauder Fixed Point Theorem

This section is dedicated to the Schauder fixed point theorem which extends the Brouwer fixed point theorem to infinite-dimensional Banach spaces. It can be proven by extending the concept of a mapping degree to Banach spaces but we will be proving it by approximating compact operators in Banach spaces by mappings with finite-dimensional range and using the Brouwer fixed point theorem on them. Let us begin by stating the said approximation theorem.

Theorem 2.20 Let $X, Y$ be normed linear spaces and let $A: S \rightarrow Y$ be a continuous operator, where $S$ is a non-empty subset of $X$ such that $A(S)$ is a relatively compact (its closure is compact) subset of $Y$. Then for all $n \in \mathbb{N}$ there exist a finite-dimensional subspace $Y_{n}$ of $Y$ and a continuous operator $A_{n}: S \rightarrow Y_{n}$ which satisfies

$$
\sup _{u \in S}\left\|A(u)-A_{n}(u)\right\| \leq \frac{1}{n}
$$

and it further holds that $A_{n}(S) \subset \operatorname{conv}(A(S))$.
Proof We will begin by constructing a finite $\frac{1}{2 n}$-net over $A(S)$. Let $n \in \mathbb{N}$ be given and let $\Phi$ be the collection of $B_{Y}\left(A(x), \frac{1}{2 n}\right)$ for all $x \in S$, we can see that $\Phi$ is a cover of $\overline{A(S)}$, indeed let $y_{1}$ be a limit point of $A(S)$, then there exists a $y_{2} \in A(S)$ with $\left\|y_{1}-y_{2}\right\|<\frac{1}{2 n}$ showing that $\Phi$ covers not only $A(S)$ but also $\overline{A(S)}$. From the assumptions of the theorem we know that $\overline{A(S)}$ is compact, thus there exists a finite subcover of $\Phi$, this along with the way that $\Phi$ is defined gives us $A u_{1}, A u_{2}, \ldots, A u_{N}$ that satisfy

$$
\begin{equation*}
\min _{k \in\{1, \ldots, N\}}\left\|A u-A u_{k}\right\| \leq \frac{1}{2 n} . \quad(\forall u \in S) \tag{2.19}
\end{equation*}
$$

Next we will define the Schauder operator $A_{n}: S \rightarrow Y$ in the following way

$$
A_{n}(u)=\frac{\sum_{k=1}^{N} a_{k}(u) A\left(u_{k}\right)}{\sum_{k=1}^{N} a_{k}(u)}, \quad(\forall u \in S)
$$

with $a_{k}: S \rightarrow \mathbb{R}$ being defined as

$$
a_{k}(u)=\max \left\{\frac{1}{n}-\left\|A(u)-A\left(u_{k}\right)\right\|, 0\right\}, \quad(k \in[1, N]) .
$$

The operator $A_{n}$ is well defined because $a_{k}$ is non-negative for all $k \in[1, N]$, from (2.19) we can see that for all $u \in S$ there exists $k \in\{1, \ldots, N\}$ s.t. $a_{k}(u)>0$ therefore there will be no division by zero taking place. Now let us discuss the continuity of $A_{n}$. As sums of continuous functions are also continuous as well as ratios of continuous functions where the denominator is greater than zero having the same property, it follows that if $a_{k}$ is continuous for all $k \in\{1, \ldots, N\}$, then $A_{n}$ is also continuous. Continuity of $a_{k}, \forall k \in\{1, \ldots, N\}$ comes from it being a composition of continuous functions. With this we have shown that $A_{n}$ is continuous. We also need to show that $A_{n}(S) \subset \operatorname{conv}(A(S))$. For all $u \in S$ and $k \in\{1, \ldots, N\}$ we have

$$
A_{n}(u)=\frac{\sum_{k=1}^{N} a_{k}(u) A\left(u_{k}\right)}{\sum_{k=1}^{N} a_{k}(u)}=\sum_{k=1}^{N} \frac{a_{k}(u)}{\underbrace{N}_{\lambda_{k}(u)} a_{l}(u)} A\left(u_{k}\right)=\sum_{k=1}^{N} \lambda_{k}(u) A\left(u_{k}\right),
$$

with $\lambda_{k}(u)=\frac{a_{k}(u)}{\sum_{l=1}^{N} a_{l}(u)}$ which for all $u \in S$ satisfies $\lambda_{k}(u) \geq 0, \forall k \in\{1, \ldots, N\}$ and $\sum_{k=1}^{N} \lambda_{k}(u)=1$. Hence $\sum_{k=1}^{N} \lambda_{k}(u) A\left(u_{k}\right)$ is a convex combination of $\left(A\left(u_{1}\right), A\left(u_{2}\right), \ldots, A\left(u_{N}\right)\right)$ giving us the inclusion

$$
A_{n}(S) \subset \operatorname{conv}\left(A\left(u_{1}\right), A\left(u_{2}\right), \ldots, A\left(u_{N}\right)\right) \subset \operatorname{conv}(A(S)) .
$$

Which is exactly what we wanted. The property that $A_{n}: S \rightarrow Y_{n}$ with $\operatorname{dim} Y_{n}<\infty$ can also be derived from this revelation as

$$
A_{n}(S) \subset \operatorname{conv}\left(A\left(u_{1}\right), A\left(u_{2}\right), \ldots, A\left(u_{N}\right)\right) \subset \operatorname{span}\left\{A\left(u_{1}\right), A\left(u_{2}\right), \ldots, A\left(u_{N}\right)\right\}=Y_{n}, \quad \operatorname{dim} Y_{n}<\infty
$$

To finish the proof we need to show that

$$
\left\|A(u)-A_{n}(u)\right\| \leq \frac{1}{n}, \quad(\forall u \in S)
$$

We have

$$
\begin{equation*}
\left\|A(u)-A_{n}(u)\right\|=\frac{\left\|\sum_{k=1}^{N} a_{k}(u)\left(A(u)-A\left(u_{k}\right)\right)\right\|}{\sum_{k=1}^{N} a_{k}(u)} \leq \frac{\sum_{k=1}^{N} a_{k}(u)\left\|A(u)-A\left(u_{k}\right)\right\|}{\sum_{k=1}^{N} a_{k}(u)} . \tag{2.20}
\end{equation*}
$$

Now from the definition of $a_{k}(u)$ we can see that

$$
\left\|A(u)-A\left(u_{k}\right)\right\| \geq \frac{1}{n} \Longrightarrow a_{k}(u)=0, \quad(\forall k \in\{1, \ldots, N\})
$$

giving us that

$$
a_{k}(u)\left\|A(u)-A\left(u_{k}\right)\right\| \leq \frac{1}{n} a_{k}(u) . \quad(\forall k \in\{1, \ldots, N\})
$$

By summing both sides we get

$$
\sum_{k=1}^{N} a_{k}(u)\left\|A(u)-A\left(u_{k}\right)\right\| \leq \frac{1}{n} \sum_{k=1}^{N} a_{k}(u)
$$

As $\sum_{k=1}^{N} a_{k}(u)>0$ (see part where we prove that $A_{n}$ is well defined) we have

$$
\frac{\sum_{k=1}^{N} a_{k}(u)\left\|A(u)-A\left(u_{k}\right)\right\|}{\sum_{k=1}^{N} a_{k}(u)} \leq \frac{1}{n} .
$$

Finally from (2.20) we get

$$
\left\|A(u)-A_{n}(u)\right\| \leq \frac{1}{n}
$$

Now we are ready to prove the Schauder fixed point theorem.

Theorem 2.21 (Schauder Fixed Point) Let $S$ be a non-empty closed convex subset of a normed linear space $X$. Then every continuous operator $A: S \rightarrow S$ with the property that $A(S)$ is relatively compact has a fixed point in $S$.

Proof Theorem 2.20 gives us the existence of a finite dimensional subspace $X_{n}$ of $X$ and a continuous operator $A_{n}: S \rightarrow X_{n}$ with $A_{n}(S) \subset \operatorname{conv}(A(S))$ for all $n \in \mathbb{N}$ where

$$
\left\|A u-A_{n} u\right\| \leq \frac{1}{n} . \quad(\forall u \in S)
$$

Now define $S_{n}=X_{n} \cap \overline{\operatorname{conv} A(S)} \subset S$, we firstly want to show that $\left.A_{n}\right|_{S_{n}}: S_{n} \rightarrow S_{n}$. This holds because (note that $S_{n} \subset S$ ) $A_{n}: S \rightarrow X_{n}$ and $A_{n}(S) \subset \operatorname{conv}(A(S))$ which is one of the properties of $A_{n}$.

In order to be able use the Brouwer fixed point theorem on $\left.A_{n}\right|_{S_{n}}$ we still have to show that $S_{n}$ is a compact convex subset of $X_{n} . S_{n}$ is the intersection of two convex sets $\overline{\operatorname{conv} A(S)}$ and $X_{n}$ (all linear subspaces are convex), hence $S_{n}$ itself is also convex.

With convexity out of the way, let us move on to boundedness of $S_{n}$. The set $A(S)$ is relatively compact and thus bounded, by extension the set $\overline{\operatorname{conv} A(S)}$ is also bounded, since $S_{n} \subset \overline{\operatorname{conv} A(S)}$ it is clear that $S_{n}$ itself is bounded.

As an intersection of two closed sets $\overline{\operatorname{conv} A(S)}$ and $X_{n}, S_{n}$ is also closed, $X_{n}$ is closed because it is a finite-dimensional subspace of $X$. We have found that $S_{n}$ is a closed and bounded subset of a finite-dimensional space $X_{n}$, so $S_{n}$ is compact.

Now applying the Brouwer fixed point theorem to $\left.A_{n}\right|_{S_{n}}: S_{n} \rightarrow S_{n}$ where $S_{n}$ is compact and convex we obtain the sequence $\left(u_{n}\right) \subset S_{n}$ with

$$
A_{n}\left(u_{n}\right)=u_{n}, \quad(\forall n \in \mathbb{N})
$$

Since the sequence $\left(A\left(u_{n}\right)\right)$ lies inside a relatively compact set $A(S)$, there exists a subsequence $\left(A\left(u_{n_{k}}\right)\right)$ of $\left(A\left(u_{n}\right)\right)$ such that

$$
\lim _{k \rightarrow \infty} A\left(u_{n_{k}}\right)=v \in X
$$

Moreover $\left(A\left(u_{n_{k}}\right)\right) \subset S$ where $S$ is closed, so $v \in S$. We also have $u_{n_{k}} \rightarrow v$ as $k \rightarrow \infty$ because

$$
\begin{aligned}
\left\|u_{n_{k}}-v\right\| & \leq\left\|u_{n_{k}}-A\left(u_{n_{k}}\right)\right\|+\left\|A\left(u_{n_{k}}\right)-v\right\|=\left\|A_{n_{k}}\left(u_{n_{k}}\right)-A\left(u_{n_{k}}\right)\right\|+\left\|A\left(u_{n_{k}}\right)-v\right\| \\
& \leq \frac{1}{n_{k}}+\left\|A\left(u_{n_{k}}\right)-v\right\| \rightarrow 0 \text { as } k \rightarrow \infty .
\end{aligned}
$$

Now using the continuity of $A$ and the fact that $u_{n_{k}} \rightarrow v$ as $k \rightarrow \infty$, we get that $A\left(u_{n_{k}}\right) \rightarrow A(v)$ as $k \rightarrow \infty$. Since $v$ and $A v$ are both limits of $A\left(u_{n_{k}}\right)$ as $k \rightarrow \infty$, we conclude that $A(v)=v$.

This result is clearly an extension of the Brouwer fixed point theorem. We would like $S$ to be assumed closed and bounded instead of compact, but unfortunately it would not be enough to guarantee a fixed point, see theorem 2.11 where we show that if $A$ is further non-expansive and $B$ uniformly convex the result holds, there we give an example of a mapping $T: \overline{B_{l^{2}}(0,1)} \rightarrow \overline{B_{l^{2}}(0,1)}$ that has no fixed points but is Lipschitz continuous with a Lipschitz constant that can be chosen arbitrarly close to 1 while remaining greater.

Theorem 2.22 (Schaefer) Let $B$ be a Banach space and let $A: B \rightarrow B$ be a continuous compact (i.e. maps bounded sets to relatively compact sets) operator, if the set

$$
\mathcal{F}=\{x \in X: x=\lambda A x\}, \quad(\lambda \in[0,1])
$$

is bounded, then $A$ has a fixed point in $B$.
Proof Let $r>\sup _{x \in \mathcal{F}}\|x\|$ and define the operator $G: B \rightarrow B$ as

$$
G x= \begin{cases}A x, & \text { if }\|A x\| \leq r \\ \frac{r \cdot A x}{\|A x\|}, & \text { if }\|A x\|>r\end{cases}
$$

We can see that $G$ is continuous and maps $\overline{B_{B}(0, r)} \subset B$ to itself, clearly if $\|A x\| \leq r$, then $\|G x\|=\|A x\| \leq r$ and in case that $\|A x\|>r$ we have $\|G x\|=\frac{r\|A x\|}{\|A x\|}=r$. The operator $G$ is further
compact as $A$ maps all bounded subsets of $B$ to relatively compact subsets of $B$ and all $G$ does is multiply images of $A$ by a number bounded above by 1 . Since $G: \overline{B_{B}(0, r)} \rightarrow \bar{B}_{0}(r)$ is continuous and compact with $\overline{B_{B}(0, r)}$ being non-empty, closed, bounded and convex, by the Schauder fixed point theorem there exists $\bar{x} \in \overline{B_{B}(0, r)}$ s.t. $G \bar{x}=\bar{x}$. We claim that $\|A \bar{x}\| \leq r$, suppose $\|A \bar{x}\|>r$ we get

$$
\bar{x}=G \bar{x}=\frac{r}{\|A \bar{x}\|} A \bar{x}=\lambda_{0} A \bar{x}, \quad\left(\lambda_{0}<1\right)
$$

implying that $\bar{x} \in \mathcal{F}$ but we also have $\|\bar{x}\|=\frac{r\|A x\|}{\|A x\|}=r$ which contradicts $r>\sup _{x \in \mathcal{F}}\|x\|$. Thus we can conclude that $G \bar{x}=A \bar{x}=\bar{x}$.

### 2.4 Kakutani Fixed Point Theorem

In this section we will discuss a fixed point theorem for point-to-set mappings which extends the Brouwer fixed point theorem and has applications for example in game theory.

Definition 2.6 (Point-to-Set Mapping) Given a non-empty set $S$, we call every mapping $\Phi$ that associates points of $S$ with subsets of $S$ (i.e., $\Phi: S \rightarrow 2^{S}$ ) a point-to-set mapping.

