THE MINIMAL LOG DISCREPANCY AND ITS APPLICATIONS IN BIRATIONAL GEOMETRY

by

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Abstract

In this article, I will discuss some recent results related to the minimal log discrepancies in dimension two and dimension three based on [HLL22] and [HL20]. I will also discuss some of their applications in birational geometry.

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Chapter 1

Introduction and Main Results

1.1 Background

The classification of algebraic varieties has been the key task for algebraic geometers. One of the most important progresses, the minimal model program, was proposed and initially tackled by Mori in 1980s, and it was later subsequently developed by many people. The minimal model program aims to classify algebraic varieties up to birational equivalence in higher dimensions, and it provides many powerful tools for constructing and studying moduli spaces. In recent years, the minimal model program played a key role in the development of the K-stability theory.

In dimension no less than 3, singularities (such as terminal, Kawamata log terminal (klt), canonical, log canonical (lc) singularities) inevitably and naturally appear in MMP, the study of these singularities and the development of MMP are deeply intertwined. Moreover, in the development of MMP, the log pair (pair for short), which contains a variety and a divisor on this variety satisfying certain properties, appears naturally. Now log pairs are the main objects that birational geometers work with.

The minimal log discrepancy, initially proposed by Shokurov, is one of the most basic but important invariants in birational geometry. One of the main concerns in MMP is the termination problem, that is, whether the minimal model program eventually terminates. Shokurov [Sho04a] proved that his ACC conjecture for mlds [Sho88, Problem 5] together with the lower-semicontinuity conjecture for mlds [Amb99, Conjecture 0.2] imply the termination of all minimal model programs. Minimal log discrepancies also play an important role in the boundedness problem of certain varieties.

1.2 Main Results

In this paper, we focus on the ACC conjecture for mlds, which is initially proposed by Shokurov:

Conjecture 1.2.1 (ACC conjecture for MLDs)**.** *Let n be a positive integer and $\Gamma \subseteq [0,1]$ a set satisfying the descending chain condition (DCC). Then the set*

$$\mathrm{Mld}(n,\Gamma) := \{\mathrm{mld}(X \ni x, B) \mid (X \ni x, B) \text{ is lc},\ \dim X = n, B \in \Gamma\}$$

satisfies the ascending chain condition (ACC).

This conjecture is only known in full generality for surfaces [Ale93] (see [Sho94b, HL20] for other proofs), toric pairs [Amb06], and exceptional singularities [HLS19]. Based on [HLL22], we will discuss the recent progresses on the ACC conjecture for mlds and some related conjectures on the minimal log discrepancies.

The minimal log discrepancy (mld) of a pair (X, B), denoted by $\mathrm{mld}(X, B)$, is defined to be the infimum of log discrepancies of all prime divisors that are exceptional over X (for a specific dfinition, see Definition 2.1.4).

Theorem 1.2.2. *Let $\Gamma \subset [0,1]$ be a DCC set. Then there exists a positive real number δ depending only on Γ, such that*

$$\{\mathrm{mld}(X, B) \mid \dim X = 3, B \in \Gamma\} \cap [1-\delta, +\infty)$$

satisfies the ACC, where $B \in \Gamma$ means that the coefficients of B belong to the set Γ.

Theorem 1.2.2 solves the conjecture for terminal threefold pairs, and it generalizes all the recent progress towards the ACC conjecture for mlds for threefolds [Kaw15b, Theorem 1.3], [Nak16, Corollary 1.5], [Jia21, Theorem 1.3]. Indeed, we prove a slightly stronger version of Theorem 1.2.2 for germs $(X \ni x, B)$ instead of pairs (X, B), see Theorem 3.6.1.

Although the MMP and the abundance conjecture are settled in dimension 3, and we even have a complete classification of terminal threefold singularities including flips as well as divisorial contractions (cf. [Mor85, Rei87, KM92, Kaw01, Kaw02, Kaw03, Kaw05, Kaw12, Yam18]), the ACC conjecture for mlds for terminal threefold pairs remains open. Thus Theorem 1.2.2 strengthens our grasp on terminal threefolds. Note that many important results in birational geometry were first observed and proved for terminal threefolds before generalizing to other larger classes of singularities and to higher dimensions, such as the existence of flips [Mor88, Sho92]. It is our hope that Theorem 1.2.2 will shed light on the study of algebraic varieties in higher dimensions.

1.2.1 Divisors computing the minimal log discrepancies

The following conjecture (see also Conjecture 6.0.2 and Question 6.0.3) is a generalization of a conjecture ([MN18, Conjecture 1.1]) proposed by Nakamura:

Conjecture 1.2.3 ([HL20, Introduction])**.** *Let d be a positive integer and $\Gamma \subset [0, 1]$ a DCC set. Then there exists a positive real number l depending only on d and Γ satisfying the following.*

Assume that $(X \ni x, B)$ is an lc pair of dimension d such that X is $\mathbb{Q}$-Gorenstein and $B \in \Gamma$. Then there exists a prime divisor E over $X \ni x$, such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$ and $a(E, X, 0) \leq l$.

The dimension two case is completely solved in [HL20] (whose proof will be given in the appendix):

Theorem 1.2.4. *Let $\Gamma \subseteq [0,1]$ be a set which satisfies the DCC. Then there exists an integer N depending only on Γ satisfying the following.*

Let $(X \ni x, B)$ be an lc surface germ such that $B \in \Gamma$. Then there exists a prime divisor E over $X \ni x$ such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$ and $a(E, X, 0) \leq N$.

For terminal threefold germs, the above conjecture is also confirmed (in fact a slightly stronger version is given). Moreover, the proof of Theorem 1.2.2 is intertwined with the proof of Theorem 1.2.5.

Theorem 1.2.5. *Let $\Gamma \subset [0,1]$ be a DCC set. Then there exists a positive integer l depending only on Γ satisfying the following. Assume that $(X \ni x, B)$ is a threefold pair such that X is terminal, $B \in \Gamma$, and $\mathrm{mld}(X \ni x, B) \geq 1$. Then there exists a prime divisor E over $X \ni x$, such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$ and $a(E, X, 0) \leq l$.*

Theorem 1.2.5 generalizes a result of Kawakita [Kaw21, Theorem 1.3(ii)], which requires X to be smooth and Γ to be a finite set. When $X \ni x$ is a fixed germ and Γ is a finite set, the existence of such a uniform bound l was predicted by Nakamura [MN18, Conjecture 1.1], and it is equivalent to the ACC conjecture for mlds for fixed germs (cf. [Kaw21, Theorem 4.6]).

Theorem 1.2.2 has many applications towards other topics on threefolds, both for local singularities and global algebraic structures. We list a few of them in the rest part of the introduction.

1.2.2 Reid's general elephant for pairs and Shokurov's boundedness of complements conjecture

For a terminal threefold singularity $x \in X$, we say that a Weil divisor H is an *elephant* of $x \in X$ if $H \in |-K_X|$ and (X, H) is canonical near x. By [Rei87, 6.4(B)], elephant exists for any terminal threefold singularity. As an application of Theorem 1.2.2, we generalize Reid's general elephant theorem to the category of pairs.

Theorem 1.2.6. *Let $\Gamma \subset [0,1] \cap \mathbb{Q}$ be a finite set. Then there exists a positive integer N depending only on Γ satisfying the following.*

Let $(X \ni x, B)$ be a threefold pair such that X is terminal, $B \in \Gamma$, and (X, B) is canonical near x. Then on a neighborhood of x, there exists an element $G \in |-N(K_X + B)|$ such that $(X, B + \frac{1}{N}G)$ is canonical near x.

We remark that in Theorem 1.2.6, if $x \in X$ is a threefold terminal singularity that is not smooth, then we can choose $G \in |-N(K_X+B)|$ such that $(X, B+\frac{1}{N}G)$ is canonical near x and $\mathrm{mld}(X \ni x, B+\frac{1}{N}G) = 1$ (see Theorem 4.2.4).

It is worth mentioning that Reid's general elephant theorem is a special case of Theorem 1.2.6 when $\Gamma = \{0\}$ and x is a closed point, where we can take $N = 1$. We refer the reader to Kollár, Mori, Prohokorv, Kawakita's previous works [KM92, Kaw02, MP08a, MP09, MP21] and reference therein for other results on general elephant for terminal threefolds.

Theorem 1.2.6 is closely related to Theorem 1.2.7, which gives an affirmative answer to Shokurov's conjecture on the boundedness of (ϵ, N)-complements ([CH21, Conjecture 1.1]; see [Sho04b, Conjecture], [Bir04, Conjectures 1.3, 1.4] for some embryonic forms) for terminal threefold germs. We refer the reader to Subsection 2.1.2 for basic notation on complements.

Theorem 1.2.7. *Let $\epsilon \geq 1$ be a real number and $\Gamma \subset [0,1]$ a DCC set. Then there exists a positive integer N depending only on ϵ and Γ satisfying the following.*

Assume that $(X \ni x, B)$ is a threefold pair, such that X is terminal, $B \in \Gamma$, and $\mathrm{mld}(X \ni x, B) \geq \epsilon$. Then there exists an N-complement $(X \ni x, B^+)$ of $(X \ni x, B)$ such that $\mathrm{mld}(X \ni x, B^+) \geq \epsilon$.

We refer the reader to Theorem 4.3.7 for a more detailed version of Theorem 1.2.7.

