A Coproduct Structure on Symplectic Cohomology

Lea Kenigsberg

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Abstract

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Lea Kenigsberg

Symplectic cohomology is an algebraic invariant which encodes dynamical information of Liouville manifolds; that is, open symplectic manifolds satisfying certain convexity conditions at infinity. In this work we define and investigate a new algebraic structure on symplectic cohomology, the coproduct. To exhibit the non triviality of this structure we study it in the case of complements of smooth divisors. Under certain technical conditions, the symplectic cohomology of such manifolds is particularly amenable to computations via a Morse-Bott model. We define the Morse-Bott coproduct and use it to illustrate that the coproduct structure on the symplectic cohomology of the cotangent bundle of a 3 sphere is not trivial.

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with "unabashed gratitude [1]"

to my advisor, Mohammed Abouzaid, who braved my speculative math & shitty drafts & ineloquent rage, and made space for me in this suspect universe called math.

[1] [1]

to Shlomo

Chapter 1: Introduction

1.1 Background

Liouville manifolds, W, are a class of open symplectic manifolds which at infinity look like the cylinder $Y \times [0, \infty)$ for some contact manifold Y. Examples of such symplectic manifolds include cotangent bundles of smooth manifolds and affine varieties.

Symplectic cohomology is the Floer theory associated to a system of Hamiltonians

$$H^\tau : S^1 \times W \to \mathbb{R}$$

which are linear of slope τ in the cylindrical end. The dynamics of such Hamiltonians are related to the periodic orbits of the Reeb flow on Y, and symplectic cohomology is one way in which they can be studied.

The symplectic cochain complex $SC^*(W, H^\tau)$ is generated by periodic orbits of the Hamiltonian vector field X_{H^τ}, and the differential counts perturbed J holomorphic cylinders interpolating between the orbits. Using continuation maps, also defined using perturbed J holomorphic cylinders, one can define

$$SH^*(W) = \varinjlim_{\tau} SH^*(W, H^\tau).$$

Symplectic cohomology admits a rich structure, including a Batalin Vilkovisky (BV) operation and a product structure. Morally speaking, the BV operation rotates the periodic orbits, and the product structure counts perturbed J holomorphic maps from the upside down pair of pants surface.

1.2 A survey of coproducts and motivation

By considering the Riemann surface, Σ, with 2 negative punctures and one positive puncture, one can define a coproduct structure:

$$V : SH^*(W) \to SH^*(W) \otimes SH^*(W)$$

by counting maps $u : \Sigma \to W$ satisfying:

$$(du - X_{H_z} \otimes \beta)^{(0,1)} = 0.$$

Here H_z is a family of Hamiltonians parameterized by Σ, and β is a 1-form satisfying certain conditions which ensure that the solutions do not escape to infinity. As was pointed out by Seidel in [2], see also [3], this coproduct structure is highly degenerate. In particular, it vanishes in all degrees $n \neq 0$. It is not completely trivial in general, however. For cotangent bundles $T^\star M$, it is proved in [3] that the coproduct satisfies

$$V([1]) = \chi(M)[a_0] \bigotimes [a_0].$$

Here $\chi(M)$ is the Euler characteristic of M, $[1]$ is the unit of $SH^*(T^\star M)$, and $[a_0]$ can be thought of as a generator of $H^n(M)$ via the inclusion $H^*(M) \to SH^*(T^\star M)$.

One can define a secondary operation parameterized by $\Sigma \times I$ as is done in [4] for cotangent bundles and more generally in [5, 6], see Figure 1.1. This operation interpolates between two degenerate operations, each of which maps one of the outputs to a constant orbit. This operation is not defined on $SH^*(W)$, however, since the non trivial boundary terms prevent it from being a chain map. Let $SH^+(W)$ be defined as the quotient of $SH^*(W)$ by the subcomplex generated by constant orbits. Then a non trivial coproduct operation, V^+, only exists on the high energy part $SH^+(W)$.

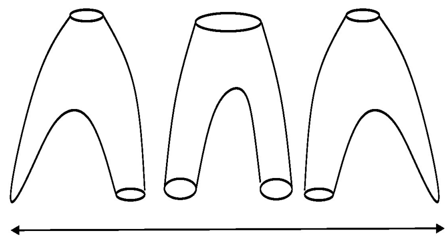

Figure 1.1: A 1 parameter family of coproduct operations.

Given the product, $\star$, and the coproduct, V^+, one would like to investigate identities such as

$$V^+(A \star B) \stackrel{?}{=} V^+(A) \star B + A \star V^+ B.$$

Note however, that the product structure on $SH^*(W)$ does not descend to a product on $SH^+(W)$, and hence the interplay between the product and coproduct can not be studied.

Remark 1. *For a spinned manifold M, Viterbo's isomorphism [7] identifies the symplectic cohomology of the cotangent bundle, $SH^*(T^\star M)$, with the homology of the free loop space, $H_*(\mathcal{L}M)$. $H_*(\mathcal{L}M)$ admits a rich structure obtained by the concatenation, cutting, and rotation of loops. Under Viterbo's isomorphism, the pair-of-pants product on symplectic homology is identified with the Chas-Sullivan [8] string topology product, and the coproduct V^+ can be identified with the Goresky-Hingston [9] coproduct on the relative homology $H_*(\mathcal{L}M/M)$.*

For loop spaces, the natural evaluation $\mathcal{L}M \to M$ gives a splitting

$$H_*(\mathcal{L}M) \cong H_*(\mathcal{L}M/M) \oplus H_*(M)$$

which allows for an extension of the coproduct to an operation

$$H_*(\mathcal{L}M) \to H_*(\mathcal{L}M) \otimes H_*(\mathcal{L}M)$$

as is done in [10]. Among other applications, such an extension allows one to study the interplay *between the product and the coproduct.*

No such splitting exists on the symplectic cohomology of Liouville domains however, which inspired a search for a Floer theory associated to Liouville domains which admits both a product and a coproduct structure. One solution was offered in [11], where it is shown that for a class of Weinstein Domains, the coproduct and product both extend to a reduced version of symplectic cohomology.

1.3 Results

The first goal of this paper is to define a new coproduct on $SH^*(W)$ by implementing the following idea: the BV operator is trivial on constant orbits. Therefore, by "gluing" this operator to the negative ends of the secondary coproduct constructed in [12], we rid of the boundary terms and consequently obtain an operation parameterized by S^3:

Theorem. *Let W be a Liouville domain. There exists a graded commutative coproduct structure of degree* $2n-3$*:*

$$\lambda : SH^*(W) \to SH^*(W) \otimes SH^*(W)$$

The second goal of this paper is to compute this structure in an example and illustrate that it is not trivial. Symplectic cohomology is notoriously difficult to compute. In [13], Diogo and Lisi use a split Morse-Bott construction to compute the symplectic cohomology of complements of smooth divisors satisfying monotonicity conditions. Such manifolds, W, can be written as $X\backslash D$ for a closed symplectic manifold (X,Ω) and a smooth symplectic submanifold D Poincaré dual to a positive multiple of Ω. In these settings the curves defining operations on symplectic cohomology can be related to Gromov-Witten invariants of D and Gromov-Witten invariants in X relative to D, which are easier to compute. The comparison of the split Morse-Bott model with the standard definition of symplectic cohomology goes through an intermediate construction, denoted the Morse-Bott symplectic cohomology.

We define the Morse-Bott coproduct and the Morse-Bott split coproduct, see Figures 5.2 and

6.2. We prove that under monotonicity conditions the Morse-Bott split coproduct moduli spaces consist of the simplest configurations:

Theorem. *Assume that* (X, Ω_X) *is spherically monotone with monotonicity constant* τ_X *and assume that* $D \subset X$ *is Poincaré dual to* $K\Omega_X$ *with* $\tau_X > K > 0$*. Then the moduli spaces of split Morse-Bott cascades contributing to the coproduct consist of Morse flowlines in the contact boundary* Y *as well as pair of pants in the symplectization* $Y \times [0, \infty)$.

We use this model to show that the coproduct does not vanish for an element of $T^\star S^3$:

Theorem. *Let* U *be a generator of* $SH^3(T^\star S^3)$*, and* $1 \in SH^0(T^\star S^3)$ *denote the unit of the ring. Then*

$$\lambda(U) = 1 \otimes 1.$$

Aside from proving that λ gives a non-trivial operation, this computation demonstrates that λ detects the constant orbits. Indeed, note that the unit of $SH^0(T^* S^3)$ can be thought of as a generator of $H^0(S^3)$ via the inclusion of constant orbits $H^*(M) \to SH^*(T^*M)$.

Chapter 2: Symplectic cohomology of Liouville manifolds

2.1 Settings

In this section we review the definition of symplectic cohomology. For an informal exposition see [2]. For more detailed expositions, see [14, 3]. Our object of study will be Liouville domains:

Definition 2 (Liouville domains)**.** *A Liouville domain, $(\overline{W}, \partial\overline{W})$, is a compact manifold with boundary together with a one form α such that $d\alpha$ is a symplectic form on $\overline{W}$, and such that the Liouville vector field, Z, defined by $d\alpha(Z,) = \alpha$ is transverse and points outwards along $\partial\overline{W}$.*

This implies that the form $\alpha|_{\partial\overline{W}}$ is a contact form on $\partial\overline{W}$ with Reeb vector field R_α defined uniquely by $\alpha(R_\alpha) = 1$ and ker $d\alpha|_{\partial\overline{W}} = R_\alpha$. Let ϕ be the flow of Z. We parameterize a neighborhood of $\partial\overline{W}$ in $\overline{W}$ by:

$$G : \partial W \times [-\delta, 0] \to W, \quad (t, w) \to \phi^t(w)$$

The completion of $\overline{W}$, denoted W, is obtained by attaching a cylindrical end: $W = \partial\overline{W} \times \mathbb{R}^+ \cup_G \overline{W}$ with symplectic form:

$$\Omega = \begin{cases} d\alpha & \text{on } \overline{W} \\ d(e^t\alpha) & \text{on } \partial\overline{W} \times \mathbb{R}^+ \end{cases}$$

with t being the $\mathbb{R}$ coordinate. We set $r = e^t$. Let

$$\mathcal{S}(\alpha) = \{T \in \mathbb{R} | \exists \text{ a closed periodic orbit of period T of } R_\alpha\}.$$

We assume that the closed Reeb orbits are transversally non-degenerate. This means that:

$$det(\mathbb{I} - d\phi_H^1|_{x_i}\xi(\gamma(0))) \neq 0$$

for all Reeb orbits x_i. Here ξ denotes the contact distribution. We now define the class of Hamiltonians that appear in the definition of symplectic cohomology:

Definition 3 (admissible Hamiltonians)**.** *Let $\mathcal{H}$ denote the class of Hamiltonians $H : S^1 \times W \to \mathbb{R}$ satisfying:*

- *$H|_{\overline{W}}$ is a fixed C^2 small Morse function.*
- *on $\partial\widehat{W} \times \mathbb{R}^+$, $H = h(t,r)$ depends only on the cylindrical variable r and the S^1 coordinate t. In addition, $\partial_r(h'(t,r)) > 0$.*
- *for $r \geqslant 1$, $H = \tau r$, $\tau \notin \mathcal{S}(\alpha)$ and $\tau \geqslant 0$.*
- *the 1- periodic orbits of H are non-degenerate.*

We will sometimes write H_t to to indicate the time dependence of H. We denote the slope of H by τ_H, and sometimes use the notation H^τ to indicate the slope. There is a partial order on admissible Hamiltonians where $H \preceq K$ if the slope of K is greater or equal to the slope of H.

For $H \in \mathcal{H}$ let $\mathcal{P}(H)$ denote the set of 1 periodic orbits of the Hamiltonian vector field X_{H_t} defined by

$$d_{H_t} = \Omega(-, X_{H_t}).$$

To such orbits one can assign a cohomological Conley-Zehnder index, defined in §2.3 and denoted by $deg(x)$.

The action of a 1 periodic orbit $x \in \mathcal{P}(H)$ is defined by:

$$\mathcal{A}_{H(x)} = \int_{S^1} -x^*\alpha + H(t, x(t))dt.$$

Let $\mathcal{J}(W)$ denote the space of time dependent almost complex structures compatible with Ω and adapted to the contact form α in the cylindrical end. The last conditions means that : $J \in \mathcal{J}(W)$ satisfies $dr \circ J = -\alpha$, and J is any time dependent $d\alpha$ compatible almost complex structure on ξ.

2.2 Symplectic cohomology

We follow the conventions and notation introduced in [14]. Given an admissible Hamiltonian H, the Floer co-chain complex associated to H is given in degree i by the formula:

$$CF^i(W; H, J) = \bigoplus_{x \in \mathcal{P}, deg(x) = i} |\mathbf{o}_x|.$$

Here the orientation line $\mathbf{o_x}$ is a one-dimensional real vector space associated to x, defined as the determinant line of a certain linearization of Floer's equation. The definition of $\mathbf{o_x}$ is recalled in §2.3 following [14, §1.7]. We let $|\mathbf{o}_x|$ denote the rank 1 free Abelian group generated by the two orientations on $\mathbf{o}_x$ modulo the relation that the sum of the orientations is zero.

We now recall the definition of the moduli spaces which in dimension 0 define the differential:

Definition 4. *Let $\widehat{\mathcal{M}}(x_-, x_+, H, J)$ denote the moduli space of finite energy maps $u : S^1 \times \mathbb{R} \to W$ satisfying the Floer Equation:*

$$(du - X_{H_t} \otimes dt)^{(0,1)} = 0, \tag{2.1}$$

and such that

$$\lim_{|s| \to \pm\infty} u(s,t) = x_\pm(t).$$

Here $x(t) \in \mathcal{P}(H)$.

For generic choice of Hamiltonian H and Almost complex structure J, the moduli space $\widehat{\mathcal{M}}(x_-, x_+, H, J)$ is smooth and admits a Gromov Floer compactification [15]. It admits an $\mathbb{R}$ action by translations, and the quotient is denoted by $\mathcal{M}(x_-, x_+, H, J)$ and is of dimension $deg(x) -$

$deg(y) - 1$. When $deg(x) = deg(y) + 1$, every solution $u \in \mathcal{M}(x_-, x_+, H, J)$ determines an isomorphism of orientation lines

$$d_u : \mathbf{o}_y \to \mathbf{o}_x$$

The differential on the cochain complex $CF^*(W; H, J)$ is defined by:

$$d(|\mathbf{o}_{x_+}|) = \bigoplus_{\substack{x_- \\ deg(x_+) = deg(x_-) - 1}} \sum_{u \in \mathcal{M}(x_-, x_+)} d_u \tag{2.2}$$

Using standard techniques in Floer theory, one can show that $d^2 = 0$ and that $HF^*(W; H, J)$ only depends on the slope of H [15, 3]. To define symplectic cohomology one defines continuation maps

$$\mathcal{K} : HF^*(W; H^+, J^+) \to HF^*(W; H^-, J^-) \tag{2.3}$$

as follows: given admissible pairs (H_t^-, J_t^-) and (H_t^+, J_t^+), with $H_t^+ \leq H_t^-$, we consider a family of Hamiltonians parameterized by the cylinder $S^1 \times \mathbb{R}$ satisfying:

$$H_{s,t} = \begin{cases} H_t^+ & \textit{for } s \gg 0 \\ H_t^- & \textit{for } s \ll 0 \end{cases}$$

subject to the condition that $\partial_s(H_{s,t}) \leqslant 0$. We also consider a family of almost complex structures parameterized by $S^1 \times \mathbb{R}$ and satisfying:

$$J_{s,t} = \begin{cases} J_t^+ & \textit{for } s \gg 0 \\ J_t^- & \textit{for } s \ll 0 \end{cases}$$

Then $\mathcal{K}(x_-, x_+, H^-, H^+)$ is the moduli space of finite energy maps $u : S^1 \times \mathbb{R} \to W$ satisfying:

$$\partial_s(u) + J_{s,t}(\partial_t u - X_{s,t}) = 0 \tag{2.4}$$

and

$$\lim_{|s|\to\infty} u(s,t) = x_{\pm}(t).$$

Here $x_{\pm}(t) \in \mathcal{P}(H^{\pm})$.

Remark 5. *The purpose of the condition $\partial_s(H_{s,t}) \leqslant 0$ is to ensure that the solutions are contained in a compact subset of W. See Lemma 11 for a more statement.*

For generic data $\mathcal{K}(x_-, x_+, H^-, H^+)$ are smooth and admit a standard Gromov-Floer compactification. The count of rigid elements define maps

$$\mathcal{K} : HF^*(H^+, J^+) \to HF^*(H^-, J^-)$$

which are independent of the choice of homotopy and complex structures. Symplectic cohomology is defined as the colimit over all linear Hamiltonians:

$$SH^*(W) = \varinjlim_{H\in\mathcal{H}^\tau} HF^*(W, H^\tau).$$

2.3 Grading conventions

We recall the construction of orientation lines and grading conventions from [14, §1.4]. A path $\Psi(t)$ in Sp(2n) starting at the identity is non-degenerate if $\Psi(1)$ does not have 1 as an eigenvalue. After parameterizing if necessary, we can associate to such a path a loop of symmetric matrices $S(t)$ satisfying:

$$\dot{\Psi}(t) = J_0 S(t)\Psi(t).$$

Such a loop is non-degenerate if $\Psi(t)$ is non-degenerate. We equip $\mathbb{C}$ with positive (resp. negative) cylindrical coordinates:

$$(-\infty, \infty) \times S^1 \to \mathbb{C}$$
$$(s,t) \to e^{\pm s \pm 2\pi i t}.$$

To a non-degenerate loop we associate the space $\mathcal{O}_{\pm}(S)$ consisting of operators of the form:

$$D_{\psi} : W^{1,p}(\mathbb{C}, \mathbb{C}^n) \to L^p(\mathbb{C}, \mathbb{C}^n)$$
$$X \to \partial_s X + J_0(\partial_t - S \cdot X)$$

where $S \in C^0(\mathbb{C}, \mathfrak{gl}(2n))$ satisfies:

$$S(e^{s+2\pi i t}) = S(t) \; for \; s \gg 0, \; if \; D_\Psi \in \mathcal{O}_+(S)$$

$$S(e^{-s-2\pi i t}) = S(t) \; for \; s \ll 0, \; if \; D_\Psi \in \mathcal{O}_-(S)$$

The spaces $\mathcal{O}_{\pm}(S)$ consist of Fredholm operators and are contractible. To each operator D_{ψ} we assign a line bundle $\det(D_{\psi}) = \det(coker^{\vee} D_{\psi}) \otimes \det(ker D_{\psi})$, and hence get a trivial line bundle over the spaces $\mathcal{O}_{\pm}(S)$.

Let $\mathcal{P}$ denote the set of 1 periodic orbits of H. We assume $c_1(W) = 0$, and hence $\mathcal{K} = \Lambda^{max}_{\mathbb{C}} T^*W$ is trivializable as a complex bundle with complex structure J. We fix one such trivialization. Given $x \in \mathcal{P}$, we choose a trivialization $\xi(t)$ of x^*TW so that the induced trivialization of $x^*\mathcal{K}$ agrees with the restriction of the trivialization above.

For every periodic orbit $x(t)$ of H the Hamiltonian flow $\psi(t)$ gives a path of symplectic matrices $D_{\psi}(t) : T_{x(0)}W \to T_{x(t)}W$. Using the trivialization above one gets:

$$\mathbb{R}^{2n} \xrightarrow{\xi(0)} T_{x(0)}W \xrightarrow{D\psi(t)} T_{x(t)}W \xrightarrow{\xi(t)^{-1}} \mathbb{R}^{2n}$$

and hence a path of symplectic matrices Ψ_x starting at the identity. The assumption that H was non degenerate implies that Ψ_x does not have 1 as an eigenvalue.

Given $x \in \mathcal{P}$, we denote by S_x the loop of symmetric matrices associated with x and by $D^{\pm}_{\psi_x}$ an element in $\mathcal{O}_{\pm}(S_x)$. We let $\delta^{\pm}_x = det(D^{\pm}_{\psi_x})$. We define

$$deg(x) = ind(D^{-}_{\psi_x}).$$

Since the spaces $\mathcal{O}$ are contractible, this determinant line is independent of choices. We let o_x (*resp.* o^+_x) be the $\mathbb{Z}$ graded Abelian group generated by the two orientations of δ^-_x (*resp.* δ^+_x) and relation that their sum vanishes. Note that there is a canonical isomorphism [14, Lemma 1.4.8]:

$$o_x \otimes o^+_x \to \mathbb{Z}. \tag{2.5}$$

Chapter 3: The Hamiltonian coproduct

In this section we introduce the operation

$$\lambda : SH^*(W) \to SH^*(W) \otimes SH^*(W),$$

and prove some of its properties.

3.1 Moduli spaces

We will construct a family of Hamiltonians parameterized by $\Sigma \times S^1 \times S^1 \times I$. Here Σ is the pair of pants surface with two negative ends and one positive end. We will use $\Theta = (\theta_1, \theta_2) \in S^1 \times S^1$ to vary the choice of asymptotic markers at the negative ends; this will induce the "gluing" of the BV operation. The parameter $\omega \in I$ is used to vary the slope of the Hamiltonians at the negative ends. That is, near the boundary of the interval we will choose the data so that the slope of the Hamiltonian will be very small at one of the negative ends. This will force a solution to be degenerate in that it will map that end to a constant orbit. In this sense, the parameter ω is used to interpolate between the degenerate operations which map one of the negative ends to a constant orbit. This construction is based on the construction of interpolating $1-$forms in [5].

Let Σ denote a sphere with one positive puncture and two negative punctures defined as follows: a cylindrical end at a puncture is a biholomorphic identification of a neighborhood of that puncture with one of the following:

- Positive interior puncture: $Z^+ = [0, \infty) \times S^1$ with complex coordinate $z = s + it$
- Negative interior puncture: $Z^- = (-\infty, 0] \times S^1$ with complex coordinate $z = s + it$

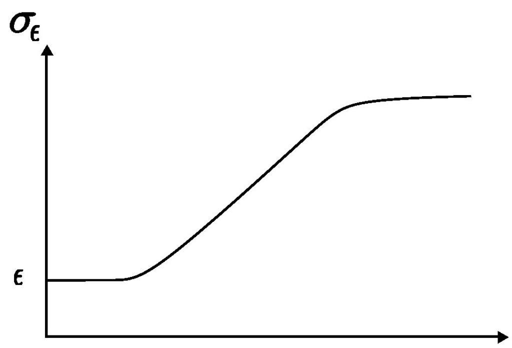

Figure 3.1: The interpolation function σ_ϵ

Let $\kappa : (0, 1] \to [0, \infty)$ be a stretching profile defined as in [5] and satisfying

$$\lim_{r \to 0^+} \kappa(r) = \infty.$$

We will use κ to force the solutions near the boundary of $S^1 \times S^1 \times I$ to degenerate and have outputs in constant orbits.

Let $\sigma_\epsilon : [0, 1] \to [0, 1]$ be a smooth function satisfying:

$$\sigma_\epsilon(s) = \begin{cases} \epsilon & \textit{for } 0 \leqslant s \leqslant \frac{1}{4} \\ s & \textit{for } \frac{1}{3} \leqslant s \leqslant \frac{2}{3} \\ 1 & \textit{for } \frac{3}{4} \leqslant s \leqslant 1 \end{cases}$$

and such that σ_ϵ is strictly increasing on $[\frac{1}{4}, \frac{1}{3}]$ and $[\frac{2}{3}, \frac{3}{4}]$, see Figure 3.1. We use σ_ϵ to ensure that the Hamiltonians near the boundary of $S^1 \times S^1 \times I$ do not have degenerate orbits.

We parameterize the unit interval with coordinates $\omega = (\omega_0, \omega_1)$ such that $\omega_0 + \omega_1 = 1, \omega_i \in (0, 1)$.

Definition 6 (Coproduct data, see Figure 3.2)**.** *To define the coproduct we fix the following Floer data:*

1. *An admissible Hamiltonian H_t^0, and admissible Hamiltonians $H_t^i, i \in \{1, 2\}$ satisfying*

$H_t^0 \preceq H_t^i$.

2. $\epsilon \ll 1$, *such that* $\epsilon\tau_i$ *is smaller than the period of any Reeb orbit of* R_α.