Definition 2.7 Let $S$ be a non-empty set and let $\Phi: S \rightarrow 2^{S}$ be a point-to-set mapping, we call $\bar{x} \in S$ a fixed point of $\Phi$ if $\bar{x} \in \Phi(\bar{x})$.

For example take $S=[0,1]$ and $\Phi(x)=\left[0, x^{2}\right]$. This mapping has two fixed points $\bar{x}_{1}=0$ and $\bar{x}_{2}=1$.

Definition 2.8 Let $S$ be a metric space and let $\Phi: S \rightarrow 2^{S}$ be a point-to-set mapping, we call $\Phi$ upper semi-continuous if $x_{n} \rightarrow x, y_{n} \rightarrow y$ and $\forall n \in \mathbb{N}: y_{n} \in \Phi\left(x_{n}\right)$ together imply that $y \in \Phi(x)$.

Definition 2.9 Let $X$ be a vector space. We call points $x_{0}, \ldots, x_{n} \in X$ with $n \in \mathbb{N}$ affinely independent if for all $\lambda_{0}, \ldots, \lambda_{n} \in \mathbb{R}$

$$
\sum_{i=0}^{n} \lambda_{i} x_{i}=0, \quad \sum_{i=0}^{n} \lambda_{i}=0
$$

imply that

$$
\lambda_{i}=0, \quad \forall i \in\{0, \ldots, n\}
$$

From this definition it is clear that all convex combinations of affinely independent points are given uniquely. Also note that if $\operatorname{dim}(X)=d$, then we can find combinations of at most $d+1$ affinely independent points in it.

Definition 2.10 $A$ set $S \subset \mathbb{R}^{d}, d \in \mathbb{N}$ defined by the affinely independent points $x_{0}, \ldots, x_{d}$ in $\mathbb{R}^{d}$ as

$$
S=\left\{\sum_{i=0}^{d} \theta_{i} x_{i} \mid \theta_{i} \geq 0(i=0, \ldots, d), \sum_{i=0}^{d} \theta_{i}=1\right\}=\operatorname{conv}\left\{x_{0}, \ldots, x_{d}\right\}
$$

is called a simplex.
A simplex in $\mathbb{R}^{2}$ is a triangle and in $\mathbb{R}^{3}$ it is a tetrahedron, in $\mathbb{R}$ it is a segment.
Now we are ready to state the Kakutani fixed point theorem. We will begin by proving it on a simplex, the proof is quite technical. Then we will go on to show that it can be extended to non-empty compact convex subsets of $\mathbb{R}^{d}$.

Theorem 2.23 (Kakutani Fixed Point on a Simplex) Let $S$ be a simplex in $\mathbb{R}^{d}$, if $\Phi: S \rightarrow 2^{S}$ is an upper semi-continuous point-to-set mapping whose images are non-empty and convex, then $\Phi$ has a fixed point in $S$.

Proof Let $S_{n}$ for $n \in \mathbb{N}$ be a division of all simplices in $S_{n-1}$ into several smaller simplices where $S_{0}=\{S\}$ such that $\max _{\triangle \in S_{n}} \operatorname{diam}(\triangle) \rightarrow 0$ as $n \rightarrow \infty$, another quality we desire is that if $E$ is an intersection of some $\triangle_{1}, \ldots, \triangle_{r} \in S_{n}$ and we denote the set of common vertices of $\triangle_{1}, \ldots, \triangle_{r}$ as $V$, then $E=\operatorname{conv}(V)$. Such a division exists, e.g., we can use the barycentric simplicial subdivision which more specifically satisfies

$$
\begin{equation*}
\max _{\Delta \in S_{1}} \operatorname{diam}(\triangle) \leq \frac{d}{d+1} \operatorname{diam}(S) \tag{2.21}
\end{equation*}
$$

Details can be found in [6], clearly applying such division recursively will yield $\max _{\triangle \in S_{1}} \operatorname{diam}(\triangle) \rightarrow$ 0 as $n \rightarrow \infty$


Figure 2.6: Barycentric simplicial subdivision.

Now if $x \in S$, then there exists $\triangle_{x} \in S_{n}$ for some fixed $n \in \mathbb{N}$, this point $x$ can be written as

$$
x=\sum_{i=0}^{d} \theta_{i} x_{i},
$$

where $x_{0}, \ldots, x_{d}$ are vertices of $\triangle_{x}$ and $\theta_{i} \geq 0(i=0, \ldots, d), \sum_{i=0}^{d} \theta_{i}=1$. For each $x_{i}, \Phi\left(x_{i}\right)$ is a set of vectors in $\mathbb{R}^{d}$, define $y_{i} \in \Phi\left(x_{i}\right)$ to be one of these vectors arbitrarily picked. The points $y_{0}, \ldots, y_{d}$ constitute some $\triangle_{y}=\operatorname{conv}\left\{y_{0}, \ldots, y_{d}\right\} \subset S$ which is not necessarily a simplex.


Figure 2.7: Relation of $x_{0}, \ldots, x_{d}$ to $y_{0}, \ldots, y_{d}$.

Let $n \in \mathbb{N}$ be fixed, for each $x \in S$ we can define a mapping $\varphi_{n}: S \rightarrow S$ in the following way, if $x$ is a vertex of some $\triangle_{x} \in S_{n}$ we set $\varphi_{n}(x)=y \in \Phi(x)$.

Now using the fact that $x$ is contained inside some simplex in $S_{n}$ with vertices $x_{0}, \ldots, x_{d}$, we define $\varphi_{n}(x)=\sum_{i=0}^{d} \theta_{i} \varphi_{n}\left(x_{i}\right)$. Such mapping is clearly well defined for vertices and interior points of simplices in $S_{n}$, we must show that it is also well defined for points that lie in two or more simplices, i.e., on the common boundary of several simplices. Let $x \in S$ be a point that lies in $\triangle_{1}, \triangle_{2}, \ldots, \triangle_{r} \in S_{n}$ and let $E=\bigcap_{k=1}^{r} \triangle_{k}$. Remember that the division we are using guarantees that the common vertices of $\triangle_{1}, \ldots, \triangle_{r}$ are all members of $E$ and that $E$ is a simplex generated by these common vertices denoted as $x_{0}, \ldots, x_{m}$ for some $m \in \mathbb{N} \cup\{0\}$, hence

$$
x=\sum_{i=0}^{m} \theta_{i} x_{i} .
$$

We also have

$$
x=\sum_{i=0}^{d} \theta_{i}^{\triangle_{k}} x_{i}^{\triangle_{k}}=\sum_{i=0}^{m} \theta_{i} x_{i}, \quad(k=0, \ldots, r)
$$

Using the fact that all convex combinations of affinely independent vectors are given uniquely we get

$$
\varphi_{n}(x)=\sum_{i=0}^{d} \theta_{i}^{\triangle_{k}} \varphi_{n}\left(x_{i}^{\triangle_{k}}\right)=\sum_{i=0}^{m} \theta_{i} \varphi_{n}\left(x_{i}\right), \quad(k=0, \ldots, r)
$$

showing that $\varphi_{n}: S \rightarrow S$ is well defined for all $x \in S$.
Next we will show that $\varphi_{n}$ is continuous, let $x_{k} \rightarrow x \in S$ as $k \rightarrow \infty$, one of following three cases can occur

1. $x \in \operatorname{int}(\triangle)$ for some $\triangle \in S_{n}$ with vertices $v_{0}, \ldots, v_{d}$. Since every neighborhood of $x$ contains all but finitely many points of $\left(x_{k}\right)$ we may assume that $\left(x_{k}\right) \subset \triangle$. We can write $x_{k}=\sum_{i=0}^{d} \theta_{i}^{k} v_{i}$ and $x=\sum_{i=0}^{d} \theta_{i} v_{i}$. We have

$$
\begin{equation*}
\sum_{i=0}^{d}\left(\theta_{i}^{k}-\theta_{i}\right) v_{i}=x_{k}-x \rightarrow 0 \in \mathbb{R}^{n} \tag{2.22}
\end{equation*}
$$

and also

$$
\begin{equation*}
\sum_{i=0}^{d}\left(\theta_{i}^{k}-\theta_{i}\right)=\sum_{i=0}^{d} \theta_{i}^{k}-\sum_{i=0}^{d} \theta_{i}=0, \quad(\forall k \in \mathbb{N}) \tag{2.23}
\end{equation*}
$$

Since $\left(\theta_{i}^{k}\right)$ are sequences inside a compact set for all $i=0, \ldots, d$, they must have a convergent subsequence $\left(\theta_{i}^{k_{l}}\right)$. Now $v_{0}, \ldots, v_{d}$ are affinely independent so (2.22), (2.23) imply that ( $\theta_{i}^{k_{l}}$ ) must converge to $\theta_{i}$. As this is the case any convergent subsequence of $\left(\theta_{i}^{k}\right)$, it follows that $\limsup _{k \rightarrow \infty} \theta_{i}^{k}=\liminf _{k \rightarrow \infty} \theta_{i}^{k}=\theta_{i}$ which implies $\lim _{k \rightarrow \infty} \theta_{i}^{k}=\theta_{i}$. Thus we have

$$
\varphi_{n}\left(x_{k}\right)=\sum_{i=0}^{d} \theta_{i}^{k} \varphi_{n}\left(v_{i}\right) \rightarrow \sum_{i=0}^{d} \theta_{i} \varphi_{n}\left(v_{i}\right)=\varphi_{n}(x) \quad(\text { as } k \rightarrow \infty) .
$$

2. Alternatively assume that $x$ is a member of $\triangle_{1}, \ldots, \triangle_{r} \in S_{n}$ and that there is a sequence $x_{k}$ such that $x_{k} \rightarrow x$. Again we may assume that $\left(x_{k}\right)$ is contained inside $\cup_{i=1}^{r} \triangle_{i}$, we also may assume that every $\triangle_{i}, i=1, \ldots, r$ contains infinitely many members of ( $x_{k}$ ) (if not we could simply discard those finitely many members). Now we can split ( $x_{k}$ ) up into subsequences $\left(x_{k}^{i}\right)$ such that for each $i \in\{1, \ldots, r\}$ it holds

$$
\left(x_{k}^{i}\right) \subset \triangle_{i},
$$

where for each $n \in \mathbb{N}$ the point $x_{k}$ is a member of the sequence $\left(x_{k}^{i}\right)$ for at least one $i \in$ $\{1, \ldots, r\}$. From our previous considerations we can see that $\varphi_{n}\left(x_{k}^{i}\right) \rightarrow \varphi_{n}(x)$ for all $i \in$ $\{1, \ldots, r\}$ (all we needed in case 1 was that the sequence was fully contained in a single simplex). In other words for any fixed $i \in\{1, \ldots, r\}$ given $\varepsilon>0$ we can find $N_{k} \in \mathbb{N}$ such that

$$
\left(k>N_{k}\right) \Longrightarrow\left\|\varphi_{n}\left(x_{k}^{i}\right)-\varphi_{n}(x)\right\|<\varepsilon .
$$

If we take $N=\max \left\{N_{1}, \ldots, N_{r}\right\}$ we get

$$
(k>N) \Longrightarrow\left\|\varphi_{n}\left(x_{k}\right)-\varphi_{n}(x)\right\|<\varepsilon
$$

this is due to the fact that for each $i \in\{1, \ldots, r\}$ the sequence $\left(x_{k}^{i}\right)$ is a subsequence of ( $x_{k}$ ) and the property that for each $k \in \mathbb{N}$ the point $x_{k}$ is a member of the sequence $\left(x_{k}^{i}\right)$. We can see that $\varphi_{n}\left(x_{k}\right) \rightarrow \varphi_{n}(x)$.

We have shown that $\varphi_{n}: S \rightarrow S$ is a continuous mapping, since $S$ is compact, convex and nonempty by the Brouwer fixed point theorem there exists a sequence $\left(x_{k}\right) \subset S$ such that $x_{k}=\varphi_{k}\left(x_{k}\right)$ for all $k \in \mathbb{N}$. Now let $\triangle_{k} \in S_{k}$ be a simplex that contains $x_{k}$, if there are more, just pick one. Let $v_{0}^{k}, \ldots, v_{d}^{k}$ denote the vertices of $\triangle_{k}$, we can write $x_{k}=\sum_{i=0}^{d} \theta_{i}^{k} v_{i}^{k}$ for some $\theta_{0}^{k}, \ldots, \theta_{d}^{k}$ with the usual properties, we also have $\varphi_{k}\left(x_{k}\right)=\sum_{i=0}^{d} \theta_{i}^{k} q_{i}^{k}$, where $q_{i}^{k}=\varphi_{k}\left(v_{i}^{k}\right) \in \Phi\left(v_{i}^{k}\right)$ for $i=0, \ldots, d$. Since $x_{k}$ is a fixed point of $\varphi_{k}$ we have

$$
x_{k}=\varphi_{k}\left(x_{k}\right)=\sum_{i=0}^{d} \theta_{i}^{k} q_{i}^{k}
$$

There exist subsequences $\left(x_{k_{l}}\right),\left(\theta_{i}^{k_{l}}\right)$ and $\left(q_{i}^{k_{l}}\right)$ that all converge as $l \rightarrow \infty$ using the same index set $\left(k_{l}\right)$, all of these sequences are contained inside compact sets so they all have convergent subsequences, now let $\left(x_{k^{\prime}}\right)$ be a convergent subsequence of $\left(x_{k}\right)$, next can find a convergent subsequence $\left(\theta_{1}^{k^{\prime \prime}}\right)$ of $\left(\theta_{1}^{k^{\prime}}\right)$, continuing this process finitely many times we will arrive at our desired index set $\left(k_{l}\right)$. We have $x_{k_{l}} \rightarrow \bar{x}, \theta_{i}^{k_{l}} \rightarrow \theta_{i}$ and $q_{i}^{k_{l}} \rightarrow q^{i}$ with

$$
\bar{x}=\sum_{i=0}^{d} \theta_{i} q_{i} .
$$

Let us also note that since $\operatorname{diam} \triangle_{k} \rightarrow 0$ as $k \rightarrow \infty$ and $x_{n} \in \triangle_{k}$ for all $k \in \mathbb{N}$, it holds $v_{i}^{k} \rightarrow \bar{x}$ for $i=0, \ldots, d$. Let us summarize some facts, we have

- $q_{i}^{k_{l}} \in \Phi\left(v_{i}^{k_{l}}\right)$ for $i=0, \ldots, d$.
- $q_{i}^{k_{l}} \rightarrow q_{i}$ for $i=0, \ldots, d$.
- $v_{i}^{k_{l}} \rightarrow \bar{x}$ for $i=0, \ldots, d$.