We remark that the boundedness of $(0, N)$-complements proved in [Bir19, HLS19] plays an important role in several breakthroughs in birational geometry including the proof of Birkar-Borisov-Alexeev-Borisov Theorem and the openness of K-semistability in families of log Fano pairs (cf. [Bir21, Xu20]), while the conjecture on the boundedness of (ϵ, N)-complements was only known for surfaces ([Bir04, Main Theorem 1.6], [CH21, Theorem 1.6]) before.

A special case of Theorem 1.2.7 gives an affirmative answer on Shokurov's index conjecture (cf. [CH21, Conjecture 7.3], [Kaw15a, Question 5.2]) for terminal threefolds.

Theorem 1.2.8. *Let $\epsilon \geq 1$ be a real number and $\Gamma \subset [0,1] \cap \mathbb{Q}$ a DCC set. Then there exists a positive integer I depending only on ϵ and Γ satisfying the following.*

Let $(X \ni x, B)$ be a threefold pair such that X is terminal, $B \in \Gamma$, and $\mathrm{mld}(X \ni x, B) = \epsilon$. Then $I(K_X + B)$ is Cartier near x.

Kawakita showed that for any canonical threefold singularity $x \in X$ with $\mathrm{mld}(X \ni x) = 1$, IK_X is Cartier for some $I \leq 6$ [Kaw15a, Theorem 1.1], hence [Kaw15a, Theorem 1.1] can be viewed as a complementary result to Theorem 1.2.8 when $\epsilon = 1$. We refer the reader to Theorem 4.1.7 for an explicit bound of I when Γ is a finite set.

1.2.3 Other Applications

As an application of our main theorems, we show the ACC for a-lc thresholds (a generalization of lc thresholds, see Definition 2.1.5) for terminal threefolds when $a \geq 1$:

Theorem 1.2.9 (ACC for a-lc thresholds for terminal threefolds)**.** *Let $a \geq 1$ be a real number, and $\Gamma \subset [0,1]$, $\Gamma' \subset [0,+\infty)$ two DCC sets. Then the set of a-lc thresholds,*

$$\{a\text{-}\mathrm{lct}(X \ni x, B; D) \mid \dim X = 3, X \textit{ is terminal}, B \in \Gamma, D \in \Gamma'\},$$

satisfies the ACC.

Theorem 1.2.9 implies the ACC for canonical thresholds in dimension 3:

Theorem 1.2.10. *Let* $\Gamma \subset [0,1]$ *and* $\Gamma' \subset [0,+\infty)$ *be two DCC sets. Then the set*

$$\mathrm{CT}(3,\Gamma,\Gamma') := \{\mathrm{ct}(X,B;D) \mid \dim X = 3, B \in \Gamma, D \in \Gamma'\}$$

satisfies the ACC.

It is worth to mention that the canonical thresholds in dimension 3 is deeply related to Sarkisov links in dimension 3 (cf. [Cor95, Pro18]). Moreover, we have a precise description of the accumulation points of $\mathrm{CT}(3,\{0\},\mathbb{Z}_{\geq 1})$:

Theorem 1.2.11. *The set of accumulation points of* $\mathrm{CT}(3,\{0\},\mathbb{Z}_{\geq 1})$ *is* $\{0\} \cup \{\frac{1}{m} \mid m \in \mathbb{Z}_{\geq 2}\}$.

Theorem 1.2.11 plays a crucial role in the proof of Theorem 1.2.2. We refer the reader to Theorem 3.5.3 for a more detailed version of Theorem 1.2.11.

[Ste11, Theorem 1.7] proved Theorem 1.2.10 when $\Gamma = \{0\}$, $\Gamma' = \mathbb{Z}_{\geq 1}$, and X is smooth, and [Che19, Theorem 1.2] proved Theorem 1.2.10 when $\Gamma = \{0\}$ and $\Gamma' = \mathbb{Z}_{\geq 1}$. [Che19, Theorem 1.3] proved that $\frac{1}{2}$ is the largest accumulation point of $\mathrm{CT}(3,\{0\},\mathbb{Z}_{\geq 1})$. We refer the reader to [Shr06, Pro08] for other related results.

Theorem 1.2.12 is another application of our main theorems:

Theorem 1.2.12. *Let* $\Gamma \subset [0,+\infty)$ *be a DCC set. Then the set of non-canonical klt threefold log Calabi-Yau pairs* (X,B) *with* $B \in \Gamma$ *forms a bounded family modulo flops.*

Theorem 1.2.12 is a generalization of [BDS20, Theorem 1.4] for threefolds. Jiang proved Theorem 1.2.12 for the case when $\Gamma = \{0\}$ [Jia21, Theorem 1.6].

We also remark that the assumption "non-canonical klt" is natural and necessary as rationally connected Calabi-Yau varieties are not canonical, and the set of $(X := Y \times \mathbb{P}^1, F_1 + F_2)$ is not birationally bounded, where Y takes all K3 surfaces and F_1, F_2 are two fibers of $X \to \mathbb{P}^1$.

Chapter 2

Preliminaries and Structure of the Proofs

2.1 Preliminaries

We adopt the standard notation and definitions in [KM98, BCHM10] and will freely use them. All varieties are assumed to be normal quasi-projective and all birational morphisms are assumed to be projective. We denote by ξ_n the n-th root of unity $e^{\frac{2\pi i}{n}}$, and denote by $\mathbb{C}[x_1,\dots,x_d]$ (resp. $\mathbb{C}\{x_1,\dots,x_d\}$, $\mathbb{C}[[x_1,\dots,x_d]]$) the ring of power series (resp. analytic power series, formal power series) with the coordinates $x_1,\dots,x_d$.

Let $\mathbb{K}=\mathbb{Q}$ or $\mathbb{R}$ be either the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal variety. A $\mathbb{K}$-*divisor* is a finite $\mathbb{K}$-linear combination $D=\sum d_iD_i$ of prime Weil divisors D_i, and d_i denotes the *coefficient* of D_i in D. A $\mathbb{K}$-*Cartier divisor* is a $\mathbb{K}$-linear combination of Cartier divisors.

We use $\sim_{\mathbb{K}}$ to denote the $\mathbb{K}$-linear equivalence between $\mathbb{K}$-divisors. For a projective morphism $X\to Z$, we use $\sim_{\mathbb{K},Z}$ to denote the relative $\mathbb{K}$-linear equivalence.

2.1.1 Pairs and singularities

Definition 2.1.1. A *contraction* is a projective morphism $f : Y \to X$ such that $f_*\mathcal{O}_Y = \mathcal{O}_X$. In particular, f is surjective and has connected fibers.

Definition 2.1.2. Let $f : Y \to X$ be a birational morphism, and $\operatorname{Exc}(f)$ the exceptional locus of f. We say that f is a *divisorial contraction* (of a prime divisor E) if $\operatorname{Exc}(f) = E$ and $-E$ is f-ample.

Definition 2.1.3 (Pairs, cf. [CH21, Definition 3.2])**.** A *pair* $(X/Z \ni z, B)$ consists of a contraction $\pi : X \to Z$, a (not necessarily closed) point $z \in Z$, and an $\mathbb{R}$-divisor $B \geq 0$ on X, such that $K_X + B$ is $\mathbb{R}$-Cartier over a neighborhood of z and $\dim z < \dim X$. If π is the identity map and $z = x$, then we may use $(X \ni x, B)$ instead of $(X/Z \ni z, B)$. In addition, if $B = 0$, then we use $X \ni x$ instead of $(X \ni x, 0)$. When we consider a pair $(X \ni x, \sum_i b_i B_i)$, where B_i are distinct prime divisors and $b_i > 0$, we always assume that $x \in \operatorname{Supp} B_i$ for each i.

If $(X \ni x, B)$ is a pair for any codimension ≥ 1 point $x \in X$, then we call (X, B) a pair. A pair $(X \ni x, B)$ is called a *germ* if x is a closed point. We say $x \in X$ is a *singularity* if $X \ni x$ is a germ.

Definition 2.1.4 (Singularities of pairs)**.** Let $(X/Z \ni z, B)$ be a pair associated with the contraction $\pi : X \to Z$, and let E be a prime divisor over X such that $z \in \pi(\operatorname{center}_X E)$. Let $f : Y \to X$ be a log resolution of (X, B) such that $\operatorname{center}_Y E$ is a divisor, and suppose that $K_Y + B_Y = f^*(K_X + B)$ over a neighborhood of z. We define $a(E, X, B) := 1 - \operatorname{mult}_E B_Y$ to be the *log discrepancy* of E with respect to (X, B).

For any prime divisor E over X, we say that E is *over* $X/Z \ni z$ if $\pi(\operatorname{center}_X E) = \bar{z}$. If π is the identity map and $z = x$, then we say that E is *over* $X \ni x$. We define

$$\operatorname{mld}(X/Z \ni z, B) := \inf\{a(E, X, B) \mid E \text{ is over } Z \ni z\}$$

to be the *minimal log discrepancy* (*mld*) of $(X/Z \ni z, B)$.

Let ϵ be a non-negative real number. We say that $(X/Z \ni z, B)$ is lc (resp. klt, ϵ-lc,ϵ-klt) if $\mathrm{mld}(X/Z \ni z, B) \geq 0$ (resp. > 0, $\geq \epsilon$, $> \epsilon$). We say that (X, B) is lc (resp. klt, ϵ-lc, ϵ-klt) if $(X \ni x, B)$ is lc (resp. klt, ϵ-lc, ϵ-klt) for any codimension ≥ 1 point $x \in X$.

We say that (X, B) is *canonical* (resp. *terminal*, *plt*) if $(X \ni x, B)$ is 1-lc (resp. 1-klt, klt) for any codimension ≥ 2 point $x \in X$.