3. *A family of Hamiltonians* $H_{z,\Theta,\omega}$ *parameterized by* $\Sigma \times S^1 \times S^1 \times (0,1)$, *linear for* $r \geqslant 1$, *and satisfying the following conditions:*

   (a) *In a neighborhood of infinity at the positive cylindrical end:*

$$H_{z,\Theta,\omega} = H_t^0$$

   (b) *In a neighborhood of infinity at the* ith *negative end, with coordinates* $(s,t) \in (-\infty, 0] \times S^1$:

$$H_{z,\Theta,\omega} = \begin{cases} \sigma_\epsilon(\omega_i) H_t^i & \text{for } -R \geqslant s \geqslant -R - \kappa(\omega_i) \\ H_{t+\theta_i}^i & \text{for } s \leqslant -R - \kappa(\omega_i) \end{cases}$$

4. *A family of almost complex structures* $J_{z,\Theta,\omega} \in \mathcal{J}$ *which only depends on* t *in a neighborhood of the positive puncture, and such that in a neighborhood of the* ith *negative puncture:*

$$J_{z,\Theta,\omega} = J_{t+\theta_i,\omega}$$

5. *A* 1 *form* β *on* Σ *satisfying* $d(H_{z,\Theta,\omega}\beta) \leqslant 0$, *and such that* β *restricts to* dt *in the neighborhood of the punctures.*

Remark 7. *The condition* $d(H_{z,\Theta,\omega}\beta) \leqslant 0$ *means that for every* $w \in W$ *and every* $v \in T\Sigma$, $d(H_{z,\Theta,\omega}(w)\beta)(v, jv) \leqslant 0$. *This condition will ensure the compactness of the moduli space. Such a family and* β *exist if we assume* $H_t^0 \preceq H_t^i$.

Note also that we use cohomological conventions: the surface Σ *receive an input at the positive end, and emits outputs at the negative punctures.*

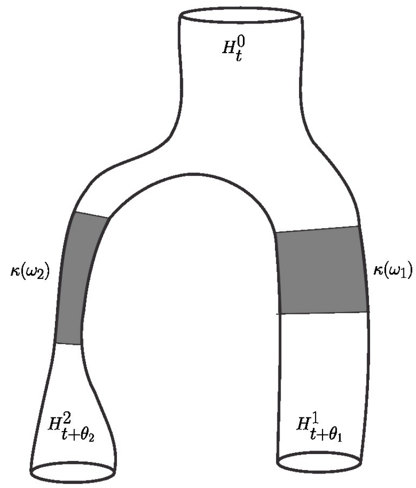

Figure 3.2: Coproduct data

Definition 8. *Given coproduct data as in 6, the moduli space of* ***coproduct curves***

$$\mathcal{M}_{\Theta,\omega}(x_0; x_1, x_2)$$

consists of finite energy maps

$$(u, \omega, \Theta) : \Sigma \to W$$

satisfying:

$$(du - X_{H_{(z,\Theta.\omega)}} \otimes \beta)^{(0,1)} = 0 \tag{3.1}$$

and

$$\lim_{s\to\infty} u(\epsilon_0(s + it)) = x_0(t)$$

in the positive puncture, and

$$\lim_{s\to-\infty} u(\epsilon_i(s + it)) = x_i(t + \theta_i)$$

in the negative punctures. Here $x_i \in \mathcal{P}(H^i)$.

The moduli spaces $\mathcal{M}_{\Theta,\omega}$ can be expressed as the zero locus of a Fredholm operator surjectivity, of which is well understood in the current setup and follows from an infinite dimensional Sard-Smale theorem, and the fact that $H_{z,\Theta,\omega}$ and $J_{z,\Theta,\omega}$ are allowed to depend on all parameters. For details see [3, §16]:

Lemma 9. *For a generic choice of data $H_{z,\Theta,\omega}$ and $J_{z,\Theta,\omega}$, and admissible Hamiltonians $H^i, i \in \{0,1,2\}$, the moduli space $\mathcal{M}_{\Theta,\omega}(x_0; x_1, x_2)$ is smooth of dimension*

$$deg(x_1) + deg(x_2) - deg(x_0) - 2n + 3$$

3.2 Orientations

Lemma 10. *If $u_{\Theta,\omega}$ is a solution of the Floer equation (3.1), then $u_{\Theta,\omega}$ induces an isomorphism*

$$det(D_{u_{\Theta,\omega}}) \otimes o_{x_0} \cong o_{x_1} \otimes o_{x_2}$$

which is canonical up to multiplication by a positive real number.

Proof. Given a solution $u_{\Theta,\omega}$ of (3.1), we can choose a trivialization of $u^*_{\Theta,\omega}TW$ so that when restricting to the neighborhoods of the three punctures we get the fixed trivialization ξ_i of x_i^*TW. With respect to this trivialization, and up to a rotation by θ_i, the linearization of (3.1) in the negative cylindrical ends agrees with $D^+_{\psi_{x_i}}$ in the complement of a disk.

We will now glue $D^+_{\Psi_{x_1}}$ and $D^+_{\Psi_{x_2}}$ to $D_{(u,\Theta,\omega)}$ to obtain an operator in $O^+(S_{x_0})$. We equip the plane with positive cylindrical coordinates and choose two negative real numbers $S_1(\omega) < -R - \kappa(\omega_1) - 1$ and $S_2(\omega) < -R - \kappa(\omega_2) - 1$. We remove the ends $\epsilon_i(S^1 \times (-\infty, S_i)), i = 1,2$ from Σ, and glue the complements to disks in $\mathbb{C}$ by identifying $S_i(\omega) \times S^1$ in the *ith* negative end to $-S_i(\omega) \times S^1 \in \mathbb{C}$. We do this identification while rotating if necessary, to account for the parameters (θ_1, θ_2).

By Lemma 2.3.12 of [14] there is a canonical isomorphism of determinant lines:

$$D^+_{\Psi_{x_2}} \# D^+_{\Psi_{x_1}} \# det(D_{(u,\Theta,\omega)}) \cong det(D^+_{\Psi_{x_2}}) \otimes det(D^+_{\Psi_{x_1}}) \otimes det(D_{(u,\Theta,\omega)}).$$

The lemma now follows by appealing to (2.5) and the fact that $O^+(S_{x_0})$ is contractible.

□

We now explain how to orient $\mathcal{M}_{\Theta,\omega}(x_0; x_1, x_2)$. Given a solution $u_{\Theta,\omega} \in \mathcal{M}_{\Theta,\omega}(x_0; x_1, x_2)$, the linearization of (3.1) gives an operator:

$$D_{u_{\Theta,\omega}} : W^{1,p}(\Sigma, u^*_{\Theta,\omega}TW) \to \mathbb{R} \oplus L^P(\Sigma, u^*_{\Theta,\omega}TW \otimes \Omega^{0,1}(\Sigma)).$$

Taking the derivative with respect to the parameterizing variables (Θ, ω) gives a Fredholm map:

$$T_{\theta_1}S^1 \oplus T_{\theta_2}S^1 \oplus T_\omega\mathbb{R}^2 \oplus W^{1,p}(\Sigma, u^*_{\Theta,\omega}TM) \to \mathbb{R} \oplus L^P(\Sigma, u^*_{\Theta,\omega}TM \otimes \Omega^{0,1}(\Sigma)). \tag{3.2}$$

Here we think of the unit interval as embedded in $\mathbb{R}^2$ with coordinates $\omega = (\omega_0, \omega_1)$ and tangent space the kernel of the map $\mathbb{R}^2 \to \mathbb{R}$, $(x_0, x_1) \to x_0 + x_1$.

The Floer data $H_{z,\Theta,\omega}$ and $J_{z,\Theta,\omega}$ is said to be regular if (3.2) is surjective. This means that

$$T_{\theta_1}S^1 \oplus T_{\theta_2}S^1 \oplus T_\omega\mathbb{R}^2 \to \mathbb{R} \oplus coker(D_{u_{\Theta,\omega}})$$

is surjective which can be achieved for generic data.

The tangent space of $\mathcal{M}_{\Theta,\omega}$ at $u_{\Theta,\omega}$ is the kernel of the operator in (3.2), hence there is a short exact sequence:

$$T_{u_{\Theta,\omega}}\mathcal{M}_{\Theta,\omega} \to T_{\theta_1}S^1 \oplus T_{\theta_2}S^1 \oplus T_\omega\mathbb{R}^2 \oplus ker(D_{u_{\Theta,\omega}}) \to \mathbb{R} \oplus coker(D_{u_{\Theta,\omega}})$$

which induces an isomorphism:

$$\begin{aligned} &det(T_{u_{\Theta,\omega}}\mathcal{M}_{\Theta,\omega}) \otimes det(\mathbb{R}) \otimes det(coker(D_{u_{\Theta,\omega}})) \\ \to & det(T_{\theta_1}S^1) \otimes det(T_{\theta_2}S^1) \otimes det(T_\omega\mathbb{R}^2) \otimes det(ker(D_{u_{\Theta,\omega}})), \end{aligned}$$

and hence an isomorphism:

$$det(T_{u_{\Theta,\omega}}\mathcal{M}_{\Theta,\omega}) \otimes det(\mathbb{R}) \to det(T_{\theta_1}S^1) \otimes det(T_{\theta_2}S^1) \otimes det(T_\omega\mathbb{R}^2) \otimes det(D_{u_{\Theta,\omega}}). \tag{3.3}$$

To obtain an orientation of $T_{u_{\Theta,\omega}}\mathcal{M}_{\Theta,\omega}$ we use the orientation for $det(D_{u_{\Theta,\omega}})$ from the gluing above and fix an orientation for $T_{\theta_i}S^1$ and $T_\omega\mathbb{R}^2$ which induces an orientation on $T\mathbb{R}$.

Assuming regularity, the moduli space $\mathcal{M}_{\Theta,\omega}(x_0; x_1, x_2)$ is hence smooth of dimension

$$deg(x_0) - deg(x_1) - deg(x_2) + 2n - 3.$$

When $deg(x_0) = deg(x_1) + deg(x_2) - 2n + 3$ the moduli spaces have dimension 0 and (3.7) descends to an isomorphism:

$$\lambda : o_{x_0} \cong o_{x_1} \otimes o_{x_2}$$

3.3 Compactness

In this subsection we first state a maximum principle for coproduct curves, which shows that coproduct curves are contained in a compact set. We then describe the Gromov-Floer compactification of $\mathcal{M}_{\Theta,\omega}$.

Lemma 11 (Maximum Principle, following [16]). *Finite energy solutions to the Floer equations (3.1) are contained in the compact set of $r \leqslant 1$ in W.*

Proof. For $u : \Sigma \to W$ a solution of the Floer equation the energy is defined as

$$E(u) = \int_\Sigma \|du - X_{H_{z,\Theta,\omega}} \otimes \beta\|^2$$

where the norm is with respect to the metric $\omega(-, J)$. Alternatively,

$$E(u) = \int_\Sigma u^*\omega - u^* dH_{z,\Theta,\omega} \wedge \beta.$$

Note that because of the dependence of $H_{z,\Theta,\omega}$ on $z \in \Sigma$, we have the expression:

$$u^* dH_{z,\Theta,\omega} = d(u^* H_{z,\Theta,\omega}) - dH_{z,\Theta,\omega}.$$

Where with the second term on the right we mean the derivative of $H_{z,\Theta,\omega}$ with respect to the parameterizing variable z. Now let $r' > 1$ be a regular value of the function $r(u(z))$ such that

$$\Sigma' = \{z \in \Sigma | r(u(z)) \geqslant r'\}$$

is non empty. Note that since the periodic orbits lie below $r < 1$, Σ' is a compact Riemann surface with boundary. Then:

$$
\begin{aligned}
E(u)|_{\Sigma'} &= \int_{\Sigma'} u^*\omega - u^* dH_{z,\Theta,\omega} \wedge \beta \\
&= \int_{\Sigma'} u^*\omega - d(u^* H_{z,\Theta,\omega}) \wedge \beta + dH_{z,\Theta,\omega} \wedge \beta \\
&\leqslant \int_{\Sigma'} u^*\omega - d(u^* H_{z,\Theta,\omega}) \wedge \beta + dH_{z,\Theta,\omega} \wedge \beta - d(H_{z,\Theta,\omega}\beta) \\
&= \int_{\Sigma'} u^*\omega - d(u^* H_{z,\Theta,\omega}\beta) \\
&= \int_{\partial\Sigma'} u^* r\lambda - u^* H_{z,\Theta,\omega}\beta \\
&= \int_{\partial\Sigma'} u^* r\lambda - u^*(a_{z,\Theta,\omega} r)\beta \\
&= r' \int_{\partial\Sigma'} \lambda(du - X_{z,\Theta,\omega} \otimes \beta) \\
&= r' \int_{\partial\Sigma'} J \circ \lambda(du - X_{H_{z,\Theta,\omega}} \otimes \beta) \circ (-i) \\
&\leqslant 0.
\end{aligned}
$$

The first inequality follows from the condition $d(H_{z,\Theta,\omega}\beta) \leqslant 0$. The last inequality follows from the fact that $\lambda \circ J = dr$, $dr(X_{H_{z,\Theta,\omega}}) = 0$, and that if $v \in \partial\Sigma'$ is a positively oriented tangent vector, then $-iv$ points outwards and hence $dr \circ du(-iv) \leqslant 0$.

It follows that the energy of u restricted to Σ' is zero, and hence u is contained in a regular level set. Note that this conclusion does not depend on the regular level set, which are dense in the interval $(1, r')$, and hence Σ' must be empty. □

Remark 12. *We will also need to allow for the following modifications. We will relax the condition of linearity and consider instead Hamiltonians which for some $r > r_0$ satisfy $-rH'(r) + H(r) \leqslant 0$. In this case, one still has $u^* H\beta \leqslant \lambda(X_H \otimes \beta)$, [3, Lemma 19.5].*

One can also allow Σ' to have negative punctures x_i, in which case the additional contribution to $\int_{\partial\Sigma'} u^* r\lambda - u^* H_{z,\Theta,\omega}\beta$ is given by $\sum \mathcal{A}_{H(x_i)} \leqslant 0$.

Gromov Floer compactification:

The moduli spaces $\mathcal{M}_{\Theta,\omega}(x_0; x_1, x_2)$ admit a Gromov-Floer compactification by broken solutions. Away from the ends, the energy estimates force local C^∞ hence uniform convergence, thus breaking can only occur at the ends. At an end, a Floer trajectory can break off leading to boundary strata consisting of fiber products of Floer cylinders and coproduct curves. There will also be terms coming from the limits $\omega \to (0,1), (1,0)$. These boundary strata correspond to moduli spaces of degenerate coproduct curves, where the slope of the Hamiltonian is small at one of the negative ends, followed by a hybrid continuation-$\mathbf{\Pi}$ cylinders. The later being solutions $u : S^1 \times \mathbb{R} \to W$ satisfying a continuation equation for $s \gg 0$, and the BV equation for $s \ll 0$. See Figure 3.3.

As usual in Floer theory, we will use moduli spaces of dimension 1 to show that the coproduct curves define a chain map and descends to an operation on symplectic cohomology. In order to achieve this, we will show that the boundary strata that correspond to the compactification of the interval are empty in low dimensions. We now make this argument explicit by defining the relevant moduli spaces. Starting with the degenerate coproduct operation, and following with a hybrid continuation-$\mathbf{\Pi}$ operation.

3.3.1 A degenerate coproduct operation:

Definition 13. *In the following we fix $i \in \{1, 2\}$ and let $j = 3 - i$. To define the boundary terms we fix the following data:*

- *An admissible Hamiltonian H of slope τ, a small ϵ such that $\epsilon\tau$ is smaller than the period of any Reeb orbit of R_α, and a family of linear Hamiltonians H_{z,θ_i} parameterized by $\Sigma \times S^1$ satisfying:*

  1. *In a neighborhood of infinity at the positive end:*

$$H_{z,\theta_i} = H_t$$

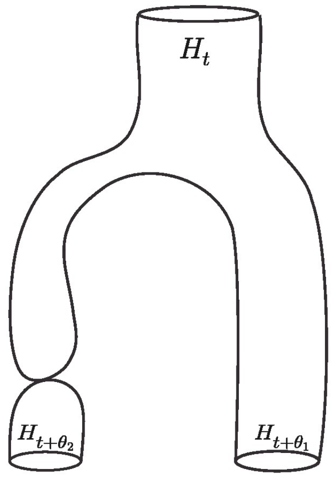

Figure 3.3: Configurations at the boundary of the interval

2. *In a neighborhood of infinity at the i'th cylindrical end:*

$$H_{z,\theta_i} = H_{t+\theta}$$

3. *In a neighborhood of infinity at the j'th end:*

$$H_{z,\theta_i} = \epsilon H$$

- *a family of admissible almost complex structures* J_{t,θ_i} *parameterized by* $\Sigma \times S^1$ *such that* $J_{z,\theta_i} = J_t$ *in a neighborhood of the positive puncture and the* $j'th$ *negative puncture, and* $J_{z,\theta_i} = J_{t+\theta_i}$ *in a neighborhood of the i'th negative puncture.*

- *A one form* β *on* Σ *satisfying* $d(H_{z,\theta_i}\beta) \leqslant 0$*, and such that* β *restricts to* dt *in the neighborhood of the punctures.*

The moduli space of ***degenerate coproduct curves***

$$\mathcal{M}_{\theta_i}(x_0; x_1, x_2)$$

consists of finite energy maps

$$(u, \omega, \theta_i) : \Sigma \to W$$

satisfying:

$$(du - X_{H_{z,\theta_i}} \otimes \beta)^{(0,1)} = 0 \tag{3.4}$$

and

$$\lim_{s\to\infty} u(\epsilon_0(s+it)) = x_0(t)$$
$$\lim_{s\to-\infty} u(\epsilon_i(s+it)) = x_i(t+\theta_i)$$
$$\lim_{s\to-\infty} u(\epsilon_j(s+it)) = x_j(t)$$

where $x_i \in \mathcal{P}(H_t)$.

The transversality results in [3, §16] apply here and show that :

Lemma 14. *For a generic choice of data H_{z,θ_i} and $\mathcal{J}_{z,\theta_1}$, the moduli space $\mathcal{M}_{\theta_i}(x_0; x_1, x_2)$ are regular for all triple of orbits x_0, x_1, x_2 and this space is smooth of dimension*

$$deg(x_1) + deg(x_2) - deg(x_0) - 2n + 1.$$

Moreover, the maximum principle Lemma 11 applies, and the solutions of (3.4) are contained in the compact set of $r \leqslant 1$ in W.

3.3.2 A Continuation-$\mathbf{\Pi}$ operation:

We now define moduli spaces of maps $u : S^1 \times \mathbb{R} \to W$ which satisfy a continuation equation for $s \gg 0$, and the BV equation for $s \ll 0$.

Definition 15. *Let*

- *$H^{\pm}$ be admissible Hamiltonians satisfying $H^- \geq H^+$*

- *$J_{s,t}$ a family of admissible almost complex structures, which are independent of s in a neighborhood of the punctures.*

- *a family of admissible Hamiltonians parameterized by $(t,s,\theta) \in \mathbb{R} \times S^1 \times S^1$ satisfying:*

$$H_{(s,t)} = \begin{cases} H_t^+ & \textit{for } s \gg 0 \\ H_{t+\theta}^- & \textit{for } s \ll 0 \end{cases}$$

and s.t. $\partial_s(H_{s,t,\theta}) \leqslant 0.$

The moduli space of ***continuation - Π cylinders***

$$\mathcal{N}(x_-, x_+)$$

consists of pairs $(u,\theta) : S^1 \times \mathbb{R} \to W$ satisfying:

$$\partial_s(u) + J_{s,t}(\partial_t u - X_{s,t,\theta}) = 0 \tag{3.5}$$

satisfying

$$\lim_{s\to\infty} u(s,t) = x_+(t)$$
$$\lim_{s\to-\infty} u(s,t,\theta) = x_-(t)$$

The methods of [15] apply here to show that:

Lemma 16. *For a generic choice of data the moduli spaces $\mathcal{N}(x_-, x_+)$ are smooth of dimension:*

$$deg(x_-) - deg(x_+) - 1.$$

The moduli spaces $\mathcal{N}(x_-, x_+)$ admit a Gromov-Floer compactification by boundary strata corresponding to fiber products $\mathcal{N}(x_-, y) \times \mathcal{M}(y, x_+)$ and $\mathcal{M}(x_-, y') \times \mathcal{N}(y', x_+)$.

Lemma 17. *For constant orbits x_+, and for regular data, the moduli spaces $\mathcal{N}(x_-, x_+)$ of dimension 0 are empty.*

Proof. An energy estimate implies that x_- is a constant orbit, and the maximum principle Lemma 11 implies that the solutions are contained in the interior of W, where the Hamiltonians $H^\pm$ are time independent and hence θ independent. By rescaling H_s if necessary, we can assume that the solutions are regular and time independent by [3, §15, claim 5], hence do not depend on θ and can't be rigid. □

Compactness in low dimensions:

Let $H_{z,\Theta,\omega}$ be the family of Hamiltonians used to define the coproduct (8). Let H_{z,θ_i} be its pointwise limit as $\omega = (\omega_1, \omega_2) \to (1, 0)$.

Lemma 18. *In dimension 1, the Gromov-Floer compactification of $\mathcal{M}_{\Theta,\omega}(x_0; x_1, x_2)$ consists of the following strata:*

1. $\mathcal{M}(x_0, x_0') \times \mathcal{M}_{\Theta,\omega}(x_0'; x_1, x_2)$
2. $\mathcal{M}_{\Theta,\omega}(x_0; x_1', x_2) \times \mathcal{M}(x_1', x_1)$
3. $\mathcal{M}_{\Theta,\omega}(x_0; x_1, x_2') \times \mathcal{M}(x_2', x_2)$

Proof. Away from the ends the energy estimates force C^∞ local and hence uniform convergence. Therefore, breaking can only happen at the ends. A Floer trajectory can break off, leading to terms as in $(1), (2), (3)$. Breaking can also happen at the boundary of the interval, where a continuation-**Π** cylinder can break off. These terms correspond to the limits $\omega = (\omega_1, \omega_2) \to (1, 0), (0, 1)$ and can be written as the fiber products $\mathcal{M}^0_{\theta_i} \times \mathcal{N}^0$, where the superscript indicates dimension. Note that when the slope of the Hamiltonian is sufficiently small, its only periodic orbits are constant and hence Lemma 17 implies that the boundary strata corresponding to the limits as $\omega \to (0, 1), (1, 0)$ are empty. □

3.4 Definition of the coproduct

As is explained in §3.2, in dimension zero each $u \in \mathcal{M}_{\Theta,\omega}$ defines an isomorphism:

$$\lambda_u : o_{x_0} \cong o_{x_1} \otimes o_{x_2},$$

which define the coproduct:

Definition 19. *Given regular data the coproduct*

$$\lambda : CF(H^0) \to CF^*(H^1) \otimes CF^*(H^2)$$

is the sum over all rigid elements:

$$\lambda = \sum_{\substack{deg(x_0)=deg(x_1)+deg(x_2)-2n+3 \\ u \in \mathcal{M}_{\Theta,\omega}(x_0;x_1,x_2)}} \lambda_u.$$

Proposition 20. *The coproduct satisfies:*

$$\lambda \circ d - d \circ \lambda = 0 \tag{3.6}$$

where by convention $d(a_1 \otimes a_2) = da_1 \otimes a_2 + (-1)^{deg(a_1)} a_1 \otimes da_2$

Proof. Consider the compactification of the moduli space in dimension 1. By Lemma (18), the boundary consists of the following strata:

1. $\mathcal{M}(x_0, x_0') \times \mathcal{M}_{\Theta,\omega}(x_0'; x_1, x_2)$
2. $\mathcal{M}_{\Theta,\omega}(x_0; x_1', x_2) \times \mathcal{M}(x_1', x_1)$
3. $\mathcal{M}_{\Theta,\omega}(x_0; x_1, x_2') \times \mathcal{M}(x_2', x_2)$

We interpret the first contribution as $\lambda \circ d$, while the last two contribute to $d \circ \lambda$. To show (3.6), we just need to confirm the signs. We consider the signs of the terms in case (3). Let $u_{\Theta,\omega} \# v$

be a broken curve lying in the boundary of $\mathcal{M}_{\Theta,\omega}(x_0; x_1, x_2)$. Floer's gluing map parameterizes a neighborhood of this curve by a gluing parameter $\rho \gg 0$. Let $D_{u_{\Theta,\omega}}$ and D_v denote the linearization of the Floer equation at $u_{\Theta,\omega}$ resp. v. We check the sign of the glued operator $D_{u_{\Theta,\omega}} \# D_v$ with the signs of a nearby curve. Given an element $u_{\Theta,\omega}$ of $\mathcal{M}_{\Theta,\omega}(x_0; x_1, x_2')$, $det(D_{u_{\Theta,\omega}})$ is oriented to that there is an isomorphism:

$$det(D_{u_{\Theta,\omega}}) \otimes det(D^-_{\Psi_{x_0}}) \to \det(D^-_{\Psi_{x_1}}) \otimes det(D^-_{\Psi_{x_2'}}) \tag{3.7}$$

or equivalently,

$$det(D^+_{\Psi_{x_1}}) \otimes det(D_{u_{\Theta,\omega}}) \otimes det(D^-_{\Psi_{x_0}}) \to \det(D^-_{\Psi_{x_2'}}) \tag{3.8}$$

The differential gives a map:

$$det(D_v) \otimes det(D^-_{\Psi_{x_2'}}) \to det(D^-_{\Psi_{x_2}}). \tag{3.9}$$

We substitute (3.8) in (3.9) to obtain:

$$det(D_v) \otimes det(D^+_{\Psi_{x_1}}) \otimes det(D_{u_{\Theta,\omega}}) \otimes det(D^-_{\Psi_{x_0}}) \to det(D^-_{\Psi_{x_2}}). \tag{3.10}$$

We transpose $det(D_v) \otimes det(D^+_{\Psi_{x_1}})$ which introduces a sign change of $deg(x_1)$, and obtain:

$$det(D_v) \otimes det(D_{u_{\Theta,\omega}}) \otimes det(D^-_{\Psi_{x_0}}) \to det(D^-_{\Psi_{x_1}}) \otimes det(D^-_{\Psi_{x_2}}). \tag{3.11}$$

We can now compare this to an element $w = u_{\Theta,\omega} \#_\rho v$. The orientation on D_w is given by

$$det(D_w) \otimes det(D^-_{\Psi_{x_0}}) \to det(D^-_{\Psi_{x_1}}) \otimes det(D^-_{\Psi_{x_2}}) \tag{3.12}$$

The curve $u_{\Theta,\omega}$ is rigid so $det(D_{u_{\Theta,\omega}})$ is canonically trivial, but $det(D_v)$ is spanned by the translation vector ∂_s. The gluing procedure sends ∂_s to $\partial_\rho(u_{\Theta,\omega} \#_\rho v)$. This gives rise to a tangent vector which points outwards in $\mathcal{M}_{\Theta,\omega}$ and hence shows up with a negative sign. A similar analysis for the other two elements proves the claim. □

Proposition 21. *The coproduct*

$$\lambda : CF(H_t^0) \to CF^*(H_t^1) \otimes CF^*(H_t^2)$$

does not depend on the choice of Floer data.

Proof. This is a cobordism argument using the fact that the space of coproduct data is connected. Indeed, the space of almost complex structures is contractible so we can interpolate between any two choices of families. In the complement of the infinite ends the condition $d(H_{z,\omega}\beta) \leqslant 0$ is local in z and convex in both β and $H_{z,\omega}$. Over the negative ends, we can again simply interpolate between any two choices of coproduct data by a straight line homotopy. □

Given regular data as in (8) it follows from the two propositions above that the coproduct descends to a well defined operation:

$$\lambda : HF^*(H^0) \to HF^*(H^1) \otimes HF^*(H^2).$$

Remark 22. *The data defining the moduli spaces $\mathcal{M}_{\Theta,\omega}$ is parameterized by $\Sigma \times S^1 \times S^1 \times (0,1)$. We can choose our data so that in the region where the slope of the Hamiltonians is very small the Hamiltonians are independent of (s,t). By [5, Lemma 3.2], in this situation one of the negative ends maps to a constant orbit, i.e. a solution will lie in the interior of W where H is a fixed Morse function. So we can in fact choose the data so that near each boundary of the interval, and in the corresponding cylindrical end where the slope is small, the Hamiltonians and almost complex structures are independent of t and hence of θ. This corresponds to collapsing the tori $S^1 \times S^1 \times 0$ and $S^1 \times S^1 \times 0$ to $\star \times S^1 \times 0$ and $S^1 \times \star \times 1$, respectively. We thus obtain a moduli space parameterized by $\Sigma \times S^3$.*

The coproduct on symplectic cohomology

In order to show that the coprodcut descends to an operation

$$SH^*(W) \to SH^*(W) \otimes SH^*(W),$$

we need to investigate its interaction with continuation maps.

To show that the diagram

$$\begin{array}{ccc} HF^*(H_0) & \xrightarrow{c} & HF^*(K_0) \\ & \searrow^{\lambda} & \downarrow \lambda \\ & & HF^*(K_1) \otimes HF^*(K_2) \end{array} \tag{3.13}$$

commutes we choose a family of coproduct data $H^u_{z,\Theta,\omega}$, $J^u_{z,\Theta,\omega}$, parameterized by $u \in [0, \infty)$ such that:

- at $u = 0$, $H^u_{z,\Theta,\omega}$ is a choice of coproduct data

$$\lambda : HF^*(H^0) \to HF^*(K_1) \otimes HF^*(K_2).$$

- For each $u > 0$, there exist S_u tending to ∞ with u, such that in the positive cylindrical end

$$H^u_{z,\Theta,\omega} = \begin{cases} H_0 \, for \ s \geqslant S_u \\ K_0 \, for \ s \leqslant S_u - 1 \end{cases}$$

We consider the moduli space $\mathcal{M}^u_{\Theta,\omega}(x_0; x_1, x_2)$ of finite energy solutions of

$$(du - X_{H^u_{(z,\Theta.\omega)}} \otimes \beta)^{(0,1)} = 0 \tag{3.14}$$

which is a smooth manifold with boundary. In dimension 1, the boundary strata consist of elements in $\mathcal{M}_{\Theta,\omega}(x_0; x_2, x_3)$ representing the operation

$$\lambda : HF^*(H^0) \to HF^*(K_1) \otimes HF^*(K_2),$$

as well as broken curves (u, v) in the fiber product of $\mathcal{K}(x_0, \tilde{x}_0) \times \mathcal{M}_{\Theta,\omega}(\tilde{x}_0; x_2, x_3)$ representing the composition

$$HF^*(H_0) \xrightarrow{c} HF^*(K_0) \xrightarrow{\lambda} HF^*(K_1) \otimes HF^*(K_2).$$

Since the continuation and coproduct operations are independent of choices, we obtain that diagram (3.13) commutes. This shows that the co-product descends to a map:

$$SH^*(W) \to HF^*(K_1) \otimes HF^*(K_2).$$

Similarly, by concatenating the data and Hamiltonians we can obtain a commutative diagram:

$$\begin{array}{ccc} HF^*(H_0) & \xrightarrow{\lambda} & HF^*(H_1) \otimes HF(K_2) \\ \downarrow \lambda & \searrow^{\lambda} & \downarrow c\otimes 1 \\ HF^*(K_1) \otimes HF^*(H_2) & \xrightarrow{1\otimes c} & HF^*(K_1) \otimes HF^*(K_2) \end{array} \tag{3.15}$$

The bottom triangle show that the co-product descends to a map

$$HF^*(H_0) \to HF^*(K_1) \otimes SH^*(W)$$

while the top triangle shows that the coproduct descends to a map:

$$HF^*(H_0) \to SH^*(W) \otimes HF^*(K_2)$$

Together, we obtain that the coproduct commutes with continuation maps, and hence descends to $SH^*(W)$.

Graded co-commutativity

The sphere with three punctures at $0, 1, \infty$ admits a rotation which fixes 0 and switches 1 and ∞. Given Floer data parameterizing the coproduct

$$\lambda : CF^*(H_0) \to CF^*(H_1) \otimes CF^*(H_2),$$

we pull back this data under the rotation. Note that this rotation does not fix the choice of asymptotic marker at the positive puncture, but all such choices are homotopic, and since the space of coproduct data is connected, we obtain the operation

$$\lambda : CF^*(H_0) \to CF^*(H_1) \otimes CF^*(H_2).$$

In order to determine the signs, we consider (3.3). The orientation of each $T_{\theta_i}S^1$ factor is preserved but the order is flipped. On the other hand, the orientation of (0,1) is reversed. The sign contribution of these two cancel. In addition we compare the orientation on $det(D_u)$ using (3.7). Permuting $det(D_{x_1})$ and $det(D_{x_2})$ introduces a sign of $(-1)^{deg(x_1)deg(x_2)}$ together we obtain that

$$\lambda = \tau \circ \lambda$$

where $\tau(a \otimes b) = (-1)^{deg(a)deg(b)} b \otimes a$

3.5 On co-associativity and co-Jacobi identities

In this section we will consider whether the coproduct is co-associative and satisfies a co-Jacobi identity. Co-associativity in this context means that the following diagram commutes:

$$\begin{array}{ccc} SH^*(W) & \xrightarrow{\quad\lambda\quad} & SH^*(W) \otimes SH^*(W) \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle 1\times\lambda} \\ SH^*(W) \otimes SH^*(W) & \xrightarrow{\lambda\times 1} & SH^*(W) \otimes SH^*(W) \otimes SH^*(W) \end{array} . \qquad (3.16)$$

The graded co-Jacobi identity given by:

$$0 = (1 + \sigma + \sigma^2) \circ ((Id \times \lambda) \circ \lambda),$$

where σ is the permutation which carries the labels $(i, i+1, i+2)$ to $(i+1, i+2, i)$.

We will define an operation, $\aleph$, whose non vanishing will be an obstruction for co-associativity and co-Jacobi to hold. The arguments follow the ideas in [14, §2.5].

To define $\aleph$, we consider the open 2-simplex, consisting of $(\omega_1, \omega_2, \omega_3)$, $\sum \omega_i = 1$, and denoted by Δ. We fix a Riemann surface S with four punctures z_0, z_1, z_2, z_3, along with a choice of a positive cylindrical end at z_0, and negative cylindrical ends at z_1, z_2, z_3. We choose the following Floer data:

- An admissible Hamiltonian H^0, and admissible Hamiltonians $H^i, i \in \{1, 2, 3\}$, satisfying $H^0 \leq H^i$.

- ϵ small such that $\epsilon\tau_i$ is smaller than the period of any Reeb orbit of R_α.

- A family of Hamiltonians $H_{z,\Theta,\omega}$ parameterized by $M \times (S^1)^4 \times \Delta$ satisfying the following conditions:

  1. In a neighborhood of infinity at the positive cylindrical end:

$$H_{s+it,\Theta,\omega} = H^0_{t+\theta_0}$$

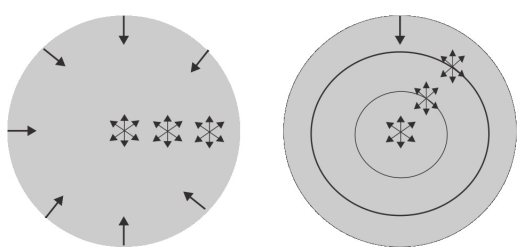

Figure 3.4: Two representations of families of Floer data defining $\aleph$

2. In a neighborhood of infinity at the ith cylindrical end, with coordinates (s,t)

$$H_{z,\Theta,\omega} = \begin{cases} \sigma_\epsilon(\omega_i)H^i_t & \textit{for } -R \geqslant s \geqslant -R-\kappa(\omega_i) \\ H^i_{t+\theta_i} & \textit{for } s \leqslant -R-\kappa(\omega_i) \end{cases}$$

- A family of almost complex structures $J_{z,\Theta,\omega} \in \mathcal{J}$ which near the positive puncture satisfy:

$$J_{z,\Theta,\omega} = J_{t+\theta_i},$$

and in a neighborhood of the ith negative puncture:

$$J_{z,\Theta,\omega} = J_{t+\theta_i,\omega}.$$

- A one form β on Σ satisfying $d(H_{z,\Theta,\omega}\beta) \leqslant 0$, and such that β restricts to dt in the neighborhood of the punctures.

Definition 23. *Given Floer data as above, the moduli space $\mathcal{M}_{\Theta,\omega}(x_0;x_1,x_2,x_3)$ consists of finite energy maps*

$$(u,\omega,\Theta) : S \to W$$

satisfying:

$$(du - X_{H_{(z,\Theta.\omega)}} \otimes \beta)^{(0,1)} = 0 \tag{3.17}$$

and

$$\lim_{s\to\pm\infty} u(\epsilon_i(s+it)) = x_i(t+\theta_i).$$

Here $x_i \in \mathcal{P}(H^i)$.

The methods of subsections 3.3 and 3.4 extend to show that these moduli spaces admit a Gromov-Floer compactification, and descend to a well defined operation:

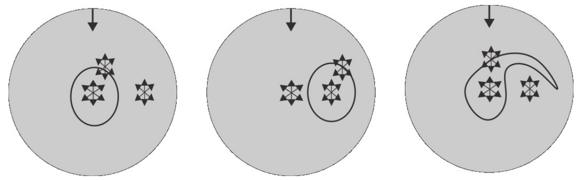

Figure 3.5: A decomposition of $\aleph$

$$\aleph : SH^*(W) \to SH^*(W) \otimes SH^*(W) \otimes SH^*(W).$$

Lemma 24. *With $\mathbb{Z}/2$ coefficients, the operation $\aleph$ satisfies:*

$$\aleph = (1 + \sigma + \sigma^2) \circ ((Id \times \lambda) \circ \lambda),$$

where σ is the permutation which carries the labels $(i, i+1, i+2)$ to $(i+1, i+2, i)$.

Sketch of proof: The operation $\aleph$ is obtained by counting solutions to the parameterized Floer equation on the complement of any 4 points on the sphere, with asymptotic markers moving at each puncture. All such choices are equivalent, so we fix one such choice with a positive puncture z_0 at ∞, and 3 negative punctures (z_1, z_2, z_3). We represent this on the plane with the negative punctures on the positive x axis. The $(S^1)^4$ family of asymptotic markers is represented by a family of arrows. See Figure 3.4, left.

By rotating the plane, we get an equivalent family of Riemann surfaces, where the asymptotic marker at ∞ is constant, and z_2, z_3 each rotates once around the origin, see Figure 3.4, right. This family decomposes as the sum of 3 operations, as in Figure 3.5. (Technically there are more terms, corresponding to an $S^1 \times S^1$ choice of asymptotic markers at a given puncture, but those yield operations which are trivial on $SH^*(W)$).

Note that the first operation on the left of Figure 3.5 is further equivalent to the operation in Figure 3.6, which we will relate to $(\lambda \times Id) \circ \lambda$. The main difference is that the data for the composition $(\lambda \times Id) \circ \lambda$ is parameterized by the open cube

$$\{(s_1, s_2, s_3, s_4) \in I \times I, s_1 + s_2 = 1, s_3 + s_4 = 1\}$$

while $\aleph$ is parameterized by the open simplex Δ. However, there is a natural homeomorphism $I \times I \to \Delta$ given by $(s_1, s_2, s_3, s_4) \to (s_1, s_2 s_3, s_2 s_4)$. The underlying Riemann surface defining the operation $(\lambda \times Id) \circ \lambda$ lies in the boundary of the space of complex structures on the complement of 4 marked points in S^2. This space is connected, and we choose an interval connecting it to the Riemann surface defining the operation in 3.6, while simultaneously interpolating between the Floer data, as is done in [5, §2.4] (for weighted 1 forms). A cobordism argument now implies that the two operations are chain homotopic. Applying the same argument to the remaining terms in 3.5, and counting with $\mathbb{Z}/2$ coefficients, gives the desired equation:

$$\aleph = (1 + \sigma + \sigma^2) \circ ((Id \times \lambda) \circ \lambda). \tag{3.18}$$

□

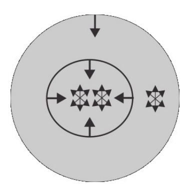

Figure 3.6: A further simplification

Remark 25. *For complements of smooth divisors satisfying the monotonicity conditions of §4, it follows from Lemma 61 that all the operations on the right in (3.18) are in fact trivial, so the coproduct is co-associative for trivial reasons.*

Chapter 4: Set up for Morse-Bott cohomologies

We now recall from [13] how to compute symplectic cohomology in certain cases. The computation involves the construction of Morse-Bott and Morse-Bott split symplectic cohomologies. The class of symplectic manifolds considered are Liouville domains with a contact boundary which admits a Reeb vector field generating an S^1 action. For these contact manifolds, the Reeb orbits are easy to identify, and come in Morse-Bott families.

In order to define the Morse-Bott symplectic complex one considers time independent Hamiltonians which are 0 in the interior of W, and in the cylindrical end only depend on the cylindrical coordinate, see Figure 4.1. Such Hamiltonians pick up the Morse-Bott non-degenerate families of Reeb orbits, as well as a degenerate family of constant orbits in the interior of W. This degenerate critical manifold is further perturbed by a C_2 small Morse function. Roughly speaking, the Morse-Bott chain complex is generated by the critical points of this Morse function as well as the Morse-Bott manifolds of non constant orbits. The differential counts moduli spaces of Morse-Bott cascades, see Figure 5.1 .

The generators of the Morse-Bott split chain complex are the same as the ones in the non-split case. However, the moduli spaces defining the differential are augmented Morse-Bott cascades, see Figure 6.1. These are obtained from the Morse-Bott cascades by a neck stretching type procedure.

The curves arising in the differential of the split Morse-Bott chain complex can then related to relative Gromov-Witten invariants and are more amenable to explicit computations.

We now precisely define the classes of Liouville manifolds, almost complex structures, and Hamiltonians for which invariants in symplectic cohomology can be computed via Morse-Bott split symplectic cohomology.

Liouville domains

We consider Liouville domains $(\overline{W}, d\alpha)$ such that α restricted to $Y = \partial\overline{W}$ has a Reeb vector field generating a free S^1 action. Such Liouville domains arise as complements of smooth divisors. More precisely, let D be the quotient of $\partial\overline{W}$ by the S^1 action, and Ω_D be the symplectic form induced from $d\alpha$. Then by [13, Lemma 2.2] there exists a compact symplectic manifold (X, Ω_X), with $D \hookrightarrow X$, such that $\Omega_X|_D = \Omega_D$, $(X \setminus D, \Omega_X)$ is symplectomorphic to $(\bar{W} \setminus \partial\bar{W}, d\alpha)$ and $[D] \in H_{2n-2}(X; \mathbb{Q})$ is Poincaré dual to $[K\Omega_X] \in H^2(X; \mathbb{Q})$ for some $K > 0$.

We will restrict to the case where (X, D) is a monotone pair. This implies that there exists a constant $\tau_X > K$ such that for each spherical homology class A we have $\Omega_X(A) = \tau_X \langle c_1(TX), A\rangle$.

Almost complex structures and neck stretching:

Let ND denote the symplectic normal bundle to D in X. Given a hermitian connection on ND, we can lift an almost complex structure J_D on D to an almost complex structure on ND denoted by a *bundle almost complex structure*. By [13, Lemma 2.2] there exist a neighbourhood $\mathcal{U}$ of the 0-section in ND and a symplectic embedding $\varphi \colon \mathcal{U} \to X$.

Definition 26 (admissible complex structures)**.**

- *An almost complex structure J_Y on the symplectization $\mathbb{R} \times Y$ is admissible if J_Y is cylindrical and Reeb-invariant and is compatible with the symplectic form $de^t\alpha$.*

- *An Ω-compatible almost complex structure J_W on $(W, d\alpha)$ is admissible if J_W is cylindrical and Reeb-invariant on $W \backslash \overline{W}$, and is compatible with the symplectic form $de^t\alpha$.*

- *An admissible almost complex structure J_X on X is an almost complex structure which is compatible with Ω and its restriction to $\phi(\mathcal{U})$ is a lift (using the hermitian connection) of an almost complex structure J_D.*

By [13, Lemma 2.5, 2.6] there exists a diffeomorphism $\psi : W \to X \setminus D$ which relates admissible complex structures on X, W, and D as follows: the pushforward $\psi_* J_W$ extends to an admissible almost complex structure on X and conversely $\psi^* J_X$ is an admissible almost complex structure on W. Moreover, if J_X is an extension of $\psi_* J_W$, then $J_X|_D$ is given by restricting J_W to a copy $c \times Y$ and taking the quotient by the Reeb action.

For the purpose of relating the split and non split Morse-Bott moduli spaces, [13, Lemma 2.7] constructs a family of almost complex structures J_κ on W, which are cylindrical outside of a compact set, and such that (W, J_κ) converges as $\kappa \to \infty$ to a split manifold whose upper level is $(\mathbb{R} \times Y, J_Y)$ and lower level is (W, J_W).

Remark 27. *For the purpose of defining the split Morse-Bott coproduct curves, we will also consider almost complex structures J_Y on $\mathbb{R} \times Y$ which are admissible outside of a compact set K, and in a compact set commute with the projection map: $\pi_D(J_Y \xi) = J_D \pi_D(\xi)$ for an almost complex structure J_D on D and any vector ξ.*

Admissible Hamiltonians:

Instead of using autonomous Hamiltonians with quadratic growth as in [13], we will use autonomous Hamiltonians with linear growth. In order to import the computations from [17], we will construct a cofinal system of Hamiltonians which approximate a quadratic, see Figure 4.1. More specifically, let $h_\tau \colon (0, +\infty) \to \mathbb{R}$ be a smooth function satisfying:

1. $h_\tau(\rho) = 0$ for $\rho < 2$;
2. $h'_\tau(\rho) > 0$ for $\rho > 2$;
3. $h''_\tau(\rho) > 0$ for $2 < \rho < \tau + 2$;
4. $h'_\tau(\rho) = \tau$ for $\rho > \tau + 2$;
5. for $\tau_0 < \tau_1$, $h_{\tau_1} = h_{\tau_0}$ for $\rho < \tau_0 + 2$

An *admissible Hamiltonian* $H_\tau \colon \mathbb{R} \times Y \to \mathbb{R}$ is given by $H_\tau(r, y) = h_\tau(\mathrm{e}^r)$ for $\tau \notin \mathcal{S}(\lambda)$. We also allow for positive scalings of H_τ.

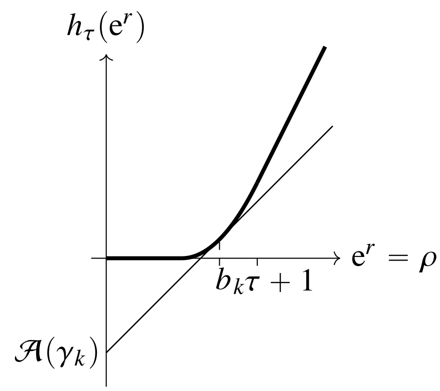

Figure 4.1: Admissible Hamiltonians, and the action of periodic orbits

Since the Hamiltonian $H_\tau(r, y) = 0$ for all $r \leqslant \ln 2$, we can extend it by 0 in the interior of W. We denote the limits as $\tau \to \infty$ by h_∞ and H_∞, respectively. Note that H_∞ is a Hamiltonian with quadratic growth.

The Hamiltonian vector field X_{H_∞} is given by $h'_\infty(e^r)\mathcal{R}$ where $\mathcal{R}$ is the Reeb vector field generated by α. Hence the periodic orbits of H_∞ come in two flavors:

1. Morse-Bott critical submanifolds of periodic orbits: for each $k \in \mathbb{Z}_+$ we obtain a Y family of non constant periodic orbits Y_k contained in the sublevel $b_k \times Y$, where b_k satisfies $h'_\infty(e^{b_k}) = k$.

2. Constant orbits:

$$W_0 = \{w \in W | dH(w) = 0\}.$$

The orbits are Morse-Bott non-degenerate, except at the boundary of the support of dH. We denoted by ∂W_0 the degenerate orbits, and by $\dot{W}_0$ the interior of W_0 .

Auxiliary data

In order to define the Morse-Bott chain complexes we will need to perturb the Morse-Bott manifolds of orbits. In order to do so, we fix a Morse function $f_D : D \to \mathbb{R}$ and a gradient like vector field Z_D. The flow of Z_D is denoted by Φ_t. Given a critical point p, its stable and unstable manifolds, with respect to the negative gradient, are defined by:

$$W^s(p) = \{q \in D | \lim_{t\to\infty} \Phi_{-t}(q) = p\}, W^u(p) = \{q \in D | \lim_{t\to\infty} \Phi_t(q) = p\}.$$

The moduli spaces defining the various operations will be constructed as fiber products over evaluation maps. In order to ensure transversality, we will consider suitable perturbations of the function f_D. For f_D a Morse function, we consider the space of perturbations $C^\infty_\epsilon(f_D)$ consisting of functions $f : D \to \mathbb{R}$ satisfying $||f||_\epsilon < \infty$ and $f(x) = 0$ in a neighborhood of the critical points of f_D, as in [18, Ch. 8]. For ϵ sufficiently small, the critical points of $f_D + f$ coincide.

The contact distribution ξ defines a connection on the S^1 bundle $S^1 \to Y \to D$, and we can lift the vector field Z_D to a horizontal vector field $\pi_* Z_D$ on Y. We consider a Morse function $f_Y : Y \to \mathbb{R}$ and a gradient like vector field Z_Y, so that the flowlines of Z_Y project to flowlines of Z_D. To construct such functions f_Y we perturb $\pi^* f_D$ in small neighborhoods of the critical S^1 manifolds. It follows that the critical points of f_Y lie above the critical points of f_D, and that the gradient flow of f_Y preserves the fiber. We assume for simplicity that f_Y has only two critical points for each critical point of f_D. We denote the space of all choices f_Y by $\mathcal{X}(f_D)$. For a fixed f_Y, we denote the space of perturbations of f_Y, projecting to a perturbation of f_D, by $C^\infty_\epsilon(f_Y, f_D)$.

For each critical point p of f_D we denote the two critical points above p by $\hat{p}$ and $\check{p}$, where the fiberwise Morse index of $\hat{p}$ is 1 and of $\check{p}$ is 0.

We also fix a Morse function f_W and a gradient like vector field Z_W such that Z_W restricted to $Y \times (-\epsilon, \infty)$ is the constant vector field ∂r.

Chapter 5: Morse-Bott theory

5.1 Morse-Bott symplectic cohomology

We first recall the main constructions and ideas underlying the Morse-Bott symplectic cohomology. For thorough details, see [19].