By upper semi-continuity of $\Phi$, these three properties give us that $q_{i} \in \Phi(\bar{x})$. Finally since $\Phi(\bar{x})$ is convex $\bar{x}=\sum_{i=0}^{d} \theta_{i} q_{i} \in \Phi(\bar{x})$.

Theorem 2.24 (Kakutani Fixed Point) Let $S$ be a non-empty compact convex subset of $\mathbb{R}^{d}$, if $\Phi: S \rightarrow 2^{S}$ is an upper semi-continuous point-to-set mapping whose images are non-empty and convex, then $\Phi$ has a fixed point in $S$.

Proof This proof will pretty much shadow the way we extended the Brouwer fixed point theorem to more general sets by using the fact that the result holds for closed balls.

Since $S$ is compact, there exists a simplex $S_{\triangle}$ that contains $S$. By theorem 2.15 there exists a continuous retraction $R: \mathbb{R}^{d} \rightarrow \operatorname{conv}(S)=S$, i.e., $R(x)=x$ for all $x \in S$. We set $\tilde{\Phi}=\Phi \circ R$, we have $\tilde{\Phi}: \mathbb{R}^{d} \rightarrow 2^{S}$. By the Kakutani fixed point theorem for simplices applied to $\tilde{\Phi}_{\mid S_{\Delta}}: S_{\Delta} \rightarrow 2^{S} \subset 2^{S_{\Delta}}$ there exists $\bar{x} \in S_{\triangle}$ such that $\bar{x} \in \tilde{\Phi}(\bar{x})$. Since range $(\tilde{\Phi}) \subset 2^{S}$, we necessarily have $\tilde{\Phi}(\bar{x}) \subset S$.

Remark 2.9 Let us see if all the assumptions are necessary

- Breaking the condition of compactness by taking an open set: Let $S=(0,1)$ and let $\Phi(x)=$ $\left[x^{3}, x^{2}\right]$, then $\Phi$ is upper semi-continuous, assume $x_{n} \rightarrow x, y_{n} \rightarrow y$ with $y_{n} \in \Phi\left(x_{n}\right)$. We have

$$
x_{n}^{3} \leq y_{n} \leq x_{n}^{2}
$$

and by sending $n \rightarrow \infty$ we get

$$
x^{3} \leq y \leq x^{2}
$$

implying that $y \in \Phi(x)$. But $\Phi$ has no fixed points. It is easy to see that no other conditions of theorem 2.24 other than compactness have been broken.

- Breaking the condition of compactness by taking an unbounded set: Let $S=[2, \infty)$, clearly $S$ is closed, take $\Phi(x)=\left[x^{2}, x^{3}\right]$ by the same reasoning as above $\Phi$ is upper semi-continuous and $\Phi$ has no fixed points. No other conditions of theorem 2.24 other than compactness have been broken.
- Convexity of $S$ : Let $S=\bar{B}_{\mathbb{R}^{2}}(0,1) \backslash B_{\mathbb{R}^{2}}\left(0, \frac{1}{2}\right)$, such $S$ is compact. Define $\Phi(x)=\{-x\}$ and assume that $x_{n} \rightarrow x, y_{n} \rightarrow y$ and $\forall n \in \mathbb{N}: y_{n} \in \Phi\left(x_{n}\right)$, we have

$$
y_{n}=-x_{n},
$$

sending $n \rightarrow \infty$ we get

$$
y=-x
$$

meaning that $y \in \Phi(x)$, showing that $\Phi$ is upper semi-continuous. Clearly $\Phi$ has no fixed points in $S$. All conditions of theorem 2.24 other then convexity of $S$ have been satisfied.

- Convexity of $\Phi(x)$ : Let $S=[0,1]$ and define $\Phi(x)=[0,1] \backslash\left(x-\frac{1}{10}, x+\frac{1}{10}\right)$, such $\Phi$ is upper semi-continuous, assume $x_{n} \rightarrow x, y_{n} \rightarrow y$ with $y_{n} \in \Phi\left(x_{n}\right)$, we have

$$
\left|y_{n}-x_{n}\right| \geq \frac{1}{10}
$$

Letting $n \rightarrow \infty$ we get

$$
|y-x| \geq \frac{1}{10}
$$

implying that $y \in \Phi(x) . \Phi(x)$ is also non-empty for all $x \in S$ and there are no fixed points of $\Phi$ in $S$. No other conditions of theorem 2.24 other than convexity of $\Phi(x)$ have been broken.

- Semi-continuity of $\Phi$ : Let $S=[0,1]$ and take

$$
\Phi(x)= \begin{cases}{\left[\frac{2}{3}, 1\right]} & \text { if } x \leq \frac{1}{2} \\ {\left[0, \frac{1}{3}\right]} & \text { if } x>\frac{1}{2}\end{cases}
$$

Clearly $\Phi(x)$ is non-empty and convex for all $x \in S$. Let $x_{n}=\frac{1}{2}+\frac{1}{n+1}$, then $x_{n} \rightarrow \frac{1}{2} \equiv x$, assume that $y_{n} \rightarrow y$ with $y_{n} \in \Phi\left(x_{n}\right)$ for all $n \in \mathbb{N}$. This means

$$
y_{n} \in\left[0, \frac{1}{3}\right] \quad(\forall n \in \mathbb{N})
$$

Since $\left[0, \frac{1}{3}\right]$ is closed, $y$ must also lie in $\left[0, \frac{1}{3}\right]$. However $\Phi(x)=\Phi\left(\frac{1}{2}\right)=\left[\frac{2}{3}, 1\right]$, so $y \notin \Phi(x)$, implying that $\Phi$ is not upper semi-continuous. Clearly $\Phi$ has no fixed points in $S$ and all the other conditions of theorem 2.24 have been satisfied.

## Chapter 3

## Applications

This chapter will feature applications of theorems we have introduced. We will begin with a few statements regarding solvability of systems of ordinary differential equations.

### 3.1 Picard-Lindelöf Theorem

Definition 3.1 Let

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t} x_{1}(t) & =f_{1}\left(t, x_{1}, \ldots, x_{n}\right),  \tag{3.1}\\
\frac{\mathrm{d}}{\mathrm{~d} t} x_{2}(t) & =f_{2}\left(t, x_{1}, \ldots, x_{n}\right), \\
& \vdots \\
\frac{\mathrm{d}}{\mathrm{~d} t} x_{n}(t) & =f_{n}\left(t, x_{1}, \ldots, x_{n}\right)
\end{align*}
$$

be a system of ordinary differential equations where $f_{1}, \ldots, f_{n}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ are defined over some set $G \subset \mathbb{R}^{n+1}$ and $x_{1}, \ldots, x_{n}: \mathbb{R} \rightarrow \mathbb{R}$ are defined over some interval $I$. We can also write this system in vector form as $\frac{\mathrm{d}}{\mathrm{d} t} x(t)=f(t, x)$, where $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n}$ and $x: \mathbb{R} \rightarrow \mathbb{R}^{n}$ are defined as

$$
x(t)=\left(x_{1}(t), \ldots, x_{n}(t)\right),
$$

and

$$
f(t, x)=\left(f_{1}(t, x), \ldots, f_{n}(t, x)\right) .
$$

We will also associate with this system the points $t_{0} \in I$ and $x_{0} \in \mathbb{R}^{n}$ s.t. $\left(t_{0}, x_{0}\right) \in G$, together

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t} x(t) & =f(t, x),  \tag{3.2}\\
x\left(t_{0}\right) & =x_{0}
\end{align*}
$$

constitute an initial value problem. We call $x$ a solution of the initial value problem (3.2) on some interval $J \subset I$ if

- $(\forall t \in J):\left(t, x_{1}(t), \ldots, x_{n}(t)\right) \in G$.
- $x$ is differentiable on $J$.
- $(\forall t \in J): \frac{\mathrm{d}}{\mathrm{d} t} x(t)=f(t, x(t))$.
- $x\left(t_{0}\right)=x_{0}$.

Theorem 3.1 (Picard-Lindelöf - version 1) Let $t_{0} \in \mathbb{R}, x_{0} \in \mathbb{R}^{n}$ and $a, b>0$, let us denote $I=\left[t_{0}, t_{0}+a\right]$ and $D=\left\{x \in \mathbb{R}^{n}:\left\|x-x_{0}\right\| \leq b\right\}$. If there exists $L>0$ such that $\forall x, y \in D$ the continuous function $f: I \times D \rightarrow \mathbb{R}^{n}$ satisfies

$$
\begin{equation*}
\|f(t, x)-f(t, y)\| \leq L\|x-y\|, \quad(\forall t \in I) \tag{3.3}
\end{equation*}
$$

then there exists a unique solution of the initial value problem (3.2) on the interval $J=\left[t_{0}, t_{0}+\delta\right]$, where $\delta \in \mathbb{R}^{+}$is chosen so that $\delta \leq a, \delta M \leq b$ and $\delta L<1$, given $M=\max _{(t, x) \in I \times D}\|f(t, x)\|$, such exists.

Proof Let $\mathcal{F}$ be the metric space of all continuous mappings $x: J \rightarrow \mathbb{R}^{n}$ that for all $t \in J$ satisfy $\left\|x(t)-x_{0}\right\| \leq b$ endowed with the metric

$$
d(x, y)=\max _{t \in J}\|x(t)-y(t)\|
$$

Such metric space is complete since $\mathcal{F}$ is a closed subsets of a metric space that is known to be complete (that is the metric space of all continuous functions over a closed interval endowed with the metric $d$ ). Define the mapping $T: \mathcal{F} \rightarrow \mathcal{F}$ as

$$
T(x)(t)=x_{0}+\int_{t_{0}}^{t} f(s, x(s)) \mathrm{d} s
$$

$T$ maps $\mathcal{F}$ into itself since for all $t \in J$ we have

$$
\begin{equation*}
\left\|T(x)(t)-x_{0}\right\|=\left\|\int_{t_{0}}^{t} f(s, x(s)) \mathrm{d} s\right\| \leq \int_{t_{0}}^{t}\|f(s, x(s))\| \mathrm{d} s \leq M \underbrace{\left(t-t_{0}\right)}_{\leq a} \leq M \delta \leq b . \tag{3.4}
\end{equation*}
$$

If $\bar{x} \in \mathcal{F}$ is a fixed point of $T$, it satisfies $\bar{x}(t)=x_{0}+\int_{t_{0}}^{t} f(s, \bar{x}(s))$ for all $t \in J$ hence it is a solution of the initial value problem (3.2) on $J$. Conversely any solution of (3.2) on $J$ is a fixed point of $T$. If we can show that there exists a unique fixed point of $T$ the proof is finished. For all $x, y \in \mathcal{F}$ and $t \in J$ we have

$$
\|T(x)(t)-T(y)(t)\| \leq \int_{t_{0}}^{t}\|f(s, x(s))-f(s, y(s))\| \mathrm{d} s
$$

using (3.3) we get

$$
\int_{t_{0}}^{t}\|f(s, x(s))-f(s, y(s))\| \mathrm{d} s \leq L \int_{t_{0}}^{t}\|x(s)-y(s)\| \mathrm{d} s \leq L \max _{s \in J}\|x(s)-y(s)\|\left(t-t_{0}\right) \leq L \delta \rho(x, y)
$$

implying that $\rho(T(x), T(y)) \leq L \delta \rho(x, y)$. Our assumptions about $\delta$ guarantee that $T$ is a contraction, thus by the Banach fixed point theorem the mapping $T$ has a unique fixed point. This completes the proof.

By adjusting the metric used in the proof we can find out that the interval on which we are guaranteed a unique solution actually does not depend on $L$. The following result is this improved version of theorem 3.1.

Theorem 3.2 (Picard-Lindelöf - version 2) Let $t_{0} \in \mathbb{R}, x_{0} \in \mathbb{R}^{n}$ and $a, b>0$, let us denote $I=\left[t_{0}, t_{0}+a\right]$ and $D=\left\{x \in \mathbb{R}^{n}:\left\|x-x_{0}\right\| \leq b\right\}$. If there exists $L>0$ such that $\forall x, y \in D$ the continuous function $f: I \times D \rightarrow \mathbb{R}^{n}$ satisfies

$$
\|f(t, x)-f(t, y)\| \leq L\|x-y\|, \quad(\forall t \in I)
$$

then there exists a unique solution of the initial value problem (3.2) on the interval $J=\left[t_{0}, t_{0}+\delta\right]$, where $\delta \in \mathbb{R}^{+}$is chosen so that $\delta \leq a$ and $\delta M \leq b$, given $M=\max _{(t, x) \in I \times D}\|f(t, x)\|$.

Proof Take $\mathcal{F}$ defined as in the previous proof but we will swap out the metric $d$ with

$$
\tilde{d}(x, y)=\max _{t \in J}\left(\mathrm{e}^{-K\left(t-t_{0}\right)}\|x(t)-y(t)\|\right),
$$

where $K>L$ is chosen arbitrarily. This metric is equivalent with $d$, the lower bound for $\tilde{d}$ comes from the relation

$$
\mathrm{e}^{-K\left(t-t_{0}\right)}\|x(t)-y(t)\| \geq \mathrm{e}^{-K \delta}\|x(t)-y(t)\|, \quad(\forall t \in J)
$$

giving us

$$
\tilde{d}(x, y) \geq \mathrm{e}^{-K \delta} d(x, y)
$$

and to get the upper bound for $\tilde{d}$ we can use the fact that $\mathrm{e}^{-K} \overbrace{\left(t-t_{0}\right)}^{\geq^{0}} \leq 1$, this gives us

$$
\mathrm{e}^{-K\left(t-t_{0}\right)}\|x(t)-y(t)\| \leq\|x(t)-y(t)\|
$$

implying that

$$
\tilde{d}(x, y) \leq d(x, y)
$$

bounding $\tilde{\rho}(x, y)$ from both sides. So $\mathcal{F}$ with the metric $\tilde{\rho}$ is a complete metric space. All that is left to do is show that $T$, defined the same way as in the previous proof is a contraction in this metric. Indeed for all $x, y \in \mathcal{F}$ and all $t \in J$ we have

$$
\begin{aligned}
\mathrm{e}^{-K\left(t-t_{0}\right)}\|T(x)(t)-T(y)(t)\| & \leq \int_{t_{0}}^{t} \mathrm{e}^{-K\left(t-t_{0}\right)}\|f(s, x(s))-f(s, y(s))\| \mathrm{d} s \\
& \leq L \int_{t_{0}}^{t} \mathrm{e}^{-K\left(t-t_{0}\right)}\|x(s)-y(s)\| \mathrm{d} s=L \int_{t_{0}}^{t} \mathrm{e}^{-K(t-s)} \mathrm{e}^{-K\left(s-t_{0}\right)}\|x(s)-y(s)\| \mathrm{d} s \\
& \leq L \max _{s \in J}\left(\mathrm{e}^{-K\left(s-t_{0}\right)}\|x(s)-y(s)\|\right) \int_{t_{0}}^{t} \mathrm{e}^{-K(t-s)} \mathrm{d} s \\
& =L \tilde{\rho}(x, y)\left(K^{-1}-K^{-1} \mathrm{e}^{-K\left(t-t_{0}\right)}\right) \leq L K^{-1} \tilde{\rho}(x, y) .
\end{aligned}
$$

Since $K$ was chosen to be greater than $L$ this shows that $T$ is a contraction, so by the Banach fixed point theorem $T$ has a unique fixed point, finishing the proof. All we needed to assume about $\delta$ was that $\delta \leq a$ and $M \delta \leq b$ to guarantee that $T$ would map $\mathcal{F}$ into itself as can be seen in (3.4).