For any (not necessarily closed) point $x \in X$, we say that (X, B) is lc (resp. klt, ϵ-lc, ϵ-klt, canonical, terminal) near x if (X, B) is lc (resp. klt, ϵ-lc, ϵ-klt, canonical, terminal) in a neighborhood of x. If X is (resp. klt, ϵ-lc, ϵ-klt, canonical, terminal) near a closed point x, then we say that $x \in X$ is an lc (resp. klt, ϵ-lc, ϵ-klt, canonical, terminal) singularity. We remark that if $(X \ni x, B)$ is lc, then (X, B) is lc near x.

Definition 2.1.5. Let a be a non-negative real number, $(X \ni x, B)$ (resp. (X, B)) a pair, and $D \geq 0$ an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We define

$$a\text{-}\mathrm{lct}(X \ni x, B; D) := \sup\{-\infty, t \mid t \geq 0, (X \ni x, B + tD) \text{ is } a\text{-}lc\}$$

$$(\text{resp. } a\text{-}\mathrm{lct}(X, B; D) := \sup\{-\infty, t \mid t \geq 0, (X, B + tD) \text{ is } a\text{-}lc\})$$

to be the *a-lc threshold* of D with respect to $(X \ni x, B)$ (resp. (X, B)). We define

$$\mathrm{ct}(X \ni x, B; D) := 1\text{-}\mathrm{lct}(X \ni x, B; D)$$

$$(\text{resp. } \mathrm{ct}(X, B; D) := \sup\{-\infty, t \mid t \geq 0, (X, B + tD) \text{ is canonical}\})$$

to be the *canonical threshold* of D with respect to $(X \ni x, B)$ (resp. (X, B)). We define $\mathrm{lct}(X \ni x, B; D) := 0\text{-}\mathrm{lct}(X \ni x, B; D)$ (resp. $\mathrm{lct}(X, B; D) := 0\text{-}\mathrm{lct}(X, B; D)$) to be the *lc threshold* of D with respect to $(X \ni x, B)$ (resp. (X, B)).

Lemma 2.1.6. *Let $(X \ni x, B)$ be a pair such that X is terminal and $\dim x = \dim X - 2$. Let E_1 be the exceptional divisor obtained by blowing up $x \in X$. If $\operatorname{mult}_x B \leq 1$, then*

$$\operatorname{mld}(X \ni x, B) = a(E_1, X, B) = 2 - \operatorname{mult}_x B \geq 1.$$

Moreover, $\operatorname{mld}(X \ni x, B) \geq 1$ *if and only if* $\operatorname{mult}_x B \leq 1$.

Proof. Since X is terminal, by [KM98, Corollary 5.18], X is smooth in codimension 2. Since $\dim x = \dim X - 2$, possibly shrinking X to a neighborhood of x, we may assume that X is smooth. By [KM98, Lemma 2.45], there exists a sequence of blow-ups

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 := X,$$

such that for any $1 \leq i \leq n$,

- f_i is a blow-up of X_{i-1} at a point x_{i-1} of codimension at least 2 with the exceptional divisor E_i,
- X_i is smooth, $\bar{x}_{i-1} := \operatorname{center}_{X_{i-1}} E_n$, and
- $a(E_n, X, B) = \operatorname{mld}(X \ni x, B)$.

In particular, x_{i-1} dominates x, $x_0 = x$, and $\dim x_{i-1} = \dim X - 2$ for $1 \leq i \leq n$. For any $0 \leq i \leq n$, we let B_i be the strict transform of B on X_i. For any $1 \leq i \leq n$, we have

$$f_i^* B_{i-1} = B_i + (\operatorname{mult}_{x_{i-1}} B_{i-1}) E_i.$$

Let U_{i-1} be an open neighborhood of x_{i-1} such that U_{i-1} and $\bar{x}_{i-1}|_{U_{i-1}}$ are both smooth. Then $f_i^{-1}(U_{i-1}) \cap E_i$ is covered by smooth rational curves that are contracted by f_i and whose intersection numbers with E_i are all equal to -1 (cf. [Har77, §2, Theorem 8.24(c)]), from which we may choose a general curve and denote it by C_i, such that $C_i \not\subset \operatorname{Supp} B_i$ and $C_i \cap \bar{x}_i \neq \emptyset$ when $i \neq n$. Thus

$$0 = f_i^* B_{i-1} \cdot C_i = (B_i + (\operatorname{mult}_{x_{i-1}} B_{i-1}) E_i) \cdot C_i = B_i \cdot C_i - \operatorname{mult}_{x_{i-1}} B_{i-1},$$

which implies that $\operatorname{mult}_{x_{i-1}} B_{i-1} = B_i \cdot C_i$. Since x_i dominates x_{i-1} and $\dim x_i = \dim X - 2$ for $1 \leq i \leq n-1$, we may choose C_i so that $C_i \not\subset \bar{x}_i$. For any $1 \leq i \leq n-1$, let $\bar{C}_i$ be the birational transform of C_i on X_{i+1}. We have $\bar{C}_i \not\subset E_{i+1}$. By the projection formula,

$$\operatorname{mult}_{x_{i-1}} B_{i-1} = B_i \cdot C_i = f_{i+1}^* B_i \cdot \bar{C}_i \geq (\operatorname{mult}_{x_i} B_i) E_{i+1} \cdot \bar{C}_i \geq \operatorname{mult}_{x_i} B_i.$$

By induction on i, we have $1 \geq \operatorname{mult}_x B \geq \operatorname{mult}_{x_i} B_i$ for any $0 \leq i \leq n-1$, thus

$$a(E_1, X, B) = 2 - \operatorname{mult}_x B \leq 2 - \operatorname{mult}_{x_{n-1}} B_{n-1} = a(E_n, X_{n-1}, B_{n-1}). \tag{2.1.1}$$

Moreover, since

$$K_{X_i} + B_i = f_i^*(K_{X_{i-1}} + B_{i-1}) + (1 - \operatorname{mult}_{x_{i-1} B_{i-1}}) E_i \geq f_i^*(K_{X_{i-1}} + B_{i-1})$$

for $1 \leq i \leq n$, by induction, we have $K_{X_{n-1}} + B_{n-1} \geq (f_1 \circ \cdots \circ f_{n-1})^*(K_X + B)$, hence

$$a(E_n, X_{n-1}, B_{n-1}) \leq a(E_n, X, B) = \operatorname{mld}(X \ni x, B) \leq a(E_1, X, B). \tag{2.1.2}$$

Lemma 2.1.6 now follows from Inequalities (2.1.1) and (2.1.2). □

2.1.2 Complements

Definition 2.1.7. Let n be a positive integer, ϵ a non-negative real number, $\Gamma_0 \subset (0, 1]$ a finite set, and $(X/Z \ni z, B)$ and $(X/Z \ni z, B^+)$ two pairs. We say that $(X/Z \ni z, B^+)$ is an $(\epsilon, \mathbb{R})$-*complement* of $(X/Z \ni z, B)$ if

- $(X/Z \ni z, B^+)$ is ϵ-lc,

- $B^+ \geq B$, and
- $K_X + B^+ \sim_{\mathbb{R}} 0$ over a neighborhood of z.

We say that $(X/Z \ni z, B^+)$ is an (ϵ, n)-*complement* of $(X/Z \ni z, B)$ if

- $(X/Z \ni z, B^+)$ is ϵ-lc,
- $nB^+ \geq \lfloor (n+1)\{B\} \rfloor + n\lfloor B \rfloor$, and
- $n(K_X + B^+) \sim 0$ over a neighborhood of z.

A $(0, \mathbb{R})$-complement is also called an $\mathbb{R}$-*complement*, and a $(0, n)$-complement is also called an n-*complement*. We say that $(X/Z \ni z, B)$ is $(\epsilon, \mathbb{R})$-complementary (resp. (ϵ, n)-complementary, $\mathbb{R}$-complementary, n-complementary) if $(X/Z \ni z, B)$ has an $(\epsilon, \mathbb{R})$-complement (resp. (ϵ, n)-complement, $\mathbb{R}$-complement, n-complement).

We say that $(X/Z \ni z, B^+)$ is a *monotonic* (ϵ, n)-*complement* of $(X/Z \ni z, B)$ if $(X/Z \ni z, B^+)$ is an (ϵ, n)-complement of $(X/Z \ni z, B)$ and $B^+ \geq B$.

We say that $(X/Z \ni z, B^+)$ is an (n, Γ_0)-*decomposable* $\mathbb{R}$-*complement* of $(X/Z \ni z, B)$ if there exist a positive integer k, $a_1, \dots, a_k \in \Gamma_0$, and $\mathbb{Q}$-divisors $B_1^+, \dots, B_k^+$ on X, such that

- $\sum_{i=1}^k a_i = 1$ and $\sum_{i=1}^k a_i B_i^+ = B^+$,
- $(X/Z \ni z, B^+)$ is an $\mathbb{R}$-complement of $(X/Z \ni z, B)$, and
- $(X/Z \ni z, B_i^+)$ is an n-complement of itself for each i.

Theorem 2.1.8 ([HLS19, Theorem 1.10])**.** *Let d be a positive integer and $\Gamma \subset [0, 1]$ a DCC set. Then there exists a positive integer n and a finite set $\Gamma_0 \subset (0, 1]$ depending only on d and Γ and satisfy the following.*

Assume that $(X/Z \ni z, B)$ is a pair of dimension d and $B \in \Gamma$, such that X is of Fano type over Z and $(X/Z \ni z, B)$ is $\mathbb{R}$-complementary. Then $(X/Z \ni z, B)$ has an (n, Γ_0)-decomposable $\mathbb{R}$-complement. Moreover, if $\bar{\Gamma} \subset \mathbb{Q}$, then $(X/Z \ni z, B)$ has a monotonic n-complement.