The chain complex:

The Morse-Bott chain complex associated to H_∞ is given by:

$$SC_*(W, H_\infty) = \left(\bigoplus_{k>0} \bigoplus_{p \in \mathrm{Crit}(f_D)} \mathbb{Z}\langle \check{p}_k, \hat{p}_k \rangle \right) \oplus \left(\bigoplus_{x \in \mathrm{Crit}(f_W)} \mathbb{Z}\langle x \rangle \right). \tag{5.1}$$

Grading:

To be consistent with the grading conventions introduced above we will differ from the conventions introduced by [19]. Our grading is n minus the grading in [19]. Please see [§3.1 13] for an explanation of the following formulas.

Let $M(p)$ denote the Morse index of a critical point p of f_D and $\tilde{M}(\tilde{p})$ the Morse index of a lift to a critical point of f_Y, where $\tilde{p}$ is $\check{p}$ or $\hat{p}$.

Recall the constant K such that $[D] \in H_{2n-2}(X; \mathbb{Q})$ is Poincaré dual to $[K\Omega_X] \in H^2(X; \mathbb{Q})$ for some $K > 0$, and the constant $\tau_X > K$ such that for each spherical homology class A we have $\Omega_X(A) = \tau_X \langle c_1(TX), A \rangle$.

For a critical point $\tilde{p}$ of f_Y of multiplicity k we define:

$$|\tilde{p}| = 2n - M(\tilde{p}) - 1 - 2\frac{\tau_X - K}{K}k \tag{5.2}$$

where τ_X is the monotonicity constant of X.

For x a critical point of f_W we define:

$$|x| = -M(x)$$

We also have gradings for Reeb orbits for which Floer curves converge at augmentation punctures. If γ is a k fold cover of a the fiber $S^1 \to Y \to D$ then its index is defined as:

$$|\gamma| = -2 + 2\frac{\tau_X - K}{K}k \tag{5.3}$$

The differential:

We now recall from [13] the construction of the Morse-Bott differential which counts Morse-Bott cascades. Recall that we have Morse-Bott non-degenerate orbits as well as degenerate ones.

Definition 28. *Let J_W be an admissible almost complex structure and H_τ an admissible Hamiltonian. Let S_-, S_+ denote Morse-Bott families of orbits. We require that $S_+ \neq S_-$ unless $S_+ = S_- = \dot{W}_0$. The moduli space of* ***Morse-Bott Floer cylinders***

$$\widehat{\mathcal{M}}(S_-, S_+, H_\tau)$$

consists of finite energy non trivial maps $v : \mathbb{R} \times S^1 \to W$ satisfying:

$$\partial_s v + J_W(\partial_t v - X_{H_\tau}) = 0 \tag{5.4}$$

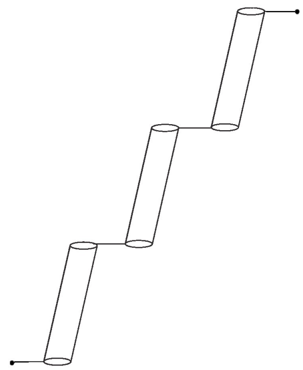

Figure 5.1: A Morse-Bott differential cascade

and

$$\lim_{s\to\infty} v(s,t) \in S_+$$
$$\lim_{s\to-\infty} v(s,t) \in S_-$$

There are two evaluation maps:

$$ev_\pm : \widehat{\mathcal{M}}(S_-, S_+, H_\tau) \to S_\pm \tag{5.5}$$

given by $v \to v(\pm\infty, 0)$. We will refer to both maps as ev to ease notation.

In the following M is one of the Morse-Bott manifolds of orbits Y or $\dot{W}_0$. Let Δ denote the diagonal $M \times M$. Given a Morse function f, we also consider the flow diagonal Δ_f in $M \times M$ defined as:

$$\Delta_f = \{(m_1, m_2) \in (M \backslash crit(f))^2 | \exists t > 0, \phi_t(m_1) = m_2\}$$

(In the notation Δ_f, f stands for flow).

Definition 29. *Let q_-, q_+ be generators of the chain complex. The moduli space of* ***Morse-Bott***

cylinders with zero cascades

$$\widehat{\mathcal{M}}_0(q_+, q_-, H_\tau)$$

consists of a Morse flowline of f_Y or f_W asymptotic to q_+ at $+\infty$ and to q_- at $-\infty$.

The moduli space of ***Morse-Bott cylinders with N cascades***

$$\widehat{\mathcal{M}}_N(q_-, q_+, H_\tau)$$

is the fiber product over the evaluation maps:

$$W^s(q_-) \underset{\Delta}{\times} \widehat{\mathcal{M}}(S_-, S_{N-2}, H_\tau) \underset{\Delta_f}{\times} \dots \underset{\Delta_f}{\times} \widehat{\mathcal{M}}(S_1, S_+, H_\tau) \underset{\Delta}{\times} W^u(q_+).$$

We also have cascades which are missing their initial or final flowline, or both, and will be used in the definition of the coproduct:

$\widehat{\mathcal{M}}_N(q_-, S_1), H_\tau)$ *is the fiber product*

$$W^s(q_-) \underset{\Delta}{\times} \widehat{\mathcal{M}}(S_-, S_{N-2}, H_\tau) \underset{\Delta_f}{\times} \dots \underset{\Delta_f}{\times} \widehat{\mathcal{M}}(S_2, S_1, H_\tau).$$

$\widehat{\mathcal{M}}_N(S_{N-1}, q_+ H_\tau)$ *is the fiber product*

$$\widehat{\mathcal{M}}(S_{N-1}, S_{N-2}, H_\tau) \underset{\Delta_f}{\times} \dots \underset{\Delta_f}{\times} \widehat{\mathcal{M}}(S_2, S_+, H_\tau) \underset{\Delta}{\times} W^u(q_+).$$

and $\widehat{\mathcal{M}}_N(S_{N-1}, S_0, H_\tau)$ *is the fiber product*

$$\widehat{\mathcal{M}}(S_{N-1}, S_{N-2}, H_\tau) \underset{\Delta_f}{\times} \dots \underset{\Delta_f}{\times} \widehat{\mathcal{M}}(S_1, S_0, H_\tau).$$

Remark 30. *In the above definition the choice of Morse function f_Y is fixed, and the fiber product is taken over evaluation maps to the diagonal and flow diagonal of f_Y. In [13], transversality is proved for the configurations arising in the Morse-Bott differential cascades for a fixed such choice. In the definition of the coproduct we will instead consider a choice of perturbation f_Y^i of*

f_Y for each ith sublevel. In this case the ith fiber product is taken over the diagonals and flow diagonals of f_Y^i.

Moduli spaces of 0 cascades correspond to Morse flow lines and admit an $\mathbb{R}$ action. The moduli spaces of N cascades admits an action of $\mathbb{R}^N$ by domain translations. We denote the quotient by these actions by $\mathcal{M}_0(q_-, q_+, H_\tau), \mathcal{M}_N(q_-, q_+, H_\tau)$ respectively.

The following Lemma follows from the transversality techniques in [19]. More precisely, transversality for evaluation maps and curves which are contained in the symplectization is proved in [19, §5]. Transversality for curves which aren't contained in the symplectization is obtained by perturbing J_W in the interior.

Lemma 31. *For a generic choice of an admissible almost complex structure the moduli spaces $\mathcal{M}_N(q_-, q_+, H_\tau)$ are smooth of dimension $|q_+| - |q_-| - 1$.*

We now recall the convergence properties of the Morse-Bott moduli spaces in the presence of degenerate orbits. For Morse-Bott non-degenerate orbits, the maximum principle and the asymptotic behavior at the cylindrical ends, gives that finite energy curves will converge exponentially to Hamiltonian orbits at the positive and negative ends, and the limit is unique.

However, this is not the case for degenerate orbits: given any sequence $s_k \to \pm\infty$, there exists a subsequence, also denoted by s_k, and a Hamiltonian orbit $\gamma(t)$ such that $v(\epsilon(s_k, t) \to \gamma(t)$. This limit may depend on the initial sequence, and any two limits are connected by a family of periodic orbits of the same action, see [19, Lemma 4.3] which is stated without proof. Hence degenerate orbits could pose a problem.

In [19] it is implicitly assumed that the moduli spaces of Morse-Bott cascades are compact. Assuming such compactness, we now recall from [13, §4] some properties of our setup which rule out degenerate orbits as asymptotic orbits of sublevels.

In the following let $v : S \to W$ be a finite energy solution of

$$(du - X_{H_\Psi} \otimes \beta)^{(0,1)} = 0 \tag{5.6}$$

where S is a Riemann surface with one positive cylindrical puncture, and m negative punctures, Ψ is any parameterizing family of Hamiltonians, and β satisfies $d(H_\Psi \beta) \leqslant 0$. This description includes the moduli spaces defining the differential, as well as moduli spaces defining continuation maps, coproduct curves, and the BV operator defined below.

Using energy considerations, the following lemma shows that there are no non-trivial finite energy curves satisfying (5.6) which are asymptotic, in a positive cylindrical end, to a degenerate constant orbit. We denote a positive parameterization of a cylindrical end by $\epsilon_+(s,t) : [0,\infty] \times S^1 \to S$.

Lemma 32. *[13, Lemma 4.6] Let v be a finite energy solution of (5.6). Suppose that for $s_k \to \infty$, $v(\epsilon_+(s_k,t)) \to x \in W_0$. Then v is constant.*

The following [13, Lemma 4.8] uses the properties of the flow and shows that a cascade does not contain any levels which are asymptotic to degenerate orbits in a negative end. This Lemma applies to the Floer cascades defining the differential, as well as BV cascades and coproduct cascades.

Lemma 33. *Floer cascades do not contain sublevels v_i with $\lim_{s\to-\infty} v_i(\epsilon_-(s,t)) = x_i \in \partial W_0$.*

Given q_-, q_+, with $|q_+| = 1 + |q_-|$, generators of the chain complex for H_τ, the differential

$$\partial : CF^*(H_\tau, J_W) \to CF^*(H_\tau, J_W)$$

is defined as the $\mathbb{Z}/2$ count of 0-dimensional cascades:

$$\partial q_+ = \sum_N \#\mathcal{M}_N(q_-, q_+, H_\tau) q_-. \tag{5.7}$$

Continuation cascades:

Given two admissible Hamiltonians $H_1 \leq H_0$, to define the moduli spaces corresponding to the continuation map we choose a family of admissible Hamiltonians parameterized by $\mathbb{R}$ such that

$$H_s = \begin{cases} H_1 & \text{for } s \gg 0 \\ H_0 & \text{for } s \ll 0. \end{cases} \tag{5.8}$$

Definition 34. *Let S_{H_i} denote the spaces of connected manifolds of orbits for H_i. The moduli space of* ***Morse-Bott continuation cylinders***

$$\mathcal{K}(S_1, S_0, H_1, H_0)$$

consists of finite energy maps satisfying:

$$\partial_s v + J_Y(\partial_t v - X_{H_s}) = 0; \tag{5.9}$$

such that $\lim_{s\to\infty} v(s,t) \in S_{H_0}$*;* $\lim_{s\to-\infty} v(s,t) \in S_{H_1}$.

Definition 35. *Let $q_1 \in S_1, q_0 \in S_0$ be generators of the chain complex for H_i, the moduli space of* ***continuation cylinders with 1 cascade*** *$\mathcal{K}_1(q_1, q_0, H_1, H_0)$ is the fiber product:*

$$W^s_Y(q_1) \underset{\Delta}{\times} \mathcal{K}(S_1, S_0, H_1, H_0) \underset{\Delta}{\times} W^s_Y(q_0)$$

The moduli space of ***continuation cylinders with N cascades***

$$\mathcal{K}_N(q_+, q_-, H)$$

is the fiber product:

$$W^s_Y(q_1) \underset{\Delta}{\times} \mathcal{K}(S_{N_1}, S_{N_2}, H_1, H_0) \underset{\Delta_f}{\times} \mathcal{M}_{N_0}(S_{N_1}, q_0, H_0)$$

with $N_1 + N_0 = N - 1$

Continuation cylinders are already considered in [13] where it is shown [13, Lemmas 5.3, and 5.4] that the $\mathbb{Z}/2$ count of 0 dimensional cascades defines an operation:

$$\mathcal{K} : HF^*(H_1, J_W) \to HF^*(H_0, J_W)$$

As above, $SH^*(W)$ is defined as the colimit over all admissible Hamiltonians H_τ :

$$SH^*(W) = \varinjlim_{H \in \mathcal{H}_\tau} HF^*(W, H_\tau)$$

The fact that this definition of $SH^*(W)$ using linear Hamiltonians agrees with the one given using quadratic Hamiltonians given in [13] follows from [3, Lemma 18.1]. The main point is that we are using a cofinal system of Hamiltonians which approximate a quadratic, and that the curves defining the differential lie in the region where the Hamiltonians are in fact quadratic.

5.2 The BV operator

It will be constructive to consider the moduli spaces corresponding to the Morse-Bott BV operator, $\mathbf{\Pi}$.

Definition 36. *Let S_+, S_- denote Morse-Bott families of orbits. The moduli space of* ***Morse-Bott BV cylinders***

$$\widehat{\mathcal{M}}_\theta(S_-, S_+, H_\tau)$$

consists of pairs (θ, v) where $v : \mathbb{R} \times S^1 \to W$ is a finite energy non constant map satisfying:

$$\partial_s v + J_W(\partial_t v - X_{H_\tau}) = 0$$

and

$$\lim_{s\to\infty} v(s,t) \in S_+$$
$$\lim_{s\to-\infty} v(s,t+\theta) \in S_-$$

Note that $\widehat{\mathcal{M}}_\theta(S_-, S_+, H_\tau)$ admit an $\mathbb{R}$ action which is free unless all maps are constant.

Definition 37. *Let q_-, q_+ be generators of the chain complex. The moduli space of Morse-Bott* ***BV cylinders with 1 cascade***

$$\mathcal{M}_{\theta,1}(q_-, q_+, H_\tau)$$

is the fiber product:

$$W^s(q_-) \underset{\Delta}{\times} \mathcal{M}_\theta(S_-, S_+, H_\tau) \underset{\Delta}{\times} W^u(q_+)$$

The moduli space of Morse-Bott ***BV cylinders with N cascades****, $N \geqslant 2$*

$$\widehat{\mathcal{M}}_{\theta,N}(q_-, q_+, H_\tau)$$

is the fiber product:

$$W^s_Y(q_-) \underset{\Delta}{\times} \mathcal{M}_\theta(S_-, S_1, H_\tau) \underset{\Delta_f}{\times} \mathcal{M}_{N-1}(S_1, q_+, H_\tau)$$

Assuming the Gromov-Floer compactification of the Morse-Bott cascades moduli spaces, Lemmas (32) and (33) above apply here to rule out cascades with levels which are asymptotic to degenerate constant orbits.

We prove transversality for the split **Π** moduli spaces below. The proof extends to Morse-Bott cascades that are contained in the symplectization. For curves and cascades which are not contained in the symplectization, the proof is easier (by perturbing J in the interior of W). Combining the two, we obtained transversality for the Morse-Bott moduli spaces.

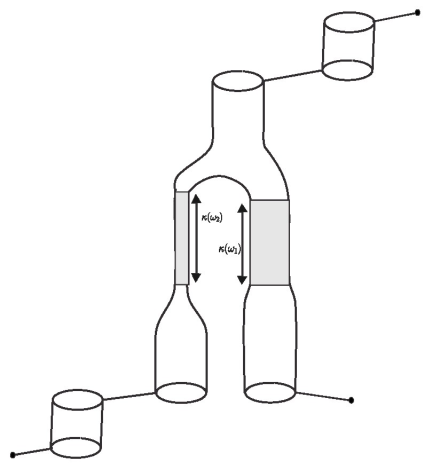

Figure 5.2: A coproduct cascade

The moduli spaces $\widehat{\mathcal{M}}_{\theta,N}(q_-,q_+,H_\tau), N \geqslant 1$, admit a free $\mathbb{R}$ action by domain translations, unless $\widehat{\mathcal{M}}_\theta$ consists of constant maps. Let $\mathcal{M}_{\theta,N}(q_-,q_+,H_\tau)$ be $\widehat{\mathcal{M}}_{\theta,N}(q_-,q_+,H_\tau)$ mod the $\mathbb{R}$ action when appropriate.

Given q_-,q_+, with $|q_+|+1 = |q_-|$, generators of the chain complex for H_τ, we define the BV operator, $\mathbf{\Pi}$, (5.1) as the $\mathbb{Z}/2$ count of rigid curves:

$$\mathbf{\Pi}\, q_+ = \sum_N \#\left(\mathcal{M}_{N,\theta}(q_-,q_+;H_\tau)\right)q_-. \tag{5.10}$$

5.3 Coproduct

Moduli spaces

We describe the moduli spaces of cascades that contribute to the Morse-Bott coproduct. Recall the functions κ, σ_ϵ:

$$\kappa : (0,1] \to [0,\infty)$$
$$\lim_{r\to 0^+} \kappa(r) = \infty$$

and $\sigma_\epsilon : [0,1] \to [0,1]$ satisfying:

$$\sigma_\epsilon(s) = \begin{cases} \epsilon & for\; 0 \leqslant s \leqslant \frac{1}{4} \\ s & for\; \frac{1}{3} \leqslant s \leqslant \frac{2}{3} \\ 1 & for\; \frac{3}{4} \leqslant s \leqslant 1 \end{cases}$$

and such that σ_ϵ is strictly increasing on $[\frac{1}{4}, \frac{1}{3}]$ and $[\frac{2}{3}, \frac{3}{4}]$. As above, $\omega = (\omega_1, \omega_2)$, satisfy $\omega_1 + \omega_2 = 1, \omega_i \in (0,1)$, and $\Theta = (\theta_1, \theta_2) \in S^1 \times S^1$.

Let H_τ be an admissible Hamiltonian of slope τ, and J an admissible almost complex structure. Let $\epsilon \ll 1$, such that $\epsilon\tau$ is smaller than the period of any Reeb orbit of R_α. To define the coproduct curves we will fix:

1. A family of Hamiltonians $H_{z,\omega}$ parameterized by $\Sigma \times (0,1)$ satisfying:

    - In a neighborhood of infinity at the positive cylindrical end:

$$H_{z,\omega} = H_\tau$$

- In a neighborhood of infinity at the ith cylindrical end:

$$H_{z,\omega} = \begin{cases} \sigma_\epsilon(\omega_i)H_\tau & for\ -R \geqslant s \geqslant -R-\kappa(\omega_i) \\ H_\tau & for\ -R-\kappa(\omega_i)-1 \geqslant s > -\infty \end{cases}$$

- $H_{z,\omega}(r, y)$ only depends on the coordinate r.

2. A family of almost complex structures J_z, parameterized by $z \in \Sigma$, and admissible for $r < 0$ and $r > \tau + 2$.

3. A one form β on Σ satisfying $d(H_{z,\omega}\beta) \leqslant 0$, and such that β restricts to dt in the neighborhood of the punctures.

Recall that in the definition of the coproduct we glue the BV operation to the negative ends. This corresponds to varying the asymptotic marker in the cylindrical ends. When we consider Morse-Bott cascades, we need to allow for potential Morse-Bott breaking before gluing the BV operator. Consequently, we have three types of coproduct moduli spaces that will contribute to the Morse-Bott coproduct cascades.

Definition 38. *Let $S_i, i \in \{0, 1, 2\}$, denote Morse-Bott families of orbits of H_τ.*

- $\mathcal{M}_\omega(S_1, S_2, S_0, H_{z,\omega})$ *is the space of pairs (v, ω) where v is a finite energy map $v : \Sigma \to W$ satisfying*

$$(du - X_{H_{z,\omega}} \otimes \beta)^{(0,1)} = 0 \tag{5.11}$$

and

$$\lim_{|s|\to\infty} v(\epsilon_i(s, t) \in S_i$$

- $\mathcal{M}_{\omega,\Theta}(S_1, S_2, S_0, H_{z,\omega})$ *is the space of tuples (v, Θ, ω) where v is a finite energy map $v : \Sigma \to$*

W satisfying

$$(du - X_{H_{z,\omega}} \otimes \beta)^{(0,1)} = 0 \tag{5.12}$$

and

$$\lim_{s\to\infty} v(\epsilon_0(s,t) \in S_0$$
$$\lim_{s\to-\infty} v(\epsilon_i(s,\theta_i) \in S_i$$

- *Let* $i \in \{1,2\}$, $j = 3 - i$. $\mathcal{M}_{\omega,\theta_i}(S_1, S_2, S_0, H_{z,\omega})$ *is the space of a tuples* (θ_i, v, ω) *such that:* $v : \Sigma \to W$ *is a finite energy map satisfying*

$$(du - X_{H_{z,\omega}} \otimes \beta)^{(0,1)} = 0,$$

and

$$\lim_{s\to\infty} v(\epsilon_0(s,t)) \in S_0$$
$$\lim_{s\to-\infty} v(\epsilon_i(s,\theta_i) \in S_i$$
$$\lim_{s\to-\infty} v(\epsilon_j(s,t) \in S_j,$$

The definition of the Morse-Bott coproduct with N cascades will involve fiber products over evaluation maps. To ensure transversality to evaluation maps we will perturb the Morse functions by considering perturbations of f_Y which project to perturbations of f_D.

Definition 39. *Let* $q_i, i = 0,1,2$ *be generators of the Morse-Bott chain complex of* H_τ. *For* $j \in \{0,\ldots,N+1\}$ *we fix a choice of perturbation* $f_Y^j \in C_\infty^\epsilon(f_Y, f_D)$. *The moduli space of* ***coproducts with N cascades***

$$\mathcal{M}_{\Theta,N}(q_0; q_1, q_2)$$

consists of the fiber products:

1.

$$W_1^s(q_1) \underset{\Delta}{\times} W_2^s(q_2) \underset{\Delta}{\times} \mathcal{M}_{\omega,\Theta}(S_1, S_2, S_0, H_{z,\omega}) \underset{\Delta_f}{\times} \mathcal{M}_{N-1}(S_0, q_0, H_\tau)$$

2.

$$\mathcal{M}_{\theta_1,N_1}(q_1, S_1, H_\tau) \underset{\Delta_f}{\times} \mathcal{M}_{\theta_2,N_2}(q_2, S_2, H_\tau) \underset{\Delta_f}{\times} \mathcal{M}_{\omega}(S_1, S_2, S_0, H_{z,\omega}) \underset{\Delta_f}{\times} \mathcal{M}_{N_0}(S_0, q_0, H_\tau)$$

with $N_0 + N_1 + N_2 = N - 1$

3.

$$W_i^s(q_i) \underset{\Delta}{\times} \mathcal{M}_{\theta_j,N_j}(q_j, S_j, H_\tau) \underset{\Delta_f}{\times} \mathcal{M}_{\omega,\theta_i}(S_i, S_j, S_0, H_{z,\omega}) \underset{\Delta_f}{\times} \mathcal{M}_{N_0}(S_0, q_0, H_\tau)$$

with $N_0 + N_j = N - 1$

Where the stable/unstable manifolds of q_i *are given with respect to the perturbations* f_Y^i*, and the* jth *fiber product is taken over the diagonal/flow diagonal of* f_Y^j*.*

Remark 40. *One should also consider coproduct curves with outputs in constant orbits in the interior of W (for which the same transversality and compactness results apply). However, it is* *clear that there are no such rigid curves since the S^1 action fixes the constant orbits, and so we do not define them.*

5.3.1 Definition of the Morse-Bott coproduct

We discuss the regularity and compactness of the Morse-Bott coproduct moduli spaces.

Transversality for the split coproduct moduli spaces is proved in §7. The proof extends to Morse-Bott cascades that are contained in the symplectization. For curves and cascades which are not contained in the symplectization, the proof is easier (by perturbing J in the interior of W). Combining the two, we obtained transversality for the Morse-Bott moduli spaces.

In order to establish compactness, we need to consider the convergence properties of cascades in the presence of degenerate orbits. The results of this section are contingent upon the compactness of the moduli space introduced in [19], in particular [19, Lemma 4.3]. Once such compactness is established, Lemmas 32 and 33 apply to rule out degenerate orbits as asymptotic orbits, and breakings of cascades at degenerate orbits.

As in the Hamiltonian case in order to establish that the coproduct moduli spaces define a chain map, we also need to consider terms coming from the compactification of the interval, consisting of fiber products of moduli spaces of degenerate coproduct curves, where the slope of the Hamiltonian is small at one of the negative ends, followed by a hybrid continuation-$\mathbf{\Pi}$ cylinders. We now define the relevant moduli spaces. Starting with the degenerate coproduct curves, and following with a hybrid continuation-$\mathbf{\Pi}$ moduli spaces.

degenerate Morse-Bott coproduct curves

Let H_τ be an admissible Hamiltonian. In the following we fix $i \in \{1, 2\}$ and let $j = 3 - i$. To define the degenerate coproduct operation we fix:

1. A family of admissible Hamiltonians H_z^i parameterized by Σ satisfying:

    - In a neighborhood of infinity at the positive cylindrical end as well as the ith cylindrical end:

$$H^i_z = H_\tau$$

- In a neighborhood of infinity at the jth cylindrical end:

$$H^i_z = \epsilon H_\tau$$

2. A family of almost complex structures J_z, parameterized by $z \in \Sigma$, which are admissible for $r < 0$ and $r \geqslant \tau + 2$.

3. A one form β on Σ satisfying $d(H_z\beta) \leqslant 0$, and such that β restricts to dt in the neighborhood of the punctures.

Definition 41. *In the following we fix $i \in \{1, 2\}$ and let $j = 3 - i$. Let S_0 and S_i denote Morse-Bott families of orbits of H_τ, and let S_j denote $\dot{W}_0$.*

- *$\mathcal{M}(S_1, S_2, S_0, H^i_z)$ is the space finite energy maps $v : \Sigma \to W$ satisfying*

$$(du - X_{H^i_z} \otimes \beta)^{(0,1)} = 0 \tag{5.13}$$

  and

$$\lim_{|s| \to \infty} v(\epsilon_k(s, t)) \in S_k$$

- *$\mathcal{M}_{\theta_i}(S_1, S_2, S_0, H^i_z)$ is the space of tuples (v, θ_1) where $v : \Sigma \to W$ is a finite energy map satisfying*

$$(du - X_{H^i_z} \otimes \beta)^{(0,1)} = 0 \tag{5.14}$$

and

$$\lim_{s\to\infty} v(\epsilon_0(s,t)) \in S_0$$

$$\lim_{s\to-\infty} v(\epsilon_i(s,\theta_i) \in S_i$$

$$\lim_{s\to-\infty} v(\epsilon_j(s,t) \in S_j$$

A Morse-Bott continuation-Π operation:

To define hybrid continuation-**Π** Morse-Bott cylinders. We fix:

- $H^{\pm}$ be admissible Hamiltonians satisfying $H^- \geq H^+$
- a family of admissible Hamiltonians parameterized by $s \in \mathbb{R}$ satisfying:

$$H_s = \begin{cases} H^+ & \text{for } s \gg 0 \\ H^- & \text{for } s \ll 0 \end{cases}$$

and *s.t.* $\partial_s(H_s) \leqslant 0$.

Definition 42. *Let S_i denote the spaces of connected manifolds of orbits for H_i. We denote by $\mathcal{N}(S_1, S_0)$ the moduli space of pairs (θ, v) where v is a finite energy map satisfying:*

$$\partial_s v + J_Y(\partial_t v - X_{H_s}) = 0; \tag{5.15}$$

and

$$\lim_{s\to\infty} v(s,t) \in S_1,$$

$$\lim_{s\to-\infty} v(s,t+\theta) \in S_2$$

A degenerate coproduct operation:

Definition 43. *Fix $i \in \{1,2\}$ and let $j = 3-i$. Let q_0 and q_i be generators of the Morse-Bott chain complex of H_τ and q_j be a generator in the interior of W. Let S_0 and S_i denote Morse-Bott families of orbits of H_τ, and let S_j denote the interior of W. The moduli space of* ***degenerate Morse-Bott coproduct curves with N cascades***

$$\mathcal{M}^j_{\Theta,N}(q_0; q_1, q_2)$$

consists of the fiber products:

1.

$$W^s(q_i) \underset{\Delta}{\times} W^s(q_j) \underset{\Delta}{\times} \mathcal{N}(W, S_j) \underset{\Delta}{\times} \mathcal{M}_{\theta_i}(S_i, S_j, S_0, H^i_z) \underset{\Delta_f}{\times} \mathcal{M}_{N-2}(S_0, q_0, H_\tau)$$

2.

$$\mathcal{M}_{\theta_i,N_i}(q_i, S_i, H_\tau) \underset{\Delta_f}{\times} W^s(q_j) \underset{\Delta}{\times} \mathcal{N}(W, S_j) \underset{\Delta}{\times} \mathcal{M}(S_i, S_j, S_0, H^i_z) \underset{\Delta_f}{\times} \mathcal{M}_{N_0}(S_0, q_0, H_\tau)$$

with $N_0 + N_i = N - 2$

Assuming the convergence properties of cascades in the presence of degenerate orbits, Lemmas 32 and 33 apply to rule out degenerate orbits as asymptotic orbits. As in the Hamiltonian case, the moduli spaces $\mathcal{M}^j_{\Theta,N}(q_0; q_1, q_2)$ are empty in low dimensions, and the analogue of Lemma 18 holds.

With these assumptions, the moduli spaces of coproduct cascades yield a well defined operation

$$\lambda : CF^*(H_\tau) \to CF^*(H_\tau) \otimes CF^*(H_\tau)$$

given by the $\mathbb{Z}/2$ count of rigid elements:

$$\lambda = \sum_{deg(q_0)=deg(q_1)+deg(q_2)+2n+3} \#\mathcal{M}_{\Theta,N}(q_1,q_2,q_0)$$

Chapter 6: Morse-Bott split theory

6.1 Morse-Bott split symplectic cohomology

We recall the construction of the Morse-Bott split symplectic complex. The generators for the chain complex are given as in the Morse-Bott construction:

$$SC_*(W, H_\infty) = \left(\bigoplus_{k>0} \bigoplus_{p \in \mathrm{Crit}(f_D)} \mathbb{Z}\langle \check{p}_k, \hat{p}_k \rangle \right) \oplus \left(\bigoplus_{x \in \mathrm{Crit}(f_W)} \mathbb{Z}\langle x \rangle \right), \tag{6.1}$$

with grading conventions as described in §5.

The split Morse-Bott differential:

We recall from [13] the definition of the moduli spaces corresponding to the Morse-Bott split differential, the idea is as follows: the Morse-Bott split moduli spaces are obtained from the Morse-Bott moduli spaces by stretching the neck along a copy of Y. Consequently one should consider cascades of augmented Floer curves, with an upper level in the symplectization $\mathbb{R} \times Y$ and augmentation planes in W.

At a negative puncture, the upper level curves might be asymptotic to a Reeb orbit at negative infinity where the Hamiltonian is zero. In this region, we need to use the SFT definition for the energy of a curve, rather than the Hamiltonian definition. Consequently, we get the notion of hybrid energy which we recall now from [13]:

Definition 44. *Consider a Hamiltonian $H : \mathbb{R} \times Y \to \mathbb{R}$ so that dH has support in $[R, \infty) \times Y$, for some $R \in \mathbb{R}$.*

Let ϑ_R be the set of all non-decreasing smooth functions $\psi : \mathbb{R} \to [0, \infty)$ such that $\psi(r) = \mathrm{e}^r$ for $r \geqslant R$.

*The **hybrid energy** E_R of $v\colon \mathbb{R}\times S^1\backslash\Gamma \to \mathbb{R}\times Y$ solving Floer's equation (6.3) is given by:*

$$E_R(v) = \sup_{\psi\in\vartheta} \int_{\mathbb{R}\times S^1} v^*\left(d(\psi\alpha) - dH\wedge dt\right). \tag{6.2}$$

We recall the definition of augmented curves appearing in the definition of the split Morse-Bott differential cascades. There are two definitions reflecting the fact that a negative output can be in the symplectization $\mathbb{R}\times Y$ or in the interior W.

Definition 45. *Let S_-, S_+ denote Morse-Bott families of orbits of H_τ in $\mathbb{R}\times Y$. The moduli space of **augmented Floer curves***

$$\widehat{\mathcal{M}}^a(S_-, S_+, H_\tau)$$

consists of tuples $(v, [U_i], \Gamma)$ where

1. *$v\colon \mathbb{R}\times S^1\backslash\Gamma \to \mathbb{R}\times Y$, is a finite energy map satisfying*

$$\partial_s v + J_Y(\partial_t v - X_H(v)) = 0; \tag{6.3}$$

2. *$\Gamma = \{z_1,\dots,z_i\} \subset \mathbb{R}\times S^1$ is a (possibly empty) finite subset.*

3. *A choice of cylindrical ends $\varphi_i : (-\infty, 0]\times S^1 \to \mathbb{R}\times S^1\backslash\{z_1,\dots,z_k\}$ of neighborhoods of the z_i, and such that*

$$\lim_{s\to-\infty} v(\varphi_i(s,.)) = (-\infty, \gamma_i(.))$$

where the γ_i are periodic Reeb orbits in Y;

4. *for each Reeb orbit γ_i above, $U_i : \mathbb{C}\to W$ is asymptotic to $(+\infty, \gamma_i)$. We consider U_i up to the action of* $\mathrm{Aut}(\mathbb{C})$.

5.

$$\lim_{s \to \infty} v(s,t) \in S_+$$

$$\lim_{s \to -\infty} v(s,t) \in S_-$$

We denote by $\widehat{\mathcal{M}}^k(S_-, S_+, H_\tau)$ the space of unaugmented upper levels in the symplectization consisting of pairs (v, Γ) and satisfying the above conditions (but missing the augmentation planes).

Definition 46. *Let S_+ denote a Morse-Bott family of orbits in $\mathbb{R} \times Y$. The moduli space of* ***augmented Floer curves a plane in*** *W*

$$\widehat{\mathcal{M}}^a(W, S_+, H_\tau)$$

consists of tuples $(v_0, v_1, [U_i], \Gamma)$ where

1. *$(v_1, [U_i], \Gamma)$ satisfy conditions $1-4$ of Definition 45*
2. *$v_0 \colon \mathbb{R} \times S^1 \to W$ is J_W-holomorphic;*
3. *$\lim_{s\to+\infty} v_1(s,t) \in S_+$;*
4. *$\lim_{s\to-\infty} v_1(s,.) = (-\infty, \gamma(\cdot))$, for some Reeb orbit γ in Y;*
5. *$\lim_{s\to+\infty} v_0(s,.) = (+\infty, \gamma(\cdot))$, where γ is the same Reeb orbit;*
6. *$\lim_{s\to-\infty} v_0(s,.) \in W$*

The definition of split Floer cylinders with cascades is now:

Definition 47. *Let q_-, q_+ be generators of the chain complex. The moduli space of* ***split cylinders with zero cascades***

$$\widehat{\mathcal{M}}_0(q_+, q_-, H)$$

consists of a Morse flowline of f_Y or f_W asymptotic to q_+ at $+\infty$ and to q_- at $-\infty$.

The moduli space of ***split cylinders with N cascades***

$$\widehat{\mathcal{M}}^a_N(q_-, q_+, H_\tau)$$

consists of elements of the fiber product:

$$W^s(q_-) \underset{\Delta}{\times} \widehat{\mathcal{M}}^a(S_-, S_{N-2}, H_\tau) \underset{\Delta_f}{\times} \ldots \underset{\Delta_f}{\times} \widehat{\mathcal{M}}^a(S_1, S_+, H_\tau) \underset{\Delta}{\times} W^u(q_+)$$

Notice that if $S_- = W$, then the bottom-most level is a split Floer cylinder as in Definition (46).

$\widehat{\mathcal{M}}^a_N(q_-, q_+, H_\tau)$ admits an $\mathbb{R}^N$ action if q_- is an orbit in $\mathbb{R} \times Y$, and an $\mathbb{R}^{N+1}$ action if q_- is a constant orbit. To see this, note that a cascade connecting a non constant orbit to a constant one contains an extra cylinder in W, refer to Figure 6.1, left. We denote the quotient by the appropriate action by $\mathcal{M}^a_N(q_-, q_+, H_\tau)$.

Given q_-, q_+, with $|q_+| = 1 + |q_-|$, generators of the chain complex for H, the Morse- Bott split differential

$$\partial : CF^*(H_\tau) \to CF^*(H_\tau)$$

is defined as the $\mathbb{Z}/2$ count of 0 dimensional cascades:

$$\partial q_+ = \sum_N \# \mathcal{M}^a_N(q_-, q_+, H_\tau) q_- \tag{6.4}$$

Split Morse-Bott continuation cascades

The moduli spaces defining the split continuation cascades are defined analogously to the split differential cascades. We use a parameterizing family of Hamiltonians H_s: given two admissible Hamiltonians $H_1 \preceq H_0$, choose a family of admissible Hamiltonians parameterized by $\mathbb{R}$ such that

$$H_s = \begin{cases} H_1 & \textit{for } s \gg 0 \\ H_0 & \textit{for } s \ll 0 \end{cases} \tag{6.5}$$

and satisfying $\partial_s(H_s) \leqslant 0$. We first define continuation moduli spaces with inputs and outputs in non-constant orbits.

Definition 48. *Let S_i denote spaces of connected manifolds of orbits for H_i in $\mathbb{R} \times Y$. We denote by $\mathcal{K}^a(S_1, S_0, H_1, H_0)$ the moduli space of tuples $(v, [U_i], \Gamma)$ where*

- *$v \colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$ satisfying the continuation equation*

$$\partial_s v + J_Y(\partial_t v - X_{H_s}) = 0; \tag{6.6}$$

- *$([U_i], \Gamma)$ satisfy conditions $2 - 4$ of Definition 45*
- *$\lim_{s \to \infty} v(s,t) \in S_0$; $\lim_{s \to -\infty} v(s,t) \in S_1$.*

We now define continuation cylinders with outputs in constant orbits:

Definition 49. *Let S_1 denote a space of connected manifolds of orbits for H_1 in $\mathbb{R} \times Y$. We denote by $\mathcal{K}^a(W, S_1, H_1, H_0)$ tuples $(v_0, v_1[U_i], \Gamma)$, where*

- *$v_1 : \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$ is a finite energy map satisfying the continuation equation*

$$\partial_s v + J_Y(\partial_t v - X_{H_s}) = 0. \tag{6.7}$$

- *$([U_i], \Gamma)$ satisfy conditions $2 - 4$ of Definition 45*
- *$v_0 \colon \mathbb{R} \times S^1 \to W$ is J_W–holomorphic;*
- *$\lim_{s \to +\infty} v_1(s,t) \in S_1$;*
- *$\lim_{s \to -\infty} v_1(s,.) = (-\infty, \gamma(\cdot))$, for some Reeb orbit γ in Y;*
- *$\lim_{s \to +\infty} v_0(s,.) = (+\infty, \gamma(\cdot))$, where γ is the same Reeb orbit;*
- *$\lim_{s \to -\infty} v_0(s,t) \in W$*

Definition 50. *Let $q_1 \in S_1, q_0 \in S_0$ be generators of the chain complex for H_i, the moduli space of **split continuation cylinders with 1 cascade***

$$\mathcal{K}_1^a(q_1, q_0, H_1, H_0)$$

is the fiber product:

$$W_Y^s(q_1) \underset{\Delta}{\times} \mathcal{K}^a(S_1, S_0, H_1, H_0) \underset{\Delta}{\times} W_Y^s(q_0)$$

*The moduli space of **split continuation cylinders with N cascades***

$$\widehat{\mathcal{K}}_N^a(q_+, q_-, H_1, H_0)$$

is the fiber product:

$$\widehat{\mathcal{M}}_{N_1}(q_1, S_{N_1}, H_1) \underset{\Delta_f}{\times} \mathcal{K}^a(S_{N_1}, S_{N_2}, H_1, H_0) \underset{\Delta_f}{\times} \widehat{\mathcal{M}}_{N_0}(S_{N_1}, q_0, H_0)$$

with $N_1 + N_0 = N - 1$

The following is proved in §7:

Proposition 51. *For a generic choice of data J_W and H_s, the moduli space $\widehat{\mathcal{K}}_N^{*,a}(q_+, q_-, H_1, H_0)$ is a manifold of dimension $|q_+| - |q_-| + N - 1$.*

The moduli spaces $\widehat{\mathcal{K}}_N^{*,a}(q_+, q_-, H_1, H_0)$ admit a free $\mathbb{R}^{N-1}$ action by domain translations on the non-continuation levels and we denote the quotient by $\mathcal{K}_N^{*,a}(q_+, q_-, H_1, H_0)$.

As above, the $\mathbb{Z}/2$ count of rigid elements defines a map:

$$\mathcal{K} : HF^*(H_1) \to HF^*(H_0)$$

and $SH^*(W)$ is defined as the colimit over admissible Hamiltonians H_τ :

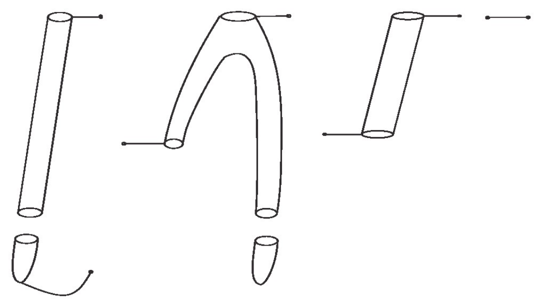

Figure 6.1: Configurations contributing to the differential in the monotone case. The line segments represent flowlines of a Morse function in the contact boundary Y or in interior of W.

$$SH^*(W) = \varinjlim_{H \in \mathcal{H}_\tau} HF^*(W, H_\tau).$$

In [19] it is proved that in the monotone case the split differential has 3 contributions. Given $p, q \in crit(f_D)$ and $x \in crit(f_W)$ the following are potential contributions to the differential (see Figure 6.1):

Theorem 52. *[19, §6] Assume that (X, Ω_X) is spherically monotone with monotonicity constant τ_X and assume that $D \subset X$ is Poincaré dual to $K\Omega_X$ with $\tau_X > K > 0$.*

Then, $SH^*(W)$ well-defined, and does not depend on choices of data.

Any Floer cascade appearing in the differential is of the following configuration:

1. *A Morse trajectory in Y or in W;*

2. *A Floer cascade in the symplectization connecting two nonconstant orbits. It has length 1 and no augmentation punctures: the cylinder in $\mathbb{R} \times Y$ projects to a non-trivial sphere in Σ, and is asymptotic to Hamiltonian orbits at $\pm\infty$.*

3. *A Floer cascade in the symplectization with one augmentation puncture connecting two nonconstant orbits. The cylinder in $\mathbb{R} \times Y$ projects to a trivial sphere in D. It is asymptotic to Hamiltonian orbits at $\pm\infty$ and to a Reeb orbit γ in $-\infty \times Y$ at the augmentation puncture.*

The augmentation plane has index zero.

4. *A Floer cascade of length* 1 *connecting a nonconstant orbit to a constant orbit. The cylinder in* $\mathbb{R} \times Y$ *is asymptotic to a Hamiltonian orbit at* $+\infty$ *and to a Reeb orbit* γ *in* $-\infty \times Y$*. It has no augmentation punctures and projects to a trivial sphere in* D*. It is followed by a holomorphic plane in* W *of index* 1 *which converges to the same orbit* γ *at* $+\infty$*.*

6.2 The Morse-Bott split BV operator

As in the Morse-Bott split differential, the moduli spaces defining the BV operator come in two flavors: the ones connecting two non constant orbits in the symplectization, and those that connect a nonconstant orbit in the symplectization, and a constant orbit in the W. We will see in Lemma (57) that the only relevant moduli spaces are in fact those of constant curves.

Definition 53. *Let* S_-, S_+ *denote Morse-Bott families of orbits of* H_τ *in* $\mathbb{R} \times Y$*. The moduli space of* ***augmented BV curves***

$$\widehat{\mathcal{M}}^a_\theta(S_-, S_+, H_\tau)$$

is the space of tuples $(v, [U_i], \Gamma, \theta)$ *where* $(v, [U_i], \Gamma)$ *satisfy conditions* $1-4$ *of Definition 45 and*

$$\lim_{s\to\infty} v(s,t) \in S_+ \tag{6.8}$$

$$\lim_{s\to-\infty} v(s,t+\theta) \in S_- \tag{6.9}$$

We denote by $\widehat{\mathcal{M}}^k_\theta(S_-, S_+, H_\tau)$ *the space of unaugmented upper levels in the symplectization consisting of tuples* (v, Γ, θ) *satisfying the above conditions.*

Definition 54. *Let* S_+ *be a Morse-Bott family of orbits in* $\mathbb{R} \times Y$*. The moduli space of* ***augmented BV curves with a plane in W***

$$\widehat{\mathcal{M}}^a_\theta(W, S_+, H_\tau)$$

is the space of tuples $(v_0, v_1, [U_i], \Gamma, \theta)$ *where* $(v_0, v_1, [U_i], \Gamma)$ *satisfy conditions* $1-5$ *of (46) and*

$$\lim_{s \to -\infty} v_0(s, t+\theta) \in W$$

We can now define the moduli space of BV-cascades:

Definition 55. *Given* $q_- \in S_-, q_+ \in S_+$ *generators of the chain complex, the moduli space of* ***BV cylinders with 1 cascade***

$$\widehat{\mathcal{M}}^a_{\theta,1}(q_-, q_+, H_\tau)$$

is the fiber product:

$$W^s(q_-) \underset{\Delta}{\times} \widehat{\mathcal{M}}^a_\theta(S_-, S_+, H_\tau) \underset{\Delta}{\times} W^u(q_+)$$

The moduli space of ***BV cylinders with N cascades****,* $N \geqslant 2$

$$\widehat{\mathcal{M}}^a_{\theta,N}(q_-, q_+, H_\tau)$$

is the fiber product:

$$W^s_Y(q_-) \underset{\Delta}{\times} \widehat{\mathcal{M}}^a_\theta(S_-, S_1, H_\tau) \underset{\Delta_f}{\times} \widehat{\mathcal{M}}^a_{N-1}(S_1, q_+, H_\tau)$$

Transversality for simple Morse-Bott differential cascades is discussed in [19, §5]. It consists of proving transversality for augmented curves in the symplectization, as well as transversality for evaluation maps. There are no new phenomena in the moduli spaces arising in the definition of the the BV-cascades. In the monotone case, the simple moduli spaces, denoted $\widehat{\mathcal{M}}^{*,a}_{\theta,N}(q_-, q_+, H_\tau)$, are sufficient for defining the BV operator. We review the definition and prove the following Lemma in §7.

Lemma 56. *For a generic choice of almost complex structure* J_W*,* $\widehat{\mathcal{M}}^{*,a}_{\theta,N}(q_-, q_+, H_\tau)$ *is a manifold of dimension* $|q_+| - |q_-| + N$ *if* q_- *is a generator in* $\mathbb{R} \times Y$ *and* $|q_+| - |q_-| + N + 1$ *if* q_- *is a* *generator in the interior of W.*

Let q_- be a nonconstant orbit. Then the moduli space $\widehat{\mathcal{M}}^a_{\theta,N}(q_-, q_+, H_\tau)$ admits a free $\mathbb{R}^N$ action by domain translations if $S_- \neq S_1$; otherwise it admits a free $\mathbb{R}^{N-1}$ action. In both cases we denote the quotient by $\mathcal{M}^a_{\theta,N}(q_-, q_+, H_\tau)$. If q_- is a constant orbit, the moduli spaces $\widehat{\mathcal{M}}^a_{\theta,N}(q_-, q_+, H_\tau)$ admit an $\mathbb{R}^{N+1}$ action, and we denote the quotient by $\mathcal{M}^a_{\theta,N}(q_-, q_+, H_\tau)$.

As in the case of the differential, a slight modification of SFT compactness and gluing arguments imply that the moduli spaces of split coproduct cascades admit a compactification by broken Morse flow lines and Floer curves and SFT type buildings.

We define the BV operator, $\mathbf{\Pi}$, on the chain complex (5.1) as the $\mathbb{Z}/2$ count of rigid curves:

$$\mathbf{\Pi}\, q_+ = \sum_{|q_+|=|q_-|-1} \#\left(\mathcal{M}_{N,\theta}(q_-, q_+; H_\tau)\right) x. \tag{6.10}$$

This following Lemma is a slight generalization of [19, Prop 6.2]:

Lemma 57. *The $\mathbf{\Pi}$ structure on (5.1) is given by :*

$$\mathbf{\Pi}(\check{p}_k) = k\hat{p}_k, \mathbf{\Pi}(\hat{p}_k) = 0$$

Proof. Lemma 6.1 of [19] establishes that the Fredholm index of a plane in W appearing as an augmentation plane is non negative, and is at least 2 if multiply covered. Consider a cascade with N levels and k augmentation planes appearing in a rigid BV-cascade from q_0 to q_1. We first assume that this is not a 1 cascade projecting to a constant sphere in D.

Let $A_1, \ldots, A_N \in H_2(D)$ denote the homology classes of the projections to D, let $B_1, \ldots, B_k \in H_2(X)$ denote the homology classes corresponding to the augmentation planes. Let $\gamma_i, i = 2, \ldots, k$ denote the limits at the augmentation punctures, and let k_i denote their multiplicities. Let $A = \sum_{i=1}^N A_i$.

We have $k_0 - \sum_{j=1}^k k_j = K\Omega_X(A) = K\frac{\langle c_1(TD), A\rangle}{\tau_X - K}$. We also have for $j \geqslant 2$, $k_j = B_j \bullet D =$

$K\Omega_X(B_j)$. Notice then that $|\gamma_i|_0 = 2\langle c_1(TX), B_j\rangle - 2B_j \bullet D - 2$. We therefore have:

$$\begin{aligned}
0 &= |q_0| - |q_1| + 1 \\
&= i(q_0) + M(q_0) - i(q_1) - M(q_1) + 2\frac{\tau_X - K}{K}(k_0 - \sum_{j=1}^{k} k_j) \\
&+ 2\frac{\tau_X - K}{K}\sum_{j=2}^{k} k_j + 1 \\
&= i(q_0) + M(p) - i(q_1) - M(q_1) + 2\langle c_1(TD), A\rangle + 2k + \sum_{j=2}^{k} |\gamma_j|_0 + 1.
\end{aligned} \tag{6.11}$$

Recall that for each $j = 2, \dots, k$, $|\gamma_j|_0 \geqslant 0$.

Consider the chain of pearls in D obtained by projecting this cascade to D. By transversality for chain of pearls Lemma 71 after [13, §5.3], if this is a simple chain of pearls, it has Fredholm index

$$I_D := M(q_0) + 2\langle c_1(TD), A\rangle - M(q_1) + 2k + N - 1.$$

If the chain of pearls is not simple, by monotonicity, we have that the index is at least as large as the index of the underlying simple chain of pearls.

Let N_0 be the number of Floer cylinders that project to constant curves in D, let $N_{\mathbf{\Pi}}$ denote the number of BV cylinders projecting to constant curves, and let N_1 be the number of cylinders that project to non-constant curves in D. $N = N_0 + N_1 + N_{\mathbf{\Pi}}$. Each Floer cylinder that projects to a constant curve in D must have at least one augmentation puncture. Hence $N_0 \leqslant k$.

By transversality for simple chains of pearls, we obtain the inequality

$$I_D \geqslant 2N_1 + 2k + N_{\mathbf{\Pi}}$$

by considering the 2-dimensional automorphism group for the N_1 non-constant spheres, the $2k$-parameter family of moving augmentation marked points on the domains, and moving the constant BV sphere along the flow of D.

Combining with Equation (6.11), we obtain

$$
\begin{aligned}
0 &= i(q_0) - i(q_1) + (I_D - N + 1) + \sum_{j=2}^{k} |\gamma_j|_0 + 1 \\
&\geqslant (i(q_0) - i(q_1)) + 2N_1 + 2k - N + \sum_{j=2}^{k} |\gamma_j|_0 + 2 \\
&= (i(q_0) - i(q_1) + 1) + N_1 + k + (k - N_0) + 1 + \sum_{j=2}^{k} |\gamma_j|_0.
\end{aligned}
$$

Observe now that each term on the right-hand-side of the inequality is non-negative and we obtain $0 \leqslant 1$, which is impossible. This means that the only possible configurations are those projecting to constant spheres. In particular, we must have $N_1 = k = N_0 = 0$. $N_{\mathbf{\Pi}} = 1$, and $i(q_0) = 0, i(q_1) = 1$. It follows that $q_- = \hat{p}_k$ and $q_+ = \check{p}_k$.

To justify the coefficients, note that given a constant solution u mapping to an orbit of multiplicity k, there are exactly k values $\theta_1, \cdots, \theta_k$ such that

$$
\begin{aligned}
u(s, 0) &= \check{p}_k \\
u(s, \theta_k) &= \hat{p}_k
\end{aligned}
$$

□

The fact the BV operator descends to cohomology can be proved by considering the boundary of dimension 1 moduli spaces. Alternatively, in the monotone case, the Morse-Bott split differential has a lower triangular form [13, Theorem 9.1]:

$$
\begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & * \end{bmatrix} \tag{6.12}
$$

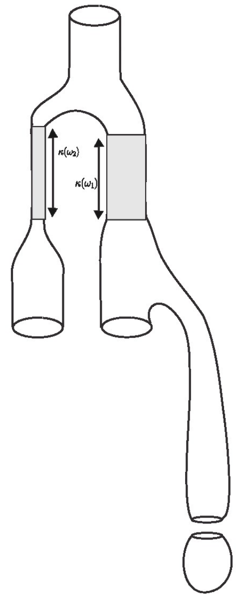

Figure 6.2: An augmented coproduct curve

with respect to the splitting of the chain complex as

$$SC_*(W, H_\infty) = \left(\bigoplus_{k>0} \bigoplus_{p \in \mathrm{Crit}(f_D)} \mathbb{Z}\langle \check{p}_k \rangle \right) \oplus \left(\bigoplus_{k>0} \bigoplus_{p \in \mathrm{Crit}(f_D)} \mathbb{Z}\langle \hat{p}_k \rangle \right) \oplus \left(\bigoplus_{x \in \mathrm{Crit}(f_W)} \mathbb{Z}\langle x \rangle \right) \quad (6.13)$$

So we can see immediately that the terms $\partial \circ \mathbf{\Pi}$ and $\mathbf{\Pi} \circ \partial$ both vanish.

6.3 Morse-Bott split coproduct

We now describe the moduli spaces of cascades that contributes to the split coproduct. The definitions use the families of admissible Hamiltonians $H_{z,\omega}$ and almost complex structures as above. As usual, we parameterize the unit interval with coordinates $\omega = (\omega_0, \omega_1)$ such that $\omega_0 + \omega_1 = 1, \omega_i \in (0, 1)$. Recall that $\Theta = (\theta_1, \theta_2) \in S^1 \times S^1$.

Let H_τ be an admissible Hamiltonian of slope τ. Let $\epsilon \ll 1$, such that $\epsilon\tau$ is smaller than the period of any Reeb orbit of R_α. To define the coproduct curves we will fix:

1. A family of Hamiltonians $H_{z,\omega}$ parameterized by $\Sigma \times (0, 1)$ satisfying:

- In a neighborhood of infinity at the positive cylindrical end:

$$H_{z,\omega} = H_\tau$$

- In a neighborhood of infinity at the ith cylindrical end:

$$H_{z,\omega} = \begin{cases} \sigma_\epsilon(\omega_i)H_\tau & for \ -R \geqslant s \geqslant -R - \kappa(\omega_i) \\ H_\tau & for \ -R - \kappa(\omega_i) - 1 \geqslant s > -\infty \end{cases}$$

- $H_{z,\omega}(r, y)$ only depends on the coordinate r.

2. A family of almost complex structures $J_{z,\omega}$ parameterized by $\Sigma \times (0,1)$, and satisfying the following conditions:

   - In a neighborhood of the punctures $J_{z,\omega} = J$
   - $J_{z,\omega}$ is admissible outside of a compact set K of $\mathbb{R} \times Y$.
   - $J_{z,\omega}$ is a lift of an almost complex structure J_D. That is, $J_{z,\omega}$ commutes with the projection map: $\pi(J_{z,\omega}\xi) = J_D\pi(\xi)$.

3. A one form β on Σ satisfying $d(H_{z,\omega}\beta) \leqslant 0$, and such that β restricts to dt in the neighborhood of the punctures.

Remark 58. *In the choice of almost complex structures, we need to let the compact set K be arbitrarily large. As in [3, §18], we will need to impose extra conditions on J_Y in order to ensure the maximum principle. One such solution is to impose that there exists a sequence $r_m \to \infty$ such that J satisfies the contact type condition along the hypersurfaces $r = r_m$.*

Definition 59. *Let $S_i, i \in \{0,1,2\}$ denote Morse-Bott families of orbits of $H_\tau \in \mathbb{R} \times Y$. We define several types of* split pair of pants co-product *as follows. In all of the following we require that $([U_i], \Gamma)$ satisfy conditions $2-4$ of Definition 45.*

1. $\mathcal{M}^a(S_1, S_2, S_0, H_{z,\omega})$ *is the space of tuples* $(v, \omega, [U_i], \Gamma)$ *where* $v : \Sigma\backslash\Gamma \to \mathbb{R} \times Y$ *is a finite energy map satisfying*

$$(du - X_{H_{z,\omega}} \otimes \beta)^{(0,1)} = 0 \tag{6.14}$$

and

$$\lim_{|s|\to\infty} v(\epsilon_i(s,t) \in S_i$$

2. $\mathcal{M}^a_\Theta(S_1, S_2, S_0, H_{z,\omega})$ *is the space of tuples* $(v, \Theta, \omega, [U_i], \Gamma)$ *where* $v : \Sigma\backslash\Gamma \to \mathbb{R} \times Y$ *is a finite energy map satisfying*

$$(du - X_{H_{z,\omega}} \otimes \beta)^{(0,1)} = 0$$

and

$$\lim_{s\to\infty} v(\epsilon_0(s,t) \in S_0$$

$$\lim_{s\to-\infty} v(\epsilon_i(s,\theta_i) \in S_i$$

3. $\mathcal{M}^a_{\theta_i}(S_1, S_2, S_0, H_{z,\omega})$ *is the space of a tuples* $(\theta_i, v, \omega, [U_i], \Gamma)$ *such that:* $\tilde{v} : \Sigma\backslash\Gamma \to \mathbb{R} \times Y$ *is a finite energy map satisfying*

$$(du - X_{H_{z,\omega}}\beta)^{(0,1)} = 0$$

and

$$\lim_{s\to\infty} v(\epsilon_0(s,t)) \in S_0$$

$$\lim_{s\to-\infty} v(\epsilon_i(s,\theta_i) \in S_i$$

$$\lim_{s\to-\infty} v(\epsilon_j(s,t) \in S_j$$

We denote by $\mathcal{M}^k(S_1, S_2, S_0, H_{z,\omega})$, the space of unaugmented upper levels in the symplectization consisting of tuples (v, Γ) satisfying the above conditions. We similarly define $\mathcal{M}^k_{\Theta}(S_1, S_2, S_0, H_{z,\omega})$ and $\mathcal{M}^k_{\theta_i}(S_1, S_2, S_0, H_{z,\omega})$.

The coproduct with N cascades is defined as the following fiber products:

Definition 60. *Let $q_i, i = 0, 1, 2$ be generators of the Morse-Bott split chain complex of H_τ. For $j \in \{0, \ldots, N+1\}$ we fix a choice of perturbation $f_Y^j \in C^\epsilon_\infty(f_Y, f_D)$. The moduli space of* ***split coproducts with N cascades***

$$\widehat{\mathcal{M}}^a_{\Theta,N}(q_0; q_1, q_2)$$

consists of the fiber products:

1.

$$\begin{aligned}&W^s_1(q_1) \underset{\Delta}{\times}\\ &W^s_2(q_2) \underset{\Delta}{\times} \mathcal{M}^a_{\omega,\Theta}(S_1, S_2, S_0, H_{z,\omega}) \underset{\Delta_f}{\times} \widehat{\mathcal{M}}^a_{N-1}(S_0, q_0, H_\tau)\end{aligned}$$

2.

$$\begin{aligned}&\widehat{\mathcal{M}}^a_{\theta_1,N_1}(q_1, S_1, H_\tau) \underset{\Delta_f}{\times}\\ &\widehat{\mathcal{M}}^a_{\theta_2,N_2}(q_2, S_2, H_\tau) \underset{\Delta_f}{\times} \mathcal{M}^a_{\omega}(S_1, S_2, S_0, H_{z,\omega}) \underset{\Delta_f}{\times} \widehat{\mathcal{M}}^a_{N_0}(S_0, q_0, H_\tau)\end{aligned}$$

with $N_0 + N_1 + N_2 = N - 1$

3.

$$W^s_i(q_i) \underset{\Delta}{\times}$$
$$\widehat{\mathcal{M}}^a_{\theta_j,N_j}(q_j,S_j,H_\tau) \underset{\Delta_f}{\times} \mathcal{M}^a_{\omega,\theta_i}(S_i,S_j,S_0,H_{z,\omega}) \underset{\Delta_f}{\times} \widehat{\mathcal{M}}^a_{N_0}(S_0,q_0,H_\tau)$$

with $N_0 + N_j = N - 1$

Where the stable/unstable manifolds of q_i *are given with respect to the perturbations* f^i_Y*, and the* j*th fiber product is taken over the diagonal/flow diagonal of* f^j_Y*.*

$\widehat{\mathcal{M}}^a_{\Theta,N}(q_0;q_1,q_2)$ admits an $\mathbb{R}^{N-1}$ action by domain translations, if the BV levels are non constant, and an $\mathbb{R}^{N-2}$ or $\mathbb{R}^{N-3}$ action if one or both of the BV cylinders are constant. We denote the quotient by $\mathcal{M}^a_{\Theta,N}(q_0;q_1,q_2)$.

Transversality for the coproducts cascades is proved in §7. The compactness assumptions for the Morse-Bott coproduct cascades, plus a slight modification of SFT compactness and gluing arguments imply that the moduli spaces of split coproduct cascades admit a compactification by broken Morse flow lines and Floer curves and SFT type buildings.

We define the split coproduct operator on the chain complex (5.1) as the $\mathbb{Z}/2$ count of rigid curves:

$$\lambda\, q_0 = \sum_{|q_0|=|q_1|+|q_2|-2n+3} \#\left(\mathcal{M}^a_{\Theta,N}(q_0;q_1,q_2)\right) q_1 \otimes q_2. \tag{6.15}$$

An explicit description of the coproduct

We will now see that in the monotone case it is sufficient to consider moduli spaces of curves projecting to simple curves. This is a slight generalization of [19, Prop 6.2].

Lemma 61. *Assume that* (X,Ω_X) *is spherically monotone with monotonicity constant* τ_X *and assume that* $D \subset X$ *is Poincaré dual to* $K\Omega_X$ *with* $\tau_X > K > 0$*. Then any rigid coproduct cascade* *is of the following configuration:*

$$\begin{aligned} &W_1^s(q_1) \underset{\Delta}{\times} \\ &W_2^s(q_2) \underset{\Delta}{\times} \mathcal{M}^a_{\omega,\Theta}(S_1, S_2, S_0, H_{z,\omega}) \underset{\Delta}{\times} W_2^u(q_0) \end{aligned}$$

Moreover a rigid curve has a checked orbit as input and hatted orbits as outputs.

Proof. Lemma 6.1 of [19] establishes that the Fredholm index of a plane in W appearing as an augmentation plane is non negative, and is at least 2 if multiply covered.

Consider a cascade with N levels and k augmentation planes appearing in the coproduct of $\lambda(q_0) = q_1 \bigotimes q_2$. Let $A_1, \ldots, A_N \in H_2(D)$ denote the homology classes of the projections to D, let $B_1, \ldots, B_k \in H_2(X)$ denote the homology classes corresponding to the augmentation planes.

Let $\gamma_i, i = 3, \ldots, k$ denote the limits at the augmentation punctures, and let k_i denote their multiplicities. Let $A = \sum_{i=1}^N A_i$. We have $k_0 - \sum_{j=1}^k k_j = K\Omega_X(A) = K\frac{\langle c_1(TD), A\rangle}{\tau_X - K}$. We also have for $j \geqslant 3$, $k_j = B_j \bullet D = K\Omega_X(B_j)$. Notice then that $|\gamma_i|_0 = 2\langle c_1(TX), B_j\rangle - 2B_j \bullet D - 2$.

We therefore have:

$$\begin{aligned} 0 &= |q_1| + |q_2| - |q_0| - 2n + 3 \\ &= i(q_0) + M(q_0) - \sum i(q_i) - \sum M(q_i) + 2\frac{\tau_X - K}{K}(k_0 - \sum_{j=1}^k k_j) \\ &+ 2\frac{\tau_X - K}{K}\sum_{j=3}^k k_j + 2 \\ &= i(q_0) + M(p) - \sum i(q_i) - \sum M(q_i) + 2\langle c_1(TD), A\rangle + 2k + \sum_{j=3}^k |\gamma_j|_0 + 2. \end{aligned} \tag{6.16}$$

Recall that for each $j = 3, \ldots, k$, $|\gamma_j|_0 \geqslant 0$.

Consider the chain of pearls in D obtained by projecting the upper level of this split Floer trajectory to D. If this is a coproduct chain of pearls, by Lemma 73 it has Fredholm index:

$$I_D := M(q_0) + 2\langle c_1(TD), A\rangle - \sum M(q_i) + 2k + N - 1.$$

If the chain of pearls is not simple, by monotonicity, we have that the index is at least as large as the index of the underlying simple chain of pearls.

Let N_0 be the number of sub-level Floer cylinders that project to constant curves in D, let $N_{\mathbf{\Pi}}$ denote the number of BV-cylinders projecting to constant curves, and let N_1 be the number of sub-level cylinders that project to non-constant curves in D. $N = N_0 + N_1 + N_{\mathbf{\Pi}} + 1$. Each Floer cylinder that projects to a constant curve in D must have at least one augmentation puncture. Hence $N_0 \leqslant k$.

By transversality for coproduct chains of pearls, we obtain the inequality

$$I_D \geqslant 2N_1 + 2k + N_{\mathbf{\Pi}}$$

by considering the 2-dimensional automorphism group for the N_1 non-constant spheres, the $2k$-parameter family of moving augmentation marked points on the domains, and moving the constant spheres along the flowlines of D. Note that at a constant sphere, the evaluation maps are submersions, so we can assume that the gradient trajectories preceding and following a constant sphere belong to the same function f_Y.

Combining with Equation (6.16), we obtain

$$\begin{aligned} 0 &= i(q_0) - \sum i(q_i) + (I_D - N + 1) + \sum_{j=3}^{k} |\gamma_j|_0 + 2 \\ &\geqslant (i(q_0) - \sum i(q_i)) + 2N_1 + 2k - N + \sum_{j=1}^{k} |\gamma_j|_0 + 3 \\ &= (i(q_0) - \sum i(q_i) + 2) + N_1 + k + (k - N_0) + \sum_{j=1}^{k} |\gamma_j|_0. \end{aligned}$$

Each term on the right-hand-side of the inequality is non-negative. In particular, we must have $N_1 = k = N_0 = 0$. The last claim follows from the fact that $0 \geqslant (i(q_0) - \sum i(q_i) + 2)$. □

Chain map:

The easiest way to show that the Morse-Bott coproduct descends to homology is to consider the matrix of the differential. In the monotone case, and with respect to the splitting of the chain complex as

$$SC^*(W, H_\infty) = \left(\bigoplus_{k>0} \bigoplus_{p \in \mathrm{Crit}(f_D)} \mathbb{Z}\langle \check{p}_k \rangle \right) \oplus \left(\bigoplus_{k>0} \bigoplus_{p \in \mathrm{Crit}(f_D)} \mathbb{Z}\langle \hat{p}_k \rangle \right) \oplus \left(\bigoplus_{x \in \mathrm{Crit}(f_W)} \mathbb{Z}\langle x \rangle \right) \quad (6.17)$$

the split Morse-Bott differential has a lower triangular form [13, Theorem 9.1]:

$$\begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & * \end{bmatrix} \quad (6.18)$$

This means that the differential of hatted orbits vanishes, so by (61) so do the terms $(1 \times \partial) \circ \lambda$ and $(\partial \times 1) \circ \lambda$. Moreover, the differential of a checked orbit is a hatted orbit or a constant orbit, on both of which the coproduct vanishes.

Sketch of isomorphism with the Hamiltonian coproduct

A cobordism argument first relates the Morse-Bott and split Morse-Bott moduli spaces by neck stretching. That is, we use a family of almost complex structures J_i on W such that (W, J_i) converges as $i \to \infty$ to a split manifold whose upper level is $(\mathbb{R} \times Y, J_Y)$ and lower level is (W, J_W), [19, Lemma 2.7].

Then a slight modification of SFT compactness implies that a sequence of J_i Floer curves will limit to an SFT type building with levels in $\mathbb{R} \times Y$ and W. The maximum principle implies that the top Floer curve in $\mathbb{R} \times Y$ is connected, [20, Step 1 in Proposition 5]. It follows that the only J_Y-

holomorphic levels that arise have exactly 1 positive end and multiple negative ends; transversality for all such levels is proved in [13, §5], and §7. A dimension argument now implies that there are at most two levels.

To show that the Morse-Bott and Hamiltonian complexes are quasi isomorphic one constructs continuation maps. Given an admissible Hamiltonian H_τ we consider a non degenerate Hamiltonian $\tilde{H}_\tau$, which is a C^2 Morse function in the interior of W, and is a small time dependent perturbation of H_τ in a neighborhood of the periodic orbits of X_{H_τ}.

To define the continuation cylinders, we consider a family of Hamiltonians parameterized by the cylinder $S^1 \times \mathbb{R}$ satisfying:

$$H_{s,t} = \begin{cases} H_\tau & \textit{for } s \gg 0 \\ \tilde{H}_\tau & \textit{for } s \ll 0, \end{cases}$$

and subject to the condition that $\partial_s(H_{s,t}) \leqslant 0$. In order to avoid difficulties with degenerate orbits we also impose the following conditions. In the cylindrical end $H_{s,t}(r, y) = h_\tau(s, e^r)$ for a function $h_\tau(s, \rho)$ satisfying

$$e^r \frac{\partial}{\partial \rho}(h(s, e^r) - h(s, e^r) \geqslant 0$$
$$\partial_s(e^r \frac{\partial}{\partial \rho}(h_\tau(s, e^r) - h_\tau(s, e^r)) \leqslant 0.$$

We also choose a family of almost complex structures interpolating between admissible almost complex structures used to define $HF^*(H_\tau, J_W)$ and $HF^*(\tilde{H}_\tau, J)$.

Then by the maximum principle, [19, Lemma 5.4], the moduli space of 0 dimensional continuation cascades is compact and give chain maps:

$$\kappa : HF^*(H_\tau, J_W) \to HF^*(\tilde{H}_\tau, J).$$

Lemma 62. *There exists a commutative diagram:*

$$\begin{array}{ccc} HF^*(H_\tau) & \xrightarrow{\lambda} & HF^*(H_\tau)\otimes HF^*(H_\tau) \\ \downarrow{\kappa} & & \downarrow{\kappa} \\ HF^*(\tilde{H}_\tau) & \xrightarrow{\lambda} & HF^*(\tilde{H}_\tau)\otimes HF^*(\tilde{H}_\tau) \end{array}$$

Proof. Note that the continuation maps consist of a disjoint union of isolated continuation cascades. By concatenating and gluing, as in section §3.4, we obtain new glued operations $\kappa\#\lambda$ and $\lambda\#\kappa$. For a large gluing parameter $\kappa\#\lambda = \kappa \circ \lambda$ and $\lambda\#\kappa = \lambda \circ \kappa$ on the chain level. The operations $\lambda\#\kappa$ and $\kappa\#\lambda$ agree at the positive and negative ends and we use a cobordism to interpolate between the two and show that they are chain homotopic.

Note that we only consider moduli spaces with inputs and outputs in non constant orbits, so the relevant interpolation between H_τ and $\tilde{H}_\tau$ happen above the region where the degenerate orbits are and hence they do not pose a problem (here we again assume the compactness of the moduli spaces). □

Chapter 7: Transversality

This section serves as a recap of the transversality methods in [19, §5]. We first recall from [19] the necessary function spaces and Fredholm theory required to deal with Morse-Bott asymptotics in §7.1. We then describe the decomposition of the linearized Floer operator in §7.2. In §7.3 we state and prove transversality for simple cascades. In §7.4 we define and state the analogous transversality results for chains of pearls.

7.1 Fredholm theory

Let $v : \dot{S} \to \mathbb{R} \times Y$ be a finite energy solution of

$$(du - X_{H_\Psi} \otimes \beta_\Phi)^{(0,1)} = 0, \tag{7.1}$$

where Ψ and Φ parameterize families of Hamiltonians and $1-$forms, satisfying

$$d(H_\Psi \beta_\Phi) \leqslant 0.$$

This description includes the moduli spaces defining the differential, as well as moduli spaces defining continuation maps, coproduct and BV cascades. The linearization of the Floer equation (7.1) can be identified with an operator:

$$D_v : W^{k,1,\delta}(\dot{S}, v^*T(\mathbb{R} \times Y)) \to L^{p,\delta}(\Lambda^{0,1}T^*\dot{S} \otimes v^*T(\mathbb{R} \times Y)).$$

Here $W^{k,p,\delta}$ denotes the space of sections that exponentially decay like $e^{-\delta|s|}$ near the punctures. More precisely, let $E \to \dot{S}$ be a hermitian vector bundle over a punctured Riemann surface, together with fixed trivializations at the punctures. Let $\delta > 0$. Then $W^{k,p,\delta}(\dot{S}, E)$ denotes the space of sections η of E whose representatives $f : Z_\pm \to \mathbb{C}^n$ in the cylindrical ends with the given trivializations satisfy:

$$||e^{\pm\delta s} f||_{W^{k,p}(Z_\pm)} < \infty.$$

Over each puncture z we obtain an asymptotic linear operator which in the cylindrical coordinates is given by:

$$\mathbf{A}_z = -J_z(t)\frac{d}{dt} - A_z(t) \tag{7.2}$$

where $A_z(t)$ is a loop of self adjoint matrices. Since the orbits we consider are Morse-Bott degenerate, the asymptotic operators are degenerate. By [19, Lemma 5.16] the Cauchy Riemann operator D_v with asymptotic operators $\mathbf{A}_z$ is conjugate to a Cauchy Riemann operator

$$D_v^\delta : W^{k,1}(\dot{S}, v^*T(\mathbb{R}\times Y) \to L^p(\Lambda^{0,1}T^*\dot{S} \otimes v^*T(\mathbb{R}\times Y))$$

with perturbed asymptotic operators $\mathbf{A}_z + \delta_z$ and is hence Fredholm. Its index is:

$$ind(D_v) = n\chi(\dot{S}) + 2c_1(v^*T(\mathbb{R}\times Y) + \sum_{\Gamma_+}(CZ(\mathbf{A}_z+\delta)) - \sum_{\Gamma_-}(CZ(\mathbf{A}_z+\delta) \tag{7.3}$$

where c_1 is computed with respect to the chosen trivialization, and $\Gamma_\pm$ denotes the set of positive/negative punctures.

In order to consider the entire family of solutions with asymptotic limits moving in Morse-Bott families, we consider the following enlargement. To each puncture we associate a subspace of the kernel of the corresponding asymptotic operator, which we denote by V_z and write $\mathbf{V}$ for this collection. Then, for each puncture z we associate a smooth bump function μ_z supported near and identically 1 even nearer to its puncture. We then define

$$\begin{aligned} W^{1,p,\delta}_{\mathbf{V}}(\dot{S}, E) = &\{u \in W^{1,p}_{\text{loc}}(E) \mid \exists\, c_z \in V_z \\ &\text{such that } u - \sum c_z\mu_z \in W^{1,p,\delta}(E)\}. \end{aligned} \tag{7.4}$$

Since the vector spaces V_z are in the kernel of D_v, the operator extends to an operator:

$$D_v : W^{k,1,\delta}_{\mathbf{V}}(\dot{S}, v^*T(\mathbb{R}\times Y) \to L^{p,\delta}_{\mathbf{V}}(\Lambda^{0,1}T^*\dot{S}\otimes v^*T(\mathbb{R}\times Y))$$

which is Fredholm with index [13, Theorem 5.18]:

$$\begin{aligned} ind(D_v) = n\chi(\dot{S}) + 2c_1(v^*T(\mathbb{R}\times Y) + \sum_{\Gamma_+}(CZ(A_z+\delta) + dim(V_z)) \\ - \sum_{\Gamma_-}(CZ(A_z+\delta) + codim(V_z)) \end{aligned} \tag{7.5}$$

7.2 A decomposition of the linear operator

By our choice of complex structures, for each $p \in Y$, the projection π_D gives a complex bundle isomorphism $(\xi_p, d\alpha) \cong (T_{\pi_D(p)}D, K\Omega_D)$. Let $w = \pi_D(v)$.

Let $\tilde{v} = (b, v)\colon \Sigma\backslash\Gamma \to \mathbb{R}\times Y$, and let $\zeta = (\zeta_a, \zeta_b)$ under the isomorphism $\tilde{v}^*T(\mathbb{R}\times Y) \cong (\mathbb{R}\oplus\mathbb{R}R)\oplus w^*T\Sigma$. We consider the decomposition:

$$D_{\tilde{v}}(\zeta_a,\zeta_b) = \begin{pmatrix} D_{aa} & D_{ab} \\ D_{ba} & D_{bb} \end{pmatrix}\begin{pmatrix} \zeta_a \\ \zeta_b \end{pmatrix}.$$

The following lemma is a slight generalization of [19, Lemma 5.22] by considering the more general Floer equation (7.1):

Lemma 63. *The isomorphism $v^*T(\mathbb{R}\times Y) \cong (\mathbb{R}\oplus\mathbb{R}R)\oplus w^*TD$ induces a decomposition:*

$$D_{\tilde{v}} = \begin{pmatrix} D^L_v & M \\ 0 & \dot{D}_w \end{pmatrix}$$

where $D^L_v\colon W^{1,p,\delta}_{\mathbf{V_0}}(\dot{S}, \mathbb{R}\oplus\mathbb{R}R) \to L^{p,\delta}(\Lambda^{0,1}T^*\dot{S}\otimes\mathbb{R}\oplus\mathbb{R}R)$, and M is a multiplication operator.

Proof. The nonlinear Floer operator takes the form of the left-hand side of the equation:

$$d\tilde{v} + J_{z,\omega}(\tilde{v})d\tilde{v}\circ j - J_{z,\omega}X_{z,\omega}\otimes\beta\circ j - X_{z,\omega}\otimes\beta = 0. \tag{7.6}$$

If we apply dr to the previous equation, and use the fact that $dr \circ J_z = -f(r,y)\alpha$, we get:

$$db - f(b,v)v^*\alpha \circ j + h'_{z,\omega}(\mathrm{e}^b)\beta \circ j = 0.$$

Denote by $\pi_\xi : TY \to \xi$ the projection along the Reeb vector field, we get

$$\pi_\xi d\tilde{v} + J_{z,\omega}(\tilde{v})\pi_\xi d\tilde{v} \circ j = 0. \tag{7.7}$$

Let g be the metric on $\mathbb{R} \times Y$ given by $g = dr^2 + \alpha^2 + d\alpha(\cdot, J_{z,\omega}\cdot)$. Let $\widehat{\nabla}$ be the Levi-Civita connection for g. Let ∇ be the Levi-Civita connection on $T\Sigma$ for the metric $\omega_\Sigma(\cdot, J^\Sigma_{z,\omega}\cdot)$.

In local coordinates $s + it$ near a point z in Σ :

$$\begin{aligned} 0 =& \partial_s\tilde{v}ds + \partial_t\tilde{v}dt + J_{z,\omega}(-\partial_s\tilde{v}dt + \partial_t\tilde{v}ds) - J_{z,\omega}X_{z,\omega} \otimes (-\beta_s dt + \beta_t ds) \\ & - X_{z,\omega} \otimes (\beta_s ds + \beta_t dt). \end{aligned}$$

Then, it follows that the linearization $D_{\tilde{v}}$ applied to a section ζ of $\tilde{v}^*T(\mathbb{R} \times Y)$ satisfies

$$\begin{aligned} D_{\tilde{v}}\zeta\,(\partial_s) =& \widetilde{\nabla}_s\zeta + J_{z,\omega}(\tilde{v})\widetilde{\nabla}_t\zeta + \big(\widetilde{\nabla}_\zeta J_{z,\omega}(\tilde{v})\big)\partial_t\tilde{v} \\ & -\widetilde{\nabla}_\zeta(X_{H_{z,\omega}}\beta_s)(\tilde{v}) - \widetilde{\nabla}_\zeta(J_{z,\omega}(\tilde{v})X_{H_{z,\omega}}\beta_t)(\tilde{v}) \end{aligned}$$

Notice that $\widetilde{\nabla}\partial_r = 0$ since g is a product metric. Observe also that for any vector field V in $T\Sigma$, there is a unique horizontal lift $\tilde{V}$ to Y with the property $\alpha(\tilde{V}) = 0$. For any two vector fields V and W in $T\Sigma$, since $d\alpha(\tilde{V},\tilde{W}) = K\omega_\Sigma(V,W)$, we have the following

$$[\tilde{V},\tilde{W}] = \widetilde{[V,W]} - K\omega_\Sigma(V,W)R.$$

From this, it follows that the Levi-Civita connection $\widetilde{\nabla}$ satisfies the following identities:

$$\begin{aligned}
\widetilde{\nabla}_{\tilde{V}}\tilde{W} &= \widetilde{\nabla_V W} - \frac{K}{2}\omega_\Sigma(V,W)R \\
\widetilde{\nabla}_R R &= 0 \\
\widetilde{\nabla}_R \tilde{V} &= -\frac{1}{2}J_{z,\omega}\tilde{V}.
\end{aligned}$$

We also have:

$$\begin{aligned}
\widetilde{\nabla}_s R &= \widetilde{\nabla}_{\pi_\xi v_s} R = -\frac{1}{2}J_{z,\omega}\pi_\xi v_s \\
\widetilde{\nabla}_s \zeta &= \widetilde{\nabla_{w_s}\eta} - \frac{K}{2}\omega_\Sigma(w_s,\eta)R - \frac{1}{2}\alpha(v_s)J_{z,\omega}\zeta,
\end{aligned}$$

and similarly for $\widetilde{\nabla}_t$. We then obtain the following covariant derivatives of $J_{z,\omega}$ where $\tilde{W}$ is a section of $\tilde{v}^*\xi$:

$$\begin{aligned}
(\widetilde{\nabla}_\zeta J_{z,\omega})\partial_r &= \widetilde{\nabla}_\zeta f(r,y)R - J_{z,\omega}\widetilde{\nabla}_\zeta \partial_r = -\frac{1}{2}f(r,y)J_{z,\omega}\zeta \\
(\widetilde{\nabla}_\zeta J_{z,\omega})R &= -\widetilde{\nabla}_\zeta(J_{z,\omega}\partial_r) - J_{z,\omega}\widetilde{\nabla}_\zeta R = -\frac{1}{2}\zeta \\
(\widetilde{\nabla}_\zeta J_{z,\omega})\tilde{W} &= \widetilde{\nabla}_\zeta(J_{z,\omega}\tilde{W}) - J_{z,\omega}\widetilde{\nabla}_\zeta \tilde{W} \\
&= \widetilde{\nabla_\eta J_{z,\omega}W} - \frac{K}{2}\omega_\Sigma(\eta, J_{z,\omega})R - J_{z,\omega}\left(\widetilde{\nabla_\eta W} - \frac{K}{2}\omega_\Sigma(\eta,W)R\right) \\
&= (\widetilde{\nabla_\eta J_\Sigma})W - \frac{K}{2}\omega_\Sigma(\eta, J_{z,\omega}W)R - \frac{K}{2f(r,y)}\omega_\Sigma(\eta,W)\partial_r.
\end{aligned}$$

We first consider the case when ζ is a section of $\tilde{v}^*\xi$, and is thus the lift $\zeta = \tilde{\eta}$ of a section η of

$w^*T\Sigma$. We compute:

$$\begin{aligned}
D_{\tilde{v}}\zeta\,(\partial_s) =& \widetilde{\nabla}_s\zeta + J_{z,\omega}(\tilde{v})\widetilde{\nabla}_t\zeta + \left(\widetilde{\nabla}_\zeta J_{z,\omega}(\tilde{v})\right)\partial_t\tilde{v} \\
&- \widetilde{\nabla}_\zeta(X_{H_{z,\omega}}\beta_s)(\tilde{v}) - \widetilde{\nabla}_\zeta(J_{z,\omega}(\tilde{v})X_{H_{z,\omega}}\beta_t)(\tilde{v}) \\
=& \widetilde{\nabla_{w_s}\eta} - \frac{K}{2}\omega_\Sigma(w_s,\eta)R - \frac{1}{2}\alpha(v_s)J_{z,\omega}\zeta \\
&+ J_{z,\omega}(\widetilde{\nabla_{w_t}\eta} - \frac{K}{2}\omega_\Sigma(w_t,\eta)R - \frac{1}{2}\alpha(v_t)J_{z,\omega}\zeta) \\
&- \frac{1}{2}f(r,y)b_tJ_{z,\omega}\zeta - \frac{1}{2}\alpha(v_t)\zeta \\
&+ (\widetilde{\nabla_\eta J_\Sigma})w_t - \frac{K}{2}\omega_\Sigma(\eta, J_\Sigma w_t)R - \frac{K}{2f(r,y)}\omega_\Sigma(\eta, w_t)\partial_r \\
&+ \frac{1}{2}\beta_s h'_{s+it,\omega}(e^r)J_{z,\omega}\zeta \\
=& \widetilde{\dot{D}^\Sigma_w}\eta - J_{z,\omega}K\omega_\Sigma(w_t,\eta)R - K\omega_\Sigma(w_s,\eta)R.
\end{aligned}$$

From this it follows that

$$D_{bb}\zeta(\partial_s) = \dot{D}^\Sigma_w,$$

and

$$D_{ab}\zeta(\partial_s) = -J_{z,\omega}K\omega_\Sigma(w_t,\pi_\Sigma\zeta)R - K\omega_\Sigma(w_s,\pi_\Sigma\zeta).$$

Note that D_{ab} is surjective except at critical points of the J map w, of which there are finitely many if w is non-constant.

We now consider $D_{\tilde{v}}\zeta\,(\partial_s)$, where $\zeta = \zeta_1\partial_r + \zeta_2 R$. Note that:

$$\begin{aligned}
(\widetilde{\nabla}_R J_{z,\omega})\partial_r &= (\widetilde{\nabla}_R f(r,y))R \\
(\widetilde{\nabla}_R J_{z,\omega})R &= -\widetilde{\nabla}_R(f(r,y)^{-1})\partial_r \\
(\widetilde{\nabla}_R J_{z,\omega})\tilde{W} &= \widetilde{\nabla}_R(J_{z,\omega}\tilde{W}) - J_{z,\omega}\widetilde{\nabla}_R\tilde{W} \\
&= \frac{1}{2}\tilde{W} - \frac{1}{2}\tilde{W} = 0.
\end{aligned}$$

and similarly:

$$\begin{aligned}(\widetilde{\nabla}_{\partial_r} J_{z,\omega})\partial_r &= (\widetilde{\nabla}_{\partial_r} f(r,y))R\\ (\widetilde{\nabla}_{\partial_r} J_{z,\omega})R &= -\widetilde{\nabla}_{\partial_r}(f(r,y)^{-1})\partial_r\\ (\widetilde{\nabla}_{\partial_r} J_{z,\omega})\tilde{W} &= 0.\end{aligned}$$

By the Leibniz rule it now follows that $D_{ba} = 0$, and D_{aa} is given by:

$$\begin{aligned}D_{\tilde{v}}\zeta\,(\partial_s) =& \zeta_s + J_{z,\omega}(\tilde{v})\zeta_t + \big(\widetilde{\nabla}_\zeta J_{z,\omega}(\tilde{v})\big)(\alpha(\partial_t v)R + b_t\partial_r)\\ &-\widetilde{\nabla}_\zeta(X_{H_{z,\omega}}\beta_s)(\tilde{v}) - \widetilde{\nabla}_\zeta(J_{z,\omega}(\tilde{v})X_{H_{z,\omega}}\beta_t)(\tilde{v})\end{aligned}$$

□

7.3 Transversality for simple cascades

In this section we discuss transversality for simple cascades. As in the theory of J holomorphic curves, simple here essentially means that the curves are not multiply covered. Under monotonicity conditions on X and D, these moduli spaces will be sufficient for defining the various operations. As usual, $\omega = (\omega_1, \omega_2)$, $\theta \in S^1$, and $\Theta = (\theta_1, \theta_2) \in S^1 \times S^1$. We start with the definition of simple BV cascades:

Definition 64. *Recall the moduli space of BV cylinders with N cascades $\widehat{\mathcal{M}}^a_{\theta,N}(q_-, q_+, H_\tau)$, Definition 55. Then the moduli space of* ***simple BV cylinders with N cascades***

$$\widehat{\mathcal{M}}^{*,a}_{\theta,N}(q_-, q_+, H_\tau)$$

consists of elements in $\widehat{\mathcal{M}}^a_{\theta,N}(q_-, q_+, H_\tau)$ for which the projection of each cylinder to D is either somewhere injective or a constant curve in D. If the projection of v to D is constant, and v is not a BV level then it has at least one augmentation puncture.

We now define simple coproduct cascades:

Definition 65. *Let* $\mathcal{M}^{a}_{\Theta,N}(q_0; q_1, q_2, H_{z,\omega}, J_{z,\omega})$ *be the moduli space of augmented coproduct curves with N cascades, see definition 60. The moduli space of* ***simple coproduct curves with*** N ***cascades***

$$\mathcal{M}^{*,a}_{\Theta,N}(q_0; q_1, q_2, H_{z,\omega}, J_{z,\omega})$$

consists of elements in $\mathcal{M}^{a}_{\Theta,N}(q_0; q_1, q_2, H_{z,\omega}, J_{z,\omega})$ *for which the projection of each non coproduct level to D is either somewhere injective or a constant curve. If the projection of v is constant, and v is not a coproduct or BV level it must have at least one augmentation puncture.*

Lastly, we define simple continuation cascades:

Definition 66. *Let* $\widehat{\mathcal{K}}^{a}_{N}(q_+, q_-, H_1, H_0)$ *denote the moduli space of continuation cylinders with N cascades, see Definition 50. The moduli space of* ***simple continuation cylinders with*** N ***cascades***

$$\widehat{\mathcal{K}}^{*,a}_{N}(q_+, q_-, H_1, H_0)$$

consists of elements in $\widehat{\mathcal{K}}^{a}_{N}(q_+, q_-, H_1, H_0)$ *for which the projection of each cylinder to D is either somewhere injective or a constant curve in D. If the projection to D is constant, and v is not a continuation level then it has at least one augmentation puncture.*

Recall that the moduli spaces $\mathcal{M}^{*,a}_{\Theta,N}(q_0; q_1, q_2, H_{z,\omega}, J_{z,\omega})$ of coproduct cascades are defined as fiber products over flow diagonals. For $j \in \{0, \ldots, N+1\}$ we consider a choice of perturbation $f^j_Y \in C^{\epsilon}_{\infty}(f_Y, f_D)$.

Proposition 67. *For a generic choice of almost complex structure* J_W $\widehat{\mathcal{M}}^{*,a}_{\theta,N}(q_-, q_+, H_\tau)$ *is a smooth manifold of dimension* $|q_+| - |q_-| + N$ *if* q_- *is a generator in* $\mathbb{R} \times Y$ *and* $|q_+| - |q_-| + N + 1$ *if* q_- *is a generator in the interior of W.*

For a generic choice of $J_{z,\omega}$*, a Morse function* f_Y*, and perturbation data* $f^j_Y \in C^{\infty}_{\epsilon}(f_Y, f_D)$*, the moduli spaces* $\mathcal{M}^{*,a}_{\Theta,N}(q_0; q_1, q_2, H_{z,\omega}, J_{z,\omega})$ *are manifolds of dimension* $|q_0| - |q_1| - |q_2| + 2n - 3$*.*

For a generic choice of data J_W *and* H_s*, the moduli space* $\widehat{\mathcal{K}}^{*,a}_{N}(q_+, q_-, H_1, H_0)$*, is a manifold of dimension* $|q_+| - |q_-| + N - 1$*.*

Proof. Let v be a solution of (7.1). The linearized operator

$$D_v : W^{k,1,\delta}_{\mathbf{V}}(\Sigma\backslash\Gamma, v^*T(\mathbb{R}\times Y) \to L^{p,\delta}(\Lambda^{0,1}T^*\Sigma\backslash\Gamma\otimes v^*T(\mathbb{R}\times Y))$$

decomposes with respect to the splitting $v^*T(\mathbb{R}\times Y)\cong(\mathbb{R}\oplus\mathbb{R}R)\oplus w^*TD$, see Lemma 63. Recall that $\mathbf{V}$ denotes the kernels of the asymptotic operators at each of the punctures. The diagonal terms are perturbed Cauchy Riemann type operators, while the off diagonal term is a compact perturbation. For non coproduct levels, the complex structure is cylindrical and surjectivity of the upper diagonal term

$$D^L_v : W^{k,1,\delta}_{\mathbf{V}}(\Sigma\backslash\Gamma, v^*(\mathbb{R}\bigoplus\mathbb{R}R)) \to L^{p,\delta}(\Lambda^{0,1}T^*\Sigma\backslash\Gamma\otimes v^*(\mathbb{R}\bigoplus\mathbb{R}R))$$

follows from automatic transversality [13, §5.2]. For coproduct levels surjectivity is obtained for a generic choice of almost complex structure satisfying $J_{z,\omega}\partial r = f(r,y)R$ as in [3, §16].

The lower diagonal term in the decomposition above can be identified with $\dot{D}_w$, the linearized Cauchy–Riemann operator associated to $\pi_D(v)$:

$$\dot{D}_w\colon W^{1,p,\delta}_{\mathbf{V_D}}(w^*TD) \to L^{p,\delta}(\mathrm{Hom}^{0,1}(T(\dot{S}), w^*TD)).$$

By the choice of almost complex structures and removal of singularities, w extends to to a holomorphic sphere in D. Let

$$D_w\colon W^{1,p}(w^*TD) \to L^p(\mathrm{Hom}^{0,1}(T\mathbb{CP}^1, w^*TD))$$

be the linearized Cauchy–Riemann operator in D at the holomorphic sphere w. The operator D_w is Fredholm independently of the weight, but $\dot{D}_w$ is only Fredholm when the weight δ is not an integer multiple of 2π. For δ sufficiently small these operators have the same Fredholm index and their kernels and cokernels are isomorphic by the map induced by restricting a section of w^*TD to

$\dot{S}$. For a generic choice of data D_w is surjective.

Since $D_{\tilde{v}}$ is upper triangular and both diagonal terms are surjective, this establishes transversality for unaugmented curves in the symplectization. We need to establish transversality for augmented curves, this will involve transversality statements for augmentation planes, as well as transversality for unaugmented curves together with transversality for the evaluation maps at augmentation punctures.

We start by discussing augmentation planes. Lemma 2.4 of [13] establishes a correspondence between finite energy J_W holomorphic planes in W and J_X holomorphic spheres in X with a single intersection with D. The order of contact gives the multiplicity of the Reeb orbit to which the plane converges. We recall the definition of augmentation spheres in X:

Definition 68. *Let $\widehat{\mathcal{M}}^*_X((B_1,\ldots,B_k);J_X)$ denote the moduli space of k J_X holomorphic spheres in X, where each sphere is somewhere injective, no image of a sphere is contained in the image of another sphere, the image of each sphere is not contained in the tubular neighbourhood $\varphi(\mathcal{U})$ of D, and such that each sphere intersects D only at $\infty \in \mathbb{CP}^1$ with order of contact $B_i \bullet D$.*

$\widehat{\mathcal{M}}^*_X((B_1,\ldots,B_k);J_X)$ admits an action by $Aut(\mathbb{C}\mathcal{P}^1,\infty)^k$ and the quotient is denoted by $\mathcal{M}^*_X((B_1,\ldots,B_k);J_X)$.

The constraints on the spheres arising as augmentation planes are justified in [13, Remark 6.7].

We now recall the definition of the evaluation maps. Recall the spaces of k punctured unaugmented curves in the symplectization, $\mathcal{M}^k_\Theta(S_1,S_2,S_0,H_{z,\omega},J_{z,\omega})$ for example. In a neighborhood of an augmentation puncture z_i, with negative cylindrical coordinates $(s,t)\in[-\infty,0]\times S^1$, a solution $v(s,t)\in\mathcal{M}^k_\Theta(S_1,S_2,S_0,H_{z,\omega})$ is asymptotic to a Reeb orbit in $\mathbb{R}\times Y$. Recall that Reeb orbits project to a point in the divisor D. We denote the limit

$$\lim_{s\to-\infty}\pi_D(v(s,t))$$

at the $i'th$ negative puncture by $\pi_D(v(z_i))$. This limit exists by Gromov's removal of singularities.

There are natural evaluation maps at augmentation punctures:

$$
\begin{aligned}
ev\colon & \mathcal{M}^k_\Theta(S_1, S_2, S_0, H_{z,\omega}, J_{z,\omega}) \to D^k \\
& (v, z_1, \dots, z_k) \mapsto (\pi(v(z_1)) \dots \pi(v(z_k))), \\
ev\colon & \mathcal{M}^*_X((B_1, B_2, \dots, B_k); J_D) \to D^k \\
& (v_1, \dots, v_k) \mapsto (v_1(\infty), v_2(\infty), \dots, v_k(\infty)).
\end{aligned}
$$

Then transversality for moduli spaces of augmented curves is now reduced to the transversality of $\mathcal{M}^*_X((B_1, B_2, \dots, B_k); J_X)$ and the evaluation maps above. This is proved in [13, Propsition 5.35] for a generic choice of data.

To conclude transversality for cascades it remains to consider transversality of evaluation maps to the the products of stable/unstable manifolds and flow diagonals in Y.

Recall that The contact distribution ξ defines a connection on the S^1 bundle $S^1 \to Y \to D$, and that for each $j \in \{0, \dots, N+1\}$ we consider a perturbation $f^j_Y \in C^\epsilon_\infty(f_Y, f_D)$. These are perturbations of the Morse function f_Y, which change its gradient vector field in the horizontal direction. Sard-Smale's theorem allows us to achieve transversality in the horizontal direction for a generic choice of perturbation data.

We now consider transversality in the vertical direction. Note that the evaluation maps are transverse to the flow diagonal at every level [13, Lemma 5.4]. For Floer levels it follows from the Reeb-invariance of the solutions and the requirement that the cylinders are not trivial, that the evaluation maps are submersions and hence transverse to all stable and unstable manifolds. This is discussed in detail in [19, Prop. 5.41].

By varying the asymptotic marker at the negative puncture, we obtain that the BV moduli spaces are S^1 bundles. This, coupled with the Reeb invariance of the solutions implies that the BV moduli spaces are in fact $S^1 \times S^1$ bundles and hence the evaluation maps are submersions and transverse to all stable and unstable manifolds. This proves the first statement of the proposition.

We now consider coproduct levels. Since we vary the asymptotic marker at both negative ends, the evaluation maps are transverse to all stable manifolds. It remains to ensure transversality of the evaluation maps at the positive end. This there are countably many such moduli spaces, this will hold by Sard-Smale for a generic choice of Morse function $f_Y \in \mathcal{X}(f_D)$. This proves the second statement of the proposition.

We now consider continuation levels. The only new case we need to consider is transversality in the vertical direction for constant continuation curves, when the continuation map is the identity. In this case, the evaluation maps are with respect to the stable and unstable manifolds of the same critical point, which intersect at the critical point, and the last statement of the proposition follows.

□

7.4 Augmented spheres and chains of pearls

In order to limit the configurations arising when (X, D) is a monotone pair, it will be useful to consider moduli spaces of chain of pearls in the divisor. By our choice of almost complex structures and removal of singularities, curves in the symplectization project to J_D holomorphic curves in the divisor, and augmentation planes in W correspond to planes in X with contact constraints in D.

By the choice of Morse functions on Y, a gradient like trajectory of f_Y projects to a gradient like trajectory of f_D. Consequently, an augmented cascade projects to a cascade in D with augmentation planes in X intersecting D. Such cascades are commonly denoted chain of pearls and in the following we define the unaugmented version.

As usual, $\omega = (\omega_1, \omega_2)$, satisfy $\omega_1 + \omega_2 = 1, \omega_i \in (0, 1)$.

Definition 69. *Let $A \in H_2(D, \mathbb{Z})$. Then $\mathcal{M}^*_{2,k,D}(A; J_D)$ is the moduli space of simple J_D holomorphic spheres in D with 2 distinguished marked points at 0 and ∞ plus a possibly empty collection of k marked points $\{z_0, \dots, z_k\}$. Simple here means that each sphere is either somewhere injective or constant.*

Let $J_{z,\omega}$ denote a family of almost complex structures parameterized by $(z, \omega) \in S^2 \times I$. Then $\mathcal{M}_{3,k,D}(A; J_{z,\omega})$ is the moduli space of $J_{z,\omega}$ holomorphic spheres in D with 3 distinguished marked points at 0, 1 and ∞ plus a possibly empty collection of k marked points $\{z_0, \dots, z_k\}$.

There are natural evaluation maps:

$$\begin{aligned} \mathrm{ev}\colon \mathcal{M}^*_{2,k,D}(A; J_D) &\to D \times D \\ v &\mapsto (v(0), v(\infty)). \end{aligned}$$

and

$$\begin{aligned} \mathrm{ev}\colon \mathcal{M}_{3,k,D}(A; J_z) &\to D \times D \times D \\ v &\mapsto (v(0), v(\infty), v(1)). \end{aligned}$$

We now describe the chain of pearls appearing in the definition of the projection of $\mathbf{\Pi}$ cascades. Note that these will be essentially the same configurations which arise for the differential.

Recall the space of perturbations $C^\infty_\epsilon(f_D)$ consisting of functions $f : D \to \mathbb{R}$ satisfying $||f||_\epsilon < \infty$ and $f(x) = 0$ in a neighborhood of the critical points of f_D. Recall also that $\underset{\Delta}{\times}$ denotes the fiber product over the diagonal of f while $\underset{\Delta_f}{\times}$ denotes the fiber product over the flow diagonal.

Definition 70. *Let q_- and q_+ be critical points of f_D. For $j \in \{1, \ldots, N\}$ we fix a choice of perturbation $f^j \in C^\epsilon_\infty(f_D)$. The moduli space of* ***simple chain of pearls with N cascades***

$$\mathcal{M}^*_{k,D}((A_1, \ldots, A_N); q_-, q_+p; J_D)$$

is the fiber product

$$W^s(q_-) \underset{\Delta}{\times} \mathcal{M}^*_{2,k_N,D}(A_N; J_D) \underset{\Delta_f}{\times} \ldots \underset{\Delta_f}{\times} \mathcal{M}^*_{2,k_1,D}(A_1; J_D) \underset{\Delta}{\times} W^u(q_+).$$

Here the $j'th$ fiber product is taken over the evaluation maps to the flow diagonal or stable/unstable manifold of f^j.

We will also consider simple chains of pearls missing their initial or final flowlines, for in-

stance:

$$\mathcal{M}^*_{k,D}((A_1,\ldots,A_{N_1}),q_1;J_D)$$

which is defined as the fiber product:

$$W^s(q_-)\underset{\Delta}{\times}\mathcal{M}^*_{2,k_N,D}(A_N;J_D)\underset{\Delta_f}{\times}\ldots\underset{\Delta_f}{\times}\mathcal{M}^*_{2,k_1,D}(A_1;J_D).$$

Lemma 71. *[13, Prop 5.26] For a generic choice of almost complex structure J_D the moduli space $\mathcal{M}^*_{k,D}((A_1,\ldots,A_N);q_-,q_+;J_D)$ is smooth of dimension*

$$M(q_-)+\sum_{i\in N}2\langle c_1(TD),A_i\rangle - M(q_+)+2k+N-1.$$

We now describe the moduli space of coproduct chain of pearls:

Definition 72. *Let $q_i, i=0,1,2$ be critical points of f_D. For $j\in\{0,\ldots,N+1\}$ we fix a choice of perturbation $f^j_D\in C^\epsilon_\infty(f_D)$. The moduli space of* ***simple coproduct chain of pearls with*** N ***cascades***

$$\mathcal{M}^*_{N,k,D}(C,(A_1,\ldots,A_N),q_0;q_1,q_2;J_D)$$

consists of elements in the fiber product:

$$\mathcal{M}^*_{k_1,D}((A_1,\ldots,A_{N_1}),q_1;J_D)\underset{\Delta_f}{\times}$$
$$\mathcal{M}^*_{k_2,D}((A_1,\ldots,A_{N_2}),q_2;J_D)\underset{\Delta_f}{\times}\mathcal{M}_{3,k_3,D}(C;J_D)\underset{\Delta_f}{\times}\mathcal{M}^*_{k_0,D}((A_1,\ldots,A_{N_0}),q_0;J_D)$$

with $k_1+k_2+k_3+k_0=k$.

The following lemma follows from the discussion in §7.3:

Lemma 73. *For a generic choice of J_D, and collection of Morse functions f^j_D, the moduli space*

$\mathcal{M}^*_{N,k,D}(C,(A_1,\ldots,A_N),q_0;q_1,q_2;J_D)$ *is a manifold of dimension*

$$M(q_0)+\sum_{j\in N}2\langle c_1(TD),A_j\rangle-\sum M(q_i)+2\langle c_1(TD),C\rangle+2k+N-1.$$

Chapter 8: Computation

In this section we illustrate that the coproduct structure is not trivial by computing it for a cohomology class in $SH^*(T^*S^3)$. We work with homogeneous coordinates $(z_0 : z_1 : z_2 : z_3 : z_4)$ in $\mathbb{CP}^4$. Let X be the complex projective quadric $\sum_{i=1}^4 z_i^2 = z_0^2$ equipped with the restriction of the Fubini- Study symplectic form ω_{FS}. The divisor $D \subset X$ is given by $\sum_{i=1}^4 z_i^2 = 0$ and can be identified with $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then the completion of $X \backslash D$ is symplectomorphic to T^*S^3 [21, §4].

Recall the constant K such that $[D] \in H_{2n-2}(X;\mathbb{Q})$ is Poincaré dual to $[K\Omega_X] \in H^2(X;\mathbb{Q})$ for some $K > 0$. In this example, $[D]$ is Poincaré dual to ω_{FS}, so $K = 1$. Moreover, X is monotone with monotonicity constant $\tau_X = 3$; this means that $\Omega_X(A) = 3\langle c_1(TX), A\rangle$.

We first recap the computation of $SH^*(T^*S^3)$ from [17]. Recall that the generators of the Morse-Bott chain complex associated to H^∞ are given by:

$$SC^*(W, H^\infty) = \left(\bigoplus_{k>0} \bigoplus_{p\in \mathrm{Crit}(f_D)} \mathbb{Z}\langle \check{q}_k, \hat{q}_k\rangle\right) \oplus \left(\bigoplus_{x\in \mathrm{Crit}(f_W)} \mathbb{Z}\langle x\rangle\right). \tag{8.1}$$

The divisor $\mathbb{CP}^1 \times \mathbb{CP}^1$ admits a perfect Morse function $f_{\mathbb{CP}^1\times\mathbb{CP}^1}$ with 4 critical points q_0, q_2^1, q_2^2, q_4. Here the subscript denotes the Morse index. In this case Y is the unit cotangent bundle UT^*S^3, and we choose a Morse function $f_{UT^*S^3}$ with two critical points on $\check{q}_i, \hat{q}_i$ for each critical point of $f_{\mathbb{CP}^1\times\mathbb{CP}^1}$. We also choose a Morse function with critical points e, c in the interior of T^*S^3. Consequently,

$$SC^*(T^*S^3, H^\infty) = \mathbb{Z}\langle e, c\rangle \bigoplus_{k>0} \langle \check{q}_{i,k}, \hat{q}_{i,k}, \check{q}^1_{2,k}, \hat{q}^1_{2,k}, \check{q}^2_{2,k}, \hat{q}^2_{2,k}\rangle.$$

where $i \in 0, 4$ and the subscript k reflects the fact that for each $k > 0$ we have a manifold of orbits of multiplicity k.

In this example $(\tau_X - K)/K = 2$, hence the grading is given as follows:

$$|\hat{q}_{i,k}| = -4k + 5 - i - 1$$
$$|\check{q}_{i,k}| = -4k + 5 - i$$
$$|e| = 0$$
$$|c| = 3$$

In our convention the differential decreases degree and the BV operator increases degree. In [17] Lisi computes the differential for T^*S^3 as follows:

$$\begin{aligned}
d\check{q}_{0,k+1} &= \hat{q}^1_{2,k} + \hat{q}^2_{2,k} \\
d\check{q}^1_{2,k+1} &= \hat{q}_{4,k} + \hat{q}_{0,k+1} \\
d\check{q}^2_{2,k+1} &= \hat{q}_{4,k} + \hat{q}_{0,k+1} \\
d\check{q}_{4,k} &= \hat{q}^1_{2,k} + \hat{q}^2_{2,k} \\
d\check{q}^1_{2,1} &= \hat{q}_{0,1} + e \\
d\check{q}^2_{2,1} &= \hat{q}_{0,1} + e
\end{aligned}$$

In this model we can thus specify generators for $SH^*(T^*S^3, \mathbb{Z}/2)$ as a graded Abelian group:

$$SH^*(T^*S^3, \mathbb{Z}/2) = \mathbb{Z}\langle c, \check{q}_{0,1}, e, \check{q}^1_{2,l} - \check{q}^2_{2,k}, \hat{q}^1_{2,k}, \check{q}_{0,k+1} - \check{q}_{4,k}, \hat{q}_{0,k+1}\rangle$$

8.1 Strategy for computation

We will compute the coproduct in degree -3. In this model the generators of the chain complex in degree -3 are given by: $\check{q}_{0,2}, \check{q}_{4,1}$. The coproduct increases degree by $2n - 3$. In our example $n = 3$ and hence the coproduct of an element in degree -3 will have total degree 0. We need to consider the following possibilities:

1. Curves where both outputs are in degree zero. The generators for the chain complex in degree zero are given by $e, \hat{q}_{0,1}$. There are no rigid curves with outputs in constant curves, see Remark 40, so this case is reduced to counting curves which are asymptotic to $\hat{q}_{0,1}$ at both negative ends.

2. Curves with one output in degree -3 and one in degree 3. The only generator in degree 3 is given by c but again there are no curves with outputs in constant orbits so we don't have such contributions.

3. Curves with outputs in degrees $-1, 1$. The chain complex in degree 1 is generated by $\check{q}_{0,1}$ and in degree -1 by $\check{q}^1_{2,1}$ and $\check{q}^2_{2,1}$. We ruled out curves with checked output orbits above, see Lemma 61.

Hence the only potential outputs up to a sign are $(\hat{q}_{0,1}, \hat{q}_{0,1})$ and since $k_0 - k_1 - k_2 \geqslant 0$ ([13, Lemma 5.43], we only need to consider curves which are asymptotic to $\check{q}_{0,2}$ at the positive end, and to $\hat{q}_{0,1}$ at both negative ends. These curves project to a point in D and to an orbit in Y hence are contained in $\mathbb{R} \times S^1 \in Y \times \mathbb{R}$. We will exhibit the existence of a unique transverse curve as follows:

- We will consider a modification of the coproduct operation, λ', by considering curves which do not have the continuation-BV operation at the negative ends. We we will express the coproduct, λ, as the gluing of λ' followed by continuation-BV cylinders.

- For λ', we will consider a choice of parameterizing data such that at $t = 1/2$ it is a pullback by a double cover from the cylinder.

- We will argue that solutions for $\omega \neq (1/2, 1/2)$ can not occur by the maximum principal.

- For $\omega = (1/2, 1/2)$ the constant solution is the only solution, and this solution is regular.

8.2 Another definition of the coproduct operation

As usual $\omega = (\omega_1, \omega_2)$. Let H_τ be an admissible Hamiltonian. Let $\Phi : \Sigma \to S^1 \times \mathbb{R}$ be a holomorphic double cover of the cylinder, with one interior branch point, and another at the positive puncture. We consider a family of smooth 1-forms β_ω on Σ parameterized by ω and satisfying the following conditions:

- $\beta_{(1/2,1/2)}$ is the pullback $\frac{1}{2}\Phi^* dt$
- In a neighborhood of the positive cylindrical end $\beta_\omega = dt$
- In a neighborhood of the ith cylindrical end $\beta_\omega = \omega_i dt$
- $d(H_\tau \beta_\omega) \leqslant 0$

We will also choose a family of almost complex structures $J_{z,\omega}$ parameterized by $\Sigma \times (0,1)$, and satisfying the following conditions:

- In a neighborhood of the punctures $J_{z,\omega} = J$
- $J_{z,\omega}$ is admissible outside of a compact set of $\mathbb{R} \times Y$.
- $J_{z,\omega}$ is a lift of an almost complex structure $J^D_{z,\omega}$. That is, $J_{z,\omega}$ commutes with the projection map: $\pi_D(J_{z,\omega}\xi) = J^D_{z,\omega}\pi(\xi)$.

Definition 74. *Let H_τ be an admissible Hamiltonian. Let S_0 denote Morse-Bott family of orbits of H_τ. Let $S_{\omega_i}, i \in \{1,2\}$, denote Morse-Bott families of orbits of $\omega_i H$. Then $\mathcal{M}^a(S_{\omega_1}, S_{\omega_2}, S_0, H_\tau, J_{z,\omega})$ is the space of tuples $(v, \omega, [U_i], \Gamma)$ where $([U_i], \Gamma)$ satisfy conditions $2-4$ of Definition 45 and $v : \Sigma\backslash\Gamma \to \mathbb{R} \times Y$ is a finite energy map satisfying*

$$(du - X_H \otimes \beta_\omega)^{(0,1)} = 0 \tag{8.2}$$

and

$$\lim_{|s|\to\infty} v(\epsilon_i(s,t) \in S_i$$

For a generic choice of data, transversality of $\mathcal{M}^a(S_{\omega_1}, S_{\omega_2}, S_0, H_\tau, J_{z,\omega})$ follows from §7. To prove compactness of $\mathcal{M}^a(S_{\omega_1}, S_{\omega_2}, S_0, H_\tau, J_{z,\omega})$ we need to consider one new technical detail. Note that for certain values of $\omega = (\omega_1, \omega_2)$, the Hamiltonian $\omega_i H_\tau$ may have a slope $\omega_i \tau \in \mathcal{S}(\lambda)$ and hence will not be admissible. In this case $\omega_i H_\tau$ will have an unbounded degenerate family of orbits. We need to ensure that there are no curves with asymptotics to those orbits. Note that the degenerate manifold of orbits of $\omega_i H_\tau$ lies in the region where H_τ is linear. Hence the periodic orbits of H_τ lie below the degenerate orbits of ωH_τ. The following follows from Remark 12:

Lemma 75. *For H_τ admissible, finite energy solutions of (8.2) are contained in a compact set $r \leqslant \tau + 2$.*

We now define the split augmented BV-continuation cylinders:

Definition 76. *Let S_{H_i} denote the spaces of connected manifolds of orbits for H_i in $\mathbb{R} \times Y$. The moduli space of* ***split Π-continuation maps***

$$\mathcal{N}^a(S_1, S_0, H_1, H_0)$$

consists of tuples $(\theta, v, [U_i], \Gamma)$, where

- *$v \colon \mathbb{R} \times S^1 \backslash \Gamma \to \mathbb{R} \times Y$ is a finite energy map satisfying the continuation equation*

$$\partial_s v + J_Y(\partial_t v - X_{H_s}) = 0; \tag{8.3}$$

- *$([U_i], \Gamma)$ satisfy conditions $2-4$ of Definition 45.*
- *$\lim_{s \to \infty} v(s,t) \in S_{H_0}$; $\lim_{s \to -\infty} v(s, t+\theta) \in S_{H_1}$.*

Definition 77. *Let $q_i, i = \{0,1,2\}$ be generators of the split Morse-Bott chain complex of H_τ. The moduli space of* ***split broken coproduct cascades***

$$\mathcal{M}^f_\Theta(q_0; q_1, q_2).$$

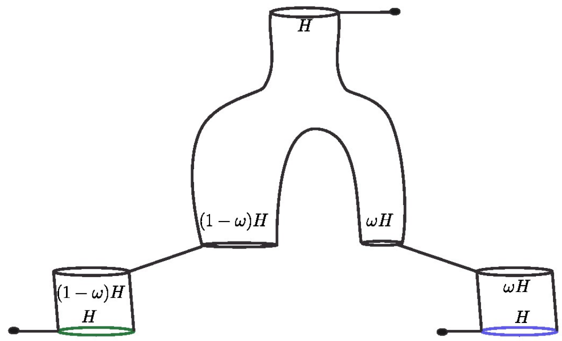

Figure 8.1: The moduli spaces defining λ'. The colors at the negative outputs indicate the S^1 family of asymptotic markers at each end.

consists of elements in the fiber product:

$$W^s_Y(q_1) \underset{\Delta}{\times} \mathcal{N}^a(S_1, S_{\omega_1}, H_\tau, \omega_1 H_\tau) \underset{\Delta_f}{\times}$$
$$W^s_Y(q_2) \underset{\Delta}{\times} \mathcal{N}^a(S_2, S_{\omega_2}, H_\tau, \omega_2 H_\tau) \underset{\Delta_f}{\times}$$
$$\mathcal{M}^a(S_{\omega_1}, S_{\omega_2}, S_0, H_\tau, J_{z,\omega})$$

The broken split coproduct operator, λ' on the chain complex (5.1) is defined as the $\mathbb{Z}/2$ count of rigid curves:

$$\lambda'(q_0) = \sum_{|q_0|=|q_1|+|q_2|-2n+3} \# \left(\mathcal{M}^f_{\Theta,N}(q_0; q_1, q_2)\right) q_1 \otimes q_2. \tag{8.4}$$

Lemma 78. *The operations λ and λ' are chain homotopic.*

Sketch of proof: This is a cobordism argument. We interpolate between the data and equations defining λ and λ'. More precisely, we consider a family of 1 forms β^r_ω, and a family of Hamiltonians $H^r_{z,\omega}$ for $r \in (-\infty, 0]$ satisfying:

- $H^0_{z,\omega}$ satisfies conditions (1)

- For $r \ll 0$, $H^r_{z,\omega} = H_\tau$ for a fixed admissible Hamiltonian H_τ.
- $d(H^r_{z,\omega}\beta^r_\omega) \leqslant 0$.
- $\beta^0_\omega = \beta$ for some fixed 1 form β.
- For $r \ll 0$, β^r_ω satisfies:
  - In the positive cylindrical end, $\beta^r_\omega = dt$.
  - in the ith negative cylindrical end with cylindrical coordinates $\epsilon_i(s,t)$ β^r_ω satisfies conditions (8.2) for $s \geqslant r$, and $\beta^r_\omega = dt$ for $s \gg r$.

The methods of section §7 apply to show that these moduli spaces are smooth. They admit a compactification by broken curves. By considering the boundary of the moduli space of dimension 1 we obtain the required chain homotopy.

□

8.3 Conclusion of computation

Without loss of generality assume that the period of a primitive Reeb orbit is 1.

Lemma 79. *Let H be an admissible Hamiltonian with slope $2+\epsilon$ for ϵ small. In dimension 0 the moduli space $\mathcal{M}^f_\Theta(\check{q}_{0,2}, \hat{q}_{0,1}, \hat{q}_{0,1})$ contains no solutions for $\omega \neq (\frac{1}{2}, \frac{1}{2})$.*

Proof. An admissible Hamiltonian H with slope $2+\epsilon$ has non constant orbits of period 1 and 2. That is,

$$SC^*(T^*S^3, H^{2+\epsilon}) = \mathbb{Z}\langle e, c\rangle \bigoplus_{k\in\{1,2\}} \langle \check{q}_{i,k}, \hat{q}_{i,k}, \check{q}^1_{2,k}, \hat{q}^1_{2,k}, \check{q}^2_{2,k}, \hat{q}^2_{2,k}\rangle.$$

A non trivial solution u of (8.2) is asymptotic at the positive end to a Reeb orbit of multiplicity 2, and at the negative ends to multiplicity 1 Reeb orbits of the Hamiltonians $\omega_i H$ and $\omega_j H$, $\omega_i + \omega_j = 1$. The main point is that the multiplicity 1 periodic orbits of $\omega_i H$ for $\omega_i > \frac{1}{2}$ occur above the multiplicity 2 orbits of H. By the maximum principle 11, see also Remark 12, there are no such solutions. Hence the only potential solution is the constant curve at $\omega = (\frac{1}{2}, \frac{1}{2})$.

Lemma 80. *In dimension* 0 *the moduli space* $\mathcal{M}^f_\Theta(\check{q}_{0,2}, \hat{q}_{0,1}, \hat{q}_{0,1})$ *consists of a single regular curve at* $\omega = (\frac{1}{2}, \frac{1}{2})$.

We will consider first the moduli space $\mathcal{M}^a(S_{\omega_1}, S_{\omega_2}, S_0, H_\tau, J_{z,\omega})$. By the choice of data at $\omega = (\frac{1}{2}, \frac{1}{2})$ a branched cover of a trivial cylinder over a primitive orbit is an element in $\mathcal{M}^a(S_{\omega_1}, S_{\omega_2}, S_0, H_\tau, J_{z,\omega})$. We now demonstrate that this solution is regular. Note that this curve projects to a constant in D, and hence the lower diagonal term in the decomposition 7.2 is surjective, [22, Lemma 6.7.6]. We now consider the upper triangular term. Recall that the formula for the Fredholm index of the operator

$$D^L_v \colon W^{1,p,\delta}_{\mathbf{V_0}}(\dot{S}, \mathbb{R} \oplus \mathbb{R}R) \to L^{p,\delta}(\Lambda^{0,1}T^*\dot{S} \otimes \mathbb{R} \oplus \mathbb{R}R)$$

is given by:

$$\begin{aligned} ind(D^L_v) = n\chi(\dot{S}) + 2c_1(v^*T(\mathbb{R} \times Y) + \sum_{\Gamma_+}(CZ(A_z + \delta) + dim(V_z)) \\ - \sum_{\Gamma_-}(CZ(A_z + \delta) + codim(V_z)), \end{aligned} \tag{8.5}$$

where in our case V_z is the subspace generate by the Reeb vector field R, and $n = 1$.

Using the computation of the Conley-Zehnder indices from [13, §5.3] in (8.5), and that $\chi(\dot{S}) = -1$ we obtain $ind(D^L_v) = 0$. We will also compute an adjusted Chern number required to apply [23, Proposition 2] . This proposition will then establish that the cokernel of D^L_v is one dimensional. For genus 0 curves the adjusted Chern number is given by

$$c_1(L) = ind(D^L_v) - 2 + \#\Gamma_0,$$

where Γ_0 is the number of positive punctures. In our case $\Gamma_0 = 2$ and hence the adjusted Chern number is 0. Proposition 2 of [23] now implies that D^L_v has at most one dimensional cokernel. An application of Sard-Smale establishes transversality for a generic choice of β_ω .

The moduli spaces $\mathcal{N}^a(S_1, S_0, H_{\tau/2}, H_\tau)$ define an operation which is a composition of two operations. The first is the continuation operation

$$\mathcal{K} : HF^*(H_{\tau/2}) \to HF^*(H_\tau)$$

and the second is the BV operation. Arguing as in Lemma 57, the continuation operation is an isomorphism onto its image. The Lemma now follows by combining Lemma 57 for the structure of the BV operation with the computation of $\mathcal{M}^a(S_{\omega_1}, S_{\omega_2}, S_0, H_\tau, J_{z,\omega})$ above.

□

Remark 81. *In the computation of indices above we used a related operator with the same formula as D^L_v but on the space of functions with exponential growth. By [13, Lemma 5.20] these operators have the same Fredholm index and their kernels and cokernels are isomorphic.*

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