### 3.2 Peano Theorem

The main result of this section guarantees existence of a solution of the initial value problem 3.2 under weaker assumptions however it does not guarantee uniqueness. We will need to supplement a few definition and an auxiliary theorem.

Definition 3.2 We call a sequence of continuous functions $\left(f_{k}\right)$ mapping some set $E$ into $\mathbb{R}^{n}$ pointwise bounded whenever

$$
(\forall x \in E)(\exists M \in \mathbb{R})(\forall k \in \mathbb{N}):\left\|f_{k}(x)\right\| \leq M
$$

Definition 3.3 We call a sequence of continuous functions $\left(f_{k}\right)$ mapping some set subset $E$ of a metric space $(X, d)$ into $\mathbb{R}^{n}$ equicontinuous whenever

$$
(\forall \varepsilon>0)(\exists \delta>0)(\forall k \in \mathbb{N})(\forall x, y \in E): d(x, y)<\delta \Longrightarrow\left\|f_{k}(x)-f_{k}(y)\right\|<\varepsilon
$$

Theorem 3.3 (Arzela-Ascoli) Let $\left(f_{n}\right)$ be a sequence of continuous functions mapping a compact set $K$ in some metric space into $\mathbb{R}^{n}$, if $\left(f_{n}\right)$ is pointwise bounded and equicontinuous then there exists a uniformly convergent subsequence $\left(f_{n_{k}}\right)$ of $\left(f_{n}\right)$.

The proof of this theorem can be found in [7], the proof is written for complex valued functions but it is also valid for functions into $\mathbb{R}^{n}$.

Theorem 3.4 (Peano) Let $t_{0} \in \mathbb{R}, x_{0} \in \mathbb{R}^{n}$ and $a, b>0$, let us denote $I=\left[t_{0}, t_{0}+a\right]$ and $D=\left\{x \in \mathbb{R}^{n}:\left\|x-x_{0}\right\| \leq b\right\}$. If the function $f: I \times D \rightarrow \mathbb{R}^{n}$ is continuous then there exists a solution of the initial value problem (3.2) on the interval $J=\left[t_{0}, t_{0}+\delta\right]$, where $\delta \in \mathbb{R}^{+}$is chosen so that $\delta \leq a$ and $\delta M \leq b$, given $M=\max _{(t, x) \in I \times D}\|f(t, x)\|$.

Proof The direction of this proof will be to verify all the assumptions of the Schauder fixed point theorem to show that the operator $T$ defined same as before has a fixed point. Let $X$ be the Banach space of all continuous functions $x: J \rightarrow \mathbb{R}^{n}$ endowed with the norm

$$
\|x\|=\max _{t \in J}\|x(t)\| .
$$

and let

$$
S=\left\{x \in X:\left\|x(t)-x_{0}\right\| \leq b \quad(\forall t \in J)\right\} .
$$

The set $S$ is clearly closed and bounded, it is also convex since for all $x, y \in S$ and $\lambda \in[0,1]$ it holds

$$
\begin{aligned}
\left\|(1-\lambda) x(t)+\lambda y(t)-x_{0}\right\| & \leq\left\|(1-\lambda)\left(x(t)-x_{0}\right)+\lambda\left(y(t)-x_{0}\right)\right\| \\
& \leq(1-\lambda)\left\|\left(x(t)-x_{0}\right)\right\|+\lambda\left\|\left(y(t)-x_{0}\right)\right\| \leq b . \quad(\forall t \in J)
\end{aligned}
$$

Now define $T: S \rightarrow S$ same as in the proof of the Picard-Lindelöf theorem, that is

$$
T(x)(t)=x_{0}+\int_{t_{0}}^{t} f(s, x(s)) \mathrm{d} s, \quad(t \in J)
$$

from (3.4) we already know that $T$ indeed maps $S$ into itself under the assumptions that $\delta \leq a$ and $M \delta \leq b$.

Let us show that $T$ is continuous. Since the function $f$ is continuous on a compact set $J \times D$, it is uniformly continuous on it, implying that

$$
(\forall \varepsilon>0)(\exists \delta>0)(\forall t \in I)(\forall x, y \in D):\|x-y\|<\delta \Longrightarrow\|f(t, x)-f(t, y)\|<\varepsilon
$$

Now suppose $\left(x_{n}\right) \subset S$ with $x_{n} \rightarrow x$, for any given $\delta$ we can find an index $N \in \mathbb{N}$ such that

$$
\left\|x_{n}-x\right\|_{X} \leq \delta
$$

if $n \geq N$. We have

$$
\left\|T\left(x_{n}\right)(t)-T(x)(t)\right\| \leq \int_{t_{0}}^{t}\left\|f\left(s, x_{n}(s)\right)-f(s, x(s))\right\| \mathrm{d} s . \quad(\forall t \in J)
$$

Now choose $\varepsilon>0$ arbitrarily and take $\delta>0$ so that for all $\alpha, \beta \in D,\|\alpha-\beta\|<\delta$ implies $\|f(s, \alpha)-f(s, \beta)\|<\varepsilon$. Taking $n$ large enough we get $\left\|x(s)-x_{n}(s)\right\|<\delta$ for all $s \in J$ (because $X$
uses the maximum norm), it follows that

$$
\int_{t_{0}}^{t}\left\|f\left(s, x_{n}(s)\right)-f(s, x(s))\right\| \mathrm{d} s \leq \varepsilon\left(t-t_{0}\right) \leq \varepsilon \delta . \quad(\forall t \in J)
$$

Because $\varepsilon>0$ was chosen arbitrarily and independently of $t$ this gives us that $T\left(x_{n}\right)(t)$ converges to $T(x)(t)$ uniformly, i.e., $\left\|T\left(x_{n}\right)-T(x)\right\|_{X} \rightarrow 0$ as $n \rightarrow \infty$, implying that $T$ is continuous.

Next we will show that $T(S)$ is relatively compact. For all $x \in S$ and $t_{1}, t_{2} \in J$ we have

$$
\left\|T(x)\left(t_{1}\right)-T(x)\left(t_{2}\right)\right\|=\left\|\int_{t_{1}}^{t_{2}} f(s, x(s)) \mathrm{d} s\right\| \leq M\left|t_{1}-t_{2}\right| .
$$

As $x$ was chosen arbitrarily this implies that all sequences in $T(S)$ are equicontinuous. All sequences in $T(S)$ are also bounded by the constant $\left\|x_{0}\right\|+b$ thus by the Arzela-Ascoli theorem if $\left(f_{n}\right)$ is a sequence in $T(S)$ it has a convergent subsequence, this means that $T(S)$ is relatively compact. In summary the continuous operator $T$ maps $S$ into $S, S$ is non-empty, closed and convex and $T(S)$ is relatively compact. By the Schauder fixed point theorem there exists a fixed point of $T$ in $S$.

### 3.3 Lax-Milgram Lemma

In this section we will use the Banach fixed point theorem to prove a fundamental result in partial differential equation theory. Let us begin by stating an auxiliary lemma

Lemma 3.5 Let $V$ be a Hilbert space, let $a, b: V \times V \rightarrow \mathbb{R}$ be bilinear forms and let $F \in V^{*}$, assume the following

- $(\exists c>0)(\forall u \in V): a(u, u) \geq c\|u\|^{2}$ (ellipticity of $a$ ).
- $(\exists k>0)(\forall u, v \in V): b(u, v) \leq k\|u\|\|v\|$ (continuity of $b$ ).
- $(\exists u \in V)(\forall v \in V): a(u, v)=F(v)$.

Then for each $\alpha \in \mathbb{R}$ with $|\alpha|<\frac{c}{k}$ it holds

$$
(\exists!u \in V)(\forall v \in V): a(u, v)+\alpha b(u, v)=F(u) .
$$

Proof Assume that $u_{1}, u_{2}$ both satisfy $(\forall v \in V): a\left(u_{1}, v\right)=F(v)=a\left(u_{2}, v\right)$, then by ellipticity of $a$ we have

$$
0=a(u_{1}-u_{2}, \underbrace{u_{1}-u_{2}}_{\in V}) \geq c\left\|u_{1}-u_{2}\right\|^{2},
$$

implying that $u_{1}=u_{2}$, hence $u$ from the third assumption is given uniquely. Let $T: V \rightarrow V$ be an operator defined so that for each $w \in V$ we have

$$
(\forall v \in V): a(T(w), v)=F(v)-\alpha b(w, v) .
$$

Since $b$ is continuous, $F(v)-\alpha b(w, v)$ for a fixed $w \in V$ is a member of $V^{*}$ giving us that $T$ is well defined (i.e. for each $w \in V$ there exists a unique $T(w) \in V$ ).

Let us show that $T$ is a contraction. Let $w_{1}, w_{2} \in V$, we have

$$
\begin{equation*}
c\left\|T\left(w_{1}\right)-T\left(w_{2}\right)\right\|^{2} \leq a\left(T\left(w_{1}\right)-T\left(w_{2}\right), T\left(w_{1}\right)-T\left(w_{2}\right)\right) \tag{3.5}
\end{equation*}
$$

and also

$$
\begin{aligned}
a\left(T\left(w_{1}\right)-T\left(w_{2}\right), T\left(w_{1}\right)-T\left(w_{2}\right)\right) & =a\left(T\left(w_{1}\right), T\left(w_{1}\right)-T\left(w_{2}\right)\right)-a\left(T\left(w_{2}\right), T\left(w_{1}\right)-T\left(w_{2}\right)\right) \\
& =-\alpha\left(b\left(w_{1}, T w_{1}-T w_{2}\right)-b\left(w_{2}, T\left(w_{1}\right)-T\left(w_{2}\right)\right)\right)=-\alpha\left(b \left(w_{1}-w_{2}, T\left(w_{1}\right)-T( \right.\right. \\
& \leq|\alpha| k\left\|w_{1}-w_{2}\right\|\left\|T w_{1}-T w_{2}\right\|
\end{aligned}
$$

Put together with (3.5) this means that

$$
\left\|T w_{1}-T w_{2}\right\| \leq|\alpha| \frac{k}{c}\left\|w_{1}-w_{2}\right\| .
$$

By our choice of $\alpha$ this gives us that $T$ is a contraction. Using the Banach fixed point theorem there exists a unique fixed point $u$ of $T$ in $V$, i.e.,

$$
(\exists!u \in V)(\forall v \in V): a(u, v)=F(v)-\alpha b(u, v)
$$

which is what we were trying to prove.

Theorem 3.6 (Lax-Milgram Lemma) Let $V$ be a Hilbert space, let $a: V \times V \rightarrow \mathbb{R}$ be a bilinear form and let $F \in V^{*}$. Assume the following

- $(\exists c>0)(\forall u \in V): a(u, u) \geq c\|u\|^{2}$ (ellipticity of $a$ ).
- $(\exists k>0)(\forall u, v \in V): a(u, v) \leq k\|u\|\|v\|$ (continuity of $a$ ).

Then

$$
(\exists!u \in V)(\forall v \in V): a(u, v)=F(v) .
$$

Proof We will begin by taking the symmetric part and antisymmetric part of $a$, that is

$$
a_{S}(u, v)=\frac{1}{2}(a(u, v)+a(v, u)),
$$

$$
a_{A}(u, v)=\frac{1}{2}(a(u, v)-a(v, u)) .
$$

Next we will define $a_{t}(u, v)=a_{S}(u, v)+t a_{A}(u, v)$ where $t \in[0,1]$. These forms have the following properties

- $a_{A}$ is continuous with the constant $k$. For all $u, v \in V$ we have $\left|a_{A}(u, v)\right| \leq \frac{1}{2}(|a(u, v)|+$ $|a(v, u)|) \leq k\|u\|\|v\|$.
- $a_{t}$ is elliptic for each $t \in[0,1]$ with the constant $c$. For all $u \in V$ it holds $a_{t}(u, u)=a(u, u) \geq$ $c\|u\|^{2}$.
- $a_{t}$ is continuous for all $t \in[0,1]$ with the constant $k$. For all $u, v \in V$ it holds

$$
\left|a_{t}(u, v)\right|=\frac{1+t}{2}|a(u, v)|+\frac{1-t}{2}|a(v, u)| \leq k\|u\|\|v\|\left(\frac{1+t}{2}+\frac{1-t}{2}\right)=k\|u\|\|v\|
$$

We claim that if

$$
\begin{equation*}
(\exists!u \in V)(\forall v \in V): a_{t}(u, v)=F(v) \tag{3.6}
\end{equation*}
$$

for some $t \in[0,1)$ then for each $\alpha \in\left(0, \frac{c}{k}\right)$ such that $t+\alpha \leq 1$ we have

$$
\begin{equation*}
(\exists!u \in V)(\forall v \in V): a_{t+\alpha}(u, v)=F(v) . \tag{3.7}
\end{equation*}
$$

This can be shown using lemma 3.5 since

$$
a_{t+\alpha}(u, v)=a_{S}(u, v)+(t+\alpha) a_{A}(u, v)=a_{t}(u, v)+\alpha a_{A}(u, v)
$$

where $a_{t}(u, v)$ is elliptic on $V$ with a constant $c$ and $a_{A}(u, v)$ is continuous with a constant $k$ and $\alpha<\frac{c}{k}$.
$a_{S}$ is an inner product on $V$ because it is symmetric, bilinear and $a_{S}(v, v)>0$ if $v$ is not the zero element of $V$ (this comes from ellipticity of $a_{0}=a_{S}$ ). Furthermore the norm induced from $a_{S}$ is equivalent to the original norm of $V$, this is a result of continuity and ellipticity of $a_{0}=a_{S}$, hence by the Riesz representation theorem we have

$$
(\exists!u \in V)(\forall v \in V): a_{S}(u, v)=F(v),
$$

which gives us that some $a_{t}$ for a larger $t$ again satisfies assumption (3.6), this constitutes a repeatable process, since the step length by which we can increase $t$ each time is constant, by repeating this process finitely many times we will arrive at

$$
(\exists!u \in V)(\forall v \in V): a_{1}(u, v)=a(u, v)=F(v) .
$$

### 3.4 Fractals

Definition 3.4 (Hausdorff Distance) Let $M$ be a metric space and let $X, Y \subset M$ be non-empty, we call

$$
d_{H}(X, Y)=\max \left\{\sup _{x \in X} d(x, Y), \sup _{y \in Y} d(y, X)\right\}
$$

the Hausdorff distance of $X$ and $Y$.
Theorem 3.7 Hausdorff distance constitutes a metric over any collection $F$ of non-empty closed bounded subsets of some metric space $M$.

## Proof

1. $d_{H}: F \times F \rightarrow \mathbb{R}$. All members of $F$ are bounded so it is clear that $d_{H}$ can only attain finite values.
2. $d_{H}(X, X)=0$ for all non-empty subsets $X$ of $M$. This is apparent from $\sup _{x \in X} d(x, X)=0$.
3. $X \neq Y \Longrightarrow d_{H}(X, Y)>0$ for all $X, Y \in F$. Both $X$ and $Y$ are closed sets with $X \neq Y$, this means that one of these sets contains a point $p$ that does not lie in the closure of the other set, we may assume that $p \in X$. Since this is the case there exists $\varepsilon>0$ such that $d(p, y) \geq \varepsilon$ for all $y \in Y$, i.e., $d(p, Y) \geq \varepsilon$, hence

$$
\sup _{x \in X} d(x, Y) \geq d(p, Y) \geq \varepsilon .
$$

From the definition of $d_{H}$ this implies that $d_{H}(X, Y) \geq \varepsilon>0$.
4. symmetry of $d_{H}$ comes from symmetry of the max operator.
5. Triangle inequality: Let $X, Y, Z$ be non-empty subsets of $M$. We will begin by showing that it holds

$$
d(x, Z) \leq d(x, y)+d(y, Z) \quad(\forall x \in X)(\forall y \in Y)
$$

For each $x \in X, y \in Y, z \in Z$ it holds

$$
d(x, z) \leq d(x, y)+d(y, z),
$$

taking a sequence $\left(z_{n}\right) \subset Z$ so that $d\left(y, z_{n}\right) \rightarrow d(y, Z)$ we get

$$
d(x, Z)=\inf _{z \in Z} d(x, z) \leq d\left(x, z_{n}\right) \leq d(x, y)+d\left(y, z_{n}\right), \quad(\forall x \in X),(\forall y \in Y)
$$

sending $n \rightarrow \infty$ yields

$$
d(x, Z) \leq d(x, y)+d(y, Z) . \quad(\forall x \in X),(\forall y \in Y)
$$

We will call this property the triangle inequality for point-to-set distance. Following up on this take a sequence $\left(y_{n}\right) \subset Y$ such that $d\left(x, y_{n}\right) \rightarrow d(x, Y)$, we have

$$
d(x, Z) \leq d\left(x, y_{n}\right)+d\left(y_{n}, Z\right) \leq d\left(x, y_{n}\right)+\sup _{y \in Y} d(y, Z) \leq d\left(x, y_{n}\right)+d_{H}(Y, Z) . \quad(\forall x \in X)
$$

sending $n \rightarrow \infty$ gives us

$$
d(x, Z) \leq d(x, Y)+d_{H}(Y, Z) . \quad(\forall x \in X)
$$

Once more we will take a sequence $\left(x_{n}\right) \in X$ such that $d\left(x_{n}, Z\right) \rightarrow \sup _{x \in X} d(x, Z)$, it holds

$$
d\left(x_{n}, Z\right) \leq d\left(x_{n}, Y\right)+d_{H}(Y, Z) \leq \sup _{x \in X} d(x, Y)+d_{H}(Y, Z) \leq d_{H}(X, Y)+d_{H}(Y, Z)
$$

Letting $n \rightarrow \infty$ we get

$$
\begin{equation*}
\sup _{x \in X} d(x, Z) \leq d_{H}(X, Y)+d_{H}(Y, Z) \tag{3.8}
\end{equation*}
$$

Note that $d_{H}(X, Z)=\max \left\{\sup _{x \in X} d(x, Z), \sup _{z \in Z} d(z, X)\right\}$, so to finish the proof we need to show that

$$
\sup _{z \in Z} d(z, X) \leq d_{H}(X, Y)+d_{H}(Y, Z),
$$

but this is nothing but swapping $X$ for $Z$ and vice versa in (3.8) which we may do since the same is assumed about both sets.

Note that if we allow a metric to attain the value of infinity, boundedness is not needed, this proof also tells us that if we do not require members of $F$ to be closed sets then $d_{H}$ is a pseudometric, i.e., has all the properties of a metric except the property that two points have zero distance implies that the two points are equal.

Theorem 3.8 If $M$ is a complete metric space then the collection of all non-empty compact sets denoted as $H(M)$ endowed with the Hausdorff metric is a complete metric space in and of itself.

The proof of this theorem can be found in [8].

Definition 3.5 (Iterated Function System) We call a family of contractions $f_{1}, \ldots, f_{m}: \mathbb{R}^{n} \rightarrow$ $\mathbb{R}^{n}$ an iterated function system and with it associate the union map $H\left(\mathbb{R}^{n}\right) \rightarrow H\left(\mathbb{R}^{n}\right)$ defined as

$$
f(X)=\bigcup_{i \in\{1, \ldots, m\}} f_{i}(X) \equiv \bigcup f_{i}(X)
$$

Theorem 3.9 Let $f_{1}, \ldots, f_{m}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be an iterated function system, then the union map $f$ associated with it is a contraction, i.e., for all $X, Y \in H\left(\mathbb{R}^{n}\right)$ it holds

$$
d_{H}(f(X), f(Y)) \leq \lambda d_{H}(X, Y) . \quad(\text { for some } \lambda<1)
$$

Proof Consider

$$
d_{H}\left(\bigcup f_{i}(X), \bigcup f_{i}(Y)\right)=\max \left\{\sup _{x_{f} \in \bigcup f_{i}(X)} d\left(x_{f}, \bigcup f_{i}(Y)\right), \sup _{y_{f} \in \bigcup f_{i}(Y)} d\left(y_{f}, \bigcup f_{i}(X)\right)\right\} .
$$

Fix $x_{f} \in \bigcup f_{i}(X)$, then $x_{f} \in f_{k}(X)$ for some $k \in\{1, \ldots, m\}$, we have

$$
d\left(x_{f}, \bigcup f_{i}(Y)\right) \leq d\left(x_{f}, f_{k}(Y)\right) \leq \sup _{x \in X} d\left(f_{k}(x), f_{k}(Y)\right) \leq \lambda_{k} \sup _{x \in X} d(x, Y)
$$

in general this gives us

$$
d\left(x_{f}, \bigcup f_{i}(Y)\right) \leq \max _{i \in\{1, \ldots, m\}}\left(\lambda_{i}\right) \sup _{x \in X} d(x, Y)
$$

implying that

$$
\sup _{x_{f} \in \bigcup f_{i}(X)} d\left(x_{f}, \bigcup f_{i}(Y)\right) \leq \max _{i \in\{1, \ldots, m\}}\left(\lambda_{i}\right) \sup _{x \in X} d(x, Y) .
$$

Note that $X$ and $Y$ here are interchangeable, so the inequality

$$
\sup _{y_{f} \in \bigcup f_{i}(Y)} d\left(y_{f}, \bigcup f_{i}(X)\right) \leq \max _{i \in\{1, \ldots, m\}}\left(\lambda_{i}\right) \sup _{y \in Y} d(y, X)
$$

also holds. In terms of Hausdorff distance this says that

$$
d_{H}(f(X), f(Y)) \leq \max _{i \in\{1, \ldots, m\}}\left(\lambda_{i}\right) d_{H}(X, Y),
$$

which is what we were trying to show.

Theorem 3.10 (Collage Theorem) Let $f_{1}, \ldots, f_{m}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be an iterated function system with contraction constants $\lambda_{1}, \ldots, \lambda_{m}$, and denote $\lambda=\max _{i \in\{1, \ldots, m\}} \lambda_{i}$, then there exists a unique fixed point $F \in H\left(\mathbb{R}^{n}\right)$ of the union map $f$ such that for all $X \in H\left(\mathbb{R}^{n}\right)$ it holds

$$
\lim _{k \rightarrow \infty} f^{k}(X) \rightarrow F
$$

Moreover for all $X \in H\left(\mathbb{R}^{n}\right)$ it holds

$$
d_{H}(X, F)<\frac{d_{H}(X, f(X))}{1-\lambda} .
$$

Proof From theorem 3.9 we know that the union map $f: H\left(\mathbb{R}^{n}\right) \rightarrow H\left(\mathbb{R}^{n}\right)$ is a contraction, hence by the Banach fixed point theorem it has a unique fixed point and for any $X \in H\left(\mathbb{R}^{n}\right)$ we have

$$
\lim _{k \rightarrow \infty} f^{k}(X)=F
$$

We can derive the convergence estimate in the following way

$$
d_{H}(X, F)=d_{H}\left(X, \lim _{k \rightarrow \infty} f^{k}(X)\right)=\lim _{k \rightarrow \infty} d_{H}\left(X, f^{k}(X)\right)
$$

the last equality comes from continuity of a metric, using the triangle inequality we get

$$
d_{H}\left(X, f^{k}(X)\right) \leq\left(d_{H}(X, f(X))+d_{H}\left(f(X), f^{2}(X)\right)+\ldots+d_{H}\left(f^{k-1}(X), f^{k}(X)\right)\right), \quad(\forall k \in \mathbb{N})
$$

using the fact that $f$ is a contraction we get

$$
d_{H}(X, F) \leq \lim _{k \rightarrow \infty} d_{H}(X, f(X))\left(1+\lambda+\ldots+\lambda^{k-1}\right)=\frac{d_{H}(X, f(X))}{1-\lambda}
$$

giving us the result.

Definition 3.6 We call $F \in H\left(\mathbb{R}^{n}\right)$ self-similar if it is a fixed point of the union map $f$ of some iterated function system $f_{1}, \ldots, f_{m}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$.

Definition 3.7 (Hausdorff Measure) Let $X \subset \mathbb{R}^{n}$ for some $n \in \mathbb{N}$, define

$$
H_{\delta}^{s}(X)=\inf \left\{\sum_{i=1}^{\infty} \operatorname{diam}\left(U_{i}\right)^{s}:\left(U_{i}\right) \text { is a } \delta \text {-cover of } X\right\}, \quad(s \geq 0)(\delta>0)
$$

where by $\delta$-cover of $X$ we mean a countable (or finite collection) of sets $U_{i} \subset \mathbb{R}^{n}$ with diameter less or equal to $\delta$ such that

$$
X \subset \bigcup_{i=1}^{\infty} U_{i} .
$$

We call

$$
H^{s}(X)=\lim _{\delta \rightarrow 0_{+}} H_{\delta}^{s}(X)
$$

the s-dimensional Hausdorff measure of $X$.

Theorem 3.11 The Hausdorff measure is well defined for every set $X \subset \mathbb{R}^{n}$, but it may be infinite.
Proof First of all for each $\delta>0$ there exists a countable or finite $\delta$-cover $\left(U_{i}\right)$ of $X \subset \mathbb{R}^{n}$, we can just take the collection of balls with diameter $\delta$ centered around each $q \in \mathbb{Q}^{n}$, as $\mathbb{Q}^{n}$ is a countable set that is dense in $\mathbb{R}^{n}$ this would give us a countable $\delta$-cover of $\mathbb{R}^{n}$ and in turn all its subsets.

Now fixing $s \geq 0, \delta>0, H_{\delta}^{s}(X)$ exists since we are taking an infimum of a set of real numbers. Now as $\delta$ decreases the set of feasible $\delta$-covers gets smaller, meaning that for each $t>0$

$$
H_{\delta+t}^{s}(X)=\inf \left\{\sum_{i=1}^{\infty} \operatorname{diam}\left(U_{i}\right)^{s}:\left(U_{i}\right) \text { is a }(\delta+t) \text {-cover of } X\right\} \leq H_{\delta}^{s}(X)
$$

if $H_{\delta}^{s}(X)$ as a function of $\delta$ is bounded above, then it converges since it non-increasing, otherwise $H_{\delta}^{s}(X)$ goes to infinity as $\delta \rightarrow 0_{+}$.

This theorem is not a reason to panic, we are not claiming that the Hausdorff measure is a measure if taken over $\mathbb{R}^{n}$, however it can be shown that it is a measure if taken over the Borel subsets of $\mathbb{R}^{n}$, the proof can be found in [9].

Theorem 3.12 Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ and let $X \subset \mathbb{R}^{n}$, if there exists $\lambda>0$ s.t. for all $x, y \in \mathbb{R}^{n}$ it holds

$$
\|f(x)-f(y)\| \leq \lambda\|x-y\|
$$

then

$$
H^{s}(f(X)) \leq \lambda^{s} H^{s}(X)
$$

Proof We will begin by proving the inequality

$$
H^{s}(f(X)) \leq \lambda^{s} H^{s}(X)
$$

Suppose that $X \subset \mathbb{R}^{n}$ and $\left(U_{i}\right)$ is a $\delta$-cover of $X$, then

$$
\operatorname{diam}\left(f\left(X \cap U_{i}\right)\right) \leq \lambda \operatorname{diam}\left(X \cap U_{i}\right) \leq \lambda \operatorname{diam}\left(U_{i}\right), \quad(\forall i \in \mathbb{N})
$$

Clearly $\left(X \cap U_{i}\right)$ is a $\delta$-cover of $X$, thus the inequality above implies that $\left(f\left(X \cap U_{i}\right)\right)$ is a $\lambda \delta$-cover of $f(X)$. We also have

$$
\sum_{i=1}^{\infty} \operatorname{diam}\left(f\left(X \cap U_{i}\right)\right)^{s} \leq \lambda^{s} \sum_{i=1}^{\infty} \operatorname{diam}\left(U_{i}\right)^{s}
$$

denote the set of all $\delta$-covers $\left(U_{i}\right)$ of $X$ as $C$, taking an infimum over this set we get

$$
\underbrace{\inf _{\left(U_{i}\right) \in C} \sum_{i=1}^{\infty} \operatorname{diam}\left(f\left(X \cap U_{i}\right)\right)^{s}}_{\geq H_{\lambda \delta}^{s}(f(X))} \leq \lambda^{s} H_{\delta}^{s}(X) .
$$

The result follows from sending $\delta \rightarrow 0_{+}$.
Theorem 3.13 (Scaling Property) Let $X \subset \mathbb{R}^{n}$ and let $\lambda>0$, it holds

$$
H^{s}(\lambda X)=\lambda^{s} H^{s}(X)
$$

We can see that this property holds just by looking at the definition of Hausdorff measure.
Definition 3.8 (Hausdorff Dimension) Hausdorff dimension is derived from Hausdorff measure, suppose that $\left(U_{i}\right)$ is a $\delta$-cover of an arbitrary set $X \subset \mathbb{R}^{n}$, and let $s, t \in \mathbb{R}^{+}$with $t>s$, it holds

$$
\sum_{i=1}^{\infty} \operatorname{diam}\left(U_{i}\right)^{t}=\sum_{i=1}^{\infty} \operatorname{diam}\left(U_{i}\right)^{t-s} \operatorname{diam}\left(U_{i}\right)^{s} \leq \delta^{t-s} \sum_{i=1}^{\infty} \operatorname{diam}\left(U_{i}\right)^{s}, \quad(\forall i \in \mathbb{N})
$$

this implies that

$$
H_{\delta}^{t}(X) \leq \delta^{t-s} H_{\delta}^{s}(X)
$$

Observe that if $H^{s}(X)<\infty$, we know that $H^{t}(X)=0$. This motivates defining Hausdorff dimension as the value of $s$ such that

$$
s=\inf \left\{t>0: H^{t}(X) \in \mathbb{R}\right\} .
$$

Notice that this way, if $H^{s}(X)$ is finite for every $s>0$, then $X$ has Hausdorff dimension 0, conversely if $H^{s}(X)=\infty$ for every $s>0$, the Hausdorff dimension of $X$ is $\infty$.

There are multiple different ways to define dimension of a fractal, under certain conditions these definitions are equivalent, the key aspect of these dimensions is the scaling property described in theorem 3.13 which we can associate it with if we do not want to delve into complicated definitions.

Definition 3.9 (Fractal) We call a self-similar set with non-integer Hausdorff dimension a fractal.

There are many ways to define a fractal, this definition covers two most common properties we expect from fractals, that is, for one fine structure or a certain roughness of the shape which is what non-integer Hausdorff dimension corresponds to. Another common feature is some form of self-similarity, we often view fractals as objects that upon zooming in repeat certain features or even stay the same.

Theorem 3.14 Let $f_{1}, \ldots, f_{m}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be an iterated function system that satisfies

$$
\begin{equation*}
d_{H}\left(f_{i}(X), f_{i}(Y)\right)=\lambda_{i}\left(d_{H}(X, Y)\right), \quad(\forall i \in\{1, \ldots, m\}) \tag{3.9}
\end{equation*}
$$

and let $F$ be the unique fixed point of the union map $f$. Suppose that there exists a non-empty open bounded set $S$ such that

$$
\begin{equation*}
\bigcup_{i=1}^{m} f_{i}(S) \subset S, \tag{3.10}
\end{equation*}
$$

where $f_{i}(S)$ for $i \in\{1, \ldots, m\}$ must be disjoint, then the Hausdorff dimension of $F$ denoted as $s$ satisfies

$$
\sum_{i=1}^{m} \lambda_{i}^{s}=1
$$

The proof can be found in [9].
This theorem lets us easily classify whether a set we produce by iterating a union map of an iterated function system is a fractal.

Let us see an example of what an iterated function system that satisfies condition (3.10) looks like. Define $f_{1}, f_{2}: \mathbb{R}^{1} \rightarrow \mathbb{R}^{1}$ so that $f_{1}(x)=\frac{1}{3} x$ and $f_{2}(x)=\frac{1}{3} x+\frac{2}{3}$, both satisfy condition (3.9) with $\lambda_{1}=\lambda_{2}=\frac{1}{3}$. Let $S=(0,1)$, then $f_{1}(S)=\left(0, \frac{1}{3}\right)$ and $f_{2}(S)=\left(\frac{2}{3}, 1\right)$, these two sets are disjoint and satisfy condition (3.10), the fixed point of the union map of these functions called is the Cantor middle third set, its Hausdorff dimension $s$ is given by

$$
\frac{1}{3^{s}}+\frac{1}{3^{s}}=1,
$$

this is equivalent to

$$
\log _{3}\left(\frac{2}{3^{s}}\right)=\log _{3}(2)-s=0
$$

hence $s=\log _{3}(2) \approx 0.6309$.
We might wonder if every self-similar set is a fractal. Define $f_{1}, f_{2}, f_{3}, f_{4}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$, as

$$
\begin{aligned}
f_{1}(x) & =\frac{1}{2} x \\
f_{2}(x) & =\frac{1}{2} x+\left(0, \frac{1}{2}\right) \\
f_{3}(x) & =\frac{1}{2} x+\left(\frac{1}{2}, 0\right) \\
f_{4}(x) & =\frac{1}{2} x+\left(\frac{1}{2}, \frac{1}{2}\right)
\end{aligned}
$$

Clearly each of these mappings satisfy condition (3.9) with $\lambda_{1}=\lambda_{2}=\lambda_{3}=\lambda_{4}=\frac{1}{2}$. Let $S=$ $(0,1) \times(0,1)$, then

$$
\bigcup_{i=1}^{4} f_{i}(S) \subset S
$$

and the images $f_{i}(S)$ with $i=1, \ldots, 4$ are disjoint. By theorem 3.14, the Hausdorff dimension $s$ of the fixed point of the union map $F$ (which is actually the square $[0,1] \times[0,1]$ ) is given by

$$
\sum_{i=1}^{4}\left(\frac{1}{2}\right)^{s}=1
$$

this implies that

$$
\log _{2}\left(4 \frac{1}{2^{s}}\right)=\log _{2}(4)-s=0
$$

implying that $s=2$, we can see that not all self-similar sets, even such that satisfy assumptions of theorem 3.14 are necessarily fractals.

Now it is time to create some fractals for ourselves. We will showcase the Koch curve generated
by functions $f_{1}, f_{2}, f_{3}, f_{4}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ defined as

$$
\begin{gathered}
f_{1}\left(x_{1}, x_{2}\right)=\left(\begin{array}{ll}
\frac{1}{3} & 0 \\
0 & \frac{1}{3}
\end{array}\right)\binom{x_{1}}{x_{2}} \\
f_{2}\left(x_{1}, x_{2}\right)=\left(\begin{array}{cc}
\frac{1}{3} \cos \left(\frac{\pi}{3}\right) & -\frac{1}{3} \sin \left(\frac{\pi}{3}\right) \\
\frac{1}{3} \sin \left(\frac{\pi}{3}\right) & \frac{1}{3} \cos \left(\frac{\pi}{3}\right)
\end{array}\right)\binom{x_{1}}{x_{2}}+\binom{\frac{1}{3}}{0} \\
f_{3}\left(x_{1}, x_{2}\right)=\left(\begin{array}{cc}
\frac{1}{3} \cos \left(-\frac{\pi}{3}\right) & -\frac{1}{3} \sin \left(-\frac{\pi}{3}\right) \\
\frac{1}{3} \sin \left(-\frac{\pi}{3}\right) & \frac{1}{3} \cos \left(-\frac{\pi}{3}\right)
\end{array}\right)\binom{x_{1}}{x_{2}}+\binom{\frac{1}{2}}{\frac{\sqrt{3}}{6}} \\
f_{4}\left(x_{1}, x_{2}\right)=\left(\begin{array}{ll}
\frac{1}{3} & 0 \\
0 & \frac{1}{3}
\end{array}\right)\binom{x_{1}}{x_{2}}+\binom{\frac{2}{3}}{0}
\end{gathered}
$$

Here finding out whether these functions satisfy the requirements of theorem 3.14 is not as easy as with in the previous examples, clearly $f_{1}, \ldots, f_{4}$ are all contractions, however the question is what open set to choose to show that these functions satisfy the open set conditions. Let $S$ be the interior of the triangle with vertices $(0,0),(1,0)$ and $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$. Since $f_{1}, \ldots, f_{4}$ are all affine mappings, for each $i \in\{1, \ldots, 4\} f_{i}(S)$ is determined by where vertices of $S$ get mapped, let us see where $f_{1}, \ldots, f_{4}$ will map these vertices.

$$
\begin{array}{lll}
f_{1}(0,0)=(0,0), & f_{1}(1,0)=\left(\frac{1}{3}, 0\right), & f_{1}\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)=\left(\frac{1}{6}, \frac{\sqrt{3}}{18}\right), \\
f_{2}(0,0)=\left(\frac{1}{3}, 0\right), & f_{2}(1,0)=\left(\frac{1}{6}+\frac{1}{3}, \frac{1}{2 \sqrt{3}}\right), & f_{2}\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)=\left(\frac{1}{3}, \frac{1}{3 \sqrt{3}}\right), \\
f_{3}(0,0)=\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), & f_{3}(1,0)=\left(\frac{1}{6}+\frac{1}{2}, 0\right), & f_{3}\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)=\left(\frac{1}{6}+\frac{1}{2},-\frac{1}{6 \sqrt{3}}+\frac{\sqrt{3}}{6}\right), \\
f_{4}(0,0)=\left(\frac{2}{3}, 0\right), & f_{1}(1,0)=(1,0), & f_{4}\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)=\left(\frac{5}{6}, \frac{\sqrt{3}}{18}\right) .
\end{array}
$$

The plot of $\bigcup_{i=1}^{4} f_{i}(S)$ will be the interior of the triangles in the image bellow


Figure 3.1: $\bigcup_{i=1}^{4} \overline{f_{i}(S)}$.

We can see that $\bigcup_{i=1}^{4} f_{i}(S) \subset S$ and that for $i \in\{1,2,3,4\}$ the sets $f_{i}(S)$ are disjoint, thus the functions $f_{1}, \ldots, f_{4}$ satisfy the requirements of theorem 3.14 and we can compute the fractal dimension of the Koch curve. In condition (3.9) functions $f_{1}$ and $f_{4}$ have $\lambda_{1}$ and $\lambda_{2}$ both equal to $\frac{1}{3}$, this is easy to show, for all $x, y \in \mathbb{R}^{2}$ it holds

$$
\left\|f_{1}(x)-f_{1}(y)\right\|=\left\|f_{4}(x)-f_{4}(y)\right\|=\left\|\frac{1}{3}(x-y)\right\|=\frac{1}{3}\|x-y\| .
$$

In fact $\lambda_{2}$ and $\lambda_{3}$ are also equal to $\frac{1}{3}$, we can omit the step where we add constants as they will get canceled out by subtraction, then each of these functions is nothing but a rotation matrix multiplied by $\frac{1}{3}$. It is well known that rotation matrices preserve distance, thus the only aspect changing distance of two points is multiplication by $\frac{1}{3}$. With this in mind, the Hausdorff dimension $s$ of the Koch curve is given by

$$
\sum_{i=1}^{4} \frac{1}{3^{s}}=4 \frac{1}{3^{s}}=1
$$

from this we can derive that

$$
s=\log _{3}(4) \approx 1.2619
$$

If we take the set $[0,1]$ as an initiator, the iterations of the union map look in the following way


Figure 3.2: Initiator - 0 iterations.


Figure 3.3: 1 iteration.


Figure 3.4: 2 iterations.


Figure 3.5: 3 iterations.

Now take the set shown in the next image as the initiator


Figure 3.6: Initiator - 0 iterations.

We get the following results


Figure 3.7: 1 iteration.


Figure 3.8: 2 iterations.


Figure 3.9: 3 iterations.


Figure 3.10: 4 iterations.

We can observe that the first iteration looks like it has nothing to do with the Koch snowflake, in the second iteration the general outline is there but the curves are still quite different. The third iterations using these two initiators look pretty much the same. The third iteration also already has so many fine details that the fourth iteration does not show any change due to the thickness of the line in the plot. Next take the square $[0,1] \times[0,1]$ as a generator, we will use a dots to plot the graph this time, the iterations look like this


Figure 3.11: Initiator - 0 iterations.


Figure 3.12: 1 iteration.


Figure 3.13: 2 iterations.


Figure 3.14: 3 iterations.

Here we can make similar observations.

### 3.5 Game Theory Applications

Definition 3.10 (Normal Form Game) $A$ game is a triplet $\left(I,\left(A_{i}\right)_{i \in I},\left(p_{i}\right)_{i \in I}\right)$, where

- $I=\{1,2, \ldots, n\}, n \in \mathbb{N}$ is called the set of players.
- For each $i \in I, A_{i}$ is the set of strategies available to player $i$, no limitation other than nonemptiness as to what $A_{i}$ contains are imposed. If for some $i \in I$, the set $A_{i}$ is finite we will denote the number of its members as $m_{i}$.

We call $A=\times_{i \in I} A_{i}$ the set of strategy profiles. Every $a \in A$ is called a strategy profile and can be written as $a=\left(a_{1}, \ldots, a_{n}\right)$ where $a_{i}$ is a strategy player $i$ has chosen.

- For each $i \in I, p_{i}: A \rightarrow \mathbb{R}$ is called the payoff function for player $i$, we call the mapping $p: S \rightarrow \mathbb{R}^{n}$ defined as $p \equiv\left(p_{1}, \ldots, p_{n}\right)$ the payoff function.

We will only consider the case where there is a finite number of players. Let us look at a classical example of a normal form game known as the prisoner's dilemma. The police have taken two suspects of grand theft auto into custody and put them in solitary confinements with no means of communicating with each other. Each of the suspects are given the option to confess or to remain silent and are told the consequences of their choices:

- If no one confesses, the suspects will both be sentences to 1 year in prison for possession of stolen goods.
- If one confesses and the other one remains silent, the one who confessed will go free and the other one will be sentences to 10 years in prison.
- If they both confess, both will be sentenced to 5 years in prison.

This game has two players, both possess the same strategy set consisting of strategies "remain silent" and "confess", the payoff function can be defined as

$$
\begin{aligned}
p(\text { "remain silent", "remain silent" }) & =(-1,-1) \\
p(\text { "remain silent", "confess" }) & =(-10,0) \\
p(\text { "confess", "remain silent" }) & =(0,-10) \\
p(\text { "confess", "confess" }) & =(-5,-5)
\end{aligned}
$$

We can visualize this problem with the following table

| Player $1 \rightarrow$ <br> Player 2 $\downarrow$ | remain <br> silent | confess |
| :--- | :--- | :--- |
| remain <br> silent | $(-1,-1)$ | $(0,-10)$ |
| confess | $(-10,0)$ | $(-5,-5)$ |

Definition 3.11 Here we define the following types of normal form games.

- Suppose that we have a normal form game $\left(I,\left(A_{i}\right)_{i \in I},\left(p_{i}\right)_{i \in I}\right)$ where for each $i \in I$, the set $A_{i}$ is finite and contains $m_{i} \in \mathbb{N}$ members. From it we can create a mixed strategy game in the following manner.
- Let $S_{i}=\Pi\left(A_{i}\right)$, where $\Pi\left(A_{i}\right)$ denotes the set of all probability distributions over $A_{i}$, we call $S_{i}$ the set of mixed strategies available to player $i$.
- Let $s_{i} \in S_{i}$, and $a_{i} \in A_{i}$ then $s_{i}\left(a_{i}\right)$ denotes the probability that pure strategy $a_{i}$ will be chosen under mixed strategy $s_{i}$.
- Suppose that $s \in \times_{i \in I} S_{i}$, we define the mixed strategy payoff function for player $i$ as

$$
u_{i}(s)=\sum_{a \in \times_{i \in I} A_{i}} p_{i}(a) \prod_{j \in I} s_{j}\left(a_{j}\right) .
$$

We call $\left(I,\left(S_{i}\right)_{i \in I},\left(u_{i}\right)_{i \in I}\right)$ a mixed strategy game. We can view this as players trying to be more unpredictable and choosing their strategy with a certain degree of randomness.

- If a game is not a mixed strategy game, we call it a pure strategy game.

We can see that the prisoners dilemma is a pure strategy game.
Since for each $i \in I$ the set $A_{i}$ is finite, the set $S_{i}$ is fully defined by a vector containing numbers $\left(\lambda_{1}, \ldots, \lambda_{m_{i}}\right)$ that satisfy

$$
\begin{equation*}
\sum_{k=1}^{m_{i}} \lambda_{k}=1,\left(\forall k \in\left\{1, \ldots, m_{i}\right\}\right): \lambda_{k} \geq 0 \tag{3.11}
\end{equation*}
$$

each $\lambda_{k}$ represents the probability that pure strategy $a_{k} \in A_{i}$ will be picked. Note that

$$
\begin{equation*}
S_{i}=\operatorname{conv}\{(\underbrace{1,0, \ldots, 0}_{\text {length }=m_{i}}),(0,1,0, \ldots, 0), \ldots,(0, \ldots, 0,1)\} . \tag{3.12}
\end{equation*}
$$

Each of the vectors composing the convex combination in (3.12) represents a pure strategy in $A_{i}$, so in some sense we can view $S_{i}$ as the set of convex combinations of pure strategies in $A_{i}$.

Definition 3.12 (Nash Equilibrium) Let $\left(I,\left(A_{i}\right)_{i \in I},\left(p_{i}\right)_{i \in I}\right)$ be a normal form game and let $a^{*}=\left(a_{1}^{*}, \ldots, a_{n}^{*}\right) \in A$ be a strategy profile. We shall establish the following notation

$$
\left(a_{i}^{*}, a_{-i}^{*}\right) \equiv\left(a_{1}^{*}, \ldots, a_{n}^{*}\right),
$$

where $a_{i}^{*} \in A_{i}$ represents the strategy player $i$ has chosen and $a_{-i}^{*} \in \times_{j \in I \backslash\{i\}} A_{j}$ contains the strategies of all players other than player $i$. We call $a^{*}$ a Nash equilibrium if for each $i \in I$ we have

$$
\begin{equation*}
u_{i}\left(a_{i}^{*}, a_{-i}^{*}\right) \geq u_{i}\left(a_{i}, a_{-i}^{*}\right), \quad\left(\forall a_{i} \in A_{i}\right) . \tag{3.13}
\end{equation*}
$$

A strategy profile $a^{*}$ being a Nash equilibrium means that all players have chosen strategies such that no one person can increase their payoff by changing their strategy. The Nash equilibrium is one of the ways we can define a solution of a normal form game. In case of the prisoners dilemma, the strategy profile ("confess", "confess") is a unique Nash equilibrium, it is even strict meaning that we can use a strict inequality in (3.13). From this we can see that in some types of games the Nash equilibrium is not the best way to define their solution.

Theorem 3.15 (Existence of a Nash Equilibrium) Let $\left(I,\left(S_{i}\right)_{i \in I},\left(u_{i}\right)_{i \in I}\right)$ be a mixed strategy game. If the set of players I is non-empty (we also already assumed I to be finite), and for each $i \in I$ the finite set $A_{i}$ of pure strategies available to each player is non-empty, then there exists a Nash equilibrium $s^{*} \in S$.

Proof Let $s \in S$ be a mixed strategy profile and analogously to the definition of Nash equilibrium let $s_{i} \in S_{i}$ to be the strategy player $i$ has chosen and let $s_{-i}=s \backslash s_{i}$ represent the strategies of the other players. For each $i \in I$ define the mapping $\Phi_{i}: S \rightarrow 2^{S_{i}}$ as

$$
\Phi_{i}(s)=\arg \max _{\tilde{s}_{i} \in S_{i}} u_{i}\left(\tilde{s}_{i}, s_{-i}\right)
$$

$\Phi_{i}$ actually does not need the whole $s$ in the input only $s_{-i}$ but defining it this way is more convenient for us, the mapping is given a strategy profile returns strategies player $i$ can use so that his payoff is optimal, keep in mind that there can be multiple strategies that have optimal payoff. Next step is defining mapping $\Phi: S \rightarrow 2^{S}$, denote the number of players as $n$ and let

$$
\begin{equation*}
\Phi(s)=\Phi_{1}(s) \times \ldots \times \Phi_{n}(s) \tag{3.14}
\end{equation*}
$$

From the definition of $\Phi_{i}$ it should be clear that if $s^{*}$ is a fixed point of $\Phi$, i.e., $s^{*} \in \Phi\left(s^{*}\right)$, then it is a Nash equilibrium. We will show that $S$ can be viewed as a subset of $\mathbb{R}^{d}$ for some $d \in \mathbb{N}$ and that $\Phi$ and $S$ satisfy the assumptions of the Kakutani fixed point theorem 2.24. Let us go by these assumptions one by one.

- $S$ is a subset of $\mathbb{R}^{d}$ for some $d \in \mathbb{N}$ : Each $s \in S$ can be written as

$$
s=\left(s_{1}, \ldots, s_{n}\right),
$$

Since for each $i \in I$, the set $A_{i}$ is finite, $s_{i}$ is given by a finite vector of non-negative real numbers $\left(\lambda_{1}, \ldots, \lambda_{m_{i}}\right), m_{i} \in \mathbb{N}$. It is easy to see that $s \in \mathbb{R}^{d}$, where $d=\sum_{i \in I} \operatorname{dim}\left(s_{i}\right)$.

- $S$ is non-empty: This follows from $I$ as well as $A_{i}$ for each $i \in I$ being non-empty.
- $S$ is compact: It is clear that for each $i \in I$ the set $S_{i}$ is compact, since $S=\times_{i \in I} S_{i}$ it is clear that $S$ is compact.
- $S$ is convex: From (3.12) we can see that for each $i \in I$ the set $S_{i}$ is convex. $S$ being the cartesian product of $S_{i}$ through all $i \in I$ is also convex, indeed suppose that $v, w \in S$ and let $t \in[0,1]$ be given arbitrarily, then

$$
(1-t) v+t w=(1-t)\left(v_{1}, \ldots, v_{n}\right)+t\left(w_{1}, \ldots, w_{n}\right)=\left((1-t) v_{1}+t w_{1}, \ldots,(1-t) v_{n}+t w_{n}\right) .
$$

But since $S_{i}$ is a convex set for each $i \in I$ the vector $(1-t) v_{i}+t w_{i} \in S_{i}$, thus $(1-t) v+t w \in S$.

- Images of $\Phi$ are non-empty: Remember that we denote the number of pure strategies in $A_{i}$ as $m_{i}$. For each $i \in I$ the set $S_{i}$ is compact and the function $u_{i}$ a composition of continuous functions, so $\arg \max _{\tilde{s}_{i} \in S_{i}} u_{i}\left(\tilde{s}_{i}, s_{-i}\right)$ always exists, implying that for all $s \in S$ the set $\Phi(s)$ is non-empty.
- Images of $\Phi$ are convex: Suppose that for some $s \in S$ we have $v, w \in \Phi(s)$, since $v, w \in S$ we may again write that

$$
\begin{aligned}
v & =\left(v_{1}, \ldots, v_{n}\right), \\
w & =\left(w_{1}, \ldots, w_{n}\right) .
\end{aligned}
$$

Now let $t \in[0,1]$ be given, we already know that

$$
(1-t) v+t w=\left((1-t) v_{1}+t w_{1}, \ldots,(1-t) v_{n}+t w_{n}\right) .
$$

In order to prove that $(1-t) v+t w \in \Phi(s)$ we need to show that for each $i \in I$ it holds

$$
(1-t) v_{i}+t w_{i} \in \Phi_{i}(s)=\arg \max _{\tilde{s}_{i} \in S_{i}} u_{i}\left(\tilde{s}_{i}, s_{-i}\right),
$$

where we use the notation $s=\left(s_{i}, s_{-i}\right)$ introduced before. Now let $i \in I$ be fixed, from the fact that

$$
\begin{aligned}
& v_{i} \in \arg \max _{\tilde{s}_{i} \in S_{i}} u_{i}\left(\tilde{s}_{i}, s_{-i}\right), \\
& w_{i} \in \arg \max _{\tilde{s}_{i} \in S_{i}} u_{i}\left(\tilde{s}_{i}, s_{-i}\right),
\end{aligned}
$$

we can see that

$$
\begin{equation*}
u_{i}\left(v_{i}, s_{-i}\right)=u_{i}\left(w_{i}, s_{-i}\right), \tag{3.15}
\end{equation*}
$$

each of these can be written as

$$
\begin{aligned}
& u_{i}\left(v_{i}, s_{-i}\right)=\sum_{a \in \times_{i=1}^{n} A_{i}} p_{i}(a) v_{i}\left(a_{i}\right) \prod_{j \in I \backslash\{i\}} s_{j}\left(a_{j}\right) \\
& u_{i}\left(w_{i}, s_{-i}\right)=\sum_{a \in \times_{i=1}^{n} A_{i}} p_{i}(a) w_{i}\left(a_{i}\right) \prod_{j \in I \backslash\{i\}} s_{j}\left(a_{j}\right)
\end{aligned}
$$

note that $a=\left(a_{1}, \ldots, a_{n}\right)$ where $a_{i}$ represents one of the pure strategies available to player $i$. These two equalities give us

$$
(1-t) u_{i}\left(v_{i}, s_{-i}\right)+t u_{i}\left(w_{i}, s_{-i}\right)=\sum_{a \in \times_{i=1}^{n} A_{i}} p_{i}(a)\left((1-t) v_{i} a_{i}+t w_{i}\left(a_{i}\right)\right) \prod_{j \in I \backslash\{i\}} s_{j}\left(a_{j}\right),
$$

but equality (3.15) implies that for the left side we have

$$
(1-t) u_{i}\left(v_{i}, s_{-i}\right)+t u_{i}\left(w_{i}, s_{-i}\right)=u_{i}\left(v_{i}, s_{-i}\right)
$$

and the right side is clearly equal to $u_{i}\left((1-t) v_{i}+t w_{i}, s_{-i}\right)$. These two observations say that

$$
u_{i}\left((1-t) v_{i}+t w_{i}, s_{-i}\right)=u_{i}\left(v_{i}, s_{-i}\right)=\max _{\tilde{s}_{i} \in S_{i}} u_{i}\left(\tilde{s}_{i}, s_{-i}\right)
$$

implying that

$$
(1-t) v_{i}+t w_{i} \in \Phi_{i}(s)
$$

- $\Phi$ is upper semi-continuous: Suppose that $\left(v^{l}\right),\left(w^{l}\right) \subset S$, with $v^{l} \rightarrow v$ and $w^{l} \rightarrow w$ where $\forall l \in \mathbb{N}: w^{l} \in \Phi\left(v^{l}\right)$, we will show that $w \in \Phi(v)$. Let $i \in I$ be given, the condition $w^{l} \in \Phi\left(v^{l}\right)$ means that

$$
w_{i}^{l} \in \arg \max _{\tilde{v}_{i} \in S_{i}} u_{i}\left(\tilde{v}_{i}, v_{-i}^{l}\right)
$$

implying that for all $\tilde{w}_{i} \in S_{i}$ it holds

$$
u_{i}\left(w_{i}^{l}, v_{-i}^{l}\right) \geq u_{i}\left(\tilde{w}_{i}, v_{-i}^{l}\right)
$$

sending $n \rightarrow \infty$ we obtain

$$
u_{i}\left(w_{i}, v_{-i}\right) \geq u_{i}\left(\tilde{w}_{i}, v_{-i}\right) . \quad\left(\forall \tilde{w}_{i} \in S_{i}\right)
$$

Since $i \in I$ was chosen arbitrarily, we conclude that $w \in \Phi(v)$.
By the Kakutani fixed point theorem there exists a fixed point of $\Phi$ in $S$, this concludes the proof.

The next theorem is a general statement that is applicable in game theory.

Theorem 3.16 (Minimax Theorem) Let mapping $f: X \times Y \rightarrow \mathbb{R}$ be continuous, where the sets $X \subset \mathbb{R}^{p}$ and $Y \subset \mathbb{R}^{q}$ for some $p, q \in \mathbb{N}$ are both non-empty, compact and convex. If for all $x_{0} \in X$ and for all $\alpha \in \mathbb{R}$ the set

$$
\left\{y \in Y: f\left(x_{0}, y\right) \leq \alpha\right\}
$$

is convex and for all $y_{0} \in Y$ and for all $\beta \in \mathbb{R}$ the set

$$
\left\{x \in X: f\left(x, y_{0}\right) \geq \beta\right\}
$$

is also convex, then it holds

$$
\max _{x \in X} \min _{y \in Y} f(x, y)=\min _{y \in Y} \max _{x \in X} f(x, y) .
$$

Proof We will begin by defining mappings $\Phi_{1}: X \rightarrow 2^{Y}$ and $\Phi_{2}: Y \rightarrow 2^{X}$ so that

$$
\Phi_{1}\left(x_{0}\right)=\left\{y_{0} \in Y: f\left(x_{0}, y_{0}\right)=\min _{y \in Y} f\left(x_{0}, y\right)\right\}
$$

and

$$
\Phi_{2}\left(y_{0}\right)=\left\{x_{0} \in X: f\left(x_{0}, y_{0}\right)=\max _{x \in X} f\left(x, y_{0}\right)\right\} .
$$

Next step is defining the mapping $\Phi: X \times Y \rightarrow 2^{X \times Y}$ as

$$
\Phi\left(x_{0}, y_{0}\right)=\Phi_{2}\left(y_{0}\right) \times \Phi_{1}\left(x_{0}\right)
$$

Clearly if some $\left(x_{0}, y_{0}\right) \in X \times Y$ satisfies $\left(x_{0}, y_{0}\right) \in \Phi\left(x_{0}, y_{0}\right)$, then it holds

$$
f\left(x_{0}, y_{0}\right)=\min _{y \in Y} f\left(x_{0}, y\right)=\max _{x \in X} f\left(x, y_{0}\right) .
$$

In order to show that such a point exists let us verify that $\Phi$ and the set $X \times Y$ satisfy the assumptions of the Kakutani fixed point theorem.

- $X \times Y$ is non-empty: This is assumed
- $X \times Y$ is compact: Both of these sets are assumed to be compact, hence their cartesian product must be compact.
- $X \times Y$ is convex: Suppose that $\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right) \in X \times Y$ and let $t \in[0,1]$ be given, then

$$
(1-t)\left(x_{1}, y_{1}\right)+t\left(x_{2}, y_{2}\right)=\left((1-t) x_{1}+t x_{2},(1-t) y_{1}+t y_{2}\right)
$$

since the sets $X, Y$ are assumed to be convex we know that

$$
\begin{aligned}
& (1-t) x_{1}+t x_{2} \in X \\
& (1-t) y_{1}+t y_{2} \in Y
\end{aligned}
$$

showing that $X \times Y$ is a convex set.

- Images of $\Phi$ are non-empty: For all $\left(x_{0}, y_{0}\right) \in X \times Y$ the sets $\Phi_{1}\left(x_{0}\right)$ and $\Phi_{2}\left(y_{0}\right)$ are non-empty because a minimum or maximum of a continuous function over a compact set always exists, so $\Phi\left(x_{0}, y_{0}\right)=\Phi_{2}\left(y_{0}\right) \times \Phi_{1}\left(x_{0}\right)$ is clearly always non-empty.
- Images of $\Phi$ are convex: Suppose that for some fixed $\left(x_{0}, y_{0}\right) \in X \times Y$ we have $\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right) \in$ $\Phi\left(x_{0}, y_{0}\right)$. We have

$$
f\left(x_{0}, y_{1}\right)=\min _{y \in Y} f\left(x_{0}, y\right)=f\left(x_{0}, y_{2}\right) \equiv \alpha,
$$

clearly $y_{1}, y_{2}$ are both members of the set

$$
\left\{y \in Y: f\left(x_{0}, y\right) \leq \alpha\right\}
$$

which is assumed to be convex. Let $t \in[0,1]$ be given, then

$$
(1-t) y_{1}+t y_{2} \in\left\{y \in Y: f\left(x_{0}, y\right) \leq \alpha\right\},
$$

implying that

$$
\begin{equation*}
(1-t) y_{1}+t y_{2} \in \Phi_{1}\left(x_{0}\right) \tag{3.16}
\end{equation*}
$$

because $\alpha=\min _{y \in Y} f\left(x_{0}, y\right)$. In a similar manner

$$
f\left(x_{1}, y_{0}\right)=\max _{x \in X} f\left(x, y_{0}\right)=f\left(x_{2}, y_{0}\right) \equiv \beta,
$$

so $x_{1}, x_{2}$ are both members of the set

$$
\left\{x \in X: f\left(x, y_{0}\right) \geq \beta\right\}
$$

also assumed to be convex, hence

$$
(1-t) x_{1}+t x_{2} \in\left\{x \in X: f\left(x, y_{0}\right) \geq \beta\right\},
$$

implying that

$$
\begin{equation*}
(1-t) x_{1}+t x_{2} \in \Phi_{2}\left(y_{0}\right) . \tag{3.17}
\end{equation*}
$$

From (3.16), (3.17) we can see that

$$
(1-t)\left(x_{1}, y_{1}\right)+t\left(x_{2}, y_{2}\right)=\left((1-t) x_{1}+t x_{2},(1-t) y_{1}+t y_{2}\right) \in \Phi_{2}\left(y_{0}\right) \times \Phi_{1}\left(x_{0}\right)=\Phi\left(x_{0}, y_{0}\right),
$$

showing that $\Phi\left(x_{0}, y_{0}\right)$ is convex.

- $\Phi$ is upper semi-continuous: Suppose that $\left(\left(x_{n}, y_{n}\right)\right),\left(\left(u_{n}, v_{n}\right)\right) \subset X \times Y$ with $\left(x_{n}, y_{n}\right) \rightarrow$ $\left(x_{0}, y_{0}\right)$ and $\left(u_{n}, v_{n}\right) \rightarrow\left(u_{0}, v_{0}\right)$, where for all $n \in \mathbb{N}$ it holds $\left(u_{n}, v_{n}\right) \in \Phi\left(x_{n}, y_{n}\right)$, we will show that $\left(u_{0}, v_{0}\right) \in \Phi\left(x_{0}, y_{0}\right)$. We have

$$
f\left(u_{n}, y_{n}\right)=\max _{u \in X} f\left(u, y_{n}\right)
$$

implying that

$$
\begin{equation*}
f\left(u_{n}, y_{n}\right) \geq f\left(u, y_{n}\right) . \quad(\forall u \in X) \tag{3.18}
\end{equation*}
$$

Similarly we have

$$
f\left(x_{n}, v_{n}\right)=\min _{v \in Y} f\left(x_{n}, v\right),
$$

meaning that

$$
\begin{equation*}
f\left(x_{n}, v_{n}\right) \leq f\left(x_{n}, v\right) . \quad(\forall v \in Y) \tag{3.19}
\end{equation*}
$$

Sending $n \rightarrow \infty$ and using continuity of $f$, equations (3.18) and (3.19) yield

$$
\begin{aligned}
& f\left(u_{0}, y_{0}\right) \geq f\left(u, y_{0}\right), \quad(\forall u \in X) \\
& f\left(x_{0}, v_{0}\right) \leq f\left(x_{0}, v\right), \quad(\forall v \in Y)
\end{aligned}
$$

in other words $u_{0} \in \Phi_{2}\left(y_{0}\right)$ and $v_{0} \in \Phi_{1}\left(x_{0}\right)$, hence

$$
\left(u_{0}, v_{0}\right) \in \Phi\left(x_{0}, y_{0}\right) .
$$

By the Kakutani fixed point theorem 2.24 there exists a fixed point $\left(x_{0}, y_{0}\right) \in X \times Y$ of $\Phi$, i.e., $\left(x_{0}, y_{0}\right) \in \Phi\left(x_{0}, y_{0}\right)$, for such point it holds

$$
f\left(x_{0}, y_{0}\right)=\min _{y \in Y} f\left(x_{0}, y\right)=\max _{x \in X} f\left(x, y_{0}\right) .
$$

We have

$$
\begin{equation*}
\min _{y \in Y} \max _{x \in X} f(x, y) \leq \max _{x \in X} f\left(x, y_{0}\right)=f\left(x_{0}, y_{0}\right)=\min _{y \in Y} f\left(x_{0}, y\right) \leq \max _{x \in X} \min _{y \in Y} f(x, y) . \tag{3.20}
\end{equation*}
$$

On the other hand we have

$$
f(\tilde{x}, \tilde{y}) \geq \min _{y \in Y} f(\tilde{x}, y), \quad(\forall \tilde{x} \in X)(\forall \tilde{y} \in Y)
$$

applying a maximum to the 1 st argument of $f$ on both sides gives us

$$
\max _{x \in X} f(x, \tilde{y}) \geq \max _{x \in X} \min _{y \in Y} f(x, y), \quad(\forall \tilde{y} \in Y)
$$

since this applies to all $\tilde{y} \in Y$ it also holds

$$
\begin{equation*}
\min _{y \in Y} \max _{x \in X} f(x, y) \geq \max _{x \in X} \min _{y \in Y} f(x, y) . \tag{3.21}
\end{equation*}
$$

This concludes the proof.

Definition 3.13 (Zero Sum Game) We call a normal form game $\left(I,\left(A_{i}\right)_{i \in I},\left(p_{i}\right)_{i \in I}\right)$ a zero sum game, if for each sa $\in A$ it holds

$$
\sum_{i \in I} p_{i}(a)=0
$$

A zero sum game has the feature that for each strategy profile $a \in A$ the sum of all player's payoffs is zero, essentially this means that one person's win is another person's loss. Note that every constant sum game (i.e., a game where for each $a \in A$, the sum of all player's payoff is always equal to the same constant) can be easily converted to a zero sum game.

Remark 3.1 In this remark we shall discuss how the minimax theorem effects zero sum games. Suppose we have a two player zero sum game, denote

$$
q_{1}=\max _{a_{1} \in A_{1}} \min _{a_{2} \in a_{2}} p_{1}\left(a_{1}, a_{2}\right),
$$

we shall call $\left(q_{1},-q_{1}\right)$ the value of the game if player 2 goes first. This corresponds to player 2 having the first turn and picking a strategy optimal to him, player 1 then chooses a strategy so that his payoff is at least $q_{1}$ thus minimizing his potential losses.

Consider the converse scenario

$$
q_{2}=\min _{a_{2} \in A_{2}} \max _{a_{1} \in A_{1}} p_{1}\left(a_{1}, a_{2}\right) .
$$

We will call $\left(q_{2},-q_{2}\right)$ the value of the game if player 1 goes first. Here player 1 first picks a strategy optimal for him and then player 2 knows this and picks his strategy so that his payoff is at least $-q_{2}$. The minimax theorem shows that under certain conditions (vaguely such as the game in question being a mixed strategy game or players being able to pick strategies from some continuous spectrum) both values of the game will be equal, i.e.,

$$
\left(q_{1},-q_{1}\right)=\left(q_{2},-q_{2}\right) .
$$

This would mean that order of play does not matter. Also note that relation (3.21) requires no assumption, thus we always have

$$
q_{2} \geq q_{1},
$$

thus having the first turn can be an advantage.
In the case that the conditions of the minimax theorem are satisfied then the strategy profile $a^{*}=\left(a_{1}^{*}, a_{2}^{*}\right)$ such that

$$
p_{1}\left(a_{1}^{*}, a_{2}^{*}\right)=\max _{a_{1} \in A_{1}} \min _{a_{2} \in a_{2}} p_{1}\left(a_{1}, a_{2}\right)=\min _{a_{2} \in A_{2}} \max _{a_{1} \in A_{1}} p_{1}\left(a_{1}, a_{2}\right)
$$

is also a Nash equilibrium. This concept can be extended for games with a finite number of players.

## Chapter 4

## Conclusion

This thesis was of the compilational nature, no new theorems or entirely new proofs were introduced, but the proofs we wrote were much more detailed than those that can be commonly found and we also introduced a lot of concepts needed in those proofs, so this text is well suited to be used as a studying material. Nearly all examples such as those showing that no assumptions of a certain theorem can be omitted we made up ourselves. Now let us summarize the contents of this thesis.

There are many fixed point theorems out of which we have picked several that are particularly useful. We started with the Banach fixed point theorem which is very far reaching even though its proof is quite simple especially compared to some we have seen later on. The Boyd-Wong theorem showed us that the condition of contractiveness can be relax a little while still maintaining the result, but we saw that if the condition of contractiveness is not satisfied, the fixed point iteration method no longer has exponential convergence. Another contraction principle we introduced was theorem 2.5 which stated that on a compact metric space $X$ we only need a mapping $T: X \rightarrow X$ to be a weak contraction, interestingly enough on compact metric spaces this result is equivalent to the Boyd-Wong theorem.

The Browder-Kirk fixed point theorem bridges the gap between contraction principles and fixed point theorems that rely on convexity and compactness such as the Schauder fixed point theorem. It provides sufficient conditions under which non-expansive mappings have a fixed point but its assumptions also include that the mapping $T$ go from a closed bounded convex and non-empty set $C$ into itself. We have shown a simple and transparent prove for the case that $C$ is a subset of a Hilbert space $H$ and later proved the general version using an auxiliary theorem that has a technical proof.

In the section about the Brouwer fixed point theorem we showed two ways it can be proven, the first way relies on the concept of mapping degree which in and of itself is a very powerful tool in dealing with fixed points and zeros of mappings, proving that mapping degree with the desired properties exists (which we did not do here) is a lot of work but if we can utilize the degree proving the Brouwer fixed point theorem itself is an easy matter. The second way we proved the

Brouwer fixed point theorem is using the Sperner lemma from graph theory, this proof is particularly interesting as it utilized two seemingly unrelated branches of mathematics.

The Schauder fixed point theorem is an extension of the Brouwer fixed point theorem into infinite-dimensional spaces, we used a proof where we approximated the operator over an infinitedimensional space by operators over finite-dimensional spaces and applied the Brouwer fixed point theorem. This proof was quite straight forward. We gave an example which showed that the condition of compactness cannot be swapped out for the set being closed and bounded. In fact if this was the case, the Browder-Kirk fixed point theorem would have been rendered useless.

The last result we introduced was the Kakutani fixed point theorem, it provides sufficient conditions under which there exists a fixed point of a point-to-set mapping. In fact the Brouwer fixed point theorem is a special case of this theorem and is used to prove this result. The proof suffers from an abundance of technical details but without these it is relatively straight forward.

We have also discussed some applications of these theorem, for applications of the Banach fixed point theorem we have used the Picard-Lindeöf theorem which provides sufficient conditions for existence of a unique solution of initial value problems of systems of ordinary differential equations, the Lax-Milgram lemma that guarantees a unique solution of certain types of operator equations that often arise when solving boundary value problems for partial differential equations and as the third application of this theorem we have introduced a method of constructing fractals using contractions in $\mathbb{R}^{n}$.

As an application of the Schauder fixed point theorem we have introduced the Peano theorem which guarantees the existence of a solution of initial value problems for systems of ordinary differential equations. This theorem is similar to the Picard-Lindeleöf theorem, it assumes less and also guarantees less.

Finally we showed some applications of the Kakutani fixed point theorem in game theory, specifically the existence theorem for a Nash equilibrium and the minimax theorem that can even be used outside of game theory. In this section we saw how easily some problems can be transformed into fixed point problems.

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