2.1.3 Index of canonical threefolds

Definition 2.1.9. Let $(X \ni x, B)$ be a pair such that $B \in \mathbb{Q}$, and $(X^{\mathrm{an}} \ni x, B^{\mathrm{an}})$ the corresponding analytic pair. The *index* (resp. *analytic index*) of $(X \ni x, B)$ is the minimal positive integer I such that $I(K_X + B)$ is (resp. $I(K_{X^{\mathrm{an}}} + B^{\mathrm{an}})$) is Cartier near x.

The following lemma indicates that the index of $X \ni x$ coincides with the analytic index of $X \ni x$. Hence we will not distinguish the index and the analytic index in our paper.

Lemma 2.1.10 ([Kaw88, Lemma 1.10])**.** *Let X be a variety and D a Weil divisor on X. Let X^{an} be the underlying analytic space of X and D^{an} the underlying analytic Weil divisor of D on X^{an}. Then D^{an} is Cartier on X^{an} if and only if D is Cartier on X.*

Theorem 2.1.11 (cf. [Kaw15a, Theorem 1.1])**.** *Let X be a canonical threefold and $x \in X$ a (not necessarily closed) point such that $\mathrm{mld}(X \ni x) = 1$. Then IK_X is Cartier near x for some positive integer $I \leq 6$.*

Proof. If $\dim x = 2$ then K_X is Cartier near x. If $\dim x = 0$, then the theorem follows from [Kaw15a, Theorem 1.1]. If $\dim x = 1$, then we let $f : Y \to X$ be the terminalization of $X \ni x$. By [KM98, Theorem 4.5], Y is smooth over a neighborhood of x. Since $K_Y = f^*K_X$, K_X is Cartier near x by the cone theorem. □

Lemma 2.1.12. *Let $(X \ni x, B)$ be a threefold germ such that $\mathrm{mld}(X \ni x, B) \geq 1$. Let $D \geq 0$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and $t := \mathrm{ct}(X \ni x, B; D)$. Then $\mathrm{mld}(X \ni x, B + tD) = 1$ if one of the following holds:*

1. $\mathrm{mult}_S(B+tD)<1$ *for any prime divisor* $S\subset \mathrm{Supp}\, D$,

2. X *is* $\mathbb{Q}$*-factorial terminal near* x *and* $x\in X$ *is not smooth, and*

3. X *is* $\mathbb{Q}$*-Gorenstein,* $x\in X$ *is not smooth, and* D *is a* $\mathbb{Q}$*-Cartier prime divisor.*

Proof. If $\mathrm{mld}(X\ni x, B+tD)>1$, then $t=\mathrm{lct}(X\ni x,B;D)$. For (1), since $\mathrm{mult}_S(B+tD)<1$ for any prime divisor $S\subset \mathrm{Supp}\, D$, there exists a curve C passing through x, such that $\mathrm{mld}(X\ni \eta_C, B+tD)=0$, where η_C is the generic point of C. By [Amb99, Theorem 0.1], $\mathrm{mld}(X\ni x, B+tD)\le 1+\mathrm{mld}(X\ni \eta_C, B+tD)=1$, a contradiction.

For (2) and (3), by (1), we may assume that there exists a $\mathbb{Q}$-Cartier prime divisor $S\subset \mathrm{Supp}\, D$ such that $\mathrm{mult}_S(B+tD)=1$. By [Sho92, Appendix, Theorem] and [Mar96, Theorem 0.1], there exists a divisor E over $X\ni x$ such that $a(E,X,0)=1+\frac{1}{I}$, where I is the index of $x\in X$. Since $\mathrm{mult}_E(B+tD)\ge \mathrm{mult}_E S\ge \frac{1}{I}$, $a(E,X,B+tD)=a(E,X,0)-\mathrm{mult}_E(B+tD)\le 1$, hence $\mathrm{mld}(X\ni x,B+tD)=1$. □

Theorem 2.1.13. *Let* $(X\ni x, B:=\sum_{i=1}^m b_iB_i)$ *be a threefold germ such that* $\mathrm{mld}(X\ni x,B)\ge 1$, X *is terminal, and each* $B_i\ge 0$ *is a* $\mathbb{Q}$*-Cartier Weil divisor. Then we have the following:*

1. *If* $X\ni x$ *is smooth, then* $\sum_{i=1}^m b_i\le 2$.

2. *If* $X\ni x$ *is not smooth, then* $\sum_{i=1}^m b_i\le 1$. *Moreover, if* $\sum_{i=1}^m b_i=1$, *then* $\mathrm{mld}(X\ni x,B)=1$.

Proof. If $X\ni x$ is smooth, then let E be the exceptional divisor of the blowing-up of X at x. Since $X\ni x$ is smooth, $\mathrm{mult}_E B_i\ge 1$ for each i. Thus

$$1\le \mathrm{mld}(X\ni x,B)\le a(E,X,B)=3-\mathrm{mult}_E B=3-\sum_{i=1}^m b_i\,\mathrm{mult}_E B_i\le 3-\sum_{i=1}^m b_i,$$

and we get (1).

If $X \ni x$ is not smooth, then let I be the index of $X \ni x$. By [Sho92, Appendix, Theorem] and [Mar96, Theorem 0.1], there exists a prime divisor E over $X \ni x$ such that $a(E, X, 0) = 1 + \frac{1}{I}$. Moreover, by [Kaw88, Lemma 5.1], IB_i is Cartier near x, and $I \operatorname{mult}_E B_i \geq 1$ for each i. Thus

$$1 \leq \operatorname{mld}(X \ni x, B) \leq a(E, X, B) = a(E, X, 0) - \operatorname{mult}_E B$$

$$= 1 + \frac{1}{I} - \sum_{i=1}^{m} b_i \operatorname{mult}_E B_i \leq 1 + \frac{1}{I} - \frac{1}{I}\sum_{i=1}^{m} b_i,$$

which implies (2). $\square$

2.1.4 Singular Riemann-Roch formula and Reid basket

Definition 2.1.14. Let $X \ni x$ be a smooth germ such that $\dim X = d$, and $\mathfrak{m}_x$ (resp. $\mathfrak{m}_x^{\mathrm{an}}$) the maximal ideal of the local ring $\mathcal{O}_{X,x}$ (resp. analytic local ring $\mathcal{O}_{X,x}^{\mathrm{an}}$). We say that $x_1, \dots, x_d \in \mathfrak{m}_x$ (resp. $x_1, \dots, x_d \in \mathfrak{m}_x^{\mathrm{an}}$) is a local coordinate system (resp. analytic local coordinate system) of $x \in X$ if the image of $x_1, \dots, x_d$ span the linear space $\mathfrak{m}_x/\mathfrak{m}_x^2$ (resp. $\mathfrak{m}_x^{\mathrm{an}}/(\mathfrak{m}_x^{\mathrm{an}})^2$). We also call $x_1, \dots, x_d$ local coordinates (resp. analytic local coordinates) of $x \in X$.

Definition 2.1.15 (Cyclic quotient singularities)**.** Let d and n be two positive integers, and $a_1, \dots, a_d$ integers. A *cyclic quotient singularity of type* $\frac{1}{n}(a_1, \dots, a_d)$ is the cyclic quotient singularity $(o \in \mathbb{C}^d)/\boldsymbol{\mu}$ given by the action

$$\boldsymbol{\mu} : (x_1, \dots, x_d) \to (\xi_n^{a_1} x_1, \xi_n^{a_2} x_2, \dots, \xi_n^{a_d} x_d)$$

on $\mathbb{C}^d$, where $x_1, \dots, x_d$ are the local coordinates of $\mathbb{C}^d \ni o$. We may also use $(o \in \mathbb{C}^d)/\frac{1}{n}(a_1, \dots, a_d)$ to represent the singularity $(o \in \mathbb{C}^d)/\boldsymbol{\mu}$, and use $(\mathbb{C}^d \ni o)/\frac{1}{n}(a_1, \dots, a_d)$ to represent the germ $(\mathbb{C}^d \ni o)/\boldsymbol{\mu}$. In particular, we may always assume that (see for example [Fuj74, §1]) the cyclic group action is *small*, that is, the action is free in codimension one.

We remark that a cyclic quotient singularity of type $\frac{1}{n}(a_1, \dots, a_d)$ is isolated if and only if $\gcd(a_i, n) = 1$ for $1 \le i \le d$ (see [Fuj74, Remark 1]).

By the terminal lemma (cf. [MS84, Corollary 1.4]), if a cyclic quotient threefold singularity $x \in X$ is terminal, then

$$(x \in X) \cong (o \in \mathbb{C}^3)/\frac{1}{n}(1, -1, b)$$

for some positive integers b, n such that $\gcd(b, n) = 1$. We say that the terminal (cyclic quotient threefold) singularity $x \in X$ is of type $\frac{1}{n}(1, -1, b)$ in this case.

Definition 2.1.16 ([Rei87, Theorem 10.2(2)])**.** For any integers $1 \le u < v$ and real numbers $s_u, \dots, s_v$, we define $\sum_{i=v}^{u} s_i := -\sum_{i=u+1}^{v-1} s_i$ when $v \ge u + 2$, and define $\sum_{i=v}^{u} s_i := 0$ when $v = u + 1$.

Let n be a positive integer and m a real number. We define

$$\overline{(m)}_n := m - \lfloor \frac{m}{n} \rfloor n.$$

Let b, n be two positive integers such that $\gcd(b, n) = 1$. Let $x \in X$ be a terminal cyclic quotient singularity of type $\frac{1}{n}(1, -1, b)$ and D a Weil divisor on X, such that $\mathcal{O}_X(D) \cong \mathcal{O}_X(iK_X)$ for some integer i near x. We define

$$c_x(D) := -i\frac{n^2 - 1}{12n} + \sum_{j=1}^{i-1} \frac{\overline{(jb)}_n(n - \overline{(jb)}_n)}{2n}.$$

We remark that $c_x(D)$ is independent of the choices of i and b by construction.

Definition-Lemma 2.1.17 ([Rei87, (6.4)])**.** Let $x \in X$ be a terminal threefold singularity. By the classification of threefold terminal singularities (cf. [Rei87, (6.1) Theorem], [Mor85, Theorems 12,23,25]), we have an analytic isomorphism $(x \in X) \cong (y \in Y)/\boldsymbol{\mu}$ for some isolated cDV singularity $(y \in Y) \subset (o \in \mathbb{C}^4)$ and cyclic group action $\boldsymbol{\mu}$ on $o \in \mathbb{C}^4$. Moreover, $y \in Y$ is defined by an equation $(f = 0) \subset (o \in \mathbb{C}^4)$ with analytic local coordinates x_1, x_2, x_3, x_4, and there exists $1 \le i \le 4$ such that f/x_i is a rational function that is invariant under the $\boldsymbol{\mu}$-action. Now we consider the 1-parameter deformation Y_λ of Y, such that Y_λ is given by $(f + \lambda x_i = 0)$. Then the deformation Y_λ is compatible with the action $\boldsymbol{\mu}$, and we let $X_\lambda := Y_\lambda/\boldsymbol{\mu}$ for each λ. For a general $\lambda \in \mathbb{C}$, the singularities of X_λ are terminal cyclic quotient singularities $Q_1, \dots, Q_m$ for some positive integer m. We have the following.

1. $Q_1, \dots, Q_m$ only rely on $x \in X$ and are independent of the choice of λ, and we define the set of *fictitious singularities* of $x \in X$ to be $I_x := \{Q_1, \dots, Q_m\}$.

2. For any $\mathbb{Q}$-Cartier Weil divisor D on X, D is deformed to a Weil divisor D_λ on X_λ. We define $c_x(D) := \sum_{j=1}^m c_{Q_j}(D_\lambda)$.

For such a general λ, X_λ is called a *Q-smoothing* of $x \in X$.

The following theorem indicates that $c_x(D)$ is well-defined.

Theorem 2.1.18 ([Rei87, Theorem 10.2(1)])**.** *Let $x \in X$ be a terminal threefold singularity and D a $\mathbb{Q}$-Cartier Weil divisor on X. Then $c_x(D)$ depends only on the analytic type of $x \in X$ and $\mathcal{O}_X(D)$ near x. Moreover, if $x \in X$ is smooth, then $c_x(D) = 0$.*

Theorem 2.1.19 ([Rei87, Theorem 10.2])**.** *Let X be a projective terminal threefold, and D a $\mathbb{Q}$-Cartier Weil divisor on X. Then*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12}D(D-K_X)(2D-K_X) + \frac{1}{12}D \cdot c_2(X) + \sum_{x \text{ is a closed point}} c_x(D).$$

Definition 2.1.20 (Reid basket for divisorial contractions)**.** Let $f : Y \to X$ be a divisorial contraction of a prime divisor F such that Y is a terminal threefold.

For any closed point $y \in F$, consider a Q-smoothing of $y \in F \subset Y$ as in Definition-Lemma 2.1.17, and let I_y be the corresponding set of fictitious singularities. For each $Q_y \in I_y$, let Y_{Q_y} be the deformed variety on which $Q_y \in Y_{Q_y}$ is a cyclic quotient terminal threefold singularity of type $\frac{1}{r_{Q_y}}(1,-1,b_{Q_y})$, and $F_{Q_y} \subset Y_{Q_y}$ the deformed divisor of $F \subset Y$. Let f_{Q_y} be the smallest non-negative integer such that $F_{Q_y} \sim f_{Q_y} K_{Y_{Q_y}}$ near Q_y. Possibly replacing b_{Q_y} with $r_{Q_y} - b_{Q_y}$, we may assume that $v_{Q_y} := \overline{(f_{Q_y} b_{Q_y})}_{r_{Q_y}} \leq \frac{r_{Q_y}}{2}$. The *Reid basket* for the divisorial contraction $f : Y \to X$ with the exceptional divisor F is defined as

$$J := \{(r_{Q_y}, v_{Q_y}) \mid y \in F, Q_y \in I_y, v_{Q_y} \neq 0\}.$$

2.1.5 Weighted blow-ups over quotient of complete intersection singularities

Definition 2.1.21. A *weight* is a vector $w \in \mathbb{Q}^d_{>0}$ for some positive integer d.

Definition 2.1.22 (Weights of monomials and polynomials)**.** Let d be a positive integer and $w = (w_1, \ldots, w_d) \in \mathbb{Q}^d_{>0}$ a weight. For any vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$, we define $\boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \ldots x_d^{\alpha_d}$, and

$$w(\boldsymbol{x}^{\boldsymbol{\alpha}}) := \sum_{i=1}^{d} w_i \alpha_i$$

to be *the weight of $\boldsymbol{x}^{\boldsymbol{\alpha}}$ with respect to* w. For any analytic power series $0 \neq h := \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d_{\geq 0}} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$, we define

$$w(h) := \min\{w(\boldsymbol{x}^{\boldsymbol{\alpha}}) \mid a_{\boldsymbol{\alpha}} \neq 0\}$$

to be *the weight of h with respect to* w. If $h = 0$, then we define $w(h) := +\infty$.

Definition 2.1.23. Let $h \in \mathbb{C}\{x_1, \ldots, x_d\}$ be an analytic power series and G a group which acts on $\mathbb{C}\{x_1, \ldots, x_d\}$. We say that h is *semi-invariant* with respect to the group action G if for any $g \in G$, $\frac{g(h)}{h} \in \mathbb{C}$. If the group action is clear from the context, then we simply say that h is semi-invariant.

Definition 2.1.24. Let $(X \ni x, B := \sum_{i=1}^k b_i B_i)$ be a threefold germ such that X is terminal and $B_i \geq 0$ are $\mathbb{Q}$-Cartier Weil divisors on X. Let d, n and $m < d$ be positive integers such that

$$(X \ni x) \cong (\phi_1 = \cdots = \phi_m = 0) \subset (\mathbb{C}^d \ni o)/\frac{1}{n}(a_1, \ldots, a_d)$$

for some semi-invariant irreducible analytic power series $\phi_1 \ldots, \phi_m \in \mathbb{C}\{x_1, \ldots, x_d\}$ such that $\operatorname{mult}_o \phi_i > 1$ for each i, here the group action on $\mathbb{C}^d$ is free outside o. By [Kaw88, Lemma 5.1], B_i can be identified with $\big((h_i = 0) \subset (\mathbb{C}^d \ni o)/\frac{1}{n}(a_1, \ldots, a_d)\big)|_X$ for some non-negative integers $a_1, \ldots, a_d$ and some semi-invariant analytic power series $h_i \in \mathbb{C}\{x_1, \ldots, x_d\}$ near $x \in X$, and we say that B_i is defined by $(h_i = 0)$ near x or B_i is locally defined by $(h_i = 0)$ for simplicity. We define the set of *admissible weights* of $X \ni x$ to be

$$\{\frac{1}{n}(w_1, \ldots, w_d) \in \frac{1}{n}\mathbb{Z}^d_{>0} \mid \text{ there exists } b \in \mathbb{Z} \text{ such that } w_i \equiv ba_i \bmod n,\, 1 \leq i \leq d\}.$$

For any admissible weight $w = \frac{1}{n}(w_1, \ldots, w_d)$, we define

$$w(X \ni x) := \frac{1}{n}\sum_{i=1}^{d} w_i - \sum_{i=1}^{m} w(\phi_i) - 1, \text{ and } w(B) := \sum_{i=1}^{k} b_i w(h_i).$$

By construction, $w(B)$ is independent of the choices of b_i and B_i, as we will explain in the following lemma.

Lemma 2.1.25. *Let $d_1, \ldots, d_m, d'_1, \ldots, d'_{m'}$ be real numbers and $D_1, \ldots, D_m, D'_1, \ldots, D'_{m'}$ $\mathbb{Q}$-Cartier Weil divisors such that*

$$\sum_{i=1}^{m} d_i D_i = \sum_{i=1}^{m'} d'_i D'_i.$$

Then $\sum_{i=1}^{m} d_i w(D_i) = \sum_{i=1}^{m'} d'_i w(D'_i)$.

Proof. Let $r_1, \ldots, r_n \in \mathbb{R}$ be a basis for the $\mathbb{Q}$-linear space spanned by the real numbers $\{d_1, \ldots, d_m, d'_1, \ldots, d'_{m'}\}$, we may write

$$\sum_{i=1}^{m} d_i D_i = \sum_{j=1}^{n} r_j \sum_{i=1}^{m} d_{i,j} D_i, \text{ and } \sum_{i=1}^{m'} d'_i D'_i = \sum_{j=1}^{n} r_j \sum_{i=1}^{m'} d'_{i,j} D'_i$$

for some rational numbers $\{d_{i,j}\}_{1\leq i\leq m, 1\leq j\leq n}$ and $\{d'_{i,j}\}_{1\leq i\leq m', 1\leq j\leq n}$. For each prime divisor D, we have

$$\operatorname{mult}_D(\sum_{i=1}^{m} d_i D_i) = \sum_{j=1}^{n} r_j \operatorname{mult}_D(\sum_{i=1}^{m} d_{i,j} D_i) = \sum_{j=1}^{n} r_j \operatorname{mult}_D(\sum_{i=1}^{m'} d'_{i,j} D'_i) = \operatorname{mult}_D(\sum_{i=1}^{m'} d'_i D'_i),$$

hence $\mathrm{mult}_D(\sum_{i=1}^m d_{i,j}D_i) = \mathrm{mult}_D(\sum_{i=1}^{m'} d'_{i,j}D'_i)$ for each j, this implies that for each j, we have

$$\sum_{i=1}^{m} d_{i,j}D_i = \sum_{i=1}^{m'} d'_{i,j}D'_i.$$

Hence

$$\sum_{i=1}^{m} d_i w(D_i) = \sum_{j=1}^{n} r_j \sum_{i=1}^{m} d_{i,j} w(D_i) = \sum_{j=1}^{n} r_j \sum_{i=1}^{m'} d'_{i,j} w(D'_i) = \sum_{i=1}^{m'} d'_i w(D'_i).$$

□

Definition 2.1.26. Let d be a positive integer, μ a real number, and $w := (w_1, \dots, w_d) \in \mathbb{Q}^d_{>0}$, $w' := (w'_1, \dots, w'_d) \in \mathbb{Q}^d_{>0}$ two weights. If $w_i \geq w'_i$ for each i, then we write $w \succeq w'$, and if $w_i = \mu w'_i$ for each i, then we write $w = \mu w'$.

Definition-Lemma 2.1.27. Under the same settings as in Definition 2.1.24. Let $f' : W \to (\mathbb{C}^d \ni o)/\frac{1}{n}(a_1, \dots, a_d)$ be the weighted blow-up at o with the (admissible) weight $w := \frac{1}{n}(w_1, \dots, w_d)$ with respect to the coordinates $x_1, \dots, x_d$ (cf. [KM92, §10] and [Hay99, §3.2]). The exceptional locus for f', denoted by E', is isomorphic to the cyclic quotient of the weighted projective space $\mathbf{P}(w_1, \dots, w_d)/\boldsymbol{\eta}$, where the cyclic group action is given by

$$\boldsymbol{\eta} : [x_1 : \dots : x_d]_w \to [\xi_n^{a_1} x_1 : \dots, \xi_n^{a_d} x_d]_w,$$

and $[x_1 : \dots : x_d]_w$ denotes the image of $(x_1, \dots, x_d) \in \mathbb{C}^d \setminus \{o\}$ under the natural quotient morphism $\mathbb{C}^d \setminus \{o\} \to \mathbf{P}(w_1, \dots, w_d)$. We remark that if the admissible weight w satisfies $w_i \equiv ba_i \mod n$ for $1 \leq i \leq d$ and some integer b such that $\gcd(b, n) = 1$, then $E' \cong \mathbf{P}(w_1, \dots, w_d)$ (cf. [Hay99, §3.2]).

Now we have an induced morphism $f : Y \to X$ by restricting f' to Y, which is the strict transform of X under f'. We call $f : Y \to X$ the *weighted blow-up with weight w at $x \in X$*, and $E := E'|_Y$ the *exceptional divisor of the weighted blow-up $f : Y \to X$ with the weight w at $x \in X$* (cf. [Hay99, §3.7]). If E is an integral scheme, then we also say f *extracts a prime divisor*.

Remark 2.1.28. The weighted blow-up constructed as above depends on the choice of local coordinates. However, we will not mention them later in this paper when the local coordinates for a weighted blow-up are clear from the context.

We will use the following well-known lemma frequently:

Lemma 2.1.29 (cf. [Mor85, the proof of Theorem 2] and [Hay99, §3.9])**.** *Under the same settings as in Definition 2.1.24. For any admissible weight w of $X \ni x$, let E be the exceptional divisor of the corresponding weighted blow-up $f : Y \to X$ at x (cf. Definition-Lemma 2.1.27). If E is a prime divisor, then*

$$K_Y = f^*K_X + w(X \ni x)E, \text{ and } f^*B = B_Y + w(B)E,$$

where B_Y is the strict transform of B on Y. In particular, $a(E, X, B) = 1 + w(X \ni x) - w(B)$.

Proof. Let $f' : W \to Z := (\mathbb{C}^d \ni o)/\frac{1}{n}(a_1, \dots, a_d)$ be the weighted blow-up with the (admissible) weight $w := \frac{1}{n}(w_1, \dots, w_d)$ with respect to the coordinates $x_1, \dots, x_d$ near o. The singular locus of W is contained in the exceptional locus $E' \subset W$ and has codimension ≥ 2. Here, $E' \cong \mathbf{P}(w_1, \dots, w_d)/\boldsymbol{\eta}$, where the cyclic group action is given by

$$\boldsymbol{\eta} : [x_1 : \dots : x_d]_w \to [\xi_n^{a_1} x_1 : \dots, \xi_n^{a_d} x_d]_w,$$

and $[x_1 : \dots : x_d]_w$ denotes the image of $(x_1, \dots, x_d) \in \mathbb{C}^d \setminus \{o\}$ under the natural quotient morphism $\mathbb{C}^d \setminus \{o\} \to \mathbf{P}(w_1, \dots, w_d)$. Since W is covered by open sets U_i with cyclic quotient singularities such that the induced cyclic group action acts diagonally on $\{x_1, \dots, x_d\}$, by [Fuj74, Lemma 1], each irreducible component of the singular locus of W coincide with a coordinate linear space $W_I := \{\cap_{i \in I}(x_i = 0)\} \subset (\mathbf{P}(w_1, \dots, w_d)/\boldsymbol{\eta} \cong E')$ for some proper subset I of $\{1, \dots, d\}$.

Let $\phi_{i,w}$ be the weighted leading terms of $\phi_{i,w}$ for $1 \leq i \leq m$ (see Definition **??**). Let X_i be the hyperplane near $(\mathbb{C}^d \ni o)/\frac{1}{n}(a_1,\dots,a_d)$ defined by $\phi_i = 0$, and Y_i the strict transform of X_i on W. Then $Y_i|_{E'} = (\phi_{i,w} = 0) \subset \mathbf{P}(w_1,\dots,w_d)/\boldsymbol{\eta}$. Since E is an integral scheme and $E \cong (\phi_{1,w} = \cdots = \phi_{m,w} = 0) \subset \mathbf{P}(w_1,\dots,w_d)/\boldsymbol{\eta}$, each $\phi_{i,w}$ is an irreducible power series whose multiplicities at $o \in \mathbb{C}^d$ is greater than two. Hence $(\phi_{i,w} = 0)$ does not contain the coordinate hyperplanes in $\mathbf{P}(w_1,\dots,w_d)/\boldsymbol{\eta}$ for $1 \le i \le m$. In fact, since $Y_J|_{E'} := \cap_{i\in J} Y_i|_{E'}$ is integral for each proper subset $J \subset \{1,\dots,m\}$, $Y_J|_{E'}$ does not contain the coordinate linear space W_I as its irreducible component for any I. This implies that Y_J does not contain any codimension one locus that is a codimension $|J|+1$ irreducible component of the singular locus of W for each proper subset $J \subset \{1,\dots,m\}$. Since $Y \cap E'$ is a local complete intersection, Y and Y_J are smooth in codimension one for each proper subset $J \subset \{1,\dots,m\}$ near Y. Hence

$$(K_W + Y_1 + \cdots + Y_m)|_{Y_1} \cdots |_{Y_m} = K_Y. \tag{2.1.3}$$

By [Rei87, (4.8)] (see also [Jia21, Propostion 2.1]),

$$K_W + Y_1 + \cdots + Y_m = f'^* K_Z + w(X \ni x)E',$$

restricting to Y_i for $i = 1,\dots,m$ successively, we are done. □

2.1.6 Newton polytope

Definition 2.1.30. Let n be a positive integer. A *Newton polytope* $\mathcal{N}$ is a subset of $\mathbb{Z}^n_{\ge 0}$ satisfying the following: for any point $\boldsymbol{x} \in \mathcal{N}$,

$$\boldsymbol{x} + \mathbb{Z}^n_{\ge 0} := \{\boldsymbol{x} + \boldsymbol{v} \mid \boldsymbol{v} \in \mathbb{Z}^n_{\ge 0}\} \subset \mathcal{N}.$$

Definition 2.1.31. Let n be a positive integer, $\mathbf{0}$ the origin of $\mathbb{Z}^n$ and $\mathcal{N} \subset \mathbb{Z}^n_{\ge 0}$ a Newton polytope. A *vertex* of $\mathcal{N}$ is a point $\boldsymbol{u} \in \mathcal{N}$, such that for any $\boldsymbol{x} \in \mathcal{N}$ and $\boldsymbol{v} \in \mathbb{Z}^n_{\ge 0}$, if $\boldsymbol{u} = \boldsymbol{x} + \boldsymbol{v}$, then $\boldsymbol{u} = \boldsymbol{x}$ and $\boldsymbol{v} = \mathbf{0}$.

Lemma 2.1.32. *Let $\{\boldsymbol{v}_j\}_{j\in\mathbb{Z}_{\ge 1}}$ be a sequence of vectors in $\mathbb{Z}^n_{\ge 0}$. Then the set $\{\mathcal{P}_j = \cup_{i=1}^{j}(\boldsymbol{v}_i + \mathbb{Z}^n_{\ge 0})\}_{j\in\mathbb{Z}_{\ge 1}}$*

satisfies the ACC under the inclusion of polytopes. Furthermore, for any Newton polytope $\mathcal{N}$ in $\mathbb{Z}^n_{\geq 0}$, there are only finitely many vertices of $\mathcal{N}$.

Proof. Suppose that $\{\mathcal{P}_j\}_{j\in\mathbb{Z}_{\geq 1}}$ does not satisfy the ACC, possibly passing to a subsequence, we may assume that $\{\mathcal{P}_j\}_{j\in\mathbb{Z}_{\geq 1}}$ is strictly increasing. As $\mathbb{Z}_{\geq 0}$ satisfies the DCC, we may find a pair (i, j) such that $i < j$ and $\boldsymbol{v}_j \in \boldsymbol{v}_i + \mathbb{Z}^n_{\geq 0}$. Thus $\mathcal{P}_j = \mathcal{P}_{j-1}$, a contradiction.

Suppose that $\mathcal{N}$ has infinitely many vertices $\boldsymbol{v}_i$, $i \in \mathbb{Z}_{\geq 1}$. Then the set $\{\mathcal{P}_j = \cup_{i=1}^{j}(\boldsymbol{v}_i + \mathbb{Z}^n_{\geq 0})\}_{j\in\mathbb{Z}_{\geq 1}}$ is strictly increasing, a contradiction. □

Theorem 2.1.33 is proved in [Ste11] based on some results from Russian literature, we give a proof here for the reader's convenience.

Theorem 2.1.33 (ACC for Newton polytopes)**.** *Let n be a positive integer, and $\{\mathcal{N}_i\}_{i\in\mathbb{Z}_{\geq 1}}$ a sequence of Newton polytopes in $\mathbb{Z}^n_{\geq 0}$. Then there exists a subsequence $\{\mathcal{N}_{i_j}\}_{j\in\mathbb{Z}_{\geq 1}}$ of Newton polytopes, such that $\mathcal{N}_{i_j} \supseteq \mathcal{N}_{i_{j+1}}$ for every positive integer j.*

Proof. Suppose that the theorem does not hold. Then there exists $i_1 \in \mathbb{Z}_{\geq 1}$ such that $\mathcal{N}_{i_1} \not\supseteq \mathcal{N}_k$ for all positive integers $k > i_1$.

Inductively, we may construct a sequence of positive integers $i_1 < i_2 < \cdots$ such that $\{\mathcal{M}_j := \mathcal{N}_{i_j}\}_{j\in\mathbb{Z}_{\geq 1}}$ satisfies that $\mathcal{M}_l \not\supseteq \mathcal{M}_m$ for all positive integers $m > l$. Then $\{\mathcal{Q}_j := \cup_{k\leq j}\mathcal{M}_k\}_{j\in\mathbb{Z}_{\geq 1}}$ is a strictly increasing sequence of Newton polytopes. So the statement is equivalent to the ACC for Newton polytopes.

By Lemma 2.1.32, any Newton polytope $\mathcal{Q}$ in $\mathbb{Z}^n_{\geq 0}$ can be written as a finite union $\mathcal{Q} = \cup_{\boldsymbol{v}}(\boldsymbol{v} + \mathbb{Z}^n_{\geq 0})$, where each $\boldsymbol{v}$ is a vertex of $\mathcal{Q}$. So we can find a sequence of vectors $\{\boldsymbol{v}_i\}_{i\in\mathbb{Z}_{>0}}$ and a sequence of positive integers $n_1 < n_2 < \cdots$, such that $\mathcal{Q}_j = \cup_{i=1}^{n_j}(\boldsymbol{v}_i + \mathbb{Z}^n_{\geq 0})$. This contradicts Lemma 2.1.32. □

2.1.7 Divisorial contractions between terminal threefold singularities

Let $f : Y \to X$ be a divisorial contraction of a prime divisor E between two terminal threefolds such that $f(E)$ is a closed point on X. Then f is classified into two types: the *ordinary type* and the *exceptional type*. Moreover, in the ordinary type case, any non-Gorenstein singularity on Y which contributes to the Reid basket of f is a cyclic quotient terminal singularity (see the paragraph after [Kaw05, Theorem 1.1]).

Definition 2.1.34. Let $h \in \mathbb{C}\{x_1, \dots, x_d\}$ be an analytic power series. Let $a_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}$ be a monomial for some $a_{\boldsymbol{\alpha}} \in \mathbb{C}$ and $\boldsymbol{\alpha} \in \mathbb{Z}^d_{\geq 0}$. By $a_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}} \in h$, we mean the monomial term $\boldsymbol{x}^{\boldsymbol{\alpha}}$ appears in the analytic power series h with the coefficient $a_{\boldsymbol{\alpha}}$.

Theorem 2.1.35 ([Kaw05, Theorem 1.2])**.** *In the statement of this theorem, d, r, r_1, r_2, α are assumed to be positive integers, λ, μ are assumed to be complex numbers, and g, p, q are assumed to be analytic power series with no non-zero constant terms.*

Let $x \in X$ be a terminal singularity that is not smooth, $f : Y \to X$ a divisorial contraction of a prime divisor E over $X \ni x$ (see Definition 2.1.2), such that Y is terminal over a neighborhood of x. Let n be the index of $X \ni x$. We may write

$$K_Y = f^*K_X + \frac{a}{n}E$$

for some positive integer a. If $f : Y \to X$ is of ordinary type, then one of the following holds:

1. *$x \in X$ is a cA/n type singularity. Moreover, under suitable analytic local coordinates x_1, x_2, x_3, x_4,*
   (a) *we have an analytic identification*
   $$(X \ni x) \cong (\phi := x_1x_2 + g(x_3^n, x_4) = 0) \subset (\mathbb{C}^4 \ni o)/\frac{1}{n}(1, -1, b, 0),$$
   where $g(x_3, x_4) \in (x_3, x_4)^2$,
   (b) *f is a weighted blow-up with the weight $w = \frac{1}{n}(r_1, r_2, a, n)$,*

(c) $x_3^{dn} \in g(x_3^n, x_4)$,

(d) $nw(\phi) = r_1 + r_2 = adn$, and

(e) $a \equiv br_1 \mod n$.

2. $x \in X$ is a cD type singularity. In this case, one of the following holds:

(2.1) Under suitable analytic local coordinates x_1, x_2, x_3, x_4,

(a) we have an analytic identification

$$(X \ni x) \cong (\phi := x_1^2 + x_1 q(x_3, x_4) + x_2^2 x_4 + \lambda x_2 x_3^2 + \mu x_3^3 + p(x_2, x_3, x_4) = 0) \subset (\mathbb{C}^4 \ni o),$$

where $p(x_2, x_3, x_4) \in (x_2, x_3, x_4)^4$,

(b) f is a weighted blow-up with the weight $w = (r+1, r, a, 1)$, where a is an odd integer,

(c) $\mu' x_3^d \in \phi$ for some $\mu' \neq 0$ and an odd integer $d \geq 3$, and if $d = 3$, then $\mu' = \mu$,

(d) $w(\phi) = w(x_2^2 x_4) = w(x_3^d) = 2r + 1 = ad$,

(e) if $q(x_3, x_4) \neq 0$, then $w(x_1 q(x_3, x_4)) = 2r + 1$, and

(f) if $d > 3$, then $\mu = \lambda = 0$.

(2.2) Under suitable analytic local coordinates x_1, x_2, x_3, x_4, x_5,

(a) we have an analytic identification

$$(X \ni x) \cong \begin{pmatrix} \phi_1 := x_1^2 + x_2 x_5 + p(x_2, x_3, x_4) = 0 \\ \phi_2 := x_2 x_4 + x_3^d + q(x_3, x_4) x_4 + x_5 = 0 \end{pmatrix} \subset (\mathbb{C}^5 \ni o),$$

where $p(x_2, x_3, x_4) \in (x_2, x_3, x_4)^4$,

(b) f is a weighted blow-up with the weight $w = (r+1, r, a, 1, r+2)$,

(c) $r + 1 = ad$ and $d \geq 2$,

(d) $w(\phi_1) = 2(r+1)$,

(e) $w(\phi_2) = r + 1$, *and*

(f) *if* $q(x_3, x_4) \neq 0$, *then* $w(q(x_3, x_4)x_4) = r + 1$.

3. $x \in X$ *is a* $cD/2$ *type singularity. In this case, one of the following holds:*

(3.1) *Under suitable analytic local coordinates* x_1, x_2, x_3, x_4,

(a) *we have an analytic identification*

$$(X \ni x) \cong (\phi := x_1^2 + x_1x_3q(x_3^2, x_4) + x_2^2x_4 + \lambda x_2x_3^{2\alpha-1} + p(x_3^2, x_4) = 0) \subset (\mathbb{C}^4 \ni o)/\frac{1}{2}(1, 1, 1, 0),$$

(b) f *is a weighted blow-up with the weight* $w = \frac{1}{2}(r + 2, r, a, 2)$,

(c) $w(\phi) = w(x_2^2x_4) = r + 1 = ad$, *where* a, r *are odd integers,*

(d) *if* $q(x_3^2, x_4) \neq 0$, *then* $w(x_1x_3q(x_3^2, x_4)) = r + 1$, *and*

(e) $x_3^{2d} \in p(x_3^2, x_4)$.

(3.2) *Under suitable analytic local coordinates* x_1, x_2, x_3, x_4, x_5,

(a) *we have an analytic identification*

$$(X \ni x) \cong \begin{pmatrix} \phi_1 := x_1^2 + x_2x_5 + p(x_3^2, x_4) = 0 \\ \phi_2 := x_2x_4 + x_3^d + q(x_3^2, x_4)x_3x_4 + x_5 = 0 \end{pmatrix} \subset (\mathbb{C}^5 \ni o)/\frac{1}{2}(1, 1, 1, 0, 1),$$

(b) f *is a weighted blow-up with the weight* $w = \frac{1}{2}(r + 2, r, a, 2, r + 4)$,

(c) $r + 2 = ad$ *and* d *is an odd integer,*

(d) $w(\phi_1) = r + 2$,

(e) $w(\phi_2) = \frac{r+2}{2}$, *and*

(f) *if* $q(x_3^2, x_4) \neq 0$, *then* $w(q(x_3^2, x_4)x_3x_4) = \frac{r+2}{2}$.

Moreover, if $a \geq 5$, *then* f *is of ordinary type. The cases are summarized in Table 2.1:*

Table 2.1: A summary of Theorem 2.1.35

Case	Type	Local coordinates	w	$w(\phi)$ or $w(\phi_i)$
(1)	cA/n	$(\phi=0)\subset\mathbb{C}^4/\frac{1}{n}(1,-1,b,0)$	$\frac{1}{n}(r_1,r_2,a,n)$	$\frac{r_1+r_2}{n}$
(2.1)	cD	$(\phi=0)\subset\mathbb{C}^4$	$(r+1,r,a,1)$	$2r+1$
(2.2)	cD	$(\phi_1=\phi_2=0)\subset\mathbb{C}^5$	$(r+1,r,a,1,r+2)$	$2(r+1)$ and $r+1$
(3.1)	$cD/2$	$(\phi=0)\subset\mathbb{C}^4/\frac{1}{2}(1,1,1,0)$	$\frac{1}{2}(r+2,r,a,2)$	$r+1$
(3.2)	$cD/2$	$(\phi_1=\phi_2=0)\subset\mathbb{C}^4/\frac{1}{2}(1,1,1,0,1)$	$\frac{1}{2}(r+1,r,a,2,r+4)$	$r+2$ and $\frac{r+2}{2}$

Proof. Most part of this theorem are identical to [Kaw05, Theorem 1.2] but with small differences for further applications. For the reader's convenience, we give a proof here.

Since $f: Y \to X$ is a divisorial contraction of ordinary type and $x \in X$ is not smooth, by [Kaw05, Theorem 1.2], we have the following possible cases.

$x \in X$ is of type cA/n, then we are in case (1). (1.a) and (1.b) follow from [Kaw05, Theorem 1.2(i)]. By [Kaw05, Theorem 1.2(i.a)], (1.e) holds, and we may pick a positive integer d such that $r_1 + r_2 = adn$. By [Kaw05, Theorem 1.2(i.c)] and [Kaw05, Theorem 1.2(i.d)], (1.c) and (1.d) hold.

$x \in X$ is of type cD or $cD/2$, then we are in either case (2) or case (3). Now (2.1.a), (2.1.b), (3.1.a), (3.1.b) follow directly from [Kaw05, Theorems 1.2(ii.a) and 1.2(ii.a.1)], and (2.2.a), (2.2.b), (3.2.a), (3.2.b) follow directly from [Kaw05, Theorem 1.2(ii.b)]. (2.1.d), (3.1.c) follow from [Kaw05, Theorems 1.2(ii.a.1) and 1.2(ii.a.2)], and (2.2.c), (3.2.c) follow from [Kaw05, Theorem 1.3(ii.b.1)]. (2.1.e), (3.1.d) follow from [Kaw05, Theorem 1.2(ii.a.2)], and (2.2.d), (2.2.e), (2.2.f), (3.2.d), (3.2.e), (3.2.f) follow from [Kaw05, Theorem 1.2(ii.b.2)]. For (2.1.f), if $d > 3$, by (2.1.d) we have $\mu = 0$. Assume that $\lambda \neq 0$. By (2.1.d), $w(x_2x_3^2) = r + 2a \geq 2r + 1$, hence $2a \geq r + 1$ and $2r + 1 = ad > \frac{1}{4}(2r + 1)d$. It follows that $d < 4$, a contradiction.

(2.1.c), (3.1.e) are not contained in the statement of [Kaw05, Theorem 1.2], however, they are implied by the proofs of the corresponding results. To be more specific, (2.1.c) is stated in [Che15, §4, Case Ic] and

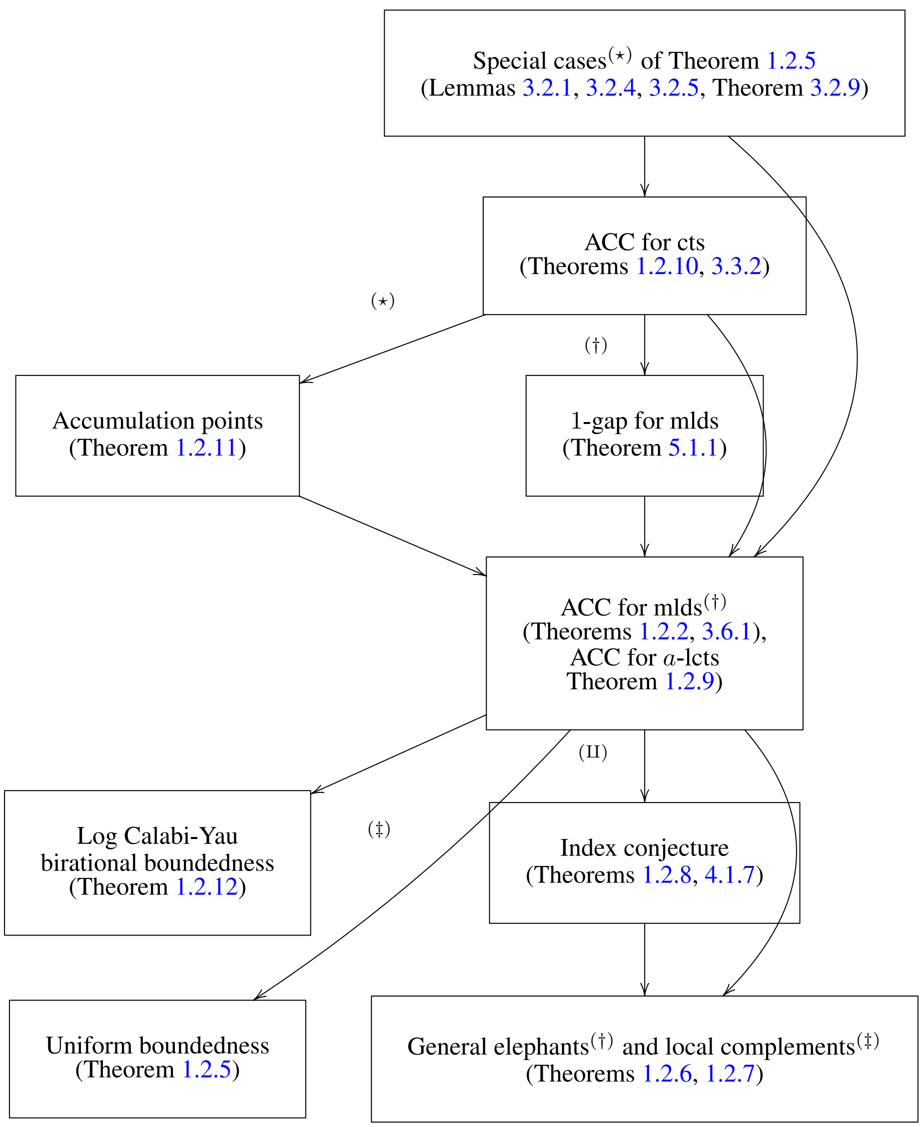
(3.1.e) is stated in [CH11, Page 13, Line 10].

For the moreover part, the Theorem follows directly from [Kaw05, Theorem 1.3]. $\square$

2.2 Sketch of the Proofs

The proof of Theorem 1.2.2 is intertwined with the proof of Theorem 1.2.5. For simplicity, we only deal with the interval $[1,+\infty)$. Suppose on the contrary, there exists a sequence of threefold germs $\{(X_i \ni x_i, B_i)\}_{i=1}^{\infty}$, where X_i is terminal and $B_i \in \Gamma$ for each i, such that $\{\mathrm{mld}(X_i \ni x_i, B_i)\}_{i=1}^{\infty} \subset (1,+\infty)$ is strictly increasing. Then the index of $X_i \ni x_i$ is bounded from above. We may assume that there exists a finite set Γ_0, such that $\lim_{i\to+\infty} B_i = \bar{B}_i$ and $\bar{B}_i \in \Gamma_0$. By the ACC for threefold canonical thresholds (Theorem 1.2.10) and [Nak16, Corollary 1.3], we may assume that $\mathrm{mld}(X_i \ni x_i, \bar{B}_i) = \alpha \geq 1$. for some constant α. In order to derive a contradiction, it suffices to show a special case of Theorem 1.2.5, that is, there exists a prime divisor $\bar{E}_i$ over $X_i \ni x_i$, such that $a(\bar{E}_i, X_i, \bar{B}_i) = \mathrm{mld}(X_i \ni x_i, \bar{B}_i) = \alpha$, and $a(\bar{E}_i, X_i, 0) \leq l$ for some constant number l, see Step 3 of the proof of Theorem 3.6.1. If $\alpha > 1$, and $x_i \in X_i$ is neither smooth nor of cA/n type for each i, then we show the special case by the uniform canonical rational polytopes (Theorem 3.4.3) and the accumulation points of the set of canonical thresