

**An Analysis of Covariational Reasoning Pedagogy for the Introduction of Derivative in
Selected Calculus Textbooks**

Yixiong Chen

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Abstract

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Yixiong Chen

Covariational reasoning is a cognitive activity that attends to two or more varying quantities and how their changes are related to each other. Previous studies indicate that covariational reasoning seems to have levels.

Content analysis was used to examine the pedagogy and development of covariational reasoning levels in the sections that conceptually introduce derivatives in four calculus textbooks. One widely used calculus textbook was selected for the study in each of the four categories: U.S. college, U.S. high school, China college, and China high school. Two qualified investigators and I conducted the study. We used a framework of five developmental levels for covariational reasoning.

The conceptual analysis of four calculus textbooks found that the U.S. college and the U.S. high school textbooks emphasize the average and instantaneous rate of change. However, both lack development of the direction and magnitude of change. On the other hand, this study's Chinese high school calculus textbook has a greater degree of development in the direction and

magnitude of change while having a deficit in the average rate of change. This study's Chinese college calculus textbook does not have any meaningful development regarding covariational reasoning pedagogy.

The relational analysis of the concepts previously identified in the conceptual analysis phase revealed that this study's U.S. college calculus textbooks provide abundant examples and exercises to transition between the average and instantaneous rate of change. On the other hand, all other calculus textbooks in this study lack any significant transition among passages that stimulate covariational reasoning.

The textbook analysis in this study provides insights into the current focus of calculus textbooks in both the U.S. and China. In addition, the study has implications for learning and teaching calculus at both high school and college, as well as future editions of calculus textbooks. Finally, limitations and recommendations are discussed.

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Dedication

To *I Ching*, better known as 易经, literally means “Oracle of Change.” And to Gottfried Wilhelm Leibniz.

Chapter I: Introduction

Covariation is the simultaneous changes of two quantities. Covariational reasoning is "the cognitive activities in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002; p. 354). A covariational relationship is a relationship among the changes in quantities. And covariational reasoning is reasoning with changes and how they are related. There are two elements in covariational reasoning; one is the perception of change, and the other is how the changes are linked or correspond to each other.

Human perception of change can be dated back to at least the Axial age (around the 8th to 3rd century B.C.), a term coined by German philosopher Karl Jaspers (1883 -1969). As documented in writing, in ancient China, Confucius (551-479 BC) stood by the riverside and said, "time passes by just like how this river flows, regardless of day or night." (子在川上曰 "逝者如斯夫! 不舍昼夜。") In the west, the Greek philosopher Heraclitus (around 500 BC) expressed his view of change, saying, "Everything is in flux." and "You cannot step twice into the same river." In today's language, these two great philosophers expressed their views of change relative to a continuous passing of time, analogous to a river's continuous water flow. What goes unsaid in this view is the independence of the passing time. The view that time is absolute and independent but that everything that happens in this universe can be expressed as something relative to time resembles the modern concept of the independent variable in the mathematical concept of function.

The perception of the correspondence relationship between variables can be dated back to the Babylonian tables of reciprocals (Youschkevitch, 1976) in 2000 B.C.

The concept of function evolves. But today, there is "no single generally accepted definition of the concept of a function" (Medvedev, 1991). Among diverse views of this concept from mathematicians and mathematics educators, two widely used concepts are set and correspondence. Moreover, in the modern view of function, the variable change (continuously or not) may not be an indispensable element. Nevertheless, some parts of the historical development of the functional relationship can be useful in considering the covariational relationship between variables.

Calculus education can serve many purposes. Modern calculus, an invention or discovery attributed mainly to Isaac Newton (1642-1726) and Gottfried Wilhelm Leibniz (1646-1716), was initially invented to understand change. Newton came to calculus as part of his investigations in physics and geometry and viewed calculus as the mathematical articulation of the change of motion and magnitudes. Leibniz focused on the tangent problem of the curves in analytical geometry and believed that calculus was a metaphysical explanation of the change. Thus, calculus education inherently has the potential to help foster students' covariational reasoning ability. However, the pedagogy of covariational reasoning is an area that is not emphasized in current calculus textbooks. According to the Mathematical Association of America's course area study group, the emphasis within calculus textbooks has been traditionally on "derivatives as the slopes of tangent lines and the integrals as areas – a very static interpretation that makes it difficult for many students to transfer these tools to dynamical situations." (MAA, 2021, p. 2). More recent work on calculus education recognizes the centrally important concept of covariation, i.e., understanding how the change in a variable is reflected in the shift in other linked variables (MAA, 2021).

Recent progress in the study of the cognitive actions involved in covariational reasoning has revealed that covariational reasoning capacity can be developmental and can have a hierarchy (Carlson et al., 2002; Cetin, 2009; Kertil et al., 2022; Moore et al., 2013; Paoletti et al., 2017; Şen-Zeytun et al., 2010; Tallman et al., 2021), i.e., people with higher levels of covariational reasoning capacity can automatically reason in the lower levels but not the opposite. According to the overall image students appear to exhibit in their problem-solving process, Carlson et al. (2002) proposed a theoretical construct of five mental actions for covariational reasoning abilities. These five mental actions correspond to five levels of thinking about change. The first level is identifying whether there is a change. The second level is identifying the direction of the change. The third level is determining the amount of change. The fourth level is finding out the average rate of change over an interval. And the fifth level is establishing the instantaneous rate of change.

Need for the Study

Covariational reasoning capacity is essential in understanding dynamic situations and modeling that understanding mathematically. However, recent studies showed that it was common to see calculus students not fully develop their covariational reasoning abilities (Carlson et al., 2002). For example, Carlson et al. (2002) used the theoretical construct of five levels of mental actions to investigate high-performing 2nd-semester calculus students' covariational reasoning ability. It was found that "observed trends suggested that this collection of calculus students have difficulty constructing images of a continuously changing rate" (p. 372). Subsequent studies used the same framework to investigate students' and teachers' mental processes regarding covariational reasoning capacity in several different situations. These studies

yielded insights that were consistent with the initial findings by Carlson et al. (Cetin, 2009; Hobson, 2017; Moore et al., 2013; Şen-Zeytun et al., 2010; Thompson & Carlson, 2017a; Thompson et al., 2017b).

School textbooks can heavily influence students' learning outcomes. According to the school learning model proposed by Carroll (1963), student learning depends on internal factors, such as students' aptitude, ability, and perseverance, as well as external factors, such as time for learning, opportunity to learn, and quality of instruction. Quality of instruction implies that "the various aspects of the learning task must be presented in such an order and with such detail that, as far as possible, every step of the learning is adequately prepared for by a previous step" (Carroll, 1963). And such quality "applies not only to the performance of a teacher but also to the characteristics of textbooks" (Carroll, 1963). Furthermore, the quality of textbooks affects students' opportunities to learn. The opportunity to learn is primarily defined as the amount of time available for learning, both in class and when doing homework. Since many homework assignments are taken from textbooks and involve substantial study time outside of formal lectures, particularly in the college setting, the quality of homework in the textbook and its pedagogical connections with formal lectures are an indispensable part of students' opportunities to learn.

Generally, the written curriculum, of which textbooks are a part, can significantly influence teacher practice and students learning (Kilpatrick et al., 2001; Stein et al., 2007). Studies suggest that curricular material influences teachers' classroom practice (Remillard & Bryans, 2004). On the students' side, studies show that teachers' understanding of a mathematical idea is an essential factor in how students understand it (Thompson, 2013). In short, textbooks have the power to provide an "organized sequence of ideas and information" to structured

teaching and learning, which guide readers' "understanding, thinking, and feeling" as well as "access to knowledge which is personally enriching and politically empowering" (Sosniak et al., 1990, p. 440).

Calculus education evolves with time. For example, the development of the calculus curriculum at the secondary level in the last hundred years in most European countries seems to have gone through three major phases (Törner et al., 2014). While university mathematicians primarily determined the first two phases of calculus education, focusing on filling the gap between secondary and college mathematics education, the last phase was informed by the mathematics education perspective (Törner et al., 2014; Toh, 2021). Many calculus educators now recognize that seeing the derivative as the instantaneous rate of change is more valuable than emphasizing its role as a method of finding slopes. Similarly, it can be more helpful to view an integral as measuring accumulation rather than emphasizing its role in finding area (MAA, 2021). Research has shown that covariational reasoning is essential in interpreting models of dynamic events and understanding major concepts of calculus and differential equations (Carlson et al., 2002; Rasmussen, 2001; Thompson, 1994a; Zandieh, 2000). It has been found that "conventional curricula have not been effective in promoting this reasoning ability in students" (Carlson et al., 2002, p. 356). Thompson and Carlson (2017a) offered a review of the research that supports the fact that "the U. S. curriculum and instruction are failing to develop students' quantitative and covariational reasoning abilities" (p. 456), which are essential for understanding calculus and modeling dynamically changing events in science and engineering.

Substantial progress has been made in the research concerning mathematics textbooks over the last few decades, focusing on textbook analysis, comparison, and use (Fan et al., 2013; Bressoud, 2016). There are several recent studies of calculus textbooks and curriculum

development in both the United States and internationally (Almeida, 2018; Bergwall et al., 2017; Lithner, 2004; Radmehr et al., 2017; Tallman et al., 2021; Toh, 2021; Yoon et al., 2020). Toh (2021) investigated the school calculus curriculum at the upper secondary and pre-university levels in Singapore and found three key features: an intuitive approach to calculus, an emphasis on technique, and a stressing of procedural over conceptual knowledge. Lithner (2004) and Tallman et al. (2021) focused their studies on strategies to solve calculus textbook exercises and characteristics of calculus I final exams. Some studies focus on the role of covariational thinking in calculus learning. Weber and Thompson (2014) found that covariational reasoning allows students to generalize their understanding of the functions in order to visualize the corresponding graphs, and that not thinking covariationally may hinder such generalizations. Tallman et al. (2021) found that "students are rarely required to interpret functions or situations in terms of covariational reasoning such as coordinating changes in output for successive equal changes in input" (p. 582). Ely (2021) argued that a differential-based approach could provide students with an intuitive sense of the fundamental theorem of calculus. Toh (2021) made a general statement that covariational reasoning was not emphasized in the two calculus textbooks he analyzed. However, in his article, Toh (2021) did not specify how covariational reasoning was not emphasized, whether it was not mentioned at all, lacked specific elements, or lacked consistency or continuity from different parts of the textbook. Based on my direct communication with Toh, his observation was that the syllabus and the textbooks in Singapore only "discuss the shape of the graph, without mentioning how the shape of the graph is related to the change of one variable when the other changes." This general statement indicates that the textbooks Toh analyzed lack the process view of functions and how function values covary. Toh's observation of how the concept of functions is used and presented in calculus textbooks is consistent with what other

researchers in the U.S. have found. For example, Thompson and Carlson (2017a) reviewed "the concept of function" from several perspectives. As the authors put it, the phrase "concept of function" begs the question of who might have it. A mathematician, a teacher, a student, and mathematical education researcher can have different views of functions. Thompson and Carlson (2017a) went on to elaborate that the idea of covariation as a foundation for function in mathematics is essential, and "the meanings of calculus that are grounded in covariational reasoning also fit precisely with the ways of thinking that science educators complain is lacking in their students' mathematics" (p. 453). From my observation in teaching advanced calculus courses, such as vector calculus and differential equations, it is not uncommon to see students who do well in the whole sequence of calculus courses but retain only the procedural knowledge or rote memory of derivatives and integration. Students' inability to mentally picture the derivative as the rate of change often necessitates that I circle back to the concept of the derivative before explaining how a vector function can be differentiable and can be integrable. Thompson and Carlson (2017a) identified the challenge of moving ahead to emphasize quantitative and covariational reasoning in the calculus curriculum. The task ahead involves more than determining whether the curriculum's focus should be shifted toward covariational reasoning but rather how to provide and "scaffold curriculum experience that is effective in getting students to spontaneously use these ways of thinking when confronted with novel tasks that require them" (Thompson et al., 2017a, p.458). In order to provide a practical experience to guide students on the way to developing covariational thinking capabilities, we need to understand how the current textbooks have done so up to the present.

Calculus is taught at both secondary and college levels. However, the role of calculus can be very different in secondary and college education. While calculus is considered an advanced

placement course in secondary education, credits in calculus are a prerequisite for many majors in college, such as science, engineering, economics, etc. Therefore, it is instructive to study and compare the pedagogy of covariational reasoning in textbooks for both educational levels.

The history and culture of a people influence curriculum development (Howson et al., 1981). Textbooks from different countries can offer distinct insights. China and the U.S. produce significant numbers of STEM graduates every year, and calculus is instrumental in STEM education. In contrast to the U.S., systematic STEM-based industrialization in China only happened recently—in the last few decades. Also in contrast to the U.S., China has a centralized education system. Therefore, there are only a few textbooks, and they are widely used. It will be instructive to analyze and compare calculus textbooks from China with those in the U.S.

Four major topics appear in most calculus textbooks: limits, the ratio of change (derivative), accumulation (integral), and series (Bressoud, 2019). Some topics may focus more on derivation techniques, such as the quotient rule of finding derivatives. Some topics may demand more conceptual buildup and explanations, such as derivative as the rate of change, integral as accumulation, and the fundamental theorem of calculus as a connection between differential and integral calculus.

According to research pioneered by Carlson et al. (2002), there are five levels of covariational thinking. Not all five levels of covariational reasoning naturally exist in all four calculus textbook topics. For example, the average rate of change in the partial sum of sequences may not be an essential topic for students who first encounter sequences and series. On the other hand, the conceptual introduction of the derivative as an instantaneous rate of change is a subject that all calculus textbooks cover. Furthermore, all five levels of covariational reasoning can naturally be used to build up the mental image of the change, from whether there is a change to

the instantaneous rate of change. I aim to demonstrate how this subject is conceptually introduced, how the supporting material is organized, and how the pedagogy concerning covariational reasoning levels are developed in the four textbooks.

In summary, analyzing calculus textbooks from four categories can shed light on two out of three major external factors that affect students' learning, students' opportunity to learn, and the quality of instruction. In addition, the analysis can provide evidence to help educators better understand the cause of the current deficit in calculus students' covariational reasoning capacities.

Purpose of the Study

This dissertation investigates how covariational reasoning levels are developed in the section that conceptually introduces derivatives in selected secondary and college calculus textbooks through an analysis of textbooks from China and the U.S. This dissertation will seek to answer the following research questions specifically:

1. How are covariational reasoning levels built up in the conceptual introduction of derivatives in two widely used calculus textbooks, one at the secondary level and one at the college level in the U.S.?
2. How are covariational reasoning levels built up in the conceptual introduction of derivatives in two widely used calculus textbooks, one at the secondary level and one at the college level in China?
3. What similarities and differences can we find by analyzing the pedagogy of derivatives in the preceding four calculus textbooks?

Procedures of the study

The research design for this study is content analysis, which uses written, audio, or visual material to determine the meaning reflected in the materials under study. Conceptual analysis and relational analysis are two general types of content analysis, and both are used in this study. Conceptual analysis determines the existence and frequency of concepts in a text, while relational analysis develops the conceptual analysis further by examining the relationships among concepts in a text.

This study comprised three phases. The first phase was conducted by me alone. In it, I selected one widely used calculus textbook from each category. The four categories were books written for high schools in China, colleges in China, high schools in the U.S., and colleges in the U.S. After selecting four textbooks, I identified the chapter and section of each that first introduces the concept of derivative.

The second phase was a conceptual analysis. Two qualified investigators and the author conducted research in this phase. The author sourced potential investigators via personal networks and social media. A set of criteria were used to screen and train potential investigators for the pilot study. After the pilot study, two investigators conducted research on their own and independently of each other. There were several steps in this phase. In the first step, the two investigators considered all passages selected in phase one, including the motivation, side notes, examples, and exercises that promote various levels of covariational thinking, identifying and coding them according to a covariational reasoning framework. The second step consisted of putting the results from the first step together, and the author conducted it. In the third step, the two independent investigators were given the results from the second step to have a second look and classify each item again according to five covariational reasoning levels. In the fourth and

final step, the author conducted an intersection of the two sets of results from the previous step as the final result of the conceptual analysis.

The third phase was relational analysis. There were two sub-phases in this relational analysis. The first sub-phase was a relational analysis within a given textbook and the second sub-phase was a pattern comparison between different textbooks. For each occurrence of an explicit promotion of covariational thinking in the passage, the same two investigators who conducted research in the previous phase and I determined whether the occurrence was a transition or a continuity from the previous occurrence. The process was the independent determination by two investigators and the author, followed by a majority vote to determine the final result. Three levels of connection were categorized: none, simple, and strong. Simple connections were defined as connections among concepts. Strong connections were defined as simple connections situated in the same context.

The second sub-phase of relational analysis was pattern comparison between different textbooks. The tables from the previous sub-phase were compared cell by cell to identify similarities and differences. I conducted this phase alone.

Chapter II: Review of Literature

The purpose of analyzing four calculus textbooks in this study is mainly to explore students' learning opportunities, and the literature review is centered around this purpose. In addition, two of the four textbooks in this study are written in Chinese and widely used in China, so a brief overview of mathematics education in China is given. The review of literature is organized into six sections. First, the role of mathematics textbooks in mathematics education is reviewed with an emphasis on the readers' (or students') perspective. Second, how readers with different identities and backgrounds might respond to the same text is reviewed through the lens of reader-oriented theory (Rosenblatt, 1995). Third, a brief overview of the history and current status of calculus textbooks from the US and China is presented. Fourth, content analysis as a research method is summarized, and some examples in mathematics education are reviewed. Fifth, a brief review of covariational reasoning pedagogy and the rationale for the framework are presented. And finally, some background information on mathematics education in China is reviewed.

Opportunities to Learn Offered by Mathematics Textbooks

Students' learning outcome depends on internal and external factors (Carroll, 1963). The internal factors are students' aptitude, ability, and perseverance, which schools have little control over but can try to cultivate over time. External factors are time for learning, opportunity to learn, and quality of instruction. School curriculum is central to external factors.

Quality of instruction includes the various aspects of the learning task that need to be organized so that a previous step adequately prepares for a later step in every learning process. Also, it "applies not only to the performance of a teacher but also to the characteristics of

textbooks" (Carroll, 1963). Furthermore, the quality of the curriculum and textbooks affects students' opportunities to learn. The opportunity to learn is primarily defined as the amount of time available for learning. The time here cannot be simplified as the "elapsed time" in school or lectures. It is when a person is oriented to a learning task by active listening and actively engaging in explaining, reading, or writing on the subject of focus. Thus, the opportunity to learn includes time students spend both in class and with homework set by the curriculum.

Calculus Curriculum and Textbooks

A curriculum is a sequence of planned experiences where students practice and achieve proficiency in content and skills. Analysis of the calculus curriculum has been traditionally focused on calculus textbooks and instruction (Bressoud, 2016). However, with the advent of interactive and visual technology, calculus curriculum reform also investigated the effects of technology on the teaching and learning of calculus, focusing on the relationship between intuitive and analytic thoughts concerning the basic ideas of calculus—infinitesimal, approximation process, change, etc. (Bressoud, 2016). Nevertheless, calculus textbooks still play a central role in the curriculum (Garcin et al., 2021; Rezat, 2021; Stein et al., 2007).

Effect of Mathematics Textbooks on Teachers

School textbooks are unique among curriculum materials in their influence on individual teachers' work because they are already "scaled up and part of the routine of school" (Ball & Cohen, 1996, p.6). To answer the question of how teachers interact with textbooks, Olsher and Cooper (2021) investigated the teacher-textbook relationship utilizing didactic tagging – a methodology where teachers assign metadata to textbook tasks. By analyzing the data from four teachers' tagged tasks to an entire precalculus high school textbook, non-trivial findings were generated from each tagger's interaction with the textbook. They proposed this method to assess

teachers' enactment of textbook curricula as an alternative to the traditional observation of classroom teaching. Randahl (2016) studied how a teacher of first-year engineering students perceived and used the calculus textbook to explore teachers' decision-making process regarding both choice of material and presentation order. Using data from lecture observations, interviews, and informal talks, Randahl found that teachers followed textbook material closely in terms of the sequence of concepts and the formal introduction of derivatives. For how the instructors may use the additional textbook features, Mesa et al. (2021) analyzed different ways university mathematics instructors utilized a feature in textbooks inviting students to get acquainted with the content before the formal class. Four utilization schemes were identified: instructors completed questioning devices for their own pre-planning; instructors required students to complete the questioning devices for the purpose of lesson planning; instructors used the questioning devices for the purpose of instruction, and instructors required students to complete the questioning devices for the purpose of assessment.

Substantial resources and effort have been invested in calculus reform since the mid-1980s (Kolata, 1986). Young (1986) argued that "Calculus is our most important course, and the future of our subject ... depends upon improving it." (p. 173) and "The computer will revolutionize our subject as greatly as did Arabic numerals, the invention of algebra, and the invention of calculus itself" (p. 174). Ferrini-Mundy et al. (1991) gave an overview of the calculus reform effort and raised the question of "what helpful features should be incorporated into textbooks"(p.633). More recently, Tall et al. (2008) mentioned that "of all the areas in mathematics, calculus has received the most interest and investment in the use of technology" (p. 207). However, the current results from the calculus curriculum reform are mixed. For example, Pogorelova (2021) analyzed the mathematical content and pedagogical approach of a chapter on

functions in a reform-based college-level calculus textbook. A synthesized framework of Hwang et al. (2021) and Sood and Jitendra (2007) was used to identify elements of reform involving the big ideas, contextual features, mediated scaffolding, problem-solving opportunities, and STEM integration. It was found that the chapter under analysis integrated many reform-based principles and that there may be room for further integration. Regarding how the college mathematics instructor interacts with interactive calculus textbooks, Liakos et al. (2021) investigated how an inquiry-oriented, dynamic, open-source calculus textbook shaped one college instructor's planning. Using data from surveys, logs, and interviews and a framework proposed by Dietiker et al. (2018), their observation suggested that the textbook supported and influenced the instructor in implementing his inquiry-oriented visions and goals while the college instructor planned his lessons.

Effect of Mathematics Textbooks on Student Achievement

Mathematics textbooks play a central role in student learning. In general, textbooks can be viewed as a vehicle for change in the curriculum, exposition of mathematics content, presentation of problems (as a collection of problems with or without solutions), the centerpiece of a course, the transmission of an ideal curriculum, the definition of a particular subject, and a means of learning mathematics (Usiskin, 2013). Textbooks are primarily written for students, and teachers' manuals accompany some textbooks. For students, a mathematic textbook is mostly an exposition of mathematics content and a source of examples and problems. Mathematics students at the tertiary level cannot learn all of the required new concepts in class time alone. Substantial individual activity outside of class time is expected and necessary (Robert, 1992). Mathematics textbooks provide an avenue for students to self-study. Also, many homework assignments are taken from textbooks. With substantial study time outside formal lectures,

particularly in the college setting, the quality of homework assignments and how they relate to formal lectures has a large effect on students' learning opportunities.

Numerous studies have demonstrated the positive correlation between learning opportunities provided in mathematics textbooks and student achievement in different populations and grade levels. For example, Hadar (2017) explored how textbooks function in education by investigating how textbooks' cognitive demands correlate with scores of all 8th-grade students in an Arab community who completed the national math test in 2015. It was found that if a textbook provided the opportunity to engage in tasks demanding higher levels of understanding, students using this book would have higher scores. Sivert et al. (2021) analyzed learning opportunities presented by four German textbooks for Grade 1 and found a significant relationship between instructional quality and student achievement in students' ability to solve quantitative comparisons. Van den Ham and Heinze (2018) examined the effect of textbooks representing the same curriculum by analyzing a three-year longitudinal data set comprised of 40 primary schools from Germany and found that mathematics teachers' textbook choices had a substantial effect on the student's mathematics achievement. In addition, the effect of textbook choice could be cumulative over the school years, which means that the textbook choice has a genuine effect on students achievement in each school year and the overall effect increases over the years. Finally, Huang et al. (2022) explored the possible contributions of textbooks to students' performance in the commutative property of addition in Shanghai, China. The authors provided evidence of two instances in which students performed well on the content provided by the textbook and weaker on the approach not covered by the textbook, indicating a strong relationship between textbook content and student achievement. However, currently, there are few studies that specifically investigate how the choice of calculus textbook relates to student

achievement. Therefore, this brief review of the significant effect of mathematics textbook choice on students' achievement is limited to the secondary level. The working assumption here is that textbook choice also influences students' learning outcomes at the college level.

Different Readers of Mathematics Textbooks

Textbooks, and calculus textbooks, once they are made, are objects. How people interact with or interpret objects can vary greatly. Similarly, how people read the same textbook greatly depends on an individual's internal factors (such as motivation, knowledge base, aptitude), external factors (such as physical surroundings, historical and social context), and the presence of other controllable and uncontrollable factors.

Three Types of Readers of Mathematics Textbooks

According to reader-oriented theory (Rosenblatt, 1995), the meaning of the text does not reside in the text only but, rather, is generated through a transaction between the text and the reader. How a reader responds to a text is shaped by the reader's goals, motivation, background, and the historical and social context in which the transaction is situated.

Weinberg and Wiesner (2011) adopted this theory and formulated concepts about three types of readers: the intended reader, the empirical reader, and the implied reader. The intended reader is the reader profile in the author's mind; the empirical reader is the person who actually reads the text; and the implied reader reflects what is required of the empirical reader to interpret the text in the way intended by the author. In the writing process, the textbook writer's intended readers are typically students. When students read the textbook, they are the empirical readers of the textbook. However, the mismatch among these three types of readers complicates students' reading process. This mismatch can happen in several stages. The first mismatch occurs between

the intended reader and the implied reader. For example, suppose the textbook writer does not have a clear picture of the intended reader's knowledge base or cannot organize the content in a way that gradually scaffolds the intended reader's development. In that case, the implied reader's profile cannot be well defined, let alone match the intended reader. The second mismatch can happen between the empirical reader and the intended reader. For example, textbooks concerning the same subject may not be suitable for all grade levels. Also, students at the same grade level may not have a similar knowledge base. The mismatch between the empirical reader and the intended reader occurs more often in the field of mathematics than in other fields because the depth of subject matter can be vast even at the secondary level, as is evident in some notoriously difficult problems in plane geometry.

Mathematics and Natural Language

Before we dive into how the different types of readers of mathematics textbooks can influence the research in this dissertation, let us briefly discuss another subject, the role of natural language in mathematics writing.

The use of symbols can be seen as the most distinctive feature of mathematics text. Mathematics text, or language, can be categorized into four sub-genres (Richard, 1991). *Research mathematics* is the spoken mathematics of professional mathematicians and scientists. *Inquiry mathematics* is used by the mathematically literate person for activities such as participating in a mathematical discussion, proposing conjectures, listening to mathematical arguments, and reading and challenging mathematical content in popular articles. *Journal mathematics* is the language of mathematical publications and papers. *School mathematics* is used in the standard classroom and primarily consists of an initiation-reply-evaluation sequence. Within this framework, mathematics textbooks can be viewed as mathematics texts intended for

students using the language of school mathematics and yet written by mathematically literate authors who speak inquiry mathematics.

Similarly, mathematics teachers are mathematically literate enough to engage in conversation using inquiry mathematics. Richard (1991) further pointed out that "for mathematical inquiry to take place, students and teachers must learn to carry on a mathematical discussion – they must learn to speak *inquiry math*" (p. 17). Mathematics textbooks are tools and media operating between students and teachers. Therefore, it is appropriate for textbooks to be written to promote and facilitate the language of inquiry math among students and teachers.

Mathematics has developed a system of symbols that can be independent of all natural languages, such as Hindu-Arabic numbers, which is almost the universal way of representing natural numbers. Nevertheless, natural languages play a vital part in mathematical discourse, particularly in learning and working on tasks that need to be related to real-world situations. The relation between language and mathematics has been previously studied for different purposes and based on different understandings of language's role in mathematics. There is much debate concerning whether mathematics is a language or a means of communicating pure mathematics. There is a spectrum of viewpoints on this issue. At one end of the spectrum, some see mathematics as a language (Wakefield, 2000). At the other end of the spectrum, there are many who perceive language as a tool that is useful for communicating mathematics, which exists independent of human activity (Sato et al., 2010; Tindal, 2014). Between these two extremes, an intermediate understanding is that mathematics *has* a language (Chen et al. 2015). This view is compatible with mathematical literacy, implying that mathematical ability consists of several mathematical competencies, such as communicative and reasoning competencies.

With this viewpoint situated in the middle of two opposite extremes, several natural questions follow: how does one's mathematical ability relate to one's natural language ability? and how can the use of natural language in a textbook help to illustrate mathematical concepts? The position research occupies on the spectrum as well as the intersection of mathematics and natural language of the textbook add another layer of complexity to the analysis of mathematics textbooks. There are various claims about the linguistic properties of mathematical text in general. Osterholm and Bergqvist (2013) conducted a survey on people's claims concerning the linguistic properties of mathematical texts. They found that most claims tended to assert that mathematical texts were complex, with many claims being vague, using terms such as "unintuitive" or "synthetic." Next, they conducted a study to determine if the common claims regarding the linguistic traits of mathematical texts were valid for Swedish mathematics textbooks. For the textbooks they analyzed, in comparison with comparable history textbooks, it was found that mathematics texts "have so far never been shown to be more complex than texts from other subjects." (Osterholm & Bergqvist, 2013, p.762). Thus, the common image of the complex nature of the mathematical text may not be a valid factor contributing to poorly organized presentation of content in mathematics textbooks.

Writers and Readers of Mathematics Textbooks

Authors of mathematics textbooks, many of whom are mathematicians, are generally experts in a specific subject. At the same time, the empirical readers of textbooks, most likely to be students, do not always match the implied readers (Weinberg & Wiesner, 2011), who many authors imagine to be experienced mathematicians. When the mismatch happens, Weinberg and Wiesner (2011) pointed out that students "will be unable to use the textbook effectively as a tool for learning the mathematics intended by the author" (p. 60). They concluded that "textbooks

generally present concepts using the conventions of standard mathematical symbolism and language, but the students who read the textbooks may not possess the appropriate codes participating in this discourse" (p.61). The reading style of an expert mathematician can be very different from that of mathematics students. For example, Shannahan et al. (2011) found that the text structure, graphics, and prose tended to be equally weighted in mathematicians' efforts to limit misinterpretation. At the same time, Shannahan et al. (2011) found that expert mathematicians tended to apply a rigorous reading style and sometimes intensive rereading, emphasizing accuracy to ensure their understanding of the text.

Different Reading Styles Among Readers with Different Expertise in the Subject Matter

The experts in specific content knowledge tend to read a text using multiple flexible strategies, which suggests that expert readers' reading process is an opportunity to generate meaning and embed the text in their prior knowledge (Weinberg, 2011; Yore, 2000). On the other hand, students, or learners of a specific subject, can have a substantially different strategy for reading textbooks. For example, undergraduate mathematics students tend to read textbooks to search for worked-out examples. Then, to complete the homework problems at the end of a section, students tend to identify an example in the same section that has similar surface properties and emulate the solving process (Biza, 2016; Lithner, 2003). Although all readers participate in a transaction with the text when they read, the different reading strategies used by experts and students can produce different outcomes in terms of understanding mathematics. The reader-centered strategy, typically employed by experts, is a meaning-making process that is more likely to bridge the gap left by the author and to lead to understanding of what the author implies.

On the other hand, the text-centered strategy, typically employed by students, treats reading as a process of receiving meaning rather than making meaning. For example, Shepherd and van de Sande (2014) found that mathematics novices tended to read through lengthy passages without pausing to check for understanding, while the mathematics experts stopped nearly twice as often. They also noted that mathematics experts, such as mathematics faculty and graduate students, can have a very different reading strategy than mathematics novices, such as first-year undergraduate mathematics students. Specifically, they found three significant differences in reading strategies. First, experts are more likely to skim over the material they are familiar with and often read the meaning, instead of reading the symbols verbatim. Second, experts have more awareness of what they understand and do not understand. Third, experts explore the content by creating examples and referring to other material when needed.

Different Reading Styles Among Students

In addition, there is also a difference among reading strategies within the student body. Berger (2019) categorized five styles of reading mathematics textbooks by analyzing video transcripts of five specially chosen students studying from a mathematics textbook. Reading styles were gauged by the depth of reading, focusing on specific components of text or not, drawing connections within texts or to prior knowledge, and performance of exercises. It was found that reading style is closely related to an individual's mathematics performance. The high-performing students were close readers who read all the text carefully and frequently paraphrase and explain the text. On the other hand, average and below-average-performing students read the text only when it appears to help to solve a particular exercise. They do not paraphrase or explain worked examples but consult texts on a "need to know" basis to complete a particular exercise.

Different Reading Styles Among People Who Have a Similar Knowledge Base but Different Backgrounds in the Field of Teaching and Learning

The complexity of the transaction between the reader and the calculus textbook was further investigated by Wisner et al. (2020). Readers who have a similar level of familiarity with the subject matter but different backgrounds in other aspects can behave differently while reading the same text. Specifically, Wisner et al. (2020) sought to understand the factors affecting how individuals read by observing and comparing two readers, one 2nd-semester calculus student and one non-mathematics STEM professor. The case studies were analyzed through the lens of mathematics disciplinary literacy, including the shared way of reading, writing, thinking, and reasoning within mathematics. A calculus textbook excerpt titled "Applications to Geometry in Calculus" was given to both readers. The excerpt was an introduction, derivation, and formula for arc length calculation. Neither reader was familiar with the subject before the study. The student was an architecture major and had not studied the particular excerpt yet. The professor in this study was a chemistry professor who took two semesters of calculus some thirty years before and claimed that he remembered little calculus and did not use calculus ideas in his research or teaching. While reading, both readers experienced similar gaps and questioned similar concepts, such as the role of Riemann sums in derivation and the role of the diagram; however, they exhibited some important differences in how they bridged gaps. One major difference was their role concerning power and authority in the textbook-reader relationship. Although the professor in this study had more limited knowledge of calculus than the student, as an instructor and STEM-discipline expert, he expressed more agency and authority in the reading process. For instance, he expressed how he would modify the diagram creation, etc. In addition, his identity as a teacher appeared to give

him a perspective from a textbook writer's point of view. Wisner et al. (2020) suggested that becoming an effective reader of mathematics is important in becoming a member of the mathematical community. They concluded that mathematics instructors "must move beyond focusing on content knowledge and instead teach specific reading strategies" (Wisner et al., 2020, p.228).

Weinberg et al. (2022) expanded their previous work to analyze the reading behavior of a group of five students and a group of five non-mathematics STEM professors while they read three excerpts from a calculus textbook. Similar to the previous study (Wisner, 2020), the study was controlled for a similar level of content knowledge for all participants. The student group comprised calculus students who had not studied the subject yet. The professor group comprised non-mathematics STEM faculties who had taken calculus courses between 15 and 40 years before the study. Through detailed documentation of the observations and analysis, they found four themes in how the students and professors tended to engage in different ways: text as a product of the author, questioning the instructional purpose of the text, paying attention to teaching-related structures in the text, and the power in the textbook-reader relationship gauged by own teaching or learning role. The different ways readers interact with the same text between two participant groups can be viewed in terms of their identities and background, such as learner vs. instructor. The findings from this study have an implication for several issues, such as research, teaching, and design of didactical texts. One implication for the analysis of textbooks is that the same content could cause a different reaction in different readers even though they have similar levels of expertise in content knowledge. Thus, to ensure the validity of the content analysis for the intended readers, the investigator's identity and background must be aligned with the intended readers to the maximum extent possible.

Current Status of Mathematics Textbook Research

Mathematics textbooks are a special kind of mathematics text whose intended readers are most likely to be students using the language of *school mathematics*. Such textbooks are written by mathematically literate authors who speak the language of *inquiry mathematics* and tend to use the language of *inquiry mathematics* in writing.

The blending of *text* (meaning thing woven) and *book* only appeared together in one word as a *textbook* around the 1830s (Stray, 1994). Generally, textbook research can be categorized into three areas: ideology in textbooks, the use of textbooks, and the development of textbooks (Johnsen, 1993). Mathematics textbooks can exist in different forms. Historically, mathematics textbooks have appeared in various media, from clay tablets, papyrus, parchment, bamboo, and paper to modern interactive electronic forms (Kilpatrick, 2014). However, in recent decades, school mathematics textbooks seem more similar in content than appearance, pedagogical effort, assistance to instructors, and ways of presentation (Kilpatrick, 2014).

A survey of empirical mathematics textbook research in the last six decades conducted by Fan (2013) showed that over half of empirical textbook research was textbook analysis and comparison. Furthermore, he gave a historical overview of mathematics textbook research, focusing most particularly on the last several decades. He claimed that it was still in an early stage of development compared with other fields of research in mathematics education and called for more work on the philosophical foundation, theoretical framework, and research methods in mathematics textbook research. He believed textbook research should be treated as a scientific research field and that common ground for underlying assumptions and intellectual framework was still lacking. He classified research questions into four broad categories: descriptive questions asking what a thing is; correlational questions asking how two or more

things are related; causal questions asking whether there is a causal relationship between two things; other questions that are not in the previous three categories.

Regarding the role of textbooks from a broader perspective of mathematics education, he further classified textbook research into three areas: issues about textbooks themselves, textbooks as dependent variables affected by other issues, and textbooks as independent variables that affect other factors in mathematics education. In addition, Fan (2013) provided a critical analysis of issues and methods of mathematics textbook research and put forward a conceptual framework that treated textbooks as an intermediate variable in education. Consequently, he defined mathematics textbook research as an inquiry into mathematics textbooks and their relationship to other factors in mathematics education. Finally, he suggested that such research should go beyond focusing on textbook analysis, comparison, and usage to employ more empirical and experimental methods to gauge their effectiveness for general mathematics education.

Section Summary

In summary, textbooks' writing and reading processes are complicated. The fact that mathematical symbols have developed their own system of meanings and intersections with natural language adds another level of complexity. It is crucial to keep in mind the complex nature of mathematics textbooks while performing research in this area to make informed decisions on every step of the research process, from research design to interpretation of research results.

Calculus Textbooks

The invention or discovery of modern calculus is mainly attributed to Newton and Leibniz in the second half of the 17th century. Guillaume de l'Hôpital wrote the first calculus textbook in 1696 (Zuccheri & Zudini, 2014), which marked the start of the teaching of calculus in Europe. During the 18th and 19th centuries, many European countries had calculus curricula either at the university level, secondary school level, or both. As a result, calculus textbooks proliferated (Bressoud, 2016; Zuccheri & Zudini, 2014).

Calculus Textbooks in the US

Calculus in the United States was traditionally considered a university-level course. There is no central agency to set a curriculum on what topics should be included in calculus textbooks in the US; therefore, the market for calculus textbooks is fragmented, with many publishers lobbying as many universities as they can to use their products. Few secondary schools taught calculus during the first half of the 20th century in the US. After the Second World War, and in the early 1950s, with the establishment of the Advanced Placement Program by the College Board, more secondary schools started to teach calculus (Bressoud, 2016; Tucker, 2013). The presence of calculus on a student's high school transcript was interpreted to be highly correlated with their success at the university level. This interpretation led to more calculus enrollment in high schools. For example, the number of students who studied calculus in high schools increased fourfold from the 1960s to the early 1970s (Rash, 1977). The high school calculus enrollment growth rate exceeded 13% during the 1980s. It slowly came down to around 6%, resulting in almost a quarter of all senior high school students enrolled in calculus in the 2010s (Bressoud, 2016). Calculus textbook publishers responded to this demand and created

Advanced Placement (AP) versions of some calculus textbooks for high school teaching and learning.

Calculus Textbooks in China

The Chinese version of the textbook by American mathematician Elias Loomis (1811-1889), *Elements of Analytical Geometry and of the Differential and Integral Calculus*, initially published in English in 1851, is generally regarded as the first calculus textbook in China (Zhang, 1999). It was translated in 1859 to classic Chinese through a joint effort formed by Alexander Wylie, a missionary to China, and Shangla Li (李善兰), a Chinese mathematician. For the first thirty years after the founding of the People's Republic of China in 1949, calculus was not an official part of China's high school mathematics curriculum. The debate on whether to include the introduction of calculus in the high school mathematics curriculum was continual. Finally, in the national curriculum published in February 1978, an introduction to calculus was formally included for the first time. Five topics and allotted hours are specified: sequence and limits for 18 hours, derivatives for 18 hours, derivative application for 10 hours, indefinite integral for 12 hours, and definite integral and its application for 14 hours.

China has a centralized education system and a national unified mathematics curriculum (Li et al., 2009). Today, the People's Education Press (PEP), founded by the Ministry of Education in the 1950s, publishes national textbooks and curriculum standards in China. Up to 1988, PEP was the only official developer of textbooks. After 2000, more publishing presses were approved for publishing textbooks. However, along with its many variations, the most popular version is still directly from PEP, *General High School Curriculum Standard Experimental Textbook of Mathematics, A Ver.*, or PEP-A (Li, 2020). The PEP-A series has twenty mathematics textbooks for high school, of which five are compulsory, five elective, and

ten optional. The compulsory ones cover core material for the high school graduation exam and are used by everyone. The electives cover materials in the Chinese National College Entrance Examination. They are used by students who want to participate in the Chinese National College Entrance Examination after high school and go to college. And the optional ones are for enrichment, covering material such as mathematics history, groups, symmetry, geometry proofs, decision and risk, Markov chains, etc. For the electives, there are two series. The first series is geared towards students who want to pursue non-STEM studies and consists of two books, elective 1-1 and elective 1-2. The second series is geared toward students who want to pursue studies in science, mathematics, and engineering and consists of three books, electives 2-1, electives 2-2, and electives 2-3. An introduction to calculus appears in electives 1-1 as the third chapter and in electives 2-2 as the first chapter.

There are various curricula and textbooks for college-level calculus teaching in China. For example, there are textbooks geared toward mathematics, science and engineering, biology, social science, etc. In contrast to the US, where publishers publish textbooks, college-level calculus textbooks in China are often written collectively by university mathematics departments. Sometimes, the calculus curriculum is combined with the curriculum of linear algebra and put in the same book and titled *Higher Mathematics*.

Content Analysis

A content analysis uses written, audio, or visual material to seek the meaning reflected in the materials under consideration. Content analysis has various definitions. According to Berelson (1952), content analysis is a research technique for the objective, systematic, and quantitative descriptions of the manifest content of communication, aiming to interpret them.

Krippendorff (2004) considers content analysis a research tool for making replicable and valid inferences from data and their context. Neuman (2006) defines content analysis as a method for gathering and analyzing the content of a text, which is anything written, visual, or spoken. Finally, Berger (1991) considers content analysis a research technique based on measuring the amount of something (violence, negative portrayals of women, etc.) in the content being examined. These definitions agree that this research technique aims to make replicable, valid, and objective inferences from documents. Content analysis can be used for a variety of purposes. For example, it can be used to identify bias, prejudice, or propaganda in a text; to analyze types of error in writing; to describe prevailing practices; to discover the level of difficulty of material in the media; and to find the relative importance of, or interest in, specific topics (Krippendorff, 2004).

In practice, content analysis is often used to determine the presence of certain words, themes, or concepts within some given qualitative data, such as text, audio, and video clips. Using content analysis, researchers can quantify and analyze the presence, meanings, and relationships of certain words, themes, or concepts. By systematically evaluating texts (e.g., documents, oral communication, and graphics), qualitative data can be converted into quantitative data. Although the term "content analysis" did not appear in English until 1941, systematic analysis of text can be traced back to the 17th century, when the first known dissertation about newspapers was defended in the 1690s (Krippendorff, 2004). At the beginning of the 20th century, the boom in newspapers in the United States created a mass interest in public opinion; it led to a field of study known as "quantitative newspaper analysis," which is used to analyze a newspaper's contents. As of today, content analysis has three distinctive characteristics. It is exploratory in process and inferential in intent; it transcends traditional notions of symbols,

content, and intent; and it has been forced to develop a methodology of its own in response to larger contexts.

There are two general types of content analysis: conceptual analysis and relational analysis. Conceptual analysis determines the existence and frequency of concepts in a text, and relational analysis develops the conceptual analysis further by examining the relationships among concepts in a text. Here are the typical steps for research that use content analysis: deciding on research questions, selecting material, building the coding framework, subdividing material into coding units, trying out the coding framework, evaluating and modifying the coding framework, performing the main analysis, finding an interpretation, and presenting (Schreier, 2012). Different research questions can be asked concerning the same content. The research question is critical to the success of content analysis and guides the development of the coding framework and the process of streamlining data (Croucher, 2019). Next, for a given research question, material selection is a step that needs careful consideration. After the research question and the material under consideration are determined, the coding framework must be built to provide a clear boundary between different content categories. This process may involve the initial framework tryout, modification, coder training, or pilot study. The pilot study is a trial run to check the coding process and see what problems could emerge in the coding and analysis (Croucher, 2019). Then, there is the main analysis, the interpretation, and the presentation of the finding. The content analysis process greatly depends on the coding process to organize large quantities of text into much fewer content categories (Weber, 1990). The credibility of content analysis largely depends on the coding framework. The data collection process needs to be transparent, and the coder needs to be provided with precise coding definitions and clear coding procedures (Weber, 1990).

Content analysis has many strengths. First, it looks directly at communication via texts or transcripts and hence gets at the central aspect of social interaction. Second, it can allow for both quantitative and qualitative operations. Third, it can provide valuable historical/cultural insights over time through the analysis of texts. Fourth, it allows a closeness to text that can alternate between specific categories and relationships and statistically analyze the text's code form. Fifth, it can be used to interpret texts for purposes such as developing expert systems since knowledge and rules can be coded in explicit statements which can be used to describe the relationship between concepts. It is an unobtrusive means of analyzing interactions, which means the presence of an observer does not influence what is being observed. Sixth, it provides insight into complex human thought and language use models. And when done well, it is considered a relatively exact research method (based on hard facts, as opposed to discourse analysis). Finally, the results of content analysis can be easily replicated (Ary et al., 2013).

Content analysis has limitations. First, it is subject to error, particularly when the relational analysis is used to attain a higher level of interpretation. Second, it often lacks a theoretical base or attempts too liberally to draw meaningful inferences about the relationships and impacts a study implies. Third, it is inherently reductive, particularly when dealing with complex texts. Fourth, it tends too often to consist simply of word counts. Fifth, it often disregards the context that produced the text. Finally, it can be laborious and time-consuming (Ary et al., 2013).

Content Analysis in Mathematics Education

Content analysis can be a valuable tool in mathematics education research. Its primary use is for the analysis of textbooks, seatwork and homework material, assessment and exam

documents, etc. Since the coding framework is the key to content analysis, the following reviews are grouped according to how coding frameworks were developed and used.

Coding Frameworks Already Exist in Previous Literature

One straightforward way to develop a coding framework in content analysis is to use a verified framework from other literature. For example, Czocher et al. (2013) used the calculus content framework (CCF) built by Sofronas et al. (2011). Tallman et al. (2021) used the framework of "understanding" developed by Tallman et al. (2016). Toh (2021) used the Singapore mathematics curriculum five-dimensional framework from the Singapore Ministry of Education. Lithner (2004) used a previously constructed framework (Lithner, 2000) based on the amount of reasoning required to solve an exercise. Nagy et al. (1991) used a six-category system based on calculus guidelines from the Ontario Ministry of Education. Finally, Bateman et al. (2021) used a coding framework that Nagle et al. (2013) developed.

Czocher et al. (2013) investigated how calculus ideas are used in later coursework, such as differential equations, by analyzing two differential equations textbooks. Specifically, they set out to find what calculus concepts and skills were foundational to developing differential equation topics and how their expected understanding aligned with the expected understanding of these topics in a later course. The calculus content framework they used in their study resulted from an analysis of 24 mathematicians' identification of important calculus content (Sofronas et al., 2011). The differential equations content framework was constructed using course objectives list, the course syllabi in their university, and consultation with five engineering faculty who taught courses that needed differential equations as prerequisites, two mathematicians, and five graduate teaching associates. The coders were the authors who decomposed the texts in two differential equations textbooks relative to the lists of content in both frameworks. It was found

that differential equations material was highly skill based, and some concepts and skills, such as Riemann sums, were not prerequisites to any of the content in differential equations. In addition, it was found that the algebraic manipulation of differential equations was emphasized in the introduction of the topic, and there was little depth of knowledge of calculus required. Furthermore, the solution techniques arranged by equation type reduce the solving process to a decision tree, which again requires little depth of knowledge of calculus.

Tallman et al. (2021) investigated what meanings are assessed in collegiate calculus in the United States. They analyzed Calculus I final exams to characterize the specific meanings of foundational concepts the exams assessed, identify features of exam items that assess productive meanings, and distinguish categories of items for which students' responses are not likely to reflect their understanding. They also suggested modifications to these items to assess students' possession of more productive understandings. Two hundred fifty-four final exams from single-variable calculus courses (Calculus I) for STEM majors were collected and analyzed in two phases. The first phase categorized items in the data set based on the primary meaning they assess. Items were differentiated by whether or not the task assessed understanding. All four authors were coders of this study. Initially, the coders used Tallman et al.'s (2016) description of "understanding," which was subsequently modified to ameliorate the disagreement among initial results from different coders. The discrepancy among individual results was further discussed, and 19 further sample problems did not achieve complete agreement. The second phase analyzed items within each category to characterize the specific concepts. Thirteen categories of concepts were found: modeling, extreme values, derivatives, functions, integral applications, limits and approximation, Riemann sums, continuity, definite integration, linear approximation, mean value theorem, the average and instantaneous rate of change, and fundamental theorem of calculus. It

was found that 20% of the 4167 items met their criteria for assessing understanding. In addition, it was found that very few items required students to engage in continuous covariational reasoning, coordinate two dynamic processes, conceptualize the rate of change, etc.

Toh (2021) investigated the Singapore upper secondary and pre-university school calculus curriculum to see what features of the content steer students toward the goal of the school mathematics curriculum. The Singapore mathematics curriculum framework has five dimensions: skills, processes, concepts, attitudes, and metacognition. The examined documents were the syllabus, teaching and learning guide, two calculus textbooks, and lecture notes. From the analysis of the above documents at the upper secondary and pre-university levels, three key features were found: an intuitive approach to calculus, an emphasis on technique, and procedural knowledge over conceptual understanding. It was also found that "the process and attitude dimensions of mathematics curriculum framework are addressed through enabling students to witness the applicability and beauty of calculus in the real world through application problems" (Toh, 2021, p. 539) and that "the alignment of the calculus content to the Metacognition dimension remains implicit" (p.539).

Lithner (2004) investigated how it was possible to solve textbook exercises without considering the intrinsic mathematical properties of the component involved and what proportions of a textbook's exercises could be solved by such solution types. The previously constructed framework by the author (Lithner, 2000) was based on the amount of plausible reasoning required to solve the exercise. Thus, the framework consists of three categories of reasoning. *Identification of similarities* is a strategy choice founded on identifying similar surface properties in an example and copying the procedure from that example. *Local plausible reasoning* is a strategy choice based on recognizing significant similarities between components

in the exercise and components in a situation in the text with only a few differences and modifying a few local steps. *Global plausible reasoning* is a strategy choice mainly founded on perceiving the intrinsic mathematical properties of the exercise and using those properties to understand or solve the problem. Because the variation of plausible reasoning may be continuous and the exact classification of the borderline case may not be strict, the author acknowledges that the study's goal is to find the approximate distribution among these three categories. The 598 exercises from three calculus textbooks were analyzed according to the following headings: exercise formulation, possible solution, reasoning structure, and reasoning characteristics. It was found that a large majority of the exercises can be solved by not understanding the intrinsic mathematics involved, and the strategic choice to solve it "may normally be based on finding and copying a similar situation in the same textbook section as the exercise in question" (p.424).

Nagy et al. (1991) investigated and compared the content of what is taught and tested in a high school calculus class. The content of instruction is inferred from the seatwork and the homework. Seventeen teachers from 13 school districts participated in the study. A daily log of one calculus class for the entire semester from each of the 17 teachers was collected, including seatwork, homework, assessment activities, and the amount of time spent on different topics. The content was categorized using the six-category system based on calculus guidelines from the Ontario Ministry of Education: skills, proofs, graphing-differentiation, graphing-integration, situational problems, and optional topics. More than one rater was assigned to the same tasks to ensure the findings' objectivity, validity, and reliability. It was found that the agreement between raters who categorized the content was 97%. After the categorization, each teacher's data were used to calculate the number of problems assigned per category, the percentage of problems in each category, the percentage of points awarded for exam questions in each category for all tests,

and the difference between the percentage of points awarded and the percentage of problem assigned in each category. It was found that substantial differences existed between content coverage and testing emphasis.

Bateman et al. (2021) studied whether the concept of the slope was reviewed in 28 introductory calculus textbooks as well as how different conceptualizations of the slope were linked and categorized. The coding framework was taken from Nagle et al. (2013). The framework has five conceptualizations of slopes: ratio, behavior indicator, steepness, constant parameter, and determining relationship. The text can rely on either visual, nonvisual, or both approaches for each category. Their study found that slope is not heavily reviewed in the textbooks they selected. It was also found that some conceptualizations of slope were "sparsely represented in textbooks" and that a well-rounded, connected view of slope is generally lacking in many textbooks.

Coding Frameworks Resulting from the Material Expansion of Previous Work

Another intuitive way to develop a coding framework is to expand or modify an existing framework to fit the specific analysis situation. For example, from the original four types of knowledge in Revised Bloom's Taxonomy, Radmehr and Drake (2017) developed 11 subtypes of knowledge for integral calculus. Gracin (2018) added one dimension, the mathematics activity, to the standard four dimensions of textbook analysis and used this five-dimensional framework to investigate the requirements in mathematics textbooks.

Radmehr and Drake (2017) unpacked the knowledge dimension for Revised Bloom's Taxonomy (RBT in Anderson et al., 2001) for integral calculus. The original framework of the RBT knowledge dimension has four types: factual, conceptual, procedural, and metacognitive knowledge. To develop subtypes of knowledge specifically for integral calculus, they considered

the examples provided in the RBT handbooks, consulted research papers that defined RBT in particular disciplines, considered integral calculus teaching resources, used mathematics education literature about RBT types of knowledge and examples, and incorporated feedback from PhD supervisors. In the end, they introduced 11 subtypes of knowledge with examples from integral calculus. For factual knowledge, the subtypes were knowledge of terminology and specific details and elements. For conceptual knowledge, the subtypes were knowledge of classifications and categories; knowledge of principles and generalizations; and knowledge of theories, models, and structures. For procedural knowledge, the subtypes were knowledge of subject-specific skills and algorithms, knowledge of subject-specific techniques and methods, and knowledge of criteria for determining when to use appropriate procedures. Finally, for metacognitive knowledge, the subtypes were strategic knowledge, knowledge about cognitive tasks, understanding of when to include appropriate contextual and conditional knowledge, and self-knowledge.

Gracin (2018) investigated requirements in mathematics textbooks. In addition to the standard four dimensions of analysis in content, cognitive demands, question type, and contextual features, he added one extra dimension of analysis, i.e., mathematical activities, to construct a five-dimensional framework of textbook analysis. He applied this five-dimensional framework to analyze more than 22,000 tasks from commonly used Croatian mathematics textbooks in the 6th, 7th, and 8th grades. The author was the primary coder who examined the tasks according to the five-dimensional instrument and coded them into the corresponding categories. Samples of the tasks were checked by the experts and creators of the standards to ensure data validity and reliability. The coding result was analyzed via SPSS to find the relative frequencies within a specific mathematical topic. It was found that there was no balance between

different task types, and textbook tasks were computational with low-level cognitive demands. The finding suggested that different mathematical activities would be more likely to challenge students and help them develop their understanding.

Newly Developed Frameworks Based on Specific Content

Because the content under analysis is diverse and a readily available framework does not always exist for specific content, many investigators develop a framework specifically for the content they analyze. For example, to understand the extent of reasoning and proving opportunities in calculus textbooks, Bergwall and Hemmi (2017) developed two frameworks: one for the expository section and one for the tasks and worked examples. In addition, Mesa (2010) developed a framework to analyze the availability of strategies to perform the work in solving initial value problems.

Bergwall and Hemmi (2017) investigated expository sections in the four most frequently used Finnish and Swedish textbook series and studied the nature and extent of reasoning and proving opportunities offered in secondary-level integral calculus textbooks. Two frameworks were used in this study. For the expository section, justification of statements was classified into four categories: *general* if the statement was justified with proof; *specific* if the statement was justified using a deductive argument based on a specific case or cases; *left to students* if the justification of the statement was left for the student to complete; and *no justification* if there was none provided. Bergwall and Hemmi (2017) used a two-dimensional framework for the tasks and work examples. The first dimension was the *nature of reasoning*, which was intended to capture different elements of mathematical reasoning. The second dimension was the *type of reasoning*, differentiating between tasks about general cases or about specific cases. The coders for this study were the two authors, who conducted the analysis independently, compared notes, and

discussed with each other when an instance of ambiguity occurred. It was found that Swedish textbooks "offer few opportunities to learn proof given the near absence of general proofs" (Bergwall & Hemmi, 2017, p. 13). The Finnish textbooks offer more opportunities for learning proof. Although proof-related tasks are about the same, around 10% of the text in both Swedish and Finnish textbooks, a difference exists in how these texts prepare students for those tasks.

Mesa (2010) studied 80 examples of initial value problems in twelve calculus textbooks to analyze their strategies for controlling the work to solve a problem and verify if the answer is correct. The author constructed a coding system to analyze the control structure of the text to answer three questions. First, how does a reader know how to solve the problem? Second, how does the reader know if an answer is found? Third, how does the reader know if the answer is correct? The sentences in worked examples were parsed into four categories according to the sentences' objectives: describe what to do, indicate that an answer is found, verify the correctness of an answer, and elaborate. The coding system was tested in seven examples by three coders other than the author, and the reliability of the coding system was verified by comparing their results with the author's results. It was found that the opportunities to establish correct answers were not generally made explicit, even with a topic like initial value problems, which could provide ample opportunities.

Content Analysis as Part of the Research Method

Content analysis can be used as a stand-alone research method or in conjunction with other research methods. For example, Fan et al. (2013) used a survey study and content analysis. Yoon et al. (2021) use discourse analysis and content analysis.

Fan et al. (2013) used a survey study as a research design and content analysis as a research method to conduct a study on the development status and direction of textbook research

in mathematics education. The results of systematically searching keywords in the digital library and selected journals found that the number of publications about textbook research increased steadily in the three decades from the 1980s to the 2010s. The framework for categorizing these articles is primarily based on the focus of these articles. They categorized mathematics textbook research into four categories: the role of textbooks, textbook analysis and comparison; textbook use; and other use. They found that the major research areas were textbook analysis and textbook comparison. Coders were the authors, and a reliability check by comparing the results obtained by the study's first and third authors yielded a 0.96 inter-rater consistency.

Yoon et al. (2021) investigated how culture, research, and policy compete in shaping the calculus curriculum in South Korea. First, discourse analysis was used to identify how arguments concerning calculus education were structured in the public discourse. Then, content analysis was used to analyze how the national calculus curriculum changed, particularly in regard to definite integrals. Through the content analysis it was found that the concepts of limits of sequences and the Riemann sum were eliminated from the 2015 curriculum revision, which led textbook producers to change how they defined and explained definite integrals. Together with the result of discourse analysis, the finding implies that mathematics education research should be put into the context of policy research on mathematics education.

Section Summary

In summary, content analysis in mathematics education can yield valuable insights into the structure, organization, statistics, and relationship among content categories. It can be used as a stand-alone or in combination with other research methods. The framework in content analysis is the key to its success. The framework's source can be a pre-existing one, a modification from a pre-existing one, or one newly developed for specific content in the study.

Covariational Reasoning

A covariational relationship is a relationship between the changes of two quantities. Covariational reasoning is "the cognitive activities in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, 2002, p. 354). There are two key elements in covariational reasoning. One is the perception of changes, and the other is how changes are linked or correspond to each other. Both elements appeared and occupied an important position in the history of the function concept. The concept of function is of great importance in developing mathematical thoughts that some believe are the cornerstone of Western culture. As Schaaf (1930) put it and quoted by Kleiner (1989) that "the keynote of Western culture is the function concept, a notion not even remotely hinted at by any earlier culture." The concept of function is indispensable in almost every mathematics subfield. However, different elements of the concept of function were emphasized at different periods of history. And as of today, there is "no single generally accepted definition of the concept of a function" (Medvedev, 1991). Among diverse views of this concept from different mathematic practitioners, from professional mathematicians to mathematics learners, two widely used concepts are set and correspondence. Throughout history, the notion of variable, change, independent variable, and the dependent variable was useful for covariational reasoning. Thompson and Carlson (2017a) systematically reviewed the idea and history of covariation. They started with the historical development of mathematicians' conceptions of functions, from the understanding of proportion, which can be represented geometrically and statically, to equations that represent constrained variation in related values and to functions that explicitly represent a relationship between values.

In the following, a brief history of the development of the function concept is reviewed with an emphasis on how the historically important elements in the function concept can be useful in covariational reasoning. Next, the buildup of covariational reasoning capacities through a typical education setting is reviewed. Then, the framework for covariational reasoning is presented, and its developmental level is reviewed.

Covariational Reasoning and the History of Functional Reasoning

Human perception of change can be dated back at least to the Axial age when philosophers from the East (Confucius) and the West (Heraclitus) expressed the continuous change of time in a way similar to the continuous flowing of water in a river. Likewise, the perception of the correspondence relationship between variables can be dated back to the Babylonian tables of reciprocals, squares and square roots, and cubes and cube roots (Youschkevitch, 1976) in 2000 B.C.

The word "function" appeared for the first time in a paper published by Leibniz in 1692, where it was used to designate geometrical quantities, such as a point of a curve (Monna, 1972). In the 17th century, with the demand from the quantitative laws of nature, for example, the study of the relation between curvilinear motion and the forces affecting the motion, function representations in algebraic form and graphic form were developed by Rene Descartes (1596-1650), Pierre de Fermat (1607-1665), and Newton (Youschkevitch, 1976). In particular, "Newton chooses time as a universal argument and interprets dependent variables as continuously flowing quantities possessing some velocity of change" (Youschkevitch, 1976, p. 54). The interpretation of time as a continuously flowing quantity that provided other dependent variables with a context of change led to the study of the continuity and differentiability of

analytic functions. However, up to the Newtonian age, the universality of time as an independent variable had not evolved since ancient times.

The calculus developed by Newton and Leibniz in the 17th century was mostly a collection of methods to solve problems about curves, such as tangent to the curve, areas under curves, lengths of curves, and velocities of points moving along curves (Kleiner, 1989). The function concept at that time emphasized change and correspondence; as Leibniz asserted, "a tangent is a function of a curve" and "a function is a fact asserted by an equation" (Iacobacci, 1965, p.85).

The algebraic form of function was further developed to emphasize variables and correspondence in the 18th century. Johann Bernoulli's (1694 – 1718) widely quoted definition of function was "One calls here Function of a variable a quantity composed in any manner whatever of this variable and of constants" (Bottazzini, 1986, p.9). Leonhard Euler (1707-1783) viewed mathematical analysis as the general science of variable and their functions. In his work of 1748, *Introductio in analysin Infinitorum*, he defined "A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities" (Ruthing, 1984, p.72). Euler did not define "analytic" but tried to explain that it involved the four algebraic operations, roots, exponentials, logarithms, trigonometric functions, derivatives, and integrals (Kleiner, 1989). During that period, the variables in a function were differentiated as independent and dependent. German mathematician Peter Dirichlet (1805 - 1859) gave his definition in 1837, which is deemed by many as the first modern definition of function: "If a variable y is so related to a variable x that whenever a numerical value is assigned to x , there is a rule according to which a unique value of y is **determined**, then y is said to be a function of the independent variable x " (Britannica, 2022).

The function concept during this period, such as variables, change, and correspondence, provides great assets for covariational reasoning. However, functional reasoning at that time did not necessitate covariational reasoning. For example, the notion that variable x **determines** the variable y implies a sequence of order for variables x and y to take values, i.e., a value needs to be given to the independent variable x first before the dependent variable y can have a value.

Consequently, the dependent variable y does not have to change unless the independent x changes first. The notions of dependent and independent variables can be very helpful for thinking of one change at one time with the strict rule that the change of the dependent variable follows the change of the independent variable. The correspondence property of a functional relationship provides a context for variable changes to be related quantitatively. On the other hand, the independent and dependent notions of variables in a functional relationship imply that their changes do not occur at the same time, thus, not simultaneously.

The calculus developed by Newton and Leibniz was not the form students see today in calculus textbooks (Kleiner, 1989). The formal development of infinitesimal calculus in algebraic form and analytic expression, particularly the rigorous definition of limit, as we see in today's calculus textbooks, will need to wait for another hundred years. Previously, the expression of the derivative as the ratio of two quantities (independent variable and dependent variable) did not necessitate the thinking of two changing quantities simultaneously. With the epsilon-delta language introduced by Karl Weierstrass (1815-1897), finding derivative as the rate of two instantaneous changes necessitate thinking of two limiting process (one in the numerator and another one in the denominator) at the same time. Thus, being able to think of an instantaneous rate of change indicates that one is capable of covariational reasoning.

Many functions in today's definition are not differentiable. Thus, the instantaneous rate of change may not be meaningful in these functions. However, up to the beginning of the 19th century, it was generally believed that all functions were continuous and differentiable; the only exception was possibly at a few isolated points, such as $x = 0$ in the absolute value function. However, the evolution of the concept of function in the 19th century took a turn and gradually emphasized correspondence with less concern about continuity. It started when Joseph Fourier (1768 - 1830) considered temperature a function of time and space. Fourier conjectured, without proof, that it would be possible to obtain a representation of any function in a trigonometric series. To take on the conjecture by Fourier, Dirichlet proposed that it would be necessary to separate the concept of function from its analytic representation (Ponte, 1992). Following his vision of separating the concept from the analytic representation of the function, Dirichlet gave the well-known example of a function named after him. The Dirichlet function is discontinuous at all the points of the domain between 0 and 1. The function takes the value of 0 if x is a rational number and takes the value of 1 otherwise. In 1861, German mathematician Georg Friedrich Bernhard Riemann (1826-1866) introduced a function that is an infinite sum of a trigonometric series to further separate the continuity and differentiability of a function. Riemann claimed without proof that the function he introduced was continuous at every x but not differentiable for infinitely many values of x . In 1872, Karl Weierstrass (1815-1897) pushed the separation between continuity and differentiability to the extreme by introducing a new function that was continuous at every value of x and differentiable at none (Bressoud, 2007). These "monstrous" functions discovered by mathematicians in the late 19th century that rely only on correspondence between variables but nothing resembles the daily experience of continuity in time and space prompted a French mathematician Charles Hermite (1822-1901) to "turn away with fright and

horror from this lamentable plague of functions that do not have derivatives" (Bressoud, 2007; p.269). Hermite's reaction to these "unnatural" functions was not alone. For example, in 1889, a French mathematician Henri Poincare (1854-1912) called these newly invented functions "bizarre functions that do their best to resemble as little as possible to those honest functions that serve a useful purpose" (Bressoud, 2007; p.269) and complained that

In earlier times, when we invented a new function, it was for the purpose of some practical goal. Today, we invent them expressly to show the flaws in our forefathers' reasoning, and we draw from them nothing more than that.

The notion of function continued to evolve with the development of set theory and abstract algebra. In the 20th century, the definition of function extended to include all correspondences satisfying the uniqueness condition between sets, numerical or non-numerical. However, in general, the variable in a function is treated as a symbol representing an element of a set. Thus, in general, a correspondence relationship between two variables in a function may not always give us useful information to reason how the changes of two variables are related, as we have seen in the Dirichlet function.

For the study in this dissertation, the notion of function is restricted to the early 19th century, which is generally analytical and differentiable except for some finite points. However, even within this classic notion of analytic functions, there is one significant difference between the relationship of variables in a covariational relationship and the relationship of variables in functions. The covariational relationship among variables emphasizes the relationship among variable changes, while the variables bounded by functions deal with obtaining variable values (as output) from other variables (as input). While identifying a relationship among variable changes through observing the relationship of variables is straightforward, the opposite operation does not always have a standard method. For example, with a given explicit function of three

variables, a total differentiation of the function will yield the relationship among three variables and three variable changes. However, when an explicit function gives a relationship among three variables and their changes, there is no standard method for obtaining a relationship involving only variables but not their changes¹. In this regard, the relationship among variables governed by an explicit function can be viewed as a simplified version of the covariational relationship among variables.

The brief review above indicates a significant overlap between the concept of variables in the functional relationship and variable changes in the covariational relationship. The pre-19th-century notion of functional reasoning can be helpful in covariational reasoning. In particular, the thinking process to obtain the derivative of analytic functions, from thinking about the ratio of two quantities to the ratio of two changing variables in two related limiting processes, provides a natural scaffold to build up covariational reasoning capacity.

The modern mathematical definition of function is characterized today by one-to-one correspondence relationships. As a result of the advancement of abstract algebra, in mathematical structures such as groups, rings, and fields, thinking of a variable only as a symbol that stands for an element of a set can be more productive than thinking of a variable as having a value that varies. For professional mathematicians, the meaning of the variable in functions typically has undergone several stages during their learning process. First, they encountered it as an independent variable and dependent variable in secondary school. Second, it became something that varied in the study of calculus or analysis in upper secondary school or early college. Third, it assumed the status of a symbol that stands for an element of a set in senior

¹ Given $f(x, y, z) = 0$, a total differentiation on the function $f(x, y, z)$ will give us the relationship among x, y, z, dx, dy, dz . However, with a given relationship such as $g(x, y, z, dx, dy, dz) = 0$, finding an explicit relationship among variables x, y, z does not always have a standard method.

college years and beyond. Unless they focus on the study or the application of calculus, thinking of variables as an element that varies continuously is not a concept they often use in their daily research activities.

Covariational Reasoning Buildup and Assessment

Numerous studies have investigated how the conception of covariation was embedded in various aspects of mathematics education, the treatment of covariational thinking in different topics (such as multiplicative objects, algebra, functions, exponential growth, calculus, and trigonometry), and students' and teachers' understanding of covariation (Thompson et al., 2017a). Thompson and Harel (2021) surveyed calculus research on the impact of mathematical understanding students developed before their formal calculus learning. First, they examined the research on students' early learning about the algebraic ideas foundational to calculus. They analyzed how these ideas can be fostered before calculus so that students' understanding of them can later be relevant to learning calculus. Their survey yielded an abundance of literature regarding the importance of continuous covariational reasoning in physics, chemistry, biology, geoscience, and economics. According to their survey, students who conceive variation productively in early grades, such as learning about variables and functions, are more likely to think of covariation productively in learning calculus. However, research suggested that many students had little opportunity in early grades to build meanings for variables, function, accumulation, and rate of change in ways that would give them a greater chance to conceive ideas in calculus productively. Toh (2021) examined the Singapore school calculus curriculum at the secondary levels and concluded that the partial calculus knowledge acquired in the early secondary levels might not help students develop a complete concept of calculus at the university level. Thus, a solid preparation in precalculus mathematics can do little to aid in learning

calculus if the rich preparation is not handled in a way that promotes ideas foundational to calculus learning. This observation prompted Thompson (Thompson & Harel, 2021; and references therein) to insist that "It takes 12 years to learn calculus" (p. 513).

Frank and Thompson (2021) used two data sources to understand how likely students in secondary school are to have opportunities to construct meanings for function, variation, and rate of change in ways that are productive for calculus learning. The two data sources were meanings for the ideas supported by precalculus textbooks and meanings secondary teachers demonstrated. For the four precalculus textbooks they chose and analyzed, they found that textbooks failed to promote an image of variation, conveyed the discrete image of change, and conveyed incoherent meaning for a constant rate of change. They also found that these four precalculus textbooks conveyed geometric meaning for an average rate of change with no alternatives and conveyed average rate of change as an arithmetic mean or "smoothed out" change which can be problematic for understanding calculus. In summary, they found a disconnect between meanings productive for learning calculus and the meaning in textbooks held by U.S. high school teachers. Comparing the meanings held by U.S. and Korean teachers, they concluded that these meanings of function, variation, and rate of change were culturally embedded in the U.S. educational system.

Thompson and Milner (2019) observed that U.S. teachers' university and professional training had little influence on the meanings they had already developed in their high school. Bressoud (2021) described the growth of high school calculus enrollment since the early 1950s, the growing pressure to enroll in high school calculus, and the equity issue in the Advanced Placement program. He observed two significant downsides to the current situation of calculus teaching in the U.S. education system. He believed that the best response was to "strengthen the

high school preparation of all students so that they experience challenging mathematics, mathematics that confronts them with unfamiliar problems and builds their ability to learn through reflective struggle" (Bressoud, 2021, p. 532).

Tallman et al. (2021) analyzed 254 Calculus I final exams at U.S. colleges and universities to find out what meanings were accessed on final exams from single-variable calculus courses. In the derivatives category, Tallman et al. (2021) found that "students are rarely required to interpret functions or situations in terms of covariational reasoning such as coordinating changes in output for successive equal changes in input" (p. 582). In the extreme values, inflection points, and graphical analyses of functions category, Tallman et al. (2021) found that "the common associations of extrema with figurative features of a derivative function's graph were likely only to require static shape thinking" (p. 583). Only one out of 96 extreme value problems were identified as demanding continuous covariational reasoning. In the Riemann sums, definite integration, and the fundamental theorem of calculus category, "no items necessitated interpreting integrals dynamically or required students to reason covariationally about integrals in such a way to promote their construction of a process view of integral functions" (Tallman et al., 2021, p. 586). In summary, very few items required students to engage in continuous covariational reasoning to solve problems.

Ely (2021) described some critical elements of differentials-based calculus courses and summarized research showing that students in these courses develop robust quantitative meanings of such notations. He argued that a differential-based approach could provide students with an intuitive sense of the fundamental theorem of calculus. As an accumulation function $f(x)$ can be expressed as $f(x) = \int_a^x r_f(t)dt$, where $r_f(t)$ is the rate of change of $f(x)$ at t . A differential dt is treated as a variable whose value varies smoothly. This treatment of differential

enables students to use smooth covariational reasoning at an infinitesimal scale to make sense of the deep connections between derivative as a rate of change and integral as accumulation.

Weber and Thompson (2014) investigated how students' understanding of graphs of single-variable functions influenced their generalizations of graphs of two-variable functions and the role of covariational reasoning in those generalizations. They analyzed the videotapes and recorded laptop screen animations from two students during an intensive instruction grounded in quantitative and covariational reasoning that lasted for three weeks. They concluded that covariational reasoning allows students to generalize their understanding of functions to visualize corresponding graphs. Furthermore, not being able to think covariationally may hinder such generalizations.

Covariational Reasoning Framework

According to the overall images students exhibit in their problem-solving process, Carlson et al. (2002) proposed a theoretical construct of five mental actions for covariational reasoning abilities.

The five levels of mental actions (M.A.s) Carlson et al. (2002) proposed range from thinking about whether there is a change, the direction of the change, the amount of change, the average rate of change, and the instantaneous rate of change (see Table 2.1).

Table 2.1: The five levels of mental actions (M.A.s) of covariational reasoning proposed by Carlson et al. (2002).	
MA1	An image of two variables changing simultaneously;
MA2	A loosely coordinated image of how the variables are changing with respect to each other (e.g., increasing, decreasing);
MA3	An image of an amount of change of one variable while considering changes in discrete amounts of the other variable;
MA4	An image of rate/slope for contiguous intervals of the function's domain;
MA5	An image of continuously changing rates over the entire domain

The theoretical construct of five mental actions proposed by Carlson et al. (2002) seems to have a hierarchy which scaffolds the way to think about change, with the end goal of visualizing the instantaneous rate of change at MA5. Using the language of derivative and a one-variable function $y = f(x)$ as an example, for any point x_0 , the instantaneous rate of change $f'(x_0)$ can be any real number. In practice, Mental action 1 (MA1) questions whether there is a change, i.e., whether $f'(x_0) = 0$. Mental action 2 (MA2) questions whether the change is increasing or decreasing, i.e., whether the number $f'(x_0)$ is positive or negative. Mental action 3 (MA3) questions the amount of the change, i.e., how quick the change is, or the absolute value of $f'(x_0)$. Mental action 4 (MA4) is the transitional way to think about $f'(x_0)$ by visualizing the average rate of change as x approaching x_0 . Finally, we reach mental action 5 (MA5), the rate of instantaneous change at x_0 is represented by a real number with its meaning fully unpacked (whether it is zero, positive, or negative, as well as its absolute value).

After proposing the theoretical construct of five mental actions, Carlson et al. (2002) used it to investigate high-performing 2nd-semester calculus students' covariational reasoning ability. It was found that "observed trends suggested that this collection of calculus students have difficulty constructing images of a continuously changing rate" (p. 372). In their observations, the five different covariational reasoning mental actions seem to have levels, i.e., "A student who is classified as exhibiting level 5 (L5) covariational reasoning, relative to a specific task, is able to reason using MA5 and is also able to unpack that mental action to reason in terms of MA1 through MA4" (Carlson et al., 2002, p. 358). The study found that most students were able to apply L3 consistently and not able to apply L4 reasoning consistently; they had difficulty applying L5 reasoning at all. Since then, several studies have used this framework to investigate students' and teachers' mental processes regarding covariational reasoning capacity (Cetin, 2009;

Hobson, 2017; Moore et al., 2013; Thompson et al., 2017a; Thompson et al., 2017b). Cetin (2009) designed three real-life situational tasks to investigate students' comprehension of the rate of change and the function-derivative relationship. That study found a pattern of difficulty that was similar to the findings of Carlson et al. (2002). For instance, Cetin wrote, "70% of the subjects were unsuccessful in reacting correctly" to certain situations (p.241). Thompson et al. (2017b) designed a graph-sketching task and used it to examine covariational reasoning capacity among U.S. and South Korean secondary mathematics teachers. They found that "covariational reasoning at a mature level is present among both U.S. and South Korea samples, but not to the extent that anyone should celebrate" (Thompson et al., 2017b, p. 106). Moore et al. (2013) investigated types of thinking students engage in when graphing in the polar coordinate system by studying two students while they graphed functions such as $r = \theta^2$ and $r = \sin(2\theta)$. It is found that both students typically relied on first graphing discrete points and comparing discrete amounts of change between these points, which is prototypical of a "chunky thinker" or MA4 (average rate of change). In addition, both students exhibited behaviors suggesting that they conceived of change as smoother within these chunks. This finding is consistent with Carlson's framework regarding the transition from mental action 4 (average rate of change or chunky change) to mental action 5 (instantaneous rate of change or smooth change).

The framework of five mental actions developed by Carlson et al. was initially designed to study people's mental activity, but it has been adapted for content analysis. It is used to analyze graphical representations in science and mathematics textbooks as well as practitioner journals (Paoletti et al., 2020). The covariational mental action framework proposed by Carlson et al. seems to be "an analytical tool with which to evaluate covariational thinking to a finer

degree than has been done in the past," and "it provides a structure and language for classifying covariational thinking" (Carlson et al., 2002, p. 359).

Recently, Tasova et al. (2018) are systematically developing a framework to analyze written curricular material in calculus textbooks by adapting a shape thinking construct (Moore & Thompson, 2015) and a covariational reasoning framework (Thompson et al., 2017a). In their working paper, Tasova et al. (2018) categorize occurrences of variables in calculus textbooks as *static* and *emergent*. The first category, named *static*, appears in the texts that do not invoke a process view of quantities or relationships among them such that values vary or covary. A typical example is the statement that " $y = ax + b$ is called a linear function," which is based on its analytic form without giving attention to how the variables x and y are changed together. Similarly, stating that "in a parabola equation $y = ax^2 + b$, the size of a determines the width of the parabola, and the sign determines the direction in which the parabola opens" does not invoke a process view of how variables x and y covary, or how the coefficient a and variable y covary. This categorization of *static* echoes Thompson and Carlson's views about how "the concept of function" can be interpreted differently among different groups (Thompson et al., 2017a). On the contrary, the other category, *emergent*, as proposed by Tasova et al. (2018), is reflected in texts that explicitly express the process of different levels of covariational reasoning. For example, a statement like "As independent variable x changes, so does the dependent variable y " provides an opportunity for the reader to develop a mental picture of the coordination of change between variables x and y . In another example, the statement that " x is a variable, $f(x)$ the other variable. As the distance between x and 1 gets smaller, the distance between $f(x)$ and 2 also get smaller" provides the reader a mental picture of how the size of variation in two variables are linked. The latter sentence explicitly states that variables change as well as how the amount of change is

linked; thus, it is deemed to promote a more sophisticated level of covariational thinking than the first example.

Developmental Levels in the Covariational Reasoning Framework

The five levels of the covariational reasoning framework originally proposed by Carlson et al. (2002) are developmental in the sense that people who can reason at the higher level automatically have the capacity to reason at lower levels. The developmental nature of the covariational reasoning capabilities is similar to the developmental nature of Piaget's (1964) theory in stages of cognitive development and Van Hiele's theory (Lerman, 2014) of development for geometrical understanding. However, there are differences.

According to Jean Piaget (1964), children's cognitive development can be divided into four stages, with each of the stages representing a new way of thinking and understanding the world. Each stage is approximately correlated with age. They are sensorimotor intelligence (0-2 years), preoperational thinking (2-7 years), concrete operational thinking (7-11 years), and formal operational thinking (12 years and up). According to Piaget, regardless of social or cultural background, the intellectual development of children takes place through stages in a fixed order and cannot be skipped.

According to Dina Van Hiele-Geldof and Pierre Van Hiele (Lerman, 2014), students' geometrical understanding also progresses through various levels and cannot be skipped. These levels are visual, descriptive/analytic, abstract/relational, formal deductive, and rigor/metamathematical, ranging from informal description to formal reasoning.

Current studies of people's covariational reasoning behavior seem to confirm that the five-level framework proposed by Carlson et al. (2002) also has a hierarchy. However, the

theoretical underpinnings of the hierarchy of the covariational reasoning levels may still be open to question.

In general, we can think about a change of a variable as a vector, which has a magnitude and a direction. In the current framework, level 1 questions whether the change vector is zero. Level 2 questions the direction of the change vector. And going from levels 3 through 5 is a process that resembles the process of finding a limit value to get the instantaneous rate of change as the magnitude of the change vector. However, in terms of the image of a vector, the direction does not necessarily need to come up before the magnitude. In the case of a single variable function, the direction of change is either increased or decreased. Thus, the thinking about direction is binary, which could be simpler than the thinking process used to get the instantaneous rate of change. The simplicity of thinking of the direction relative to the magnitude could contribute to why the direction of the change is at a lower level (or easier to come by) than the magnitude of the change.

Section Summary

Covariational thinking is embedded in the concept of function, variables, change, and correspondence. However, for a professional mathematician, thinking of a variable as an element that changes continuously may not always be as productive as thinking of a variable as a symbol that stands for an element of a set.

Covariational reasoning seems to be developmental. The theoretical framework of covariational reasoning proposed by Carlson et al. (2002) has five levels, from the first level of questioning whether there is a change to the last level of picturing an instantaneous rate of change. Subsequent studies of covariational reasoning in various populations and situations seem to confirm the validity of the hierarchy of this framework. The subsequent study of this

dissertation is built on the validity of the five-level framework proposed originally by Carlson et al. (2002).

Mathematics Education in China

Education in China has a long and continuous history. The history of mathematics education in China dates back at least 3000 years. The Western view of modern mathematics education is well documented in the handbook by Karp and Schubring (2014). The Chinese approach to modern mathematics education is articulated in the book by Zhang (2013).

In this section, several brief overviews are given. First, a brief description of the Chinese culture of learning is reviewed to give the cultural context of learning and teaching in China. Next, a brief description of the current Chinese education system is given. Then mathematics education in China is briefly reviewed. Calculus is currently a part of the high school curriculum in China and is a requirement for some college majors, such as STEM and economics.

Brief Description of the Chinese Culture of Learning

Chinese traditional culture is dominated by Confucian orthodoxy with complementary ideas from Taoism and Buddhism. The principle and practices of education with Confucianism are documented in two texts: Analects (论语) and Record of Learning (学记). In principle, the aim of learning in Confucianism (Tan, 2017) is to cultivate humanity (仁) through normative behavior (礼) so that learners can broaden the true way of life (道). In practice, it promotes learning for self-realization and achievement, which has two aims: to facilitate individual human development and to "regulate the family, govern the state and lead the world into peace (齐家, 治国, 平天下)." These two purposes of learning reflect the intrinsic and extrinsic needs

satisfied by learning. These two purposes are related, integrated parts of the whole picture with a central underlying theme of harmonic coexistence among people and with nature.

There is a difference between Buddhism in general and Chinese Buddhism. Chinese Buddhism promotes learning from everyday activities, from kitchen duties to general labor, to obtain true wisdom and stop suffering (Shih, 2007). In addition, learning should be practiced through the cycle of reflection and practice, i.e., thinking and doing.

Lao Tzu's philosophy of Taoism influences Chinese culture more subtly. The central theme of Taoism is that human life is an integral part of the cosmos: the goal of human life is to experience and maintain harmony with the Tao (道). Learning is a way to understand the cosmos to provide solutions to human problems. Thus, intervening excessively in nature creates complexities and difficulties in humans' relationship with nature.

In fact, Lao Tzu's writing, the *Book of The Way* (道德经), which consists of eighty-one chapters, is the original and spiritual foundation of later works, including writings by Confucians and Chinese Buddhists. Moreover, many of the later works from different schools of thought are practical guides to gain more of an understanding of the Tao under certain specific situations (道裂百家).

Brief Description of the Current Educational System in China

Since the founding of the People's Republic of China in 1949, typical Chinese students have spent six years in elementary school, three years in middle school, and three years in high school before college. The Chinese educational system is centralized and controlled by the state. The first nine years of education (elementary and middle schools) are compulsory for everyone in China. High school is not compulsory, and admission to high school is based on a competitive entrance exam. Students who do not pass the exam may elect to enter vocational or technical

secondary schools or enter the workforce immediately. After three more years of schooling, high school graduates can choose to take the National College Entrance Examinations to gain admission to universities and colleges or to join the workforce (Mak, 2013; and references therein).

Brief Description of Mathematics Education in China

Mathematics education in ancient China has a 3000-year history. Mathematics education is always an integrated part of human development in Chinese civilization. For example, mathematics (明算科) was one of the topics in the Imperial Examination, officially instituted in the Tang dynasty (618-907 AD). The Imperial Examination was open to all people as a channel to become governmental officials who serve society, a model emulated in modern times by most societies around the world for the civil service system. However, it seems that the current mathematics education researchers and teachers in China have not fully understood and applied the mathematical thoughts created in ancient China (Qin, 2018).

Since the founding of the People's Republic of China in 1949, mathematics has been a school subject in China starting in the first grade. The study sequences are arithmetic, algebra, geometry, and trigonometry followed by advanced topics such as calculus, probability, statistics, and game theory, which are introduced in high schools. Chinese mathematics education aims to position students' mathematical development on a good foundation, which refers to three major abilities: mathematical operations, spatial imagination, and logical thinking. The current mathematics education in China is comprised of five aspects: the import of new knowledge, teacher-student interaction and trial teaching, variant teaching, repeated exercise, and refining of mathematical thinking (Duan, 2012).

China's current mathematics curriculum reform re-emphasizes two basics (basic knowledge and basic skills). In addition, with the focus on problem-solving ability and the close relationship between mathematics and daily life, mathematical modeling gets more attention in the new curriculum. In the trial version of the new curriculum published in 2014, comprised of Essential Series, Elective Series 1, and Elective Series 2, calculus is included in both Elective Series (Wang et al., 2018). Computer-assisted instruction of college calculus was implemented in the reform of calculus education of the 1980s (Lang, 1999).

Chapter III: Methodology

This research investigates what learning opportunities were made explicit in four widely used calculus textbooks to help students build covariational reasoning capacities. There are three major considerations to review before we dive into the details of the research design. From the reader's theory reviewed in the last chapter, the first consideration is the intended readers for the calculus textbooks under investigation, who can be categorized into two major groups: calculus students and front-line calculus teachers. The second consideration involves the explicitness of the language—is it enough to promote covariational thinking? This consideration is based on applying the reader's theory to common mathematics concepts, such as function. As exemplified in Thompson and Carlson's review (2017a), the same word, such as “function,” can invoke different mental pictures in different groups of the mathematics community, such as mathematics students, mathematics educators, and professional mathematicians. Finally, for comparing four categories of calculus textbooks in two languages, there is another issue about the role of natural language in the mathematics presentation of calculus textbooks. Presentations across different languages could be perceived by different investigators differently.

The method of this research falls into the category of content analysis. The commonly accepted definition of content analysis is the use of written, audio, or visual material to seek the meaning reflected in the materials under study. What is missing (or hidden) in the definition is the identity and background of the researcher conducting the content analysis. For example, suppose that content analysis only relies on counting the occurrence of certain words, general patterns in the length of sentences, etc.; it is a task that computers can accomplish in modern times, quite possibly with better accuracy than most humans. However, whenever human judgment of the meaning of the content is under consideration in the analysis, the identity and

background of the researcher need to be carefully calibrated to be aligned with the intended readers of the content. Thus, to address the first concern, the investigators must come from the pool of intended readers, ideally calculus students. However, there is another constraint for the selection criteria of investigators, i.e., they must be able to accurately understand and evaluate the five levels of covariational reasoning. The consideration for second concern is reflected in the criteria of the text requirement in this study. Only text that explicitly spells out the change of one variable relative to another variable will be considered as something that promotes covariational reasoning. The third concern regards the cross-sectional effects of natural language on mathematics content. For example, a study involving separate analysis of textbooks in English and Chinese by separate groups of investigators will bring up the issue of the different influences of subjectivity in separate groups. To address this issue, this study requires the same group of investigators to analyze all four textbooks, with each investigator independently evaluating them.

Both types of content analysis, conceptual and relational analysis, are used in this research. Conceptual analysis determines the existence and frequency of concepts in a text. Relational analysis develops the conceptual analysis further by examining the relationships among concepts in a text. Thus, this study was divided into three phases. The first phase was the textbook selection conducted by me alone. The second phase was conceptual analysis and was conducted by two independent investigators. My role in the second phase was coordination, screening, training investigators, and as a resource when they needed some clarification. Finally, the third phase was relational analysis conducted by two independent investigators and me.

Textbook Selection

One widely used calculus textbook from each of the four categories was selected. The four categories are US College, US high school, China college, and China high school. For US college, the eighth edition of *Calculus* by James Stewart was selected. According to some estimates (Baldassi, 2014), this book is used by about 70% of mathematics, science, and engineering students. I specifically searched for the Advanced Placement (AP) edition of the calculus textbook written for US high school students. Several AP editions of calculus textbooks emerged as a result of my survey, such as *Calculus of a Single Variable* by Ron Larson and Bruce H. Edwards; *Calculus Graphics, Numerical, Algebraic* by Ross L. Finney, Franklin D. Demana, Bert K. Waits, Daniel Kennedy, and David M. Bressoud; and *Calculus for AP: A Complete Course* by Stephen Kokoska and James Stewart. In the end, I chose *Calculus Graphics, Numerical, Algebraic* by Finney et al. for the US high school category. It is listed as the most common choice of AP calculus textbook on some websites (such as <https://magoosh.com/>), and it is one of the AP calculus sample textbooks listed by the College Board (<https://apcentral.collegeboard.org/courses/ap-calculus-ab/course-audit>). For the China college category, through a casual survey of people currently teaching college mathematics in China, I chose *Higher Mathematics* by the Math Department in Tongji University, a reputable university in China known for its strong programs in technology, engineering, and architecture. For the China high school category, I chose the most popular and widely used series of textbooks, published by the People's Education Press (PEP) (Li, 2020). This dissertation studies the calculus textbooks used by STEM students. Thus, *Electives 2-2* will be used to study high school calculus textbooks in China.

After selecting four textbooks, I identified the chapter and section that first introduces the concept of derivatives. Subsequent studies analyze that section only. Table 3.1 shows four calculus textbooks chosen for this study and the sections introducing the concept of the derivative.

Table 3.1: Four calculus textbooks chosen for this study, and the sections that introduces the concept of derivatives		
Textbook & Section	China	The U.S.
High School	Elective 2-2 from General High School Curriculum Standard Experimental Textbook Mathematics (2005) by People’s Education Press.	Calculus: <i>graphical, Numerical, Algebraic</i> (2014) by R. L. Finney, F. D. Demana, B. K. Waits, D. Kennedy, & D. M. Bressoud. ISBN: 0-13-331161-9.
	Chapter 1.1, Pages 1 – 11.	Section 2.4, Pages 87 – 96.
College	<i>Higher Mathematics</i> (2014) by Tongji University. ISBN: 978-7-04-039663-8.	<i>Calculus</i> (2016) by J. Stewart. ISBN: 978-1-285-74062-1
	Chapter 2.1, Pages 73 – 84.	Section 2.1, Pages 105 – 117.

Conceptual Analysis

The second phase was a conceptual analysis conducted by two investigators other than me. Any written text that *explicitly* promotes covariational thinking was classified into five categories, parallel with the five mental actions originally proposed by Carlson et al. (2002). In this study, only written text was analyzed; figures were not included in the analysis because figures in the paper textbooks were presented as static and did not necessarily invoke any image of change. Different readers may or may not come up with an image of change or covariation when they look at the figures. The divergence of responses to a static figure is similar to the divergence of responses from the different readers when they encounter a phrase such as “linear function.” Some readers may have a mental image of two variables changing together at an even pace, while others only picture a straight line. Some readers may picture an analytic expression

of the type $y = ax + b$. However, the captions of the figures are part of the text and were included in the analysis.

I used the framework of five text levels to analyze the *emergent* category (Tasova et al., 2018) of texts in the four selected calculus textbooks (see Table 3.2). As alluded to earlier, the emphasis on the explicitness of the text is a direct result of the intended readers' perspectives, a consideration of reader-oriented theory (Rosenblatt, 1995).

Table 3.2: Five levels of covariational mental activities and the five levels of passage that promote covariational thinking	
Mental Action	Text Levels that Explicitly Promote Covariational Thinking
(MA1) Coordinating the value of one variable with changes in the other variable	(T1) Covariation of variables is explicitly written.
(MA2) Coordinating the direction of change of one variable with changes in the other variable	(T2) The direction of the change of one variable is explicitly written with the change of another variable.
(MA3) Coordinating the amount of change of one variable with changes in the other variable	(T3) The amount of the change of one variable is explicitly written with the change of another variable.
(MA4) Coordinating the average rate-of-change of the function with uniform increments of change in the input variable	(T4) The average rate of the change of one variable is explicitly written with the uniform change of another variable.
(MA5) Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	(T5) The instantaneous rate of change of one variable is explicitly written with continuous or instantaneous change of another variable.

The process of searching for qualified investigators was more involved. To start with, they needed to be familiar with calculus and to be completely fluent in both Chinese and English. In the research community that publishes in English, I observed that many researchers have used the five-level framework by Carlson et al. (2002), with some of them in Asia, such as South Korea or Thailand. However, there are no existing reports of any research conducted in China

using this framework, either by students or teachers. I further reached out to knowledgeable people in Chinese mathematics education and could not find any existing experts on this topic within the Chinese-speaking community.

I decided to find people and train them to do the evaluations. The first thought I came up with was calculus students, either those currently learning calculus or those who have taken calculus before. If this idea had worked out, the profile of people who conducted this study would have matched exactly the intended readers of the textbook under study. However, this idea was quickly dismissed when I realized that there is a huge gap between the thinking process involved in doing calculus (a student's perspective) and the knowing of other people's thinking processes and how a text may promote them (an experienced educator's perspective). The next logical pool of people I looked into was calculus teachers. The first step was looking for bilingual calculus teachers who were interested in the study through previous colleagues, a network of friends, and social media. Initially, I aimed to have a pool of investigators who, collectively, had teaching experience in all four categories of schools. This step yielded five candidates including four current and one former calculus teachers. All of these candidates taught calculus at the college level and two had experience in teaching high school. All of them have formal mathematics training backgrounds up to the master's level, two have doctorates in mathematics, and one is currently in a doctorate of mathematics education program. Four of them currently reside in the US and one of them recently finished his doctoral study in mathematics and returned to China. The second step of investigator qualification started with the introduction of the framework and pilot study. I set up a Zoom meeting with each candidate separately. In the meeting I explained in detail how the framework worked, what this study was about, and how much time it could possibly take. Gauged by the candidates' interest exhibited in

the Zoom meeting and their understanding of the covariational reasoning framework in the subsequent pilot study, only two candidates were included in the formal investigation. In the following description of the investigators, pseudonyms are used.

Betty is licensed for grades 7-12 mathematics in New York State and is currently teaching in a public high school there. She came to the US from China for college and is fluent in English and Chinese. I got to know Betty when she was a mathematics graduate student in the department where I taught part-time. Betty also taught calculus at the college level when she was a graduate assistant for the mathematics department. I contacted Betty and set up a Zoom meeting. In the introductory meeting, Betty seemed to get the general idea of this project fairly quickly and drew on comparable examples from her high school teaching. We went ahead to conduct a pilot study, and Betty continued to show a high level of interest through timely communication with me and active discussion via Zoom meetings. Thus, Betty was included in all subsequent studies.

Adam is currently teaching at a private university in New York State. He previously taught high school and is also licensed for 7-12 grade mathematics in New York. Adam came to the US from China while in high school and is fluent in English and Chinese. I came to know Adam through a social media network on which I posted a general request for an investigator to work in calculus textbook analysis in English and Chinese. Adam responded to my post and seemed to be very interested in the project. In the introductory Zoom meeting, Adam immediately understood the general idea of content analysis. We went ahead with the pilot study, and his initial result came back quickly. The conversation continued, and Adam kept a heightened interest during subsequent works. Thus, Adam is included as an investigator for this study.

Here is a description of the pilot study in the qualification process. The study used a section concerning introduction to derivatives from a calculus textbook by Anton et al. (2012), which was not one of the four chosen textbooks for formal research. In the pilot study, conducted with each investigator separately, the author explained to the investigator the five levels of covariational reasoning and what to look for in the text. I put great effort in to train the investigator to get into the mindset of calculus students who are not necessarily mathematics experts and do not speak the language of inquiry mathematics. I emphasized that the sentences need to explicitly promote the five levels of thinking, but not what we, as experts on the topics, thought was implied. I further specifically spelled out the difference between what we, as mathematics teachers and mathematically literate people, might infer from the sentences and what students, as mathematics novices, may have difficulty forming a mental image of. A sample of examples that explicitly promote covariational thinking is given in Table 3.3.

Table 3.3: Examples of passages that explicitly promote different levels of covariational thinking in the pilot study		
Text Levels	Textbook Writing Examples (Anton et al., 2012, p.131-p.143)	Rationale
(T1) Covariation of variables is explicitly written.	“If we let x approach x_0 , then the point Q will move along the curve and approach the point P.” (p.131)	x is a variable, the point Q is another variable. The change of x and the change of Q is explicitly written.
(T2) The direction of the change of one variable is explicitly written with the change of another variable.	“In this case, s increases as t increases.” (p. 135)	t is one variable, and s is another variable. The direction of the change for s is explicitly written.
(T3) The size of the change of one variable is explicitly written with the change of another variable.	“For example, knowing that the speed of an aircraft is 500 mi/h tells us how fast it is going, but not which way it is moving.” (p.134)	Magnitude of change is explicitly written as 500 miles, and the change of time is one hour.

(T4) The average magnitude of the change of one variable is explicitly written with the uniform change of another variable.	“Find the average velocities of the particle over the time intervals (a) $[0, 2]$ and (b) $[2, 3]$.” (p. 135)	Average rate of change in distance is explicitly written with the change of time from 0 to 2 and 2 to 3.
(T5) The instantaneous rate of change of one variable is explicitly written with continuous or instantaneous change of another variable.	“These average velocities may be viewed as approximations to the ‘instantaneous velocity’ of the particle at time t_0 . If these average velocities have a limit as h approaches zero, then we can take that limit to be the <i>instantaneous velocity</i> of the particle at time t_0 .” (p. 136)	t is a variable that changes when the change of variable h is (or approaches) zero. The velocity is viewed as “instantaneous” (built from the concept of limit).

In the pilot study, after the initial introduction, each investigator looked for the sentences that explicitly promoted any of the five levels of covariational thinking. Once a passage was identified, the investigator took a screenshot, recorded the page number, and underlined the keywords. Next, the investigator put the screenshots in a Google Doc in a Google Drive, shared with me only. Next, the investigator categorized each item as either T1, T2, ..., or T5. After each investigator finished their round of study, I set up another Zoom meeting and went through all items with the investigator to further validate the rationale for each item. The rationale for this step in the pilot study was to clarify concepts and processes further. The pilot study for conceptual analysis took about two weeks for each investigator.

After the pilot study, I created a shared Google Drive folder for Adam and one for Betty. Adam and Betty conducted their investigations independently. They never met and communicated in any fashion. This phase of the formal conceptual study and the next phase of the relational study was conducted by Adam and Betty first and aggregated by the author from time to time to provide them with more material to investigate. Both investigators worked full

time at their jobs and participated in this study only a few hours a week after work or on weekends.

For the section that introduces the concept of the derivatives in each of the four chosen textbooks, denoted as U1 (meaning the first universe), Adam and Betty collected, in a fashion similar to that used in the pilot study, all the passages that promote covariational thinking. The sequence of the textbooks analyzed was US College, US High School, China College, and China High School. It took one to three weeks for each of the four textbooks for each investigator. For each textbook, Adam and Betty took screenshots of the passages they considered to be promoting different levels of covariational thinking and highlighted the keywords in the identified section. Afterward, they put the image of the screenshot and the page of the passage in a table and marked them as T1, T2, ..., or T5 in the shared Google Doc. The whole collection of screenshots from Adam's initial analysis is denoted as A1. Similarly, the whole collection of screenshots from Betty is denoted as B1. Once Adam and Betty had completed their first assessment separately, I went through every item in A1 and B1 and created a union of A1 with B1, denoted as U2 (meaning the second universe). It took about ten weeks for all four textbooks to go through this analysis stage. After all four textbooks were analyzed, I created a Google Sheet for each textbook that contained every item in U2 and put the sheet in the folders shared with Adam and Betty separately. Adam and Betty next re-evaluated every item in U2 to see if they met the criteria set forth in Table 3.2. The rationale for this step is two-fold. One rationale is that individual investigators may have omitted or neglected an item at the first look. Once they see what other investigators deem as a passage that promotes covariational thinking, they may spot an omission from the first look. The other rationale is that after two months, a fresh look at their own first evaluation may yield a different opinion. In the second assessment, the

investigators did not need to screenshot or highlight any new passage from the original textbook. They worked on only the Google Sheet that contains every item of U2. Each item of U2 occupies one row in the Google Sheet. Adam and Betty placed a mark on a row if they decided that item should not be part of the passage that promotes any level of covariational thinking. Next to the mark, they could write a brief reason why the item did not qualify to promote any level of covariational thinking. This second assessment was an elimination process from U2. The resulting collection of the screenshots left from Adam's elimination process is denoted as A2. Similarly, the resulting collection of the screenshots left by Betty is denoted as B2. Finally, I went through A2 and B2 item by item and created another collection of screenshots, named Z3, including each item which appeared in both A2 and B2. In other words, Z3 is the intersection of A2 and B2. The collection of items in Z3 is the final result of the conceptual analysis. Figure 3.1 shows the whole process from the section introducing the concept of derivatives to the final results of the conceptual analysis.

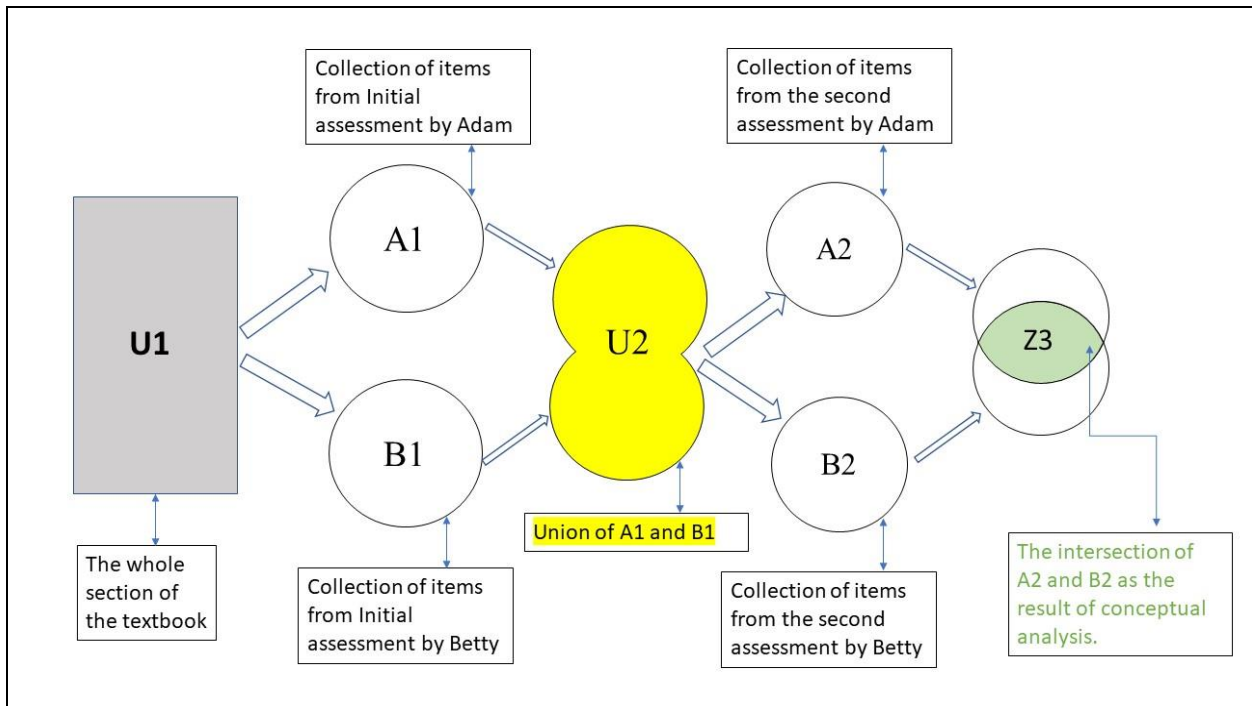


Figure 3.1: The investigation process from the section of the textbook that introduces the concept of derivatives to the results of conceptual analysis

Relational Analysis

The third phase is relational analysis. There are two sub-phases in this relational analysis. The first sub-phase is a relational analysis within a given textbook and the second sub-phase is a pattern comparison among different textbooks. In addition to Adam and Betty, I also participated in the first sub-phase. Within the collection of items from the previous conceptual analysis, Z3, each investigator decided whether an item is a transition or has continuity with another item in terms of covariational thinking. For each occurrence of a passage that promotes covariational thinking in the motivation, real-world example and application, exercise, and homework problems sections, investigators determined whether this occurrence is connected to the main expository presentation and established the nature of the connection. For example, if a question was asked in the motivation and was then explained in the main text, that can be considered a connection. If a concept was presented in the main text, and the following exercise was posed to reinforce the idea, that could also be considered a connection. Three levels of connection were used: *none*, *simple*, and *strong*. Simple connections are connections among concepts. A strong connection not only connects different levels of concept but uses the same context to do so.

The second sub-phase of relational analysis is pattern comparison among different textbooks. This sub-phase was conducted by the author alone. Using the result from the previous sub-phase, I gave an overview of the similarity and differences that exist in all four textbooks.

Chapter IV: Results

The result of the conceptual analysis and relational analysis is presented in this chapter. All textbooks have main text, examples, and exercises in the sections introducing the derivative concept. In addition, some textbooks have side notes that give some historical facts, and one textbook includes a writing project after the exercise section. Table 4.1 presents an overview of these sections.

Table 4.1: Overview of sections introducing the derivative concept in this study's four selected calculus textbooks				
Textbook	US College	US High School	China College	China High School
Main text	118 sentences	92 sentences	113 sentences	77 sentences
Examples	7 examples	8 examples	11 examples	3 examples
Figures	9 figures	5 figures	5 figures	4 figures
Exercises	61 problems	71 problems	20 problems	12 problems
Side note	11 sentences	20 sentences		5 sentences
Writing project	8 sentences and 4 references			

Results from the Conceptual Analysis

In analyzing each selected calculus textbook, there are several intermediate steps to getting the final results of the conceptual analysis - two rounds of independent analysis from two investigators, Adam and Betty. The intermediate results from the first round were abbreviated as A1 (Adam's) and B1 (Betty's). Similarly, the intermediate results from the second round were abbreviated as A2 and B2. Table 4.1.1 shows this process's intermediate and final results and their abbreviations.

Table 4.1.1: The intermediate and final results of the conceptual analysis and their abbreviations	
Abbreviation	The intermediate or final step of conceptual analysis
U1	The whole section of the selected calculus textbook
A1	The whole collection of screenshots by Adam from his initial analysis
B1	The whole collection of screenshots by Betty from her initial analysis

U2	Union of A1 and B1
A2	The resulting collection of screenshots left after Adam's eliminations from U2
B2	The resulting collection of screenshots left after Betty's eliminations from U2
Z3	The intersection of A2 and B2 as the final result of conceptual analysis

Covariational Reasoning—Conceptual Analysis of a US College Calculus Textbook

Text Occurrences That Promote Different Levels of Covariational Reasoning

The textbook for US colleges by Stewart (2016) was the first calculus textbook Adam and Betty formally analyzed. At first glance, Adam identified 15 passages that promote thinking about change (T1). However, two months later, after gaining more experience by analyzing more textbooks, he crossed out more than half of his original identifications. The core of this progression was a deeper understanding of and an adherence to the definition of T1, which requires that changes of two variables be explicitly written so as to be qualified as an explicit expression of covariation. For example, Figure 4.1.1.1 shows a screenshot of a passage that Adam initially thought should be characterized as T1.

■ Rates of Change

Suppose y is a quantity that depends on another quantity x . Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2 , then the change in x (also called the **increment** of x) is

Figure 4.1.1.1: A passage Adam rated as T1 in this initial assessment but decided was not eligible in his second assessment.

In his second look two months later, Adam realized that this sentence states only that y and x were related. But the change of x and the change of y was not explicitly spelled out. Thus, he crossed out this text in his second assessment. Betty did not include this passage as T1 in her initial assessment, and she crossed it out when she saw it in U2. Figure 4.1.1.2 shows an example Adam thought qualified as T1 in both the initial and the second assessment. However, Betty did

not think it qualified as T1 and duly noted, "it does not specify the change although people may image the change based on the understanding of function."

16. The displacement (in feet) of a particle moving in a straight line is given by $s = \frac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.

Figure 4.1.1.2: A passage Adam rated as T1 in both his initial and second assessment but which was crossed out by Betty, who did not think it qualified as T1

The final selection criterion is that both Adam and Betty needed to think an example qualified as T1 in their second assessment for the example to count. Thus, the text shown in Figure 4.1.1.2 was not included in the final result. An example of the text that both Adam and Betty thought qualified as T1 is shown in Figure 4.1.1.3. Although my opinion was not part of the selection process, I did notice that the changes of two variables (Q and x) in this sentence and the related changes were spelled out explicitly.

Then we let Q approach P along the curve C by letting x approach a . If m_{PQ} approaches a number m , then we define the *tangent* t to be the line through P with slope m . (This

Figure 4.1.1.3: A passage both Adam and Betty rated as T1

The average rate of change and the instantaneous rate of change are essential for the section that introduces the derivative concept. On page 110 of Stewart's textbook, an elaborated definition of the average rate of change is given by explicitly spelling out the change of independent variable x and the change of the function value $f(x)$ as well as their quotient. Next, the instantaneous rate of change is defined as the limit of average rates of change. Subsequently, both Adam and Betty took these two terms as academic terms that could promote covariational thinking at T4 and T5. In some occurrences of these terms, Adam classified them as T4 or T5 if

the text also refers to some variables' change. Figure 4.1.1.4 shows an example of the passage that both Adam and Betty thought promotes covariational thinking about the average rate of change. I can see that the change is spelled out explicitly with respect to the variable x .

51. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.

(a) Find the **average rate of change** of C with respect to x when the production level is changed

(i) from $x = 100$ to $x = 105$

(ii) from $x = 100$ to $x = 101$

Figure 4.1.1.4: A screenshot of the passage; both Adam and Betty thought it explicitly promotes covariational thinking in terms of the average rate of change (T4).

As a result of the whole process, three occurrences of T1, none of T2, one of T3, nine of T4, and eight of T5 were found. Table 4.1.1.1 shows all intermediate and final results, listing the number of occurrences of passages that explicitly promote each level of covariational thinking. Passages are from the section that conceptually introduces derivative in Stewart's textbook (2016).

Table 4.1.1.1: Intermediate and final results from the analysis of the introduction of derivatives in the calculus textbook by Stewart (2016). See Table 4.1.1 for the meaning of A1, B1, ..., Z3.

US College	A1	B1	U2	A2	B2	Z3
T1	15	3	15	7	3	3
T2	0	3	3	0	0	0
T3	1	3	4	1	4	1
T4	11	18	20	9	10	9
T5	6	8	10	8	10	8

Development of Covariational Reasoning Levels

After all the passages that promote the different levels of covariational reasoning were identified, we could investigate how the levels were built up as the textbook progressed. The line of presentation in this textbook is the main text, followed by the exercises. All the examples are embedded in the main text. Figures, together with sidenotes, appear on the right side of each page. The width of the sidenote column is about 30% of the width of the whole page. Table 4.1.1.2 shows the sequence of passages that promote different levels of covariational reasoning as the book progresses through the main text and into exercises.

Table 4.1.1.2: The development of levels of covariational reasoning as the passage progresses through the main text to the exercises in the introduction of derivatives section of the calculus textbook by Stewart (2016).

Source	Sub-category	Passage number in the main text	Level
Main expository presentation	Main text	4	T1
	Main text	21	T1
	Main text	29	T3
	Main text	49	T1
	Main text	64	T4
	Main text	66	T5
	Example	83	T5
	Example	102	T5
	Main text	109	T4
		Problem number in the exercises	
Exercise	Exercise	16	T4
	Exercise	16	T5
	Exercise	18	T4
	Exercise	47	T4
	Exercise	47	T5
	Exercise	48	T4
	Exercise	48	T5
	Exercise	49	T4
	Exercise	49	T5
	Exercise	50	T4
	Exercise	51	T4
	Exercise	51	T5

The development of covariational reasoning levels is shown in Figure 4.1.1.5. The x -axis represents the passage's progression, with the passages in the main expository body running from 1 to 118 (M1 to M118), followed by the exercises from 1 to 61 (E1 to E61). Low levels of covariational reasoning (T1 to T3) are developed sparsely in the first half of the main text. The second half of the main text presents a few opportunities to develop the concept of average rate and instantaneous rate of change, which are heavily developed in the exercises part of the section, with some problems presenting opportunities to develop both the concepts of average (T4) and instantaneous (T5) rate of change.

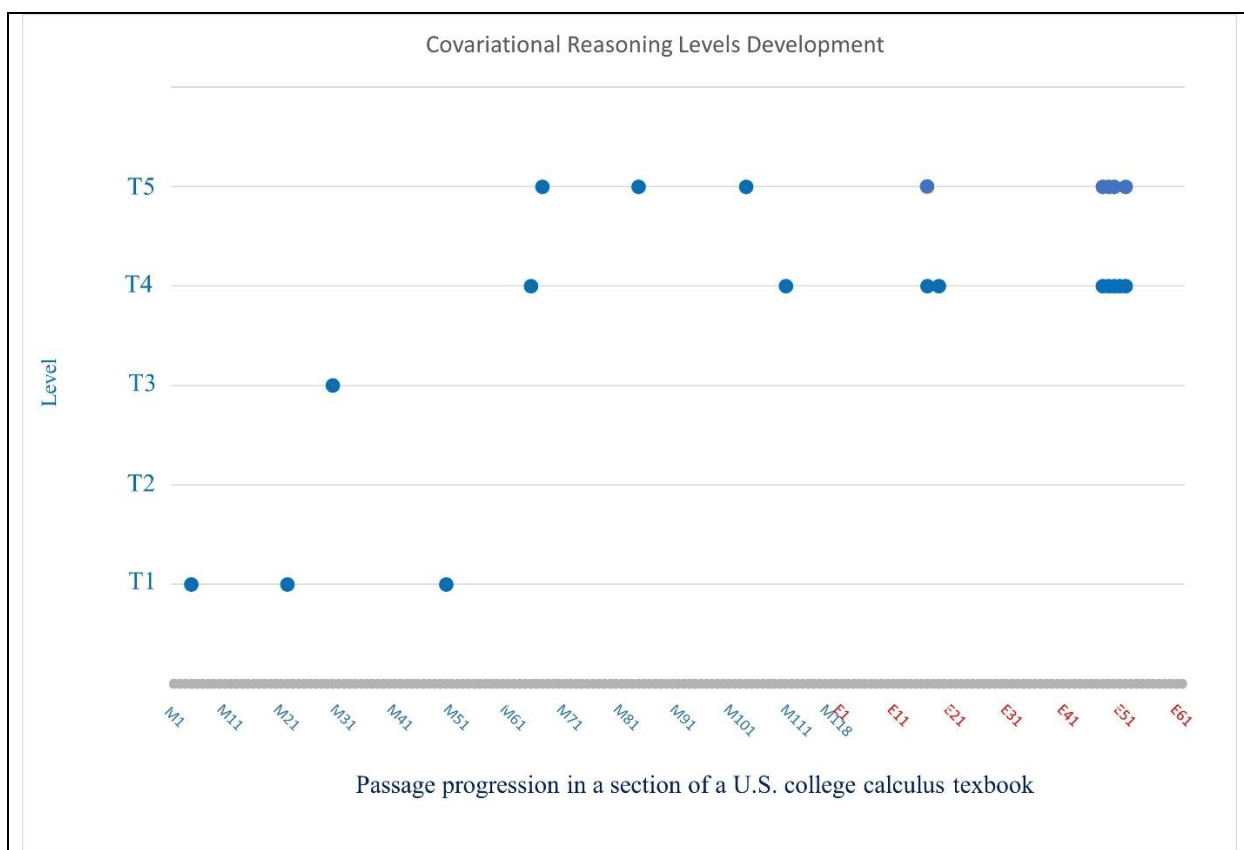


Figure 4.1.1.5: Development of covariational reasoning levels as the passage progresses through the main text (M1 to M118) to the exercises (E1 to E61) in the section that conceptually introduces derivatives in the calculus textbook by Stewart (2016).

Covariational Reasoning—Conceptual Analysis of a US High School Calculus

Textbook

The textbook for the US high schools by Finney et al. (2014) was the second textbook Adam and Betty formally analyzed. Contrary to other calculus textbooks in this study, which put the rate of change as a subsection in the derivative section, the textbook by Finney et al. puts the section that explains the rate of change in the chapter "Limits and Continuity." Then, the chapter titled "Derivatives" starts with a formal definition of derivative by using the concept of limit to define the derivative at a point and the derivative of a function. After the definition, examples of finding derivative functions from various functions are given. This study analyzed how the derivative was conceptually introduced. Thus, section 2.4, which introduces rates of change, tangent lines, and sensitivity, was analyzed.

Text Occurrences That Promote Different Levels of Covariational Reasoning

The rate of change in Finney et al.'s textbook was defined as "the amount of change divided by the time it takes." This definition is casual and oversimplifies by suggesting that all independent variables are time and that the change of other variables should all be treated with respect to time. In addition, this definition lacks a process view of the function and presents only a static picture that the rate of change is a result of a division. Despite this oversimplification of the definition, Adam and Betty still found some passages that explicitly promoted covariational thinking. Figure 4.1.2.1 shows one example that Adam and Betty agreed to designate as T1.

Sensitivity

We live in an interconnected world where changes in one quantity cause changes in another. For example, crop yields per acre depend on rainfall. If rainfall has been low, each small increase in the amount of rain creates a small increase in crop yield. For a drug

Figure 4.1.2.1: A passage in the calculus textbook by Finney et al. (2014); Adam and Betty found that it promotes covariational thinking of change (T1).

There was no passage Adam and Betty agreed on that promotes covariational thinking regarding the direction of change. However, Figure 4.1.2.2 shows a passage that Adam and Betty found to promote covariational thinking regarding the magnitude of change.

This means that when the dosage is 1 mg, a small additional dosage, ΔD mg, will result in a drop in the patient's temperature of approximately $\frac{3}{4} \Delta D$ degrees.

Figure 4.1.2.2: A passage from the calculus textbook by Finney et al. (2014) that both Adam and Betty rated as promoting covariational thinking in terms of the amount of change (T3).

The instantaneous rate of change for $f(x)$ with respect to x at $x = a$ is presented as a static picture, being the same as the slope of $y = f(x)$ at $x = a$, or the slope of the tangent to $y = f(x)$ at $x = a$, or $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$. Although the average rate of change and the instantaneous rate of change were not defined in a way that explicitly promoted covariational thinking, both Adam and Betty accepted them as academic terms that could promote covariational thinking to readers. Thus, they took all the later occurrences of these two terms in the main text and the exercises as T4 and T5. Many occurrences of T4 and T5 were located in the examples and exercises. Figure 4.1.2.3 shows a passage Adam and Betty rated as T4.

You may use a calculator with these problems.

- 1. Multiple Choice** Which of the following values is the average rate of $f(x) = \sqrt{x + 1}$ over the interval $(0, 3)$?
- (A) -3 (B) -1 (C) $-1/3$ (D) $1/3$ (E) 3

Figure 4.1.2.3: A Finney et al. passage that both Adam and Betty rated as promoting covariational thinking in terms of the average rate of change (T4).

As a result of the process, investigators found three occurrences of T1, one T3, nine T4s, and fourteen T5s. Table 4.1.2.1 shows all intermediate and final results in terms of the number of occurrences of passages that explicitly promote different levels of covariational thinking in the section that conceptually introduces derivative in the calculus textbook by Finney et al. (2014).

Table 4.1.2.1: Intermediate and final results of the analysis of the introduction of derivatives in Finney et al.'s textbook. See Table 4.1.1 for the meaning of A1, B1, ..., Z3.

US High School	A1	B1	U2	A2	B2	Z3
T1	6	3	7	6	4	3
T2	0	2	2	0	0	0
T3	1	2	3	1	3	1
T4	8	8	12	7	12	8
T5	8	8	15	12	15	12

Development of Covariational Reasoning Levels

The line of presentation in Finney et al.'s textbook is similar to that of the previous one. One difference is the placement of figures. All figures and examples are embedded in the main text, followed by the exercises. There are three types of exercise in this book, Quick Review (10 problems), Exercises (57 problems), and Quick Quiz (4 problems). Table 4.1.2.2 shows the sequence of passages that promote different levels of covariational reasoning as the book progresses through the main text and exercises.

Table 4.1.2.2: The development of levels of covariational reasoning as the passage progresses through the main text and the exercises in the introduction of derivatives section of Finney et al.'s textbook (2014)

Source	Sub-category	Passage number in the main text	Level
Main expository presentation	Example	4	T4
	Main text	23	T1
	Example	74	T5
	Main text	80	T1
	Main text	81	T1
	Example	90	T5
	Main text	92	T3

		Problem number in the exercises	
Exercise	Exercise	11 (10+1)	T4
	Exercise	23 (10+13)	T5
	Exercise	33 (10+23)	T5
	Exercise	37 (10+27)	T5
	Exercise	39 (10+29)	T5
	Exercise	40 (10+30)	T5
	Exercise	41 (10+31)	T5
	Exercise	42 (10+32)	T5
	Exercise	47 (10+37)	T5
	Exercise	50 (10+40)	T4
	Exercise	50 (10+40)	T4
	Exercise	50 (10+40)	T4
	Exercise	54 (10+44)	T4
	Exercise	66 (10+56)	T4
	Exercise	66 (10+56)	T5
	Exercise	68 (10+57+1)	T4
	Exercise	71 (10+57+4)	T5

The development of covariational reasoning levels in this section of the textbook is presented in Figure 4.1.2.4. where the x -axis represents the progression of the passage, with the passages in the main expository body running from 1 to 92 (M1 to M92) followed by the exercises from 1 to 71 (E1 to E71). We can see that this textbook offers a significant number of high-level examples (T4 and T5) of covariational reasoning in the exercise part of the section, in a way that is similar to the US college calculus textbook in this study. However, the development of the levels in the main text is more smoothly progressive. The lower levels (T1 to T3) are sparse and appear in the second half of the main text after the covariational reasoning levels are progressed and developed to high levels (T4 and T5) in the middle of the main text.

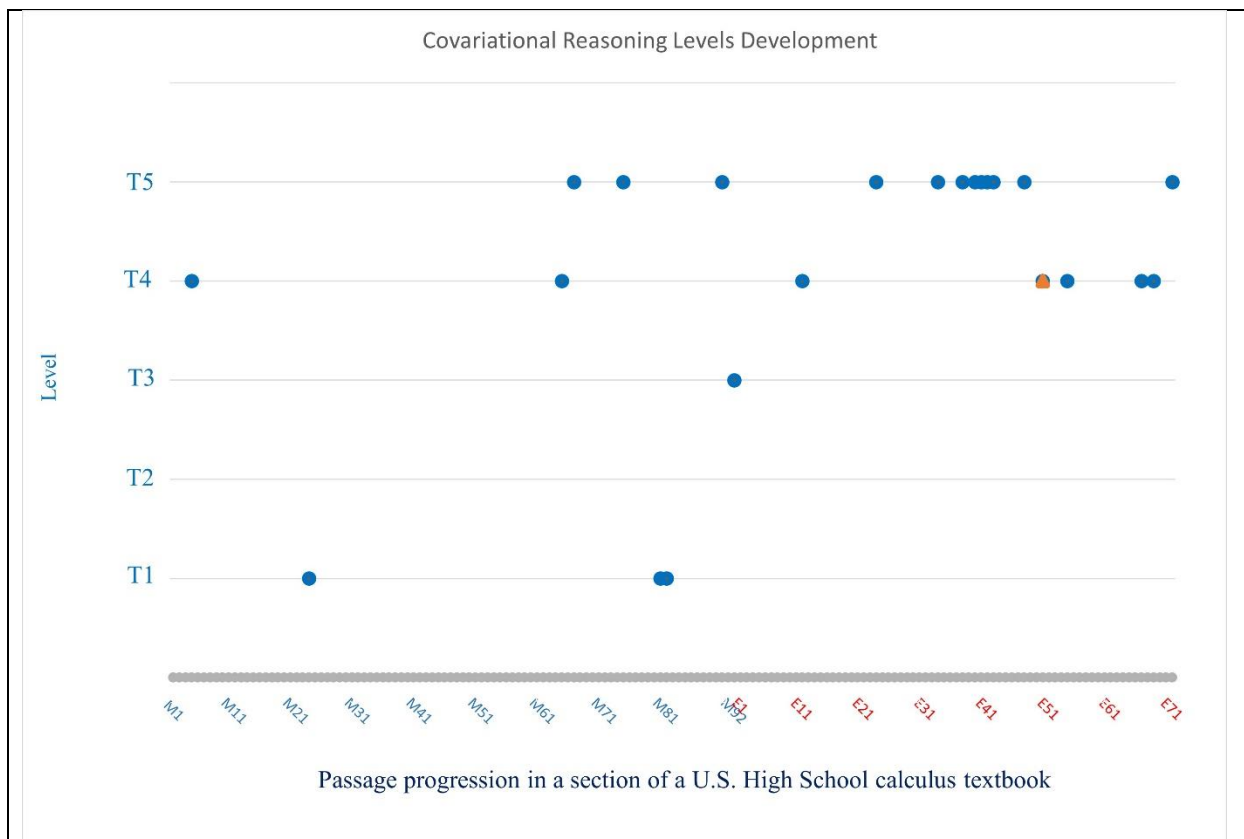


Figure 4.1.2.4: The development of levels of covariational reasoning as passages progress through the main text (M1 to M92) and the exercises (E1 to E71) in the introduction of derivatives section in Finney et al.'s textbook (2014)

Covariational Reasoning—Conceptual Analysis of a Chinese College Calculus

Textbook

The textbook for Chinese Colleges, *Higher Mathematics* by Tongji University Press (2014), is the third textbook Adam and Betty formally analyzed.

Text Occurrences That Promote Different Levels of Covariational Reasoning

In the initial assessment, Adam found no occurrence of T2 through T5. At the same time, Betty saw some passages that qualify as T2 through T5. Figure 4.1.3.1 shows an example Betty initially identified as T3.

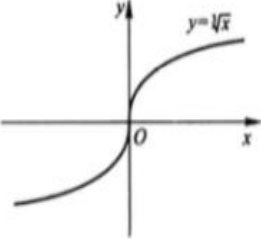
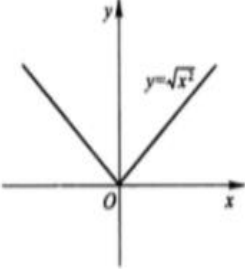
The original text in Chinese	2. 当物体的温度高于周围介质的温度时,物体就不断冷却.若物体的温度 T 与时间 t 的函数关系为 $T=T(t)$,应怎样确定该物体在时刻 t 的冷却速度?
What it means in English	When the temperature of an object is higher than its surroundings, the object is continuously cooling down. If the relationship between the temperature and the time is $T = T(t)$, <u>how can one determine the speed of cooling at time t?</u>
Figure 4.1.3.1: A <i>Higher Mathematics</i> passage Betty rated as promoting covariational thinking in terms of the magnitude of change (T3)—Adam disagreed.	

Adam did not think this passage qualified as T3 and duly noted that it "did not mention about the amount of the change of one variable." In her initial assessment, Betty identified nine occurrences of T4, and Adam rejected them all. Figure 4.1.3.2 shows a passage that Betty rated as T4; Adam rejected that rating and duly noted that "the relationship of change is not explicitly written." From my understanding, it seems that Betty took the occurrence of 导数, derivative in English, as an academic term that she regarded as a promotion of the average rate of change, and Adam disagreed.

The original text in Chinese	<p>例4 求函数 $f(x) = \sin x$ 的导数.</p> <p>解 $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$</p> $= \lim_{h \rightarrow 0} \frac{1}{h} \cdot 2 \cos\left(x + \frac{h}{2}\right) \sin \frac{h}{2}$ $= \lim_{h \rightarrow 0} \cos\left(x + \frac{h}{2}\right) \cdot \frac{\sin \frac{h}{2}}{\frac{h}{2}} = \cos x,$ <p>即 $(\sin x)' = \cos x.$</p>
What it means in English	Example 4: Find the derivative function of $f(x) = \sin x$. Solution: ...
Figure 4.1.3.2: A <i>Higher Mathematics</i> passage Betty rated as promoting covariational thinking in terms of the average rate of change (T4)—Adam disagreed.	

Most of the items which Betty rated as T5 were rejected by Adam. For example, Figure

4.1.3.3 shows a passage Betty rated as T5; Adam did not.

<p>The original text in Chinese</p>	<p>例 11 函数 $y = \sqrt{x^2}$ (即 $y = x$) 在 $(-\infty, +\infty)$ 内连续, 但在例 7 中已经看到, 这函数在 $x=0$ 处不可导. 曲线 $y = \sqrt{x^2}$ 在 <u>原点 O</u> 没有切线(图 2-5).</p> <div style="display: flex; justify-content: space-around; align-items: center;"> <div style="text-align: center;">  <p>图 2-4</p> </div> <div style="text-align: center;">  <p>图 2-5</p> </div> </div>
<p>What it means in English</p>	<p>Example 4: Function $y = \sqrt{x^2}$ (i.e., $y = x$) is continuous in $(-\infty, +\infty)$, but in example 7, it has been shown that this function is not differentiable <u>at $x = 0$</u>, function $y = \sqrt{x^2}$ does not have a tangent line <u>at the origin</u> (graph 2-5).</p>
<p>Figure 4.1.3.3: A <i>Higher Mathematics</i> passage Betty rated as promoting covariational thinking in terms of the instantaneous rate of change (T5)—Adam disagreed.</p>	

Out of eighteen items Betty initially rated as T5, Adam agreed with four of them in his second assessment. All four use academic terms such as derivative, speed, and slope. Figure 4.1.3.4 shows a passage Betty identified as T5 and which Adam accepted.

<p>The original text in Chinese</p>	<p>10. 已知物体的运动规律为 $s = t^3$ m, 求这物体在 $t = 2$ s 时的 速度.</p>
<p>What it means in English</p>	<p>10. The object's moving pattern is known to be $s = t^3$ m; find the object's <u>speed</u> at $t = 2$.</p>
<p>Figure 4.1.3.4: A <i>Higher Mathematics</i> passage Betty rated as promoting covariational thinking in terms of the instantaneous rate of change (T5)—Adam agreed.</p>	

As the result of the investigation process, six occurrences of T1 and four occurrences of T5 were found. Table 4.1.3.1 shows all intermediate and final results in terms of the number of occurrences of the passages that explicitly promote different levels of covariational thinking in the section that conceptually introduces derivative in the Chinese calculus textbook by Tongji University Press (2014).

Table 4.1.3.1: Intermediate and final results from the analysis of the introduction of derivatives section in the Chinese calculus textbook, <i>Higher Mathematics</i> . See Table 4.1.1 for the meaning of A1, B1, ..., Z3.						
China College	A1	B1	U2	A2	B2	Z3
T1	9	2	11	9	6	6
T2	0	1	1	0	1	0
T3	0	1	1	0	1	0
T4	0	9	9	0	1	0
T5	0	18	18	4	7	4

Development of Covariational Reasoning Levels

The line of presentation in this textbook is main text followed by exercises. There are no sidenotes. All figures and examples are embedded in the main text. Many of the examples are derivation of the derivative for commonly seen functions such as power functions, trigonometry functions, exponent functions, and logarithm functions. The geometric interpretation of derivative and the relationship between continuity and differentiability is also covered in this section. To illustrate the development of covariational reasoning levels, Table 4.1.3.2 shows the sequence of passages that promote different levels of covariational reasoning as the text progresses through the main text and exercises.

Table 4.1.3.2: The development of levels of covariational reasoning as passages progress through the main text and the exercises in the introduction of derivatives section in <i>Higher Mathematics</i> , a mathematics textbook used widely in Chinese colleges			
Source	Sub-category	Passage number in main text	Level
	Main text	25	T1

Main expository presentation	Main text	30	T1
	Main text	34	T1
	Main text	34	T1
	Main text	47	T1
	Main text	105	T1
		Problem number in the exercises	
Exercise	Exercise	4	T5
	Exercise	10	T5
	Exercise	12	T5
	Exercise	19	T5

The development of covariational reasoning levels in the section of the textbook that introduces derivative is presented in the Figure 4.1.3.5. where the x-axis represents the progression of the passage, with the passages in the main text running from 1 to 113 (M1 to M113) followed by the exercises from 1 to 20 (E1 to E20). We can see that this textbook has a simple pattern in terms of covariational reasoning level development. There is only T1 in the main text and only T5 in the exercises, and there is no development in between.

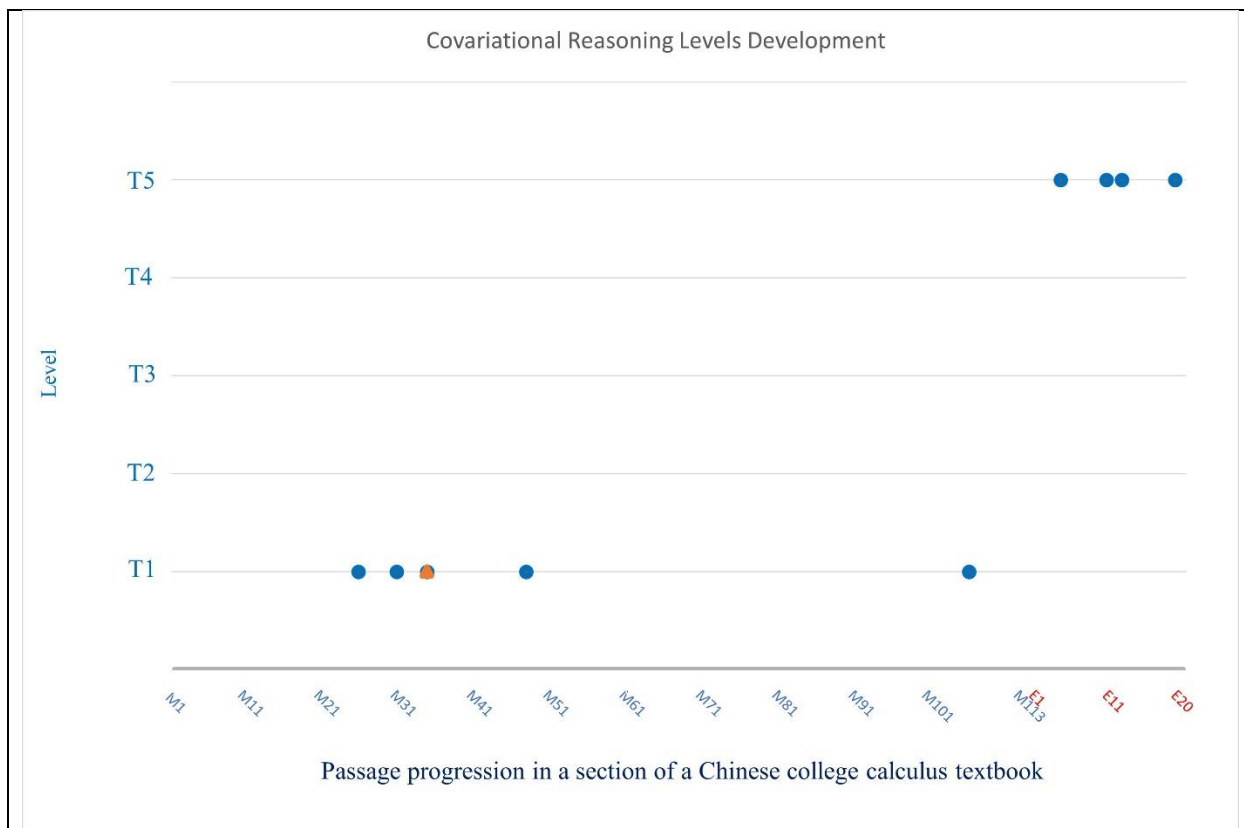


Figure 4.1.3.5: The development of levels of covariational reasoning as the textbook progresses through the main text (M1 to M113) to the exercises (E1 to E20) in the introduction of derivatives section of *Higher Mathematics*, a widely used calculus textbook in Chinese colleges

Covariational Reasoning—Conceptual Analysis of a Chinese High School Calculus

Textbook

The textbook for the Chinese high schools, *Elective 2-2* by People's Education Press, is the last textbook Adam and Betty formally analyzed.


Text Occurrences That Promote Different Levels of Covariational Reasoning

Both Adam and Betty identified several passages that promote almost every level of covariational thinking. For example, Figure 4.1.4.1 shows a passage Adam initially rated as T1 but was missed by Betty. In her second assessment, Betty agreed that it qualified as T1 as well.

The original text in Chinese	<p>问题1 气球膨胀率</p> <p>很多人都吹过气球。回忆一下吹气球的过程，可以发现，随着气球内空气容量的增加，气球的半径增加得越来越慢。从数学的角度，如何描述这种现象呢？</p>
What it means in English	<p>Question 1: Balloon inflation rate</p> <p>... as the volume of air inside the balloon <u>increases</u>, the <u>increase</u> of the radius of balloon gets slower and slower. ...</p>
<p>Figure 4.1.4.1: An <i>Elective 2-2</i> passage Adam initially rated as promoting covariational thinking of change (T1)—Betty agreed in her second assessment.</p>	

Figure 4.1.4.2 shows a passage that Betty initially rated as T1 but was missed by Adam.

In his second assessment, Adam agreed that it qualified as T1.

The original text in Chinese	 <p>如图 1.1-2，当点 $P_n(x_n, f(x_n))$ ($n=1, 2, 3, 4$) 沿着曲线 $f(x)$ 趋近于点 $P(x_0, f(x_0))$ 时，割线 PP_n 的变化趋势是什么？</p>
What it means in English	<p>Observation: As shown in graph 1.1-2, when the point $P_n(x_n, f(x_n))$ ($n=1,2,3,4$) approaches the point $P_0(x_0, f(x_0))$ along the curve $f(x)$, how does the secant line P_0P_n change?</p>
<p>Figure 4.1.4.2: An <i>Elective 2-2</i> passage Betty initially rated as promoting covariational thinking of change (T1)—Adam agreed in his second assessment.</p>	

The most differences in Adam and Betty's analysis were with items in T4 and T5. For example, Betty identified six occurrences of T4, but Adam thought none of them fit his understanding of T4. Figure 4.1.4.3 shows one case that Betty identified as T4, but Adam did not.

The original text in Chinese	<p>气球的平均膨胀率为</p> $\frac{r(1)-r(0)}{1-0} \approx 0.62 \text{ (dm/L).}$
What it means in English	<p>The balloon's average inflation rate is ...</p>
<p>Figure 4.1.4.3: An <i>Elective 2-2</i> passage Betty rated as promoting covariational thinking in terms of the average rate of change (T4)—rejected by Adam.</p>	

Adam stated in his analysis that the reason for his rejection was that it "did not mention change." Judging from Adam's analysis of other textbooks, it is likely that he meant that the variable of the change was not explicitly stated. I agree that, in principle, the inflation rate could be with respect to time, volume, etc. Thus, a simple statement of the inflation rate may only invoke an image of change but not a covariation of two variables. In the result of the investigation process, four occurrences of T1, five of T2, four of T3, and three of T5 were identified. Table 4.1.4.1 shows all intermediate and final results in terms of the number of occurrences of passages that explicitly promote different levels of covariational thinking in the section that conceptually introduces derivative in the Chinese calculus high school textbook, *Elective 2-2*, by People's Education Press. This textbook has the evenest distribution of texts that promote different levels of covariational thinking.

Table 4.1.4.1: Intermediate and final results from the analysis of the introduction of derivatives in <i>Elective 2-2</i> . See Table 4.1.1 for the meaning of A1, B1, ..., Z3.						
China High School	A1	B1	U2	A2	B2	Z3
T1	4	6	10	4	4	4
T2	5	4	9	8	6	5
T3	4	2	4	4	4	4
T4	0	6	6	0	6	0
T5	4	13	13	3	12	3

Development of Covariational Reasoning Levels

The manner of presentation in this textbook is mostly main text followed by exercises. There are 3 exercises embedded in the main text. In addition, after the main text, there are 6 problems in group A, and 3 more in group B. There are a few sidenotes, but not in a separate column from the main text. All figures and examples are embedded in the main text. For the development of covariational reasoning levels, Table 4.1.4.2 shows the sequence of passages that

promote different levels of covariational reasoning as a the text progresses through the main text and exercises.

Table 4.1.4.2: The development of levels of covariational reasoning as passages progress through the main text and the exercises in the introduction of derivatives section of *Elective 2-2* by People's Education Press, a mathematics textbook widely used in the Chinese High Schools

Source	Sub-category	Passage number in main text	Level
Main expository presentation	Main text	4	T1
	Main text	7	T3
	Main text	8	T3
	Main text	9	T2
	Main text	29	T3
	Main text	30	T1
	Main text	32	T3
	Main text	50	T1
	Main text	51	T1
	Main text	62	T2
	Main text	62	T2
	Main text	62	T2
	Main text	62	T2
		Problem number in the exercises	
Exercise	Exercise	5 (3+2)	T5
	Exercise	7 (3+4)	T5
	Exercise	10 (3+6+1)	T5

The development of covariational reasoning levels in the section of the textbook that introduces derivative is presented in the Figure 4.1.4.4. The sequence for reading the section is first skipping the embedded exercises in the main text and adding these embedded exercises to the exercise section. Because none of the three exercises embedded in the main text was identified as promoting any level of covariational reasoning, this change in sequence would have minimal effect in terms of getting an overall picture of the development of covariational reasoning levels in this textbook. In Figure 4.1.4.4, the x -axis represents the progression of the passage, with the passages in the main text running from 1 to 77 (M1 to M73) followed by the

exercises which run from 1 to 12 (E1 to E12). We can see that this textbook has a rich development of covariational reasoning at lower levels (T1 to T3) in the main text. There is no explicit development of T4 and the occurrence of T5 can only be found in the exercises.

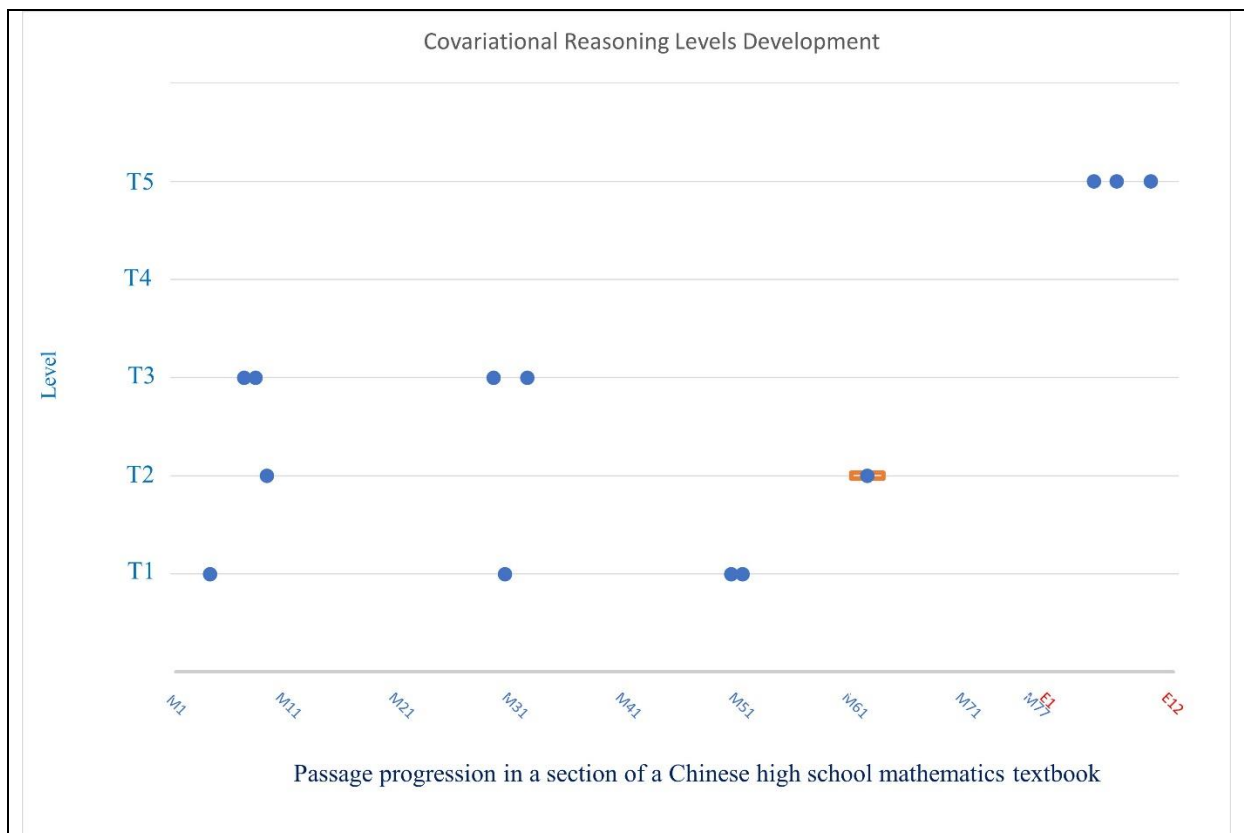


Figure 4.1.4.4: The development of levels of covariational reasoning as the passages progress through the main text (M1 to M77) and the exercise (E1 to E12) in the introduction of derivatives section in *Elective 2-2*, by People's Education Press, a widely used mathematics textbook in Chinese high schools

Results from the Relational Analysis

According to the research initially conducted by Carlson et al. (2002) and subsequent studies, it is very likely that there is an existence of an internal connection and progression among different levels of covariational reasoning activity, from thinking about whether there is a switch in the direction of the change, the magnitude of change, the average rate of the change, or the instantaneous rate of change. Therefore, the passages that promote different levels of

covariational reasoning activities could have some links and could be structured to scaffold the conceptual buildup of covariational reasoning. Therefore, two types of connections were differentiated in the relational analysis. The *simple* connection is the connection between passages that explicitly promote covariational thinking, and the *strong* connection places the simple connection in the context of the same issue or problem. In this phase of analysis, Adam, Betty, and the author looked at all the results of conceptual analysis for each of the four textbooks and independently determined whether there is any connection between occurrences of the passages that promote any level of covariational reasoning. Then I put all the results together. If at least two investigators identified a connection, it was deemed a connection.

Conceptual Relationships Among Passages That Promote Covariational Reasoning from a US College Calculus Textbook

For the simple connection, one example Adam identified was that after the average rate of change was defined, whenever the average rate of change was mentioned again later in the text, it served as a reminder of the definition; thus, a simple connection was formed with the original definition. Table 4.2.1.1 shows an example of a simple connection Adam identified among passages that promote covariational thinking in terms of the average rate of change. Adam identified seven simple connections of this type.

Table 4.2.1.1: A simple connection Adam identified among items that promote covariational thinking in terms of the average rate of changes (T4) in Stewart's textbook (2016)

<p>Definition, the first mention of the rate of change</p>	<p>■ Rates of Change</p> <p>Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the increment of x) is</p> $\Delta x = x_2 - x_1$ <p>and the corresponding change in y is</p> $\Delta y = f(x_2) - f(x_1)$ <p>The difference quotient</p> $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ <p>is called the average rate of change of y with respect to x over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 8.</p>
<p>Subsequent use of the academic term and the mention of the variable involved</p>	<p>51. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.</p> <p>(a) Find the average rate of change of C with respect to x when the production level is changed</p> <p>(i) from $x = 100$ to $x = 105$</p> <p>(ii) from $x = 100$ to $x = 101$</p>

For the strong connection, one example Betty identified was problem 47 (p.115) in the exercises section, which questioned the average rate of change (T4) and the instantaneous rate of change (T5) within the same context. Table 4.2.1.2 shows an example of the passage Betty identified as a strong connection between the average rate of change (T4) and the instantaneous rate of change (T5). In addition, Betty identified five strong connections in problems 16, 47, 48, 49, and 51. Betty identified these strong connections as similar: the problem asked for the average rate of change before the question for the instantaneous rate of change.

Table 4.2.1.2: An example of the strong connection Betty identified between the average rate of change (T4) and the instantaneous rate of change (T5) in problem 47 in Stewart's textbook

<p>T4 (a) part of the problem</p>	<p>47. Researchers measured the average blood alcohol concentration $C(t)$ of eight men starting one hour after consumption of 30 mL of ethanol (corresponding to two alcoholic drinks).</p> <table border="1" style="margin: 10px auto;"> <tr> <td>t (hours)</td> <td>1.0</td> <td>1.5</td> <td>2.0</td> <td>2.5</td> <td>3.0</td> </tr> <tr> <td>$C(t)$ (g/dL)</td> <td>0.033</td> <td>0.024</td> <td>0.018</td> <td>0.012</td> <td>0.007</td> </tr> </table> <p>(a) Find the average rate of change of C with respect to t over each time interval: (i) [1.0, 2.0] (ii) [1.5, 2.0] (iii) [2.0, 2.5] (iv) [2.0, 3.0] In each case, include the units.</p> <p>(b) Estimate the instantaneous rate of change at $t = 2$ and interpret your result. What are the units?</p>	t (hours)	1.0	1.5	2.0	2.5	3.0	$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007
t (hours)		1.0	1.5	2.0	2.5	3.0							
$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007								
<p>T5 (b) part of the problem</p>													

In this phase of relational analysis, Adam identified only simple connections among T4s, and Betty identified only strong connections between T4s and T5s. Although their results in this analysis had nothing in common, I agree with Adam and Betty separately in their analysis. As a result of this investigation, seven simple and five strong connections were identified in Stewart's textbook. Table 4.2.1.3 shows the summary of connections identified in this study phase.

Table 4.2.1.3: Numbers of identified conceptual connections among passages that promote covariational thinking in the section that conceptually introduces derivatives in Stewart's textbook (2016)	
Simple	Strong
7 (T4 with T4)	5 (T5 with T4)

Conceptual Relationship Among Passages That Promote Covariational Reasoning from a US High School Calculus Textbook

Adam saw no conceptual connection among passages identified from the previous study phase in Finney et al.'s textbook (2016). On the other hand, Betty identified the phrase "the average rate of change," which appears three times in problem 40 in the exercise section, as a strong connection. I think that the rate of change concept was not defined in a way that explicitly

promotes covariational thinking in Finney et al.'s textbook. Thus, the definition itself is not part of the text that promotes any level of covariational thinking. Subsequently, the academic term "the average rate of change" in this textbook does not automatically invoke a picture of covariational thinking, and no connection exists if there is no concept to start with. In the end, there is no connection to report in this study among the passages that promote covariational thinking in Finney et al.'s textbook.

Conceptual Relationship Among passages That Promote Covariational Reasoning from a Chinese College Calculus Textbook

Adam saw no conceptual connection among passages identified from the previous phase of study of *Higher Mathematics* by Tongji University Press (2014). However, Betty identified strong connections among three occurrences of T1: examples describing in detail what happens when one moving point in a curve approaches a fixed point. From the textbook, three things happen when the moving point moves to the fixed point. Table 4.2.3.1 shows the passages Betty identified as promoting covariational thinking about change.

Table 4.2.3.1: Strong connections among three passages that promote covariational thinking about change identified by Betty in <i>Higher Mathematics</i> (2014)	
The original text in Chinese	What it means in English
其中 φ 为割线 MN 的倾角. 当点 N 沿曲线 C 趋于点 M 时, $x \rightarrow x_0$. 如果当 $x \rightarrow x_0$ 时, 上式的极限存在, 设为 k , 即	When N is moving to M along the curve C, $x \rightarrow x_0$...
曲线 C 在点 M 处的切线. 事实上, 由 $\angle NMT = \varphi - \alpha$ 以及 $x \rightarrow x_0$ 时 $\varphi \rightarrow \alpha$ 可见 $x \rightarrow x_0$ 时 (这时 $ MN \rightarrow 0$), $\angle NMT \rightarrow 0$. 因此直线 MT 确为曲线 C 在点 M 处的切线.	... as $x \rightarrow x_0$, $\varphi \rightarrow \alpha$
曲线 C 在点 M 处的切线. 事实上, 由 $\angle NMT = \varphi - \alpha$ 以及 $x \rightarrow x_0$ 时 $\varphi \rightarrow \alpha$, 可见 $x \rightarrow x_0$ 时 (这时 $ MN \rightarrow 0$), $\angle NMT \rightarrow 0$. 因此直线 MT 确为曲线 C 在点 M 处的切线.	... as $x \rightarrow x_0$, $MN \rightarrow 0$, $\angle NMT \rightarrow 0$.

I agree with Betty in her assessment on the grounds that different perspectives to describe the same event do help the readers to form a more complete image of change. Thus, this study reports two occurrences of strong conceptual connection among T1s, and there is no simple connection among any texts that promotes covariational thinking. Table 4.2.3.2 shows the summary of connections identified in the section that conceptually introduces derivatives in *Higher Mathematics* in this phase of the study.

Table 4.2.3.2: Number of conceptual connections among passages that promote covariational thinking in the section that conceptually introduces derivatives in <i>Higher Mathematics</i> (2014)	
Simple	Strong
0	2 (T1with T1)

Conceptual Relationship Among Passages That Promote Covariational Reasoning from a Chinese High School Calculus Textbook

Adam saw no conceptual connection among passages identified from the previous study phase in the Chinese calculus textbook at the high school level, *Elective 2-2*, by People's Education Press (2005). However, Betty saw two connections, both of which were strong. One strong connection was in the context of balloon inflation. The textbook said, "as the volume of the balloon increases, the rate of diameter increase gets slower." One sentence describes this observation qualitatively, and after some computation, another sentence describes this phenomenon quantitatively. Table 4.2.4.1 shows the example that Betty identified of a strong connection between two passages that explicitly promote covariational thinking regarding the direction of the change.

Table 4.2.4.1: A strong conceptual connection identified by Betty between two passages that explicitly promote covariational thinking in terms of the direction of the change in <i>Elective 2-2</i>	
The original text in Chinese	What it means in English

<p>问题1 气球膨胀率</p> <p>很多人都吹过气球。回忆一下吹气球的过程，可以发现，随着气球内空气容量的增加，气球的半径增加得越来越慢。从数学的角度，如何描述这种现象呢？</p>	<p>... as the volume inside of the balloon increases, the average rate of diameter increase gets slower.</p>
<p>随着气球体积逐渐变大，它的平均膨胀率逐渐变小了</p>	<p>... as the volume of the balloon gradually increases, the average inflation rate gradually becomes smaller ...</p>

I agree with Betty that these two occurrences have a strong connection. Another place where Betty identified strong connections is in the passages that described the property of the slope of the tangent line and how the value of the functions behaves around the neighborhood of three different points. Table 4.2.4.2 shows a strong conceptual connection identified by Betty among three passages that explicitly promote covariational thinking in terms of the direction of change.

Table 4.2.4.2: A strong conceptual connection identified by Betty among passages that explicitly promote covariational thinking in terms of the direction of the change in <i>Elective 2-2</i>	
The original text in Chinese	What it means in English
<p>(1) 当 $t=t_0$ 时，曲线 $h(t)$ 在 t_0 处的切线 l_0 平行于 x 轴。所以，在 $t=t_0$ 附近曲线比较平坦，几乎没有升降。</p>	<p>At $t = t_0$, the tangent line of the curve is parallel with the x-axis. Thus, around $t = t_0$, the curve is flat. There is almost no increase or decrease.</p>
<p>(2) 当 $t=t_1$ 时，曲线 $h(t)$ 在 t_1 处的切线 l_1 的斜率 $h'(t_1) < 0$。所以，在 $t=t_1$ 附近曲线下降，即函数 $h(t)$ 在 $t=t_1$ 附近单调递减。</p>	<p>At $t = t_1$, the slope of the tangent line of the curve $h'(t_1) < 0$.</p>

	Thus, around $t = t_1$, the curve is monotonically decreasing.
<div style="border: 2px solid red; padding: 5px;"> <p>(3) 当 $t=t_2$ 时, 曲线 $h(t)$ 在 t_2 处的切线 l_2 的斜率 $h'(t_2) < 0$. 所以, 在 $t=t_2$ 附近曲线下降, 即函数 $h(t)$ 在 $t=t_2$ 附近也单调递减.</p> </div>	At $t = t_2$, the slope of the tangent line of the curve $h'(t_2) < 0$. Thus, around $t = t_2$ The curve is also monotonically decreasing.

Adam did not think there was any connection among these three sentences, although he put them in the T2 category. I did not think these sentences qualified as T2. My reason was that although the increase or decrease of the function value is explicitly written, the change of the independent variable is not explicitly written. The wording "around the neighborhood of certain point" does not explicitly spell out how the variable changes, whether it increases or decreases. I think that "around the neighborhood of a certain point" is very vague and could mean a point jumping around. Thus, the author did not take them as passages that promote covariational thinking; subsequently, there was no conceptual connection between them. In the end, this study concluded that there was one strong conceptual connection among T1s and no simple connection among passages promoting covariational thinking in this textbook. Table 4.2.4.3 shows the summary of connections identified in the section that conceptually introduces derivatives in this phase of the study.

Table 4.2.4.3: Number of conceptual connections among passages that promote covariational thinking identified in the section that conceptually introduces derivatives in <i>Elective 2-2</i>	
Simple	Strong
0	1 (T1with T1)

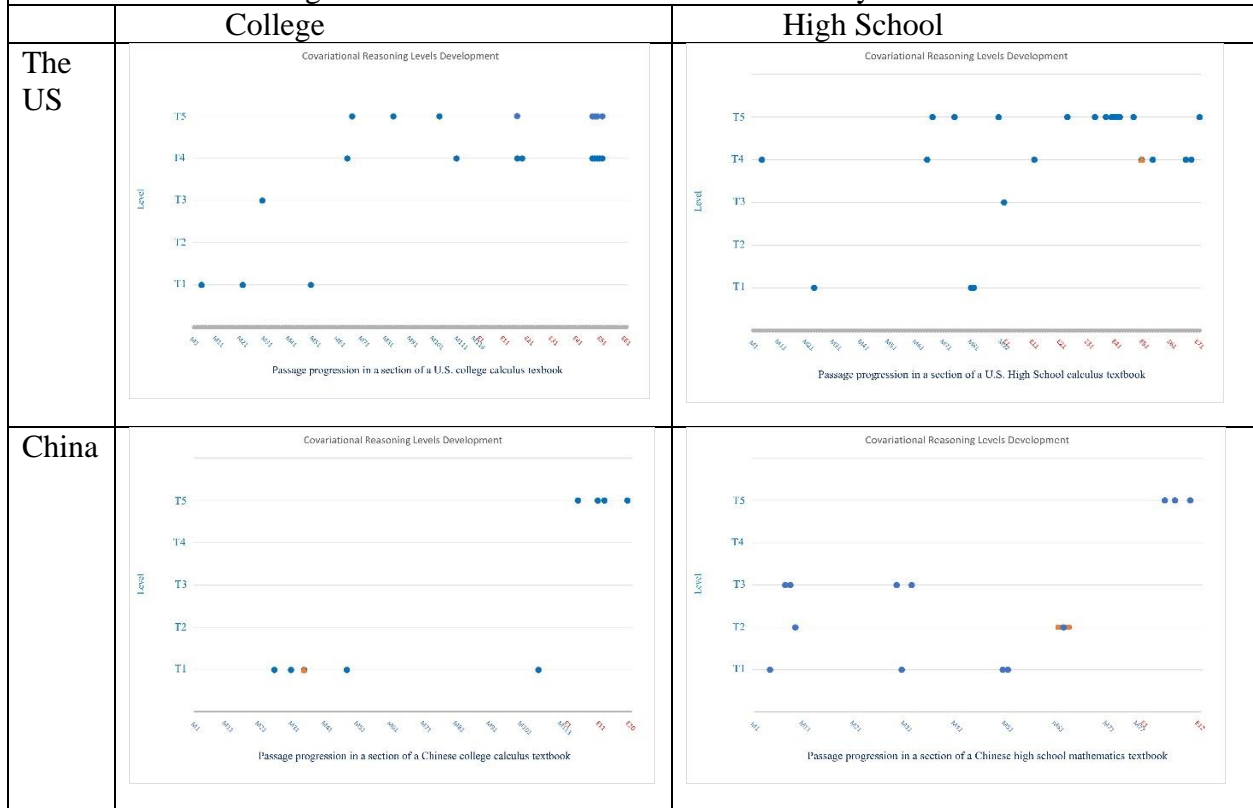
Comparison Among the Four Textbooks

Considering the occurrence of passages that promote different levels of covariational thinking and connections among them gives us some general ideas of how the pedagogy of covariational thinking is conducted in different textbooks. Table 4.3.1 shows the number of passages that promote the different levels of covariational thinking and the number of different conceptual connections among these passages for this study's four selected calculus textbooks.

Table 4.3.1: Number of passages that explicitly promote covariational thinking at different levels and number of conceptual connections identified among these passages for four selected calculus textbooks analyzed in this study				
Textbook	US College	US High School	China College	China High School
T1	3	3	6	4
T2	0	0	0	5
T3	1	1	0	4
T4	9	8	0	0
T5	8	12	4	3
Simple connection	7 (T4 with T4)	0	0	0
Strong connection	5 (T5 with T4)	0	2 (T1 with T1)	1 (T1 with T1)

In addition, putting the four graphs obtained in previous sections side by side gives us a general idea about the similarities and differences of the pedagogy of covariational reasoning levels developed in the four calculus textbooks. Finally, table 4.3.2 shows how the passages develop covariational reasoning levels in this study's four selected calculus textbooks.

Table 4.3.2: The occurrence and the development of passages that explicitly promote covariational reasoning in the four calculus textbooks in this study



There are several observations. First, all four textbooks have some occurrence of T1 and T5. It makes sense that T1 is found in the passage that explicitly spells out the change among variables and, thus, is the starting point when the book starts to deal with change systematically. And T5 can be found in the passage on the instantaneous rate of change as derivative, the ending point or the goal of the section that conceptually introduces derivative. Second, what happens between the starting and ending points differs from textbook to textbook. This study found that the Chinese college calculus textbook did not have any stimulants to covariational thinking in between, while the Chinese high school calculus textbook had the most elaborate and robust buildup of T2 and T3 before it got to the instantaneous rate of change (T5). Third, there was no development of average and instantaneous rates of change (T4 and T5) in the main expository

text in both Chinese textbooks. And the instantaneous rate of change (T5) only exists in the exercise sections of both Chinese textbooks. Fourth, both college calculus textbooks develop the covariational reasoning level mainly in one direction, i.e., the level goes up as the passage of the text progress. On the contrary, in the main expository section, both high school calculus textbooks developed the covariational levels in a circulated way, i.e., the occurrences of levels went up and down. Fifth, the calculus textbooks for US colleges and high schools are heavy in the average and the instantaneous rate of change (T4 and T5). The changing secant line between two points (one fixed and one moving) becoming the tangent line seems to be one popular example used to illustrate how the average rate of change becomes the instantaneous rate of change in the US calculus textbooks, whether they are Advanced Placement versions or not. Sixth, in terms of connections among concepts, only the US college textbook seems to make meaningful and substantial efforts. However, the scaffold only happens among T4s and T5s, i.e., the transition between the average and the instantaneous rate of change. Also, it mostly happens in the exercise section. There is no scaffold among other levels of covariational thinking in textbooks selected for this study. Seventh, a writing project in Stewart's textbook is unique among all four textbooks in this study. In the writing project, readers are asked to research Pierre Fermat's or Isaac Barrow's methods for finding the equation of the tangent line. They were pioneers in the subject of tangent lines before the explicit formulation of the ideas of limits and derivatives by Isaac Newton. This unique writing project put the concept of derivatives in a historical context, and the research opportunity it provides for college students seems to be an appropriate way to further scaffold, deepen, and enrich readers' understanding of derivatives. Finally, there is one interesting and unique feature in the Chinese high school calculus textbook, i.e., a substantial effort in building T2 and T3. Table 4.3.2 shows some examples of T2s.

Table 4.3.3: Some examples of passages that explicitly promote covariational thinking in terms of the direction of the change in <i>Elective 2-2</i>	
The original text in Chinese	What it means in English
<p>问题1 气球膨胀率</p> <p>很多人都吹过气球。回忆一下吹气球的过程，可以发现，随着气球内空气容量的增加，气球的半径增加得越来越慢。从数学的角度，如何描述这种现象呢？</p>	<p>... as the volume of the air inside the balloon increases, the <u>increase</u> of radius of the balloon gets slower and slower.</p>
<p>在第2 h与第6 h时，原油温度的瞬时变化率分别为-3与5。它说明在第2 h附近，原油温度大约以3 °C/h的速率下降；在第6 h附近，原油温度大约以5 °C/h的速率上升。</p>	<p>... in the neighborhood of 2h, the temperature of crude oil <u>decreases</u> by the speed of 3°C/h. ...</p>
<p>(2) 当$t=t_1$时，曲线$h(t)$在t_1处的切线l_1的斜率$h'(t_1)<0$，所以，在$t=t_1$附近曲线下降，即函数$h(t)$在$t=t_1$附近单调递减。</p>	<p>At $t = t_1$, the slope of the tangent line of the curve $h'(t_1) < 0$. Thus, around $t = t_1$, the curve is <u>monotonically decreasing</u>.</p>

In summary, the calculus textbook for Chinese high schools seems to have the most evenly distributed passages that explicitly promote covariational reasoning, particularly from T1 to T3. On the other hand, the calculus textbooks in the US (both for high schools and colleges) emphasize the average rate of change, the instantaneous rate of change, and the transition in between. Therefore, it is fair to say that none of the calculus textbooks in this study have a consistent way of systematically treating the pedagogy of covariational reasoning, and the development of levels is uneven in different ways.

Chapter V: Summary, Conclusions, and Recommendations

Summary

Covariation refers to the simultaneous changes of two quantities. Covariational reasoning is a cognitive activity involving mentally coordinating two variable changes and understanding how they are related. Covariational reasoning capacity plays an essential role in understanding dynamic events. From previous research, it is evident that this capacity is not well developed among students. However, calculus education can be used to cultivate students' covariational thinking skills. Covariational reasoning seems to have developmental levels, from thinking about whether there is a change, the direction of change, the magnitude of change, and the average rate of change to the instantaneous rate of change. Out of four major topics in typical calculus education (limits, derivatives, integrals, series), derivatives and how they are introduced is where all five levels of covariational reasoning naturally occur. Calculus textbooks can focus differently on how the concept of the derivative is introduced and presented. Some calculus textbooks introduce the derivative formally by using the definition of limit. In contrast, some other textbooks use a more intuitive approach to scaffold the systematic ways of thinking about change and how changes are related, i.e., covariational thinking. An analysis of the variety of pedagogical efforts on this subject can offer insights and provide a blueprint for future improvement. One widely used calculus textbook from each of four categories was selected for this study: a U.S. college, a U.S. high school, a Chinese College, and a Chinese high school.

Written text can invoke different mental pictures for readers with different backgrounds. For example, in calculus, the concept of function is the base upon which all further treatments of change are built. At the same time, professional mathematicians, mathematics teachers, students, and mathematical education researchers can have different views of "function." Thus, the same

passage may invoke different mental pictures and views of change. Two well-trained investigators and I conducted a detailed analysis of the pedagogy and development of covariational reasoning levels in the conceptual introduction of derivatives in each of the four selected calculus textbooks.

This study is the first of its kind in several aspects. First, although the covariational reasoning framework with five levels is not new, adopting and applying them to analyze written text is new. It is important to note that this framework of covariational reasoning is still in development and may be subject to more refinement as new evidence emerges. The conceptual analysis portion of this study shows that the five distinct ways of covariational reasoning can exist at various frequencies in calculus textbooks. This result does not conflict with the framework initially proposed by Carlson et al. (2002). In terms of the research process, a new investigator selection method was tried out. Typically, experts on the subject matter, who are likely to be professional mathematicians, code passages, or other material in content analysis. However, the investigators in this study were deliberately chosen to be not professional mathematicians. According to reader-oriented theory (Rosenblatt, 1995), the readers' reaction to (or interaction with) the text depends on the readers' mathematical, educational, and pedagogical background. The investigator selection process in his study was designed to make the reaction from intended readers as authentic as possible. Thus choosing investigators with profiles as close to the intended reader as I have access to was an intentional consideration. The cross-sectional comparison of pedagogy in textbooks from different grade levels and countries is also a task that must be carefully designed. Great care has been put in place to ensure the investigators have a similar level of language expertise in both languages and their educational backgrounds from either country do not form any pre-existing bias against any textbook.

Conclusions

By analyzing the conceptual introduction of derivatives in four calculus textbooks, we found that the students' opportunities to learn in terms of covariational reasoning were not made explicit and were not uniformly developed in each of the four textbooks under study.

The First Research Question

The first research question inquires how the pedagogy and development of covariational reasoning levels in the conceptual introduction of derivatives in two widely used calculus textbooks in the U.S., one for colleges and one for high schools, is organized. For the U.S. college category, the eighth edition of *Calculus* by James Stewart (2016) was selected. For the U.S. high school category, *Calculus Graphics, Numerical, Algebraic* by Finney et al. (2016) was selected. The answer to the first research question is that the organization of covariational reasoning pedagogy in the section of this study in these two textbooks is similar, and the one for the college level is more in-depth. The length of the sections from both textbooks is similar: the college one has 118 sentences, 7 examples, and 61 total problems, whereas the high school one has 92 sentences, 8 examples, and 71 problems. They both have figures (9 vs. 5) and sidenotes (10 vs. 20 sentences). One unique feature of the calculus textbook for U.S. colleges is a writing project at the end of the section. For passages that promote different modes of covariational thinking, both textbooks heavily emphasize the average rate of change (9 vs. 8) and instantaneous rate of change (8 vs. 12 occurrences). Both have minimal writing on general covariation, the direction of change, and the magnitude of change (3, 0, and 1 occurrence for both textbooks). Relational analysis was conducted to identify the conceptual connections among the passages that promote covariational thinking. It was found that the college textbook has several connections among average rates of change. In addition, it has several connections

between average and instantaneous rates of change within the same context. On the other hand, no connection among passages that explicitly promote covariational thinking was found in the high school calculus textbook in this study.

The Second Research Question

The second research question inquires how the pedagogy and development of covariational reasoning levels in the conceptual introduction of derivatives in two widely used calculus textbooks in China, one for colleges and one for high schools, is organized. *Higher Mathematics* (2014) by Tongji University was selected for the Chinese college category. For the Chinese high school category, *Elective 2-2* (2005) by People's Education Press was selected. The sections under study have a similar number of figures (5 vs. 4). The number of total exercises in both books is small compared to their U.S. peers (20 and 12 vs. 61 and 71).

The sections introducing derivatives in these two Chinese calculus textbooks are very different in terms of pedagogical effort toward covariational reasoning development.

The calculus textbook for college is significantly longer (113 vs. 77 sentences). Also, it has many more examples (11 vs. 3). However, the conceptual buildup for covariational reasoning in the section under study in the college one is minimal. Out of ten pages of the main text, only a little more than two pages are spent on two cases (speed in a linear movement and a tangent line problem) that lead to the formal definition of derivatives. After the formal definition of the derivative, all examples are devoted to the specific way to find derivative functions for various functions. In these two pages of conceptual buildup, only a general description of covariation and the instantaneous rate of change were found, with nothing in between.

On the contrary, the section under the study of the Chinese calculus textbook at the high school level spends all nine pages of the main text building the covariational concept. All three

examples concern instantaneous rates of change and their interpretation. Furthermore, the textbook contains sentences promoting almost all levels of covariational thinking. This study concluded that only explicit promotion of thinking about the average rate of change was missing in this textbook. However, this conclusion was the result of this particular research design. One investigator identified six passages that promote covariational thinking in terms of average rates of change. The other investigator rejected them all based on his observation that variable change was not mentioned explicitly when the "average rate of change" appeared in the text. Thus, the other investigator determined that a superficial appearance of the text, such as the "average rate of change," did not necessarily invoke a mental picture of covariational thinking if the changing variable was not explicitly pronounced. Because that phase of the study had only two investigators, one veto vote from one investigator made none of the passages qualify as occurrences that explicitly promote the thinking of the average rate of change. The two Chinese calculus textbooks have one thing in common: there was no effort to connect different levels of covariational reasoning concepts. Both textbooks show few conceptual connections between the various types of covariation (2 vs. 1). Moreover, the existing connections in both books do not look like the results of any conscious effort to connect concepts. The connections are more like some accidental result of elaborating on the same issue.

The Third Research Question

The third question inquires as to the similarities and differences in the pedagogy and development of covariational reasoning levels among the four textbooks. One similarity that stands out for all four calculus textbooks is the presence of passages explicitly written to address general covariation and covariational thinking in terms of the instantaneous rate of change. General covariation is a natural place to start when textbook authors address change

systematically. Also, the instantaneous rate of change is the end goal of the section introducing the concept of derivatives. Thus, the presence of general covariation and the instantaneous rate of change in all textbooks should not come as a surprise, but it is still good to see them validated as research findings. The presence of general covariation and the instantaneous rate of change is the only similarity the sections of calculus textbooks under study have. Suppose the presence of general covariation and the instantaneous rate of change is the minimal requirement for any calculus textbook section that introduces the concept of derivatives. In that case, the Chinese college textbook seems to have done the bare minimum. The other three calculus textbooks all have some additional effort toward the pedagogy of covariational thinking. The Chinese high school book has some solid work on the direction and magnitude of change: detailed explanations and examples are abundant in the direction and the magnitude of change. The sheer number of occurrences of the direction and magnitude of the changes in the Chinese calculus textbook for the high school level is another validation that the direction of change and magnitude of change are natural steps in covariational thinking before introducing the average and instantaneous rate of change. Both US calculus textbooks skip much of the topic of direction and magnitude of change but emphasize the average and instantaneous rates of change. The U.S. college textbook makes an extra effort to smoothly connect the average and the instantaneous rates of change by putting them in the same context in five different exercise problems.

Further Findings

In the process of this research, I gained some unexpected insights. First, both U.S. calculus textbooks in this study emphasize the transition from the average to the instantaneous rate of change. From a casual look at other U.S. calculus textbooks, this pattern seems to be a general case for most calculus textbooks in the U.S. This emphasis could be a result of the

Western world's consistent and obsessive intellectual effort to understand the concept of the infinitesimal, from the paradoxes of Zeno (Boyer, 1959) in Greece around twenty-five hundred years ago to Pierre Fermat and Isaac Barrow in Britain, who devised their own ways to compute tangent lines (Boyer, 1959; Kline, 1972) in the seventeenth century, and to a more current understanding of infinitesimals from the perspective of non-standard analysis (Robinson, 1966). The debate on whether calculus should be taught formally with the latest development of mathematical rigor or should be introduced to novices in a way parallel to its historical development, such as connecting several intuitive ways of thinking about change, is continual. The U.S. calculus textbooks seem to lean toward the former way.

On the other hand, the Chinese calculus textbook at the high school level has rich passages that promote almost all levels of covariational thinking. Only minor wording modification is needed to have an almost perfect scaffold of covariational reasoning pedagogy. For example, in the passage referring to the average rate of change, an improvement could be made by adding a few words to specify the independent variables.

Finally, the Chinese calculus textbook for the college level as a stand-alone textbook provides the bare minimum in the pedagogy and development of covariational reasoning. However, if this textbook is placed in the context of the Chinese education system, it could make sense. This observation is based on the fact that almost everyone who enters college in China is a graduate of a Chinese high school. Thus, whoever studies calculus in a Chinese college must have studied calculus in a Chinese high school well enough to go through all the conceptual buildup of covariational reasoning. Then logically, after they enter college, two pages on the rate of change can serve as a quick and sufficient review. If someone who attended a Chinese college had never encountered the rate of change concept, the structure and organization of the

presentation from the Chinese high school calculus textbook could provide a good reference. The textbook analysis needs to be placed in their respective educational system to reveal why the books are structured in a certain way. The case of the Chinese college calculus textbook and its role in the Chinese educational system provides an excellent example of what Fan (2013) called mathematics textbook research as a factor in overall mathematics education research. Finally, the size of the exercise section in both the U.S. textbooks is significantly larger than their Chinese counterparts and has much richer development in covariational reasoning. This finding could be interpreted as a different perspective on the role of exercise. The U.S. textbooks seem to treat the exercise section as an integrated part of textbooks, a possible reflection of student-centered pedagogy. Thus, ample opportunities for readers to get hands-on experience are essential in learning. On the contrary, the Chinese high school textbook has a rich and elaborate development of lower levels of covariational reasoning but has a smaller exercise section. It could reflect a greater focus on instructional quality with much less attention to the role of hands-on experience in solving problems. This arrangement could reflect the instructor-centered pedagogy in Chinese high schools.

More Discussion of the Findings

The significance of this study can be viewed from several perspectives. First, it is an effort to shed light on how explicitly the section that conceptually introduces derivative in calculus textbooks offer the opportunity to learn in terms of the pedagogy and development of covariational reasoning. The findings that none of the textbooks in this study have a coherent, systematic development of the covariational reasoning levels proposed by Carlson et al. (2002) in the section that conceptually introduces derivative can be interpreted in several ways. For example, it could be interpreted that the development of covariational reasoning is not a focus of

the existing textbooks, or it could be interpreted that the framework by Carlson et al. (2002) itself needs to be further modified to be adapted for the written text. Second, the findings of this study could be used to interpret different ways of thinking about change existing in different cultural and historical backgrounds. For example, the elaborate development of the direction and magnitude of change in the textbook for Chinese high school students could be interpreted as an independent validation of the universality of Carlson's framework up to the third level. At the same time, the conspicuous lack of any development in the average rate of change in both Chinese textbooks could be interpreted as carelessness exhibited by writers of similar backgrounds. Or it could be interpreted that the transition between the average and instantaneous rate of change is deemed unnecessary for writers or calculus educators with certain cultural backgrounds. Finally, the advantage and drawbacks of using a mathematics novice instead of a mathematics expert in the coding process are worth exploring. One drawback could be a significant divergence of opinions in concepts that need a judgmental call. For example, one investigator repeatedly asserted six passages that could qualify for the definition that promotes the average rate of change of two variables. In contrast, the other investigator repeatedly rejected all six occurrences. Another drawback could be the lack of time for novice investigators to build a more sophisticated understanding of research language in mathematics education. For example, the instantaneous rate of change of two variables can be deemed by mathematics education experts as a highly sophisticated process. For the expert in mathematics education researcher, a simple mention of the phrase, such as "instantaneous rate of change," may not be counted as an occurrence that promotes a mental picture of the instantaneous rate of change. However, both investigators count every occurrence of the "instantaneous rate of change," "slope," or "speed" as a passage that promotes the instantaneous rate of change. Suppose a mathematics education

research expert decides that all the simple occurrences of the wording "instantaneous rate of change" lack the buildup process to truly stimulate a mental picture of the instantaneous rate of change. In that case, the findings of this study could change dramatically. The different coding results from experts and novices of mathematics education research can be another field for further study. Moreover, the difference in coding results can also depend on the clarity of the definition that qualifies to promote an instantaneous rate of change. The divergence in the opinions due to the ambiguity of the definition is also a limitation of this study. In light of this understanding, the further study that will help to overcome this limitation is recommended in the next section.

Limitations and Recommendations

This study yielded some meaningful insights. However, this study has limitations regarding research design, implementation, and scope of the work. In light of these limitations, some recommendations are discussed based on the insight and experience gained from the study.

Limitations

This study aimed to determine how explicitly the learning opportunities in the development of covariational reasoning were presented in four calculus textbooks. In the process of finding the best way possible to accomplish this, there were choices made in the research design and implementation, both of which have limitations.

The research design in this study may skew the data for the instantaneous rate of change. The design leaned towards the survey approach, i.e., the investigators were chosen deliberately to be mathematics novices. Although the definition for each level of the text that can promote different modes of covariational reasoning is spelled out clearly, room to interpret the high level

of covariational reasoning, such as the instantaneous rate of change, still exists. It is possible for some mathematics novices to "know" the instantaneous rate of change but lack the development view of the transition from the average rate of change to the instantaneous rate of change. As documented in Bressoud (2019, p.189), many students who have seen some calculus can calculate the instantaneous rate of change. Still, almost none can correctly compute the average rate of change for a similar context. In this study, both investigators counted every occurrence of the wording, such as the instantaneous rate of change, slope, speed, etc., as an occurrence of the instantaneous rate of change. However, as discussed in the previous section, experts in mathematics education research may have a very different opinion. Thus, the occurrences of the passages deemed to promote the covariational thinking of the instantaneous rate of change may be systematically skewed upward by the research design in this study.

The implementation of the design can produce bias as well. There are two major sources of potential bias. One source of bias is my action in the implementation process, such as the screening and training process to qualify potential investigators. To limit the bias, I took great care to explain the covariational reasoning framework item by item to each potential investigator in the pilot study. In addition, after the pilot study, I excluded myself from the first phase of the study, i.e., the conceptual analysis. I responded to individual investigators' questions only to clarify the concepts and ensure maximum independent judgment from investigators. Another issue regarding my implementation is the low number of investigators who conducted the conceptual analysis. Due to the strict criteria imposed at the research design stage, I could only qualify two investigators to conduct the study. If there were more investigators, the one-vote veto rule would not apply in the conceptual analysis phase. Instead, a majority vote could have been used in the second step in the conceptual analysis phase. Another source of possible bias comes

from each investigator's action. For example, an individual investigator can be inconsistent from textbook to textbook. The counter to the inherent bias from an individual is to have a large pool of investigators and have a majority vote. For example, if there were five independent investigators, the final step of conceptual analysis could be modified to a majority vote, which would need only three out of five votes instead of a unanimous vote (two out of a total of two votes).

Even with the explicit definition in place, determining whether a certain passage can promote a certain level of covariational reasoning is subtle. The readers' interaction with the text depends on the readers' mathematical, educational, and pedagogical background. With this subjective and judgmental aspect of the study in mind, I took great care in screening and training the conceptual analysis investigators. Much emphasis was placed on the investigators examining the passages from the perspective of first-time calculus students. However, there was significant disagreement between the two investigators, as shown in the data.

More evaluation is needed to fully address the opportunities for learning covariational reasoning provided by the calculus textbooks in four categories. One limitation of this study is that only parts of the calculus textbooks were analyzed. For example, there may be more explicit covariational reasoning pedagogy in another part of some calculus textbooks, such as the application of derivatives. Furthermore, only one calculus textbook from each category was selected. Except for the Chinese high school, the other three categories have many choices. Therefore, the result and pattern found in this study cannot be seen as a representation of the whole book or the whole category. Therefore, the current result is only valid for the four sections of textbooks used in this study.

Despite all the limitations discussed above, I speculate that the general pattern of the pedagogy of covariational reasoning found in the sections of this study in four selected textbooks is valid. The general pattern that stands out in this study is the following. First, the pedagogy and development of covariational reasoning levels in both U.S. calculus textbooks are similar. They emphasize the average and instantaneous rate of change while ignoring much of the direction and magnitude of change. Second, the Chinese calculus textbooks are very different for high schools and colleges, with elaborate covariational conceptual buildup in high schools and none in colleges.

Recommendations

This study suggests that a deficit of developmental effort in covariational reasoning pedagogy is common in the section introducing derivatives in calculus textbooks. However, there are several things calculus educators can do to cope with it before the textbook deficit is corrected. For classroom teachers, they can be conscious of the variable change and how changes are related to each other through the conceptual introduction of derivatives. They could also introduce problems that differentiate how the change depends on different variables, such as speed change with respect to time and the speed change related to distance. The curriculum designers of a calculus course could emphasize more of the conceptual understanding of the derivative by allocating more classroom hours and activities to deepen the understanding of variable changes and how the changes relate to each other. Finally, calculus test makers could design conceptual questions that do not necessarily involve the analytic formula, which can be relatively easily mastered by rote memorization, but place more emphasis on the conceptual understanding of the variable changes and the relationship between changes. For example, in some conceptual questions, if in multiple-choice format, both questions and answers can be

posed as different shapes of graphs. In open questions, students could be asked to draw a sketch based on a description of variable changes and how they are related.

Covariational reasoning pedagogy should be the central theme of calculus education. Calculus textbook writers should keep this theme in mind with the organization of motivation, presentation, example, and exercise. This theme can start with a review of the concept of function. Typically, calculus textbooks start with four ways of representing a function, i.e., words, tables, algebraic, and graphical. All four ways have elements of correspondence or change or both. Using the rate of change as an example, calculus textbook writers can strengthen the covariational relationship among variables in two major aspects. First, the example used does not have to be exclusively concerned with space, time, tangent, speed, etc. There are examples of change in other fields, such as GDP change in economics and temperature change over time in medicine. Second, the variables under examination do not always need to have time or value on the x -axis involved. Out of many examples used in sections that conceptually introduce derivatives in this study, only three did not use time as the independent variable. One is from the U.S. high school calculus textbook, where the variables are drug dosages in the bloodstream and body temperature. Another one is from a Chinese high school calculus textbook, where the variables in consideration are the diameter and volume of a balloon. The third one is from the U.S. college calculus textbook, where the variables in consideration were the length of a fabric and production cost. Using time as a variable in the covariational relationship is a natural extension of daily life experiences. However, using two other variables other than time to illustrate a covariational relationship has some distinct advantages when we get to the reverse function and the relationship between the derivative of the reversed function and the derivative of the original function. For example, it will be easier to picture the reversed dependent-

independent relationship between the diameter and volume of the balloon than to picture a reversed dependent-independent relationship between time and distance.

In the process of this study, I gained some insights of potential use should a similar investigation be conducted in the future. One crucial step in the research process is investigator recruitment. Investigators recruited for this type of content analysis should have mathematics content knowledge and practical experience in professional mathematics education. Prior exposure to pedagogical training, such as professional development in high school mathematics teaching, can be very valuable. In addition, prior exposure to the practice of educational psychology and the language used in mathematics education research can help a trainee pick up the covariational reasoning framework fairly quickly. On the contrary, someone whose skill set mainly comprises mathematics content knowledge, such as a graduate teaching fellow in a mathematics department, may have difficulty understanding the reader's theory and the difference between what the instructor knows and what students think.

More study of the pedagogy and development of covariational reasoning in calculus textbooks could be done in several ways, such as by comparing different textbooks in the same category and investigating other parts of calculus textbooks, such as indefinite integral sections.

As we have seen in this study, the two Chinese calculus textbooks have a very limited number of exercises compared to their U.S. counterparts. However, it does not mean that Chinese students' access to exercises is limited to problems in the textbooks. These days, students opportunities to learn through problem-solving in mathematics should include traditional exercise books and exercises on online platforms. A study of all of the problems made available to calculus students in the same four categories of the academic setting could reveal

some interesting patterns in the relative importance of covariational reasoning development compared to other skill sets in calculus learning.

The common content analysis method is through experts' coding and rating. This study started with no existing experts and leaned conceptually toward a survey approach. Eventually, in the process of qualifying survey respondents, two front-line calculus teachers were trained to be experts. The main obstacle to using a survey approach in studies like this is the framework's complexity, which could take a long time for the average intended reader of the calculus textbook to understand. However, suppose an instrument could be developed to map readers' interaction with calculus textbooks to certain datasets and classify these reactions to levels in covariational reasoning. In that case, the result from the survey approach would be more valid than the experts' coding and rating results. The survey approach is a direction worth exploring, particularly with the advent of interactive textbooks and the fact that calculus education employs the most intensive use of technology among all mathematics subjects.

The third part of this study was a comparison across different calculus textbooks published around the same time. This inquiry questions how things are different across space dimensions with time as a fixed variable. It would be interesting to see a comparative study across time dimensions. For example, different editions of the same textbook reflect the progressive understanding of the pedagogy on the same topic. An interesting investigation would be how the pedagogy of covariational reasoning changed over time through different editions of the same textbook. The comparative study across time could yield interesting results; I envision drastic changes in Chinese calculus textbooks across time dimension. The vision is based on the pace of change in Chinese mathematics education in the last hundred years. According to research by Ma (2010), Chinese arithmetic teachers today have much more profound knowledge

than their U.S. counterparts. However, as Ma (2010) stated in the preface of her book, "a hundred years ago, most Chinese people had not even seen the Hindu-Arabic system, let alone learned how to calculate with it." Thus, the fast-paced change in Chinese mathematics education could have led to fast progress in Chinese calculus textbook writing.

References

- Almeida, L., & Silva, K. (2018) A semiotic interpretation of the derivative concept in a textbook. *ZDM Mathematics Education*, 50:881-892.
- Anderson, L.W., Krathwohl, D.R., Airasian, P.W., Cruikshank, K. A., Mayer, R. E., Pintrich, P. R., Raths, J., & Wittrock, M. C. (2001) *A taxonomy for learning, teaching, and assessing: a revision of Bloom's Taxonomy of educational objectives*. Addison Wesley Longman, Inc.
- Anton, H., Bivens, I., & Davis, S. (2012). *Calculus: Early Transcendentals* (10th ed.). John Wiley & Sons, Inc. ISBN 978-0-470-64769-1.
- Ary, D., Jacobs, L.C., Sorensen, C.K., & Walker, D.A. (2013). *Introduction to research in education* (9th Ed). Cengage Learning.
- Baldassi, J. (2014). *James Stewart and the legacy of Canada's most successful textbook author*. <https://quillandquire.com/omni/james-stewart-and-the-legacy-of-canadas-most-successful-textbook-author/>
- Ball, D. L., & Cohen, D. K. (1996). Reform by the book: what is: or might be: the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher*, 25(9), 6-14.
- Bateman, S., LaForest, K. & Moore-Russo, D. (2021). Review of slope in calculus textbooks. In A.I. Sacristán, J.C. Cortés-Zavala, & P.M. Ruiz-Arias. (Eds.) *Proceedings of the 42nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (pp. 438-447). Sinoloa, Mexico: PME-NA.
- Berelson, B. (1952). *Content analysis in communication research*. New York: The Free Press.
- Berger, A. (1991). *Media research techniques*. Newbury Park, CA: Sage.
- Berger, M. (2019). Different reading styles for mathematics text. *Educational studies in mathematics*, 100(2), 139-159.
- Bergwall, A. & Hemmi, K. (2017) The state of proof in Finnish and Swedish mathematics textbooks—Capturing differences in approaches to upper-secondary integral calculus, *Mathematical Thinking and Learning*, 19:1, 1-18.
- Biza, I., Giraldo, V., Hochmuth, R., Khakbaz, A. S., & Rasmussen, C. (2016). *Research on teaching and learning mathematics at the tertiary level: State-of-the-art and looking ahead*. Springer.

- Bottazzini, U. (1986). *The higher calculus: A history of real and complex analysis from Euler to Weierstrass*. Springer-Verlag.
- Boyer, C. B. (1959). *The history of the calculus and its conceptual development*. Dover Publications.
- Bressoud, D., Ghedamsi, I., Martinez-Luaces, V., & Törner, G. (2016). *Teaching and Learning of Calculus*. Springer.
- Bressoud, D. (2007). *A radical approach to real analysis, 2nd Ed.* The Mathematical Association of America.
- Bressoud, D. (2019). *Calculus Reordered: A History of the Big Ideas*. Princeton University Press.
- Bressoud, D. (2020). The strange role of calculus in the United States. *ZDM Mathematics Education*, 53, 521-533.
- Britannica, T. Editors of Encyclopedia (2022, August 23). *function*. *Encyclopedia Britannica*. <https://www.britannica.com/science/function-mathematics>
- Carlson, M., Jacobs, S., Coe, E., Larson, S., and Hsu E. (2002). Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study. *Journal for Research in Mathematics Education*, 33(5): 352-378.
- Carlson, M., Larsen, S., & Jacobs, S. (2001). An investigation of covariational reasoning and its role in learning the concepts of limit and accumulation. In R. Speiser, C. Maher, & C. Walter (Eds.) *Proceedings of the 23rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Col 1, pp. 145-153, Snowbird, UT: PME-NA
- Carroll, J. B. (1963). A model of school learning. *Teachers College Record*, 64 (8) :723–733. <https://doi-org.ezproxy.cul.columbia.edu/10.1177/016146816306400801>
- Çetin, N. (2009) The ability of students to comprehend the function-derivative relationship with regard to problems from their real life, *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 19(3), 232-244, DOI:10.1080/10511970701686987
- Chen, F., & Chalhoub-Deville, M. (2016). Differential and long-term language impact on math. *Language Testing*, 33(4), 577–605.
- Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, 26, 135-164.
- Croucher, S. M. and Cronn-Mills, D. (2019). *Understanding communication research methods*. Routledge.

- Czocher, J., Tague, J., & Baker, G. (2013) Where does the calculus go? An investigation of how calculus ideas are used in later coursework, *International Journal of Mathematical Education in Science and Technology*, 44:5, 673-684.
- Dietiker, L., Males, L.M., Amador, J.M., & Earnest, D. (2018). Research commentary: Curricular noticing: A framework to describe teachers' interactions with curriculum materials. *Journal for Research in Mathematics Education*, 49(5), 521–532. <https://doi.org/10.5951/jresmetheduc.49.5.0521>
- Duan, B., & Zhang, C. (2012). Mathematics education in China. In E.C. Grivoyannis (Eds.), *The new Chinese economy: Dynamic transition into the future* (pp.104-135). Palgrave Macmillan.
- Eichler, A., & Erens, R. (2014). Teachers' beliefs toward teaching calculus. *ZDM Mathematics Education*, 46: 647-659.
- Ely, R. (2021). Teaching calculus with infinitesimal and differentials. *ZDM Mathematics Education*, 53:591-604.
- Fan, L. (2013). Textbook research as scientific research: towards a common ground on issues and methods of research on mathematics textbooks. *ZDM Mathematics Education*, 45:765–777.
- Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: Development status and directions. *ZDM Mathematics Education*, 45:633-646.
- Frank, K. & Thompson, P. W. (2021). School students' preparation for calculus in the United States, *ZDM Mathematics Education*, 53:549–562.
- Ferrini-Mundy, J., & Graham, K. G. (1991). An overview of the calculus curriculum reform effort: Issues for learning, teaching, and curriculum development. *The American Mathematical Monthly*, 98(7), 627–635. <https://doi.org/10.1080/00029890.1991.11995769>
- Finney, R. L., Demana, F. D., Waits, B. K., Kennedy, D., and Bressoud, D. M. (2016). *Calculus: Graphical, numerical, algebraic*. Pearson.
- Gracin, D. (2018). Requirement in mathematics textbooks: A five-dimensional analysis of textbook exercises and examples. *International Journal of Mathematical Education in Science and Technology*, 49:7, 1003-1024.
- Gracin, D. G., & Matic, L. J. (2021). Use of textbooks and other resources in curriculum reform. A longitudinal case study. *ZDM Mathematics Education*, 53, 1373 – 1385. <https://doi.org/10.1007/s11858-021-01271-0>

- Grant, M. R., Crombie, W., Enderson, M., & Cobb, N. (2016). Polynomial calculus: Rethinking the role of calculus in high schools. *International Journal of Mathematical Education in Science and Technology*, 47(6), 823–836. <https://doi.org/10.1080/0020739X.2015.1133851>
- Hadar, L. L. (2017). Opportunities to learn: Mathematics textbooks and students' achievement. *Studies in Educational Evaluation*, 55: 153-166.
- Heid, M. K. (1988). Resequencing Skills and Concepts in Applied Calculus Using the Computer as a Tool. *Journal for Research in Mathematics Education*, 19(1): 3-25.
- Hobson, N. L. F., & Moore, K. C. (2017). Exploring experts' covariational reasoning. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the Twentieth Annual Conference on Research in Undergraduate Mathematics Education* (pp. 664-672). San Diego, CA.
- Hoffkamp, A. (2011) The use of interactive visualizations to foster the understanding of concepts of calculus: design principles and empirical results. *ZDM Mathematics Education*, 43: 359-372.
- Howson, G. (2013). The development of mathematics textbooks: Historical reflections from a personal perspective. *ZDM Mathematics Education*, 45(5), 647 – 658. <https://doi.org/10.1007/S11858-013-0511-9>
- Howson, G., Keitel, C., & Kilpatrick, J. (1981). *Curriculum development in mathematics*. Cambridge University Press.
- Huang, X., Xiao, Y., Webster, J. S., Howe, R. E., & Li, Y. (2022). Exploring Shanghai students' mathematics learning as related to content presentation in textbooks: the case of the commutative property of addition. *ZDM Mathematics Education*, 54:595-609.
- Hwang, S., Yeo, S., & Son, T. (2021). A comparative analysis of fraction addition and subtraction contents in the mathematics textbooks in the U.S. and South Korea. *International Electronic Journal of Elementary Education*, 13(4), 511 – 521. <https://doi.org/10.26822/iejee.2021.208>
- Iacobacci, R. F. (1965). *Augustin-Louis Cauchy and the development of mathematical analysis*. Ph.D. Dissertation, New York University.
- Johnsen, E. G. (1993). *Textbooks in the kaleidoscope: A critical survey of literature and research on educational texts* (L. Sivesind, Trans.). Oslo, Norway: Scandinavian University Press.
- Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *The Journal of Mathematical Behavior*, 31(3), 313-330.
- Jones, S. R. (2018) Prototype image in mathematics education: the case of the graphical representation of the definite integral. *Educational Studies in Mathematics*, 97: 215-234.

- Karp, A., & Schubring, G. (2014). *Handbooks on the history of mathematics education*. Springer.
- Kertil, M., Erbas, A. K., & Cetinkaya, B. (2022). Per-service mathematics teachers' understanding of rate of change throughout a model development sequence. *International Journal of Science and Mathematics Education*. DOI:10.1007/s10763-022-10324-z
- Kilpatrick, J. (2014). From clay tablet to computer tablet: The evolution of school mathematics textbooks. *International Conference on Mathematics Textbook Research and Development 2014*, 3-12.
- Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.
- Kidron, I. (2014). Calculus Teaching and Learning. In S. Lerman (Ed.). *Encyclopedia of Mathematics Education*. New York: Springer Science.
- Kirsch, A. (2014). The fundamental theorem of calculus: visually? *ZDM Mathematics Education* 46: 691-695.
- Kleiner, Israel (1989). "Evolution of the Function Concept: A Brief Survey". *The College Mathematics Journal*. Mathematical Association of America. 20 (4): 282–300. doi:10.2307/2686848. JSTOR 2686848.
- Kline, M. (1972). *Mathematical thoughts from ancient to modern time*. Oxford University Press.
- Kolata, G. B. (1986). Calculus reform: Is it needed? Is it possible? in L. A. Steen (ed.), *Calculus for a New Century* (pp. 89-94). (MAA Notes Number 8). Mathematical Association of America.
- Krippendorff, K. (2004). *Content analysis: An introduction to its methodology* (2nd ed.). Thousand Oaks, CA: Sage.
- Lang, X. (1999) CAI and the reform of calculus education in China, *International Journal of Mathematical Education in Science and Technology*, 30:3, 399-404, DOI: 10.1080/002073999287914
- Lerman, S. (Ed.) (2014). *Encyclopedia of mathematics education*. Springer.
- Li, H. (2010). Higher Education in China: Complement or Competition to US Universities? In C. T. Clotfelter (Ed.), *American universities in a global market* (pp. 269-304). Chicago: National Bureau of Economic Research.
- Li, S. (2020). *A comparative study of Directional Connections in Popular U.S. and Chinese High School Mathematics textbook*. Doctoral dissertation, Columbia University.

- Li, Y., Zhang, J., & Ma, T. (2009) Approaches and practices in developing school mathematics textbooks in China. *ZDM Mathematics Education*, 41:733-748.
- Liakos, Y., Gerami, S., Mesa, V., Judson, T., & Ma, Y., (2021). How an inquiry-oriented textbook shaped a calculus instructor's planning. *International Journal of Mathematical Education in Science and Technology*, DOI: 10.1080/0020739X.2021.1961171
- Lithner, J. (2000). Mathematical reasoning in task solving. *Educational studies in mathematics*, 41, 165–190.
- Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics*, 52, 29–55.
- Lithner, J. (2004). Mathematical reasoning in calculus textbook exercises. *Journal of Mathematical Behavior*, 23, 405-427.
- Luciana Zuccheri & Verena Zudini (2014). History of teaching calculus. In A. Karp & G. Schubring (Eds.), *Handbook on the history of mathematics education* (pp. 493 - 513). Springer.
- Ma, L. (2010). *Knowing and teaching elementary mathematics*. Routledge.
- MAA. Mathematical Association of America course area study groups (2021). <https://www.maa.org/sites/default/files/CalculusCASGReportFinal.pdf>
- Mak, J. (2013). *How do students from China studying at U.S. engineering graduate degree programs develop along the Perry scheme*. Doctoral dissertation, Columbia University.
- Medvedev, F. A. (1991). *Scenes from the History of Real Functions*. Birkhauser.
- Mesa, V. (2010). Strategies for controlling the work in mathematics textbooks for introductory calculus. *Research in Collegiate Mathematics Education*, 16, 235-265.
- Mesa, V., Ma, Y., Quiroz, C., Gerami, S., Liakos, Y., Judson, T., & Chamberlain, L. (2021). University instructors' use of questioning devices in mathematics textbooks: an instrumental approach. *ZDM Mathematics Education*, 53:1299-1311. <https://doi.org/10.1007/s11858-021-01296-5>
- Monna, A. F. (1972). "The concept of function in the 19th and 20th centuries, in particular with regard to the discussions between Baire, Borel and Lebesgue". *Archive for History of Exact Sciences*. 9 (1): 57–84. doi:10.1007/BF00348540. S2CID 120506760
- Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *Journal of Mathematical Behavior*, 32(3), 461–473. <http://doi.org/10.1016/j.jmathb.2013.05.002>

- Moore, K. C., & Thompson, P. A. (2015). *Shape thinking and students' graphing activity*. Paper presented at the Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education, Pittsburgh, PA: RUME.
- Nagle, C., Moore-Russo, D., Viglietti, J., & Martin, K. (2013). Calculus students' and instructors' conceptualizations of slope: A comparison across academic levels. *International Journal of Science and Mathematics Education, 11*(6), 1491-1515.
- Nagy, P., Traub, R. E., MacRury, K., & Klaiman, R. (1991). High School calculus: Comparing the content of assignments and tests. *Journal for Research in Mathematics Education, 22*(1): 69-75.
- Neuman, W. L. (2006). *Social research methods: qualitative and quantitative approaches*. Needham Heights, MA: Allyn & Bacon.
- Olsher, S., & Cooper, J. (2021). Aspects of the teacher-textbook relationship: What can we learn about teachers when they tag didactic metadata? *ZDM Mathematics Education, 53*:1347–1358. <https://doi.org/10.1007/s11858-021-01251-4>
- Österholm, M. & Bergqvist, E. (2013) What is so special about mathematical texts? Analyses of common claims in research literature and of properties of textbooks. *ZDM Mathematics Education, 45*(5), 751-763.
- Paoletti, T., Moore, K. C. (2017). The parametric nature of two students' covariational reasoning. *The journal of mathematical behavior 48*(5): 137-151.
- Paoletti, T., Lee, H. Y., Rahman, Z., Vishnubhotla, M., & Basu, D. (2020). Comparing graphical representations in mathematics, science, and engineering textbooks and practitioner journals, *International Journal of Mathematical Education in Science and Technology*, DOI: 10.1080/0020739X.2020.1847336
- Piaget, J. (1964). Cognitive Development in Children: Development and Learning. *Journal of Research in Science Teaching, 2*, 176-186. <http://dx.doi.org/10.1002/tea.3660020306>
- Pogorelova, L. (2021). How much reform? An analysis of a chapter in a reform-based calculus textbook. *Journal of Mathematics Education at Teachers College, 13*(1): 15-27.
- Ponte, J. P. (1992). "The history of the concept of function and some educational implications". *The Mathematics Educator. 3*(2): 3–8.
- Qin, D. (2018). Chinese mathematics education system and mathematics education tradition. In Y. Cao & F.K.S. Leung (Eds.), *The 21st century mathematics education in China* (pp.27-52). Springer.

- Radmehr, F. & Drake, M. (2017). Revised Bloom's taxonomy and integral calculus: Unpacking the knowledge dimension, *International Journal of Mathematical Education in Science and Technology*, 48(8), 1206-1224.
- Randahl, M. (2016). The mathematics textbook at tertiary level as curriculum material – exploring the teacher's decision-making process. *International Journal of Mathematical Education in Science and Technology*, 47(6), 897–916.
- Rash, A. M. (1977). Is calculus an appropriate high school course? *The High School Journal* 60(6), 277–283.
- Rasmussen, C. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior*, 20, 55-87.
- Remillard, J. T., & Bryans, M. (2004). Teachers' orientations toward mathematics curriculum materials: Implications for teacher learning. *Journal for Research in Mathematics Education*, 35, 352–388.
- Rezat, S., Fan, L., & Pepin, B. (2021). Mathematics textbooks and curriculum resources as instruments for change. *ZDM Mathematics Education*, 53:1189-1206.
- Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp.13–51). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Robert, A. & Schwarzenberg, R. (1992). Research into teaching and learning. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 127 -139). Kluwer Academic Publishers.
- Rosenblatt, L. (1995). *Literature as exploration*. New York.
- Robinson, A. (1966). *Non-standard analysis*. Princeton University Press.
- Ruthing, D. (1984). "Some definitions of the concept of function from Bernoulli, Joh. to Bourbaki, N.". *Mathematical Intelligencer*. 6(4): 72–77.
doi:10.1007/BF03026743. S2CID 189883712.
- Sato, E., Rabinowitz, S., Gallagher, C., & Huang, C. W. (2010). *Accommodations for English Language Learner Students: The Effect of Linguistic Modification of Math Test Item Sets*. Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education.
- Schaaf, W. L. (1930). "Mathematics and World History." *Math. Teacher*, 23: 496-503.
- Schreier, M. (2012). *Qualitative content analysis in practice*. Thousand Oaks, CA: Sage.

- Şen-Zeytun, A., Cetinkaya, B., & Erbas, A. K. (2010). Mathematics teachers' covariational reasoning levels and predictions about students' covariational reasoning. *Educational Sciences: Theory and Practice*, 10(3), 1601–1612.
- Shanahan, C., Shanahan, T., & Misschia, C. (2011). Analysis of expert readers in three disciplines: History, mathematics, and chemistry. *Journal of Literacy*, 43(4), 393–429.
- Shava, G., Hleza, S., Tlou, F., Shonhiwa, S., and Mathonsi, E. (2021). Qualitative content analysis, utility, usability and processes in educational research. *International Journal of Research and Innovation in Social Science*, 5(7), 553-558.
- Shepherd, M. D., & van de Sande, C. C. (2014). Reading mathematics for understanding—from novice to expert. *Journal of Mathematical Behavior*, 35, 74–86.
- Shih, J. (2007). Buddhist learning: A process to be enlightened. In S. B. Merriam (ed.) *Non-Western Perspectives on Learning and Knowing*. Malabar, FL: Krieger, 99-112.
- Sievert, H., van den Ham, A.-K., & Heinze, A. (2021). The role of textbook quality in first graders' ability to solve quantitative comparisons: A multilevel analysis. *ZDM Mathematics Education*. <https://doi.org/10.1007/s11858-021-01266-x>
- Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *Journal of Mathematical Behavior*, 30, 131–148.
- Sood, S., & Jitendra, A. K. (2007). A comparative analysis of number sense instruction in reform-based and traditional mathematics textbooks. *The Journal of Special Education*, 41(3), 145 – 157. <https://doi.org/10.1177/0022466907041003010>
- Sosniak, L. A., & Perlman, C. L. (1990). Secondary education by the book. *Journal of Curriculum Studies*, 22(5), 427–442.
- Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp.319 - 370). Charlotte, NC: Information Age Publishing.
- Stevens, I. E., Paoletti, T., Moore, K. C., Hobson, N. L. F., & Hardison, H. (2017). Principles for designing tasks that promote covariational reasoning. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the Twentieth Annual Conference on Research in Undergraduate Mathematics Education* (pp. 928-936). San Diego, CA.
- Stewart, J. (2016). *Calculus*. Cengage Learning. ISBN 978-1-305-27176-0.
- Stray, C. (1994). Paradigms regained: Towards a historical sociology of the textbook. *Journal of Curriculum Studies*, 26, 1–29.

- Tall, D., Smith, D., & Piez, C. (2008). Technology and calculus. In M. K. Heid & G. M. Blume (Eds), *Research on technology and the teaching and learning of mathematics* (Vol. 1, pp. 207–258).
- Tallman, M., Carlson, M., Bressoud, D., & Pearson, M. (2016). A characterization of Calculus I final exams in U.S. colleges and universities. *International Journal of Research in Undergraduate Mathematics Education*, 2(1), 105–133.
- Tallman, M., Reed, Z., Oehrtman, M., & Carlson, M. P. (2021). What meanings are assessed in collegiate calculus in the United States? *ZDM Mathematics Education*.
<https://doi.org/10.1007/s11858-020-01212-3>
- Tan, C. (2017). Confucianism and education. *Oxford Research Encyclopedias*.
<https://doi.org/10.1093/acrefore/9780190264093.013.226>
- Tang, S. (1999). Cooperation or competition: A comparison of U.S. and Chinese college students. *The Journal of Psychology*, 133(4), 413-423.
- Tasova, H. I., Stevens, I. S. & Moore, K. (2018). A framework for analyzing written curriculum from a shape-thinking and (co)variational reasoning perspective. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the Twenty-first Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1527-1533). San Diego, CA.
- Thompson, P. W. (1994a). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, 26, 229-274.
- Thompson, P. W. (1994b). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.) *Research in Collegiate Mathematics Education, I: Issues in Mathematics Education*, (Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.
- Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital directions for research in mathematics education* (pp. 57–93). New York: Springer.
- Thompson, P. W., & Carlson, M. P. (2017a). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.
- Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017b). Covariational reasoning among U.S. and South Korea secondary mathematics teachers. *Journal of Mathematical Behavior*, 48: 95-111.
- Thompson, P. W. & Harel, G. (2021). Ideas foundational to calculus learning and their links to students' difficulties. *ZDM Mathematics Education*, 53:507–519.

- Thompson, P. W., & Milner, F. A. (2019). Teachers' meanings for function and function notation in South Korea and the United States. In H. G. Weigand, W. McCallum, M. Menghini, M. Neubrand, & G. Schubring (Eds.), *The legacy of Felix Klein* (pp. 55–66). New York: Springer.
- Tindal, G. (2014). Large-scale assessments for all students: Issues and options. In G. Tindal & T. M. Haladyna (Eds.), *Large-scale assessment programs for all students: Validity, technical adequacy, and implementation* (pp. 1-24). Mahwah, New Jersey: Lawrence Erlbaum Associates, Inc., Publishers.
- Toh, T. (2021). School calculus curriculum and the Singapore mathematics curriculum framework. *ZDM Mathematics Education*, <https://doi.org/10.1007/s11858-021-01225-6>
- Törner, G., Potari, D., & Zachariades, T. (2014). Calculus in European classrooms: Curriculum and teaching in different educational and cultural contexts. *ZDM - The International Journal on Mathematics Education*, 46, 549–560.
- Tucker, A. (2013). The History of the Undergraduate Program in Mathematics in the United States. *The American Mathematical Monthly*, 120(8), 689-705.
- Usiskin, Z. (2013). Studying textbooks in an information age—a United States Perspective. *ZDM Mathematics Education*, 45, 713–723.
- Van Den Ham, A.-K., and Heinze, A. (2018). Does the textbook matter? Longitudinal effects of textbook choice on primary school students' achievement in mathematics. *Stud. Educ. Eval.* 59, 133–140. doi: 10.1016/j.stueduc.2018.07.005
- Wakefield, D. V. (2000). Math as a second language. *Educational Forum* 64(3), 272-279.
- Wang, L., Liu, Q., Du, X., & Liu, J. (2018). Chinese mathematics curriculum reform in the twenty-first century. In Y. Cao & F.K.S. Leung (Eds.), *The 21st century mathematics education in China* (pp.53-72). Springer.
- Weber, R. P. (1990). *Basic Content Analysis*. CA: Sage Beverly Hills.
- Weber, E., Tallman, M., Byerley, C., & Thompson, P. W. (2012). Understanding the derivativethrough the calculus triangle. *The Mathematics Teacher*, 106(4), 274-278. <https://doi.org/10.5951/mathteacher.106.4.0274>
- Weber, E., & Thompson, P. W. (2014). Students' images of two-variable functions and their graphs. *Educational Studies in Mathematics*, 87(1), 67–85.
- Weinberg, A. & Wiesner, E. (2011). Understanding mathematics textbooks through reader-oriented theory. *Educational Studies in Mathematics*, 76(1), 49–63.

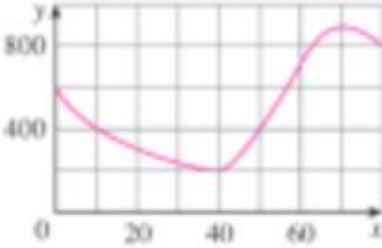
- Weinberg, A., Wiesner, E. & Fulmer, E.F. (2022). Didactical disciplinary literacy in mathematics: Making meaning from textbooks. *Int. J. Res. Undergrad. Math. Ed.* <https://doi.org/10.1007/s40753-022-00164-1>
- Wiesner, E., Weinberg, A., Fulmer, E. F., & Barr, J. (2020). The Roles of textual features, background knowledge, and disciplinary expertise in reading a calculus textbook. *Journal of Research in Mathematics Education*, 51(2), 204-233.
- Xu, B. (2013) The development of school mathematics textbooks in China since 1950. *ZDM Mathematics Education*, 45: 725-736.
- Yoon, H., Bae, Y., Lim, W., and Kwon, O. (2021). A story of the national calculus curriculum: How Culture, research, and policy compete and compromise in shaping the calculus curriculum in South Korea. *ZDM Mathematics Education*, <https://doi.org/10.1007/s11858-020-01219-w>.
- Yore, L. D. (2000). Enhancing science literacy for all students with embedded reading instruction and writing-to-learn activities. *Journal of Deaf Studies and Deaf Education*, 5(1), 105–112.
- Young, G. S. (1986) Present Problems and Future Prospects, in L. A. Steen (ed.), *Calculus for a New Century* (pp. 172-175). (MAA Notes Number 8). Mathematical Association of America.
- Youschkevitch, A. P. (1976). "The concept of function up to the middle of the 19th century". *Archive for History of Exact Sciences*. 16(1): 37- 85. doi:10.1007/BF00348305. S2CID 121038818
- Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education, IV* (Vol. 8, pp. 103-127). Providence, RI: American Mathematical Society.
- Zuccheri, L., & Zudini, V. (2014). History of teaching calculus. In A. Karp, & G. Schubring (Eds.), *Handbook on the history of mathematics education* (pp. 493–514). Springer.
- Zhang, D. (张奠宙) (1999). *The development of modern mathematics in China* (中国近现代数学的发展). ISBN:7-5375-1888-2.
- Zhang, D. (张奠宙) (2013). *Mathematics education in Chinese way* (数学教育的“中国道路”). ISBN: 9787544447188G.3738.
- Higher Mathematics* (2014). Tongji University Press. (高等数学, 2014, 同济出版社) ISBN: 978-7-04-039663-8.

Senior High School Mathematics, *Elective 2-2* (2014) (普通高中课程标准试验教科书 数学选修 2-2). People's Education Press.

Appendix A

Passages that identified to promote different level of covariational thinking from the section that introduce the derivative concept (p.105-117) in the calculus textbook by Stewart. (2015).

Level	Source	Passage number (Page number)	Passage
T1	Main text	4 (106)	Then we let Q approach P along the curve C by letting x approach a . If m_{PQ} approaches a number m , then we define the <i>tangent</i> t to be the line through P with slope m . (This
T1	Main text	21 (107)	Notice that as x approaches a , h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes
T1	Main text	49 (109)	If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a . Therefore an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is
T3	Main text	29 (108)	origin at time t . The function f that describes the motion is called the position function of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$. (See Figure 5.)
T4	Main text	64 (110)	<p>■ Rates of Change</p> <p>Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the increment of x) is</p> $\Delta x = x_2 - x_1$ <p>and the corresponding change in y is</p> $\Delta y = f(x_2) - f(x_1)$ <p>The difference quotient</p> $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ <p>is called the average rate of change of y with respect to x over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 8.</p>

T4	Main text	109 (112)	<p>In Examples 3, 6, and 7 we saw three specific examples of rates of change: the velocity of an object is the rate of change of displacement with respect to time; marginal cost is the rate of change of production cost with respect to the number of items produced; the rate of change of the debt with respect to time is of interest in economics. Here is a small sample of other rates of change: In physics, the rate of change of work with respect to time is called <i>power</i>. Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the <i>rate of reaction</i>).</p>
T4	Exercise	114	<p>16. The displacement (in feet) of a particle moving in a straight line is given by $s = \frac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.</p> <p>(a) Find the average velocity over each time interval:</p> <p>(i) $[4, 8]$ (ii) $[6, 8]$ (iii) $[8, 10]$ (iv) $[8, 12]$</p>
T4	Exercise	114	<p>18. The graph of a function f is shown.</p> <p>(a) Find the average rate of change of f on the interval $[20, 60]$.</p> <p>(b) Identify an interval on which the average rate of change of f is 0.</p> <p>(c) Which interval gives a larger average rate of change, $[40, 60]$ or $[40, 70]$?</p> <p>(d) Compute $\frac{f(40) - f(10)}{40 - 10}$. What does this value represent geometrically?</p> 

T4	Exercise	115	<p>47. Researchers measured the average blood alcohol concentration $C(t)$ of eight men starting one hour after consumption of 30 mL of ethanol (corresponding to two alcoholic drinks).</p> <table border="1" data-bbox="667 380 1386 499"> <tr> <td>t (hours)</td> <td>1.0</td> <td>1.5</td> <td>2.0</td> <td>2.5</td> <td>3.0</td> </tr> <tr> <td>$C(t)$ (g/dL)</td> <td>0.033</td> <td>0.024</td> <td>0.018</td> <td>0.012</td> <td>0.007</td> </tr> </table> <p>(a) Find the average rate of change of C with respect to t over each time interval: (i) [1.0, 2.0] (ii) [1.5, 2.0] (iii) [2.0, 2.5] (iv) [2.0, 3.0] In each case, include the units.</p> <p>(b) Estimate the instantaneous rate of change at $t = 2$ and interpret your result. What are the units?</p>	t (hours)	1.0	1.5	2.0	2.5	3.0	$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007
t (hours)	1.0	1.5	2.0	2.5	3.0										
$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007										
T4	Exercise	115	<p>48. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of October 1 are given.)</p> <table border="1" data-bbox="646 989 1346 1129"> <tr> <td>Year</td> <td>2004</td> <td>2006</td> <td>2008</td> <td>2010</td> <td>2012</td> </tr> <tr> <td>N</td> <td>8569</td> <td>12,440</td> <td>16,680</td> <td>16,858</td> <td>18,066</td> </tr> </table> <p>(a) Find the average rate of growth (i) from 2006 to 2008 (ii) from 2008 to 2010 In each case, include the units. What can you conclude?</p>	Year	2004	2006	2008	2010	2012	N	8569	12,440	16,680	16,858	18,066
Year	2004	2006	2008	2010	2012										
N	8569	12,440	16,680	16,858	18,066										
T4	Exercise	115	<p>49. The table shows world average daily oil consumption from 1985 to 2010 measured in thousands of barrels per day.</p> <p>(a) Compute and interpret the average rate of change from 1990 to 2005. What are the units?</p>												

T4	Exercise	115	<p>50. The table shows values of the viral load $V(t)$ in HIV patient 303, measured in RNA copies/mL, t days after ABT-538 treatment was begun.</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tbody> <tr> <td>t</td> <td>4</td> <td>8</td> <td>11</td> <td>15</td> <td>22</td> </tr> <tr> <td>$V(t)$</td> <td>53</td> <td>18</td> <td>9.4</td> <td>5.2</td> <td>3.6</td> </tr> </tbody> </table> <p>(a) Find the average rate of change of V with respect to t over each time interval: (i) $[4, 11]$ (ii) $[8, 11]$ (iii) $[11, 15]$ (iv) $[11, 22]$ What are the units? (b) Estimate and interpret the value of the derivative $V'(11)$.</p>	t	4	8	11	15	22	$V(t)$	53	18	9.4	5.2	3.6
t	4	8	11	15	22										
$V(t)$	53	18	9.4	5.2	3.6										
T4	Exercise	116	<p>51. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.</p> <p>(a) Find the average rate of change of C with respect to x when the production level is changed (i) from $x = 100$ to $x = 105$ (ii) from $x = 100$ to $x = 101$</p>												
T5	Main text	66 (110)	<p>smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0. The limit of these average rates of change is called the (instantaneous) rate of change of y with respect to x at $x = x_1$, which (as in the case of velocity) is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:</p>												
T5	Example	83 (111)	<p>SOLUTION</p> <p>(a) The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x.</p>												
T5	Example	102 (112)	<p>SOLUTION The derivative $D'(2000)$ means the rate of change of D with respect to t when $t = 2000$, that is, the rate of increase of the national debt in 2000.</p>												
T5	Exercise	114	<p>16. The displacement (in feet) of a particle moving in a straight line is given by $s = \frac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.</p> <p>(a) Find the average velocity over each time interval: (i) $[4, 8]$ (ii) $[6, 8]$ (iii) $[8, 10]$ (iv) $[8, 12]$</p> <p>(b) Find the instantaneous velocity when $t = 8$.</p> <p>(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a). Then draw the tangent line whose slope is the instantaneous velocity in part (b).</p>												

T5	Exercise	115	<p>47. Researchers measured the average blood alcohol concentration $C(t)$ of eight men starting one hour after consumption of 30 mL of ethanol (corresponding to two alcoholic drinks).</p> <table border="1" data-bbox="651 359 1395 478"> <tbody> <tr> <td>t (hours)</td> <td>1.0</td> <td>1.5</td> <td>2.0</td> <td>2.5</td> <td>3.0</td> </tr> <tr> <td>$C(t)$ (g/dL)</td> <td>0.033</td> <td>0.024</td> <td>0.018</td> <td>0.012</td> <td>0.007</td> </tr> </tbody> </table> <p>(a) Find the average rate of change of C with respect to t over each time interval:</p> <p>(i) [1.0, 2.0] (ii) [1.5, 2.0] (iii) [2.0, 2.5] (iv) [2.0, 3.0]</p> <p>In each case, include the units.</p> <p>(b) Estimate the instantaneous rate of change at $t = 2$ and interpret your result. What are the units?</p>	t (hours)	1.0	1.5	2.0	2.5	3.0	$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007
t (hours)	1.0	1.5	2.0	2.5	3.0										
$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007										
T5	Exercise	115	<p>48. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of October 1 are given.)</p> <table border="1" data-bbox="659 1001 1367 1142"> <tbody> <tr> <td>Year</td> <td>2004</td> <td>2006</td> <td>2008</td> <td>2010</td> <td>2012</td> </tr> <tr> <td>N</td> <td>8569</td> <td>12,440</td> <td>16,680</td> <td>16,858</td> <td>18,066</td> </tr> </tbody> </table> <p>(a) Find the average rate of growth</p> <p>(i) from 2006 to 2008 (ii) from 2008 to 2010</p> <p>In each case, include the units. What can you conclude?</p> <p>(b) Estimate the instantaneous rate of growth in 2010 by taking the average of two average rates of change. What are its units?</p> <p>(c) Estimate the instantaneous rate of growth in 2010 by measuring the slope of a tangent.</p>	Year	2004	2006	2008	2010	2012	N	8569	12,440	16,680	16,858	18,066
Year	2004	2006	2008	2010	2012										
N	8569	12,440	16,680	16,858	18,066										

T5	Exercise	115	<p>49. The table shows world average daily oil consumption from 1985 to 2010 measured in thousands of barrels per day.</p> <p>(a) Compute and interpret the average rate of change from 1990 to 2005. What are the units?</p> <p>(b) Estimate the instantaneous rate of change in 2000 by taking the average of two average rates of change. What are its units?</p> <table border="1" data-bbox="732 516 1260 877"> <thead> <tr> <th>Years since 1985</th> <th>Thousands of barrels of oil per day</th> </tr> </thead> <tbody> <tr> <td>0</td> <td>60,083</td> </tr> <tr> <td>5</td> <td>66,533</td> </tr> <tr> <td>10</td> <td>70,099</td> </tr> <tr> <td>15</td> <td>76,784</td> </tr> <tr> <td>20</td> <td>84,077</td> </tr> <tr> <td>25</td> <td>87,302</td> </tr> </tbody> </table> <p style="text-align: center;">Source: US Energy Information Administration</p>	Years since 1985	Thousands of barrels of oil per day	0	60,083	5	66,533	10	70,099	15	76,784	20	84,077	25	87,302
Years since 1985	Thousands of barrels of oil per day																
0	60,083																
5	66,533																
10	70,099																
15	76,784																
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T5	Exercise	116	<p>51. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.</p> <p>(a) Find the average rate of change of C with respect to x when the production level is changed</p> <p>(i) from $x = 100$ to $x = 105$</p> <p>(ii) from $x = 100$ to $x = 101$</p> <p>(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the <i>marginal cost</i>. Its significance will be explained in Section 2.7.)</p>														


Appendix B

Passages that identified to promote different level of covariational thinking from the section that introduce the derivative concept (p.87- 96) in the calculus textbook by Finney et al. (2014).

Level	Source	Passage number (Page number)	Passage
T1	Main text	23 (88)	<p>Tangent to a Curve</p> <p>The moral of the fruit fly story would seem to be that we should define the rate at which the value of the function $y = f(x)$ is changing with respect to x at any particular value $x = a$ to be the slope of the tangent to the curve $y = f(x)$ at $x = a$. But how are we to define the tangent line at an arbitrary point P on the curve and find its slope from the for-</p>
T1	Main text	80 (92)	<p>Sensitivity</p> <p>We live in an interconnected world where changes in one quantity cause changes in another. For example, crop yields per acre depend on rainfall. If rainfall has been low, each small increase in the amount of rain creates a small increase in crop yield. For a drug</p>
T1	Main text	81 (92)	<p>that works to lower a patient's temperature, each small increase in the amount of the drug will lower the temperature a small amount. The mathematical connection between such</p>
T3	Main text	92 (93)	<p>This means that when the dosage is 1 mg, a small additional dosage, ΔD mg, will result in a drop in the patient's temperature of approximately $3/4 \Delta D$ degrees.</p>
T4	Example	4 (87)	<p>EXAMPLE 1 Finding Average Rate of Change</p> <p>Find the average rate of change of $f(x) = x^3 - x$ over the interval $[1, 3]$.</p>

T4	Exercise	93	<h3 style="background-color: #2e8b57; color: white; padding: 5px;">Section 2.4 Exercises</h3> <p>In Exercises 1–6, find the average rate of change of the function over each interval.</p> <p>1. $f(x) = x^3 + 1$ 2. $f(x) = \sqrt{4x + 1}$ (a) $[2, 3]$ (b) $[-1, 1]$ (a) $[0, 2]$ (b) $[10, 12]$</p> <p>3. $f(x) = e^x$ 4. $f(x) = \ln x$ (a) $[-2, 0]$ (b) $[1, 3]$ (a) $[1, 4]$ (b) $[100, 103]$</p> <p>5. $f(x) = \cot x$ (a) $[\pi/4, 3\pi/4]$ (b) $[\pi/6, \pi/2]$</p> <p>6. $f(x) = 2 + \cos x$ (a) $[0, \pi]$ (b) $[-\pi, \pi]$</p>
T4	Exercise	95	<p>(a) Find the average rate of change in spending from 2008 to 2013.</p> <p>(b) Find the average rate of change in spending from 2008 to 2011.</p> <p>(c) Find the average rate of change in spending from 2011 to 2013.</p>
T4	Exercise	95	<p>(a) Find the average rate of change in spending from 2008 to 2013.</p> <p>(b) Find the average rate of change in spending from 2008 to 2011.</p> <p>(c) Find the average rate of change in spending from 2011 to 2013.</p>
T4	Exercise	95	<p>(a) Find the average rate of change in spending from 2008 to 2013.</p> <p>(b) Find the average rate of change in spending from 2008 to 2011.</p> <p>(c) Find the average rate of change in spending from 2011 to 2013.</p>

T4	Exercise	95	<p>44. Multiple Choice Find the average rate of change of $f(x) = x^2 + x$ over the interval $[1, 3]$.</p> <p>(A) -5 (B) $1/5$ (C) $1/4$ (D) 4 (E) 5</p>
T4	Exercise	96	<p>You may use a calculator with these problems.</p> <p>1. Multiple Choice Which of the following values is the average rate of $f(x) = \sqrt{x + 1}$ over the interval $(0, 3)$?</p> <p>(A) -3 (B) -1 (C) $-1/3$ (D) $1/3$ (E) 3</p>
T4	Exercise	96	<p>56. Consider the function f given in Example 1. Explain how the average rate of change of f over the interval $[3, 3 + h]$ is the same as the difference quotient of f at $a = 3$.</p>
T5	Example	74 (92)	<p>EXAMPLE 7 Investigating Free Fall</p> <p>Find the speed of the falling rock in Example 1, Section 2.1, at $t = 1$ sec.</p> <p>-----</p>
T5	Example	90 (92)	<p>EXAMPLE 8 Measuring Sensitivity to Medicine</p> <p>A patient enters the hospital with a temperature of 102°F and is given medicine to lower the temperature. As a function of the dosage, D, measured in milligrams, the patient's temperature will be</p> $T(D) = 99 + \frac{3}{1 + D}$ <p>Find and interpret the sensitivity of the patient's temperature to the medicine dosage when $D = 1$ mg. <i>continued</i></p>
T5	Exercise	94	<p>In Exercises 13 and 14, find the slope of the curve at the indicated point.</p> <p>13. $f(x) = x$ at (a) $x = 2$ (b) $x = -3$</p> <p>14. $f(x) = x - 2$ at $x = 1$</p>
T5	Exercise	94	<p>Find the instantaneous rate of change of the position function $y = f(t)$ in feet at the given time t in seconds.</p> <p>23. $f(t) = 3t - 7, t = 1$</p> <p>24. $f(t) = 3t^2 + 2t, t = 3$</p> <p>25. $f(t) = \frac{t + 1}{t}, t = 2$</p> <p>26. $f(t) = t^3 - 1, t = 2$</p>

T5	Exercise	94	<p>27. Free Fall An object is dropped from the top of a 100-m tower. Its height above ground after t sec is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?</p> <p>28. Rocket Launch At t sec after lift-off, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing after 10 sec?</p> <p>29. Area of Circle What is the rate of change of the area of a circle with respect to the radius when the radius is $r = 3$ in.?</p> <p>30. Volume of Sphere What is the rate of change of the volume of a sphere with respect to the radius when the radius is $r = 2$ in.?</p> <p>31. Free Fall on Mars The equation for free fall at the surface of Mars is $s = 1.86t^2$ m with t in seconds. Assume a rock is dropped from the top of a 200-m cliff. Find the speed of the rock at $t = 1$ sec.</p>  <p>32. Free Fall on Jupiter The equation for free fall at the surface of Jupiter is $s = 11.44t^2$ m with t in seconds. Assume a rock is dropped from the top of a 500-m cliff. Find the speed of the rock at $t = 2$ sec.</p>
T5	Exercise	94	<p>29. Area of Circle What is the rate of change of the area of a circle with respect to the radius when the radius is $r = 3$ in.?</p>
T5	Exercise	94	<p>30. Volume of Sphere What is the rate of change of the volume of a sphere with respect to the radius when the radius is $r = 2$ in.?</p>
T5	Exercise	94	<p>31. Free Fall on Mars The equation for free fall at the surface of Mars is $s = 1.86t^2$ m with t in seconds. Assume a rock is dropped from the top of a 200-m cliff. Find the speed of the rock at $t = 1$ sec.</p>

T5	Exercise	94	<p>32. Free Fall on Jupiter The equation for free fall at the surface of Jupiter is $s = 11.44t^2$ m with t in seconds. Assume a rock is dropped from the top of a 500-m cliff. Find the speed of the rock at $t = 2$ sec.</p>
T5	Exercise	95	<p>37. Sensitivity A patient's temperature T as a function of the dosage D of a medicine is given by $T(D) = 99 + 4/(1 + D)$. Find and interpret the sensitivity of the patient's temperature to the dosage when $D = 2$ mg.</p> <p>38. Sensitivity If a ball is thrown straight up with an initial velocity of v feet per second, it will reach a maximum height of $H = v^2/64$ feet. Find and interpret the sensitivity of the height to the initial velocity when the initial velocity is 40 ft/sec.</p>
T5	Exercise	96	<p>4. Free Response Let $f(x) = 2x - x^2$.</p> <p>(a) Find $f(3)$. (b) Find $f(3 + h)$.</p> <p>(c) Find $\frac{f(3 + h) - f(3)}{h}$.</p> <p>(d) Find the instantaneous rate of change of f at $x = 3$.</p>
T5	Exercise	96	<p>56. Consider the function f given in Example 1. Explain how the average rate of change of f over the interval $[3, 3 + h]$ is the same as the difference quotient of f at $a = 3$.</p>


Appendix C

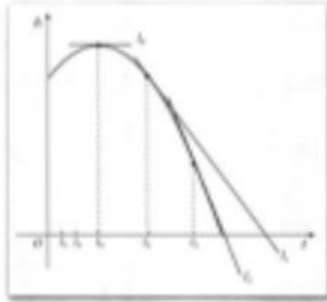
Passages that identified to promote different level of covariational thinking from the section that introduce the derivative concept (p. 73-84) in *Higher Mathematic* by Tongji University Press (2014).

Level	Source	Passage number (Page number)	Passage
T1	Main text	25 (74)	<p>当点 N 沿曲线 C 趋于点 M 时, 如果割线 MN 绕点 M 旋转而趋于极限位置 MT, 直线 MT 就称为曲线 C 在点 M 处的切线. 这里极限位置的含义是: 只要弦长 MN 趋于零, $\angle NMT$ 也趋于零.</p>
T1	Main text	30 (74)	<p>其中 φ 为割线 MN 的倾角. 当点 N 沿曲线 C 趋于点 M 时, $x \rightarrow x_0$. 如果当 $x \rightarrow x_0$ 时, 上式的极限存在, 设为 k, 即</p>
T1	Main text	34 (75)	<p>曲线 C 在点 M 处的切线. 事实上, 由 $\angle NMT = \varphi - \alpha$ 以及 $x \rightarrow x_0$ 时 $\varphi \rightarrow \alpha$ 可见 $x \rightarrow x_0$ 时 (这时 $MN \rightarrow 0$), $\angle NMT \rightarrow 0$. 因此直线 MT 确为曲线 C 在点 M 处的切线.</p>
T1	Main text	34 (75)	<p>曲线 C 在点 M 处的切线. 事实上, 由 $\angle NMT = \varphi - \alpha$ 以及 $x \rightarrow x_0$ 时 $\varphi \rightarrow \alpha$, 可见 $x \rightarrow x_0$ 时 (这时 $MN \rightarrow 0$), $\angle NMT \rightarrow 0$. 因此直线 MT 确为曲线 C 在点 M 处的切线.</p>
T1	Main text	47 (76)	<p>原因是由于 $\Delta x \rightarrow 0$ 时, 比式 $\frac{\Delta y}{\Delta x} \rightarrow \infty$, 为了方便起见, 也往往说函数 $y = f(x)$ 在点 x_0 处的导数为无穷大.</p>
T1	Main text	105 (82)	<p>由此可见, 当 $\Delta x \rightarrow 0$ 时, $\Delta y \rightarrow 0$. 这就是说, 函数 $y = f(x)$ 在点 x 处是连续的. 所以, 如果函数 $y = f(x)$ 在点 x 处可导, 那么函数在该点必连续.</p>
T5	Exercise	83	<p>4. 设 $f(x) = 10x^3$, 试按定义求 $f'(-1)$.</p>
T5	Exercise	84	<p>10. 已知物体的运动规律为 $s = t^3$ m, 求这物体在 $t = 2$ s 时的速度.</p>
T5	Exercise	84	<p>12. 求曲线 $y = \sin x$ 在具有下列横坐标的各点处切线的斜率:</p> <p style="text-align: center;">$x = \frac{2}{3}\pi, \quad x = \pi.$</p>
T5	Exercise	84	<p>19. 已知 $f(x) = \begin{cases} \sin x, & x < 0, \\ x, & x \geq 0, \end{cases}$ 求 $f'(x)$.</p>

Appendix D

Passages that identified to promote different level of covariational thinking from the section that introduce the derivative concept (p. 2-11) in a Chinese high school calculus textbook, *elective 2-2*, by People's Education Press (2005).

Level	Source	Passage number (Page number)	Passage
T1	Main text	4 (2)	<p>问题1 气球膨胀率</p> <p>很多人都吹过气球。回忆一下吹气球的过程，可以发现，随着气球内空气容量的增加，气球的半径增加得越来越慢。从数学的角度，如何描述这种现象呢？</p>
T1	Main text	30 (5)	<p>从物理的角度看，时间间隔 Δt 无限变小时，平均速度 \bar{v} 就无限趋近于 $t=2$ 时的瞬时速度。因此，运动员在 $t=2$ 时的瞬时速度是 -13.1 m/s。</p>
T1	Main text	51 (7)	<p>我们发现，当点 P_n 趋近于点 P 时，割线 PP_n 趋近于确定的位置。这个确定位置的直线 PT 称为点 P 处的切线 (tangent line)。值得关注的问题是，割线 PP_n 的斜率与切线 PT 的斜率 k 有什么关系呢？</p>
T1	Main text	50 (7)	<p> 观察</p> <p>如图 1.1-2，当点 $P_n(x_n, f(x_n)) (n=1, 2, 3, 4)$ 沿着曲线 $f(x)$ 趋近于点 $P(x_0, f(x_0))$ 时，割线 PP_n 的变化趋势是什么？</p>
T2	Main text	9 (2)	<p>随着气球体积逐渐变大，它的平均膨胀率逐渐变小了</p>

T2	Main text	62 (8)	<p>例2 如图 1.1-3, 它表示跳水运动中高度随时间变化的函数 $A(t) = -4.9t^2 + 6.52t + 1.0$ 的图象. 根据图象, 请描述、比较曲线 $h(t)$ 在 t_0, t_1, t_2 附近的变化情况.</p>  <p>解: 我们用曲线 $A(t)$ 在 t_0, t_1, t_2 处的切线, 刻画曲线 $A(t)$ 在上述三个时刻附近的变化情况.</p> <p>(1) 当 $t=t_0$ 时, 曲线 $A(t)$ 在 t_0 处的切线 l_0 平行于 x 轴. 所以, 在 $t=t_0$ 附近曲线比较平坦, 几乎没有升降.</p> <p>(2) 当 $t=t_1$ 时, 曲线 $A(t)$ 在 t_1 处的切线 l_1 的斜率 $A'(t_1) < 0$. 所以, 在 $t=t_1$ 附近曲线下降, 即函数 $A(t)$ 在 $t=t_1$ 附近单调递减.</p> <p>(3) 当 $t=t_2$ 时, 曲线 $A(t)$ 在 t_2 处的切线 l_2 的斜率 $A'(t_2) < 0$. 所以, 在 $t=t_2$ 附近曲线下降, 即函数 $A(t)$ 在 $t=t_2$ 附近也单调递减.</p> <p>从图 1.1-3 可以看出, 直线 l_1 的倾斜程度小于直线 l_2 的倾斜程度. 这说明曲线 $A(t)$ 在 t_1 附近比在 t_2 附近下降得平缓.</p>
T2	Main text	62 (8)	<p>(1) 当 $t=t_0$ 时, 曲线 $h(t)$ 在 t_0 处的切线 l_0 平行于 x 轴. 所以, 在 $t=t_0$ 附近曲线比较平坦, 几乎没有升降.</p>
T2	Main text	62 (8)	<p>(2) 当 $t=t_1$ 时, 曲线 $h(t)$ 在 t_1 处的切线 l_1 的斜率 $h'(t_1) < 0$. 所以, 在 $t=t_1$ 附近曲线下降, 即函数 $h(t)$ 在 $t=t_1$ 附近单调递减.</p>
T2	Main text	62 (8)	<p>(3) 当 $t=t_2$ 时, 曲线 $h(t)$ 在 t_2 处的切线 l_2 的斜率 $h'(t_2) < 0$. 所以, 在 $t=t_2$ 附近曲线下降, 即函数 $h(t)$ 在 $t=t_2$ 附近也单调递减.</p>
T3	Main text	7 (2)	<p>当空气容量 V 从 0 增加到 1 L 时, 气球半径增加了 $r(1) - r(0) \approx 0.62$ (dm),</p>
T3	Main text	8 (2)	<p>类似地, 当空气容量 V 从 1 L 增加到 2 L 时, 气球半径增加了 $r(2) - r(1) \approx 0.16$ (dm),</p>

T3	Main text	29 (5)	我们发现, 当 Δt 趋近于 0 时, 即无论 t 从小于 2 的一边, 还是从大于 2 的一边趋近于 2 时, 平均速度都趋近于一个确定的值 -13.1 .
T3	Main text	32 (5)	表示 “当 $t=2$, Δt 趋近于 0 时, 平均速度 \bar{v} 趋近于确定值 -13.1 ”.
T5	Exercise	10	2. 在高台跳水运动中, t s 时运动员相对于水面的高度 (单位: m) 是 $h(t) = -4.9t^2 + 6.5t + 10$. 求运动员跳水过程中运动速度在 $t=2$ s 时的瞬时速度, 并解释此时的运动情况.
T5	Exercise	10	4. 已知车轮旋转的角度与时间的平方成正比. 如果车轮启动后转动第一圈需要 0.5 s, 求转动开始到第 3.2 s 时的瞬时速度.
T5	Exercise	11	1. 在高台跳水运动中, t s 时运动员相对于水面的高度 (单位: m) 是 $h(t) = -4.9t^2 + 6.5t + 10$. 高度 h 关于时间 t 的导数是速度 v , 速度 v 关于时间 t 的导数是什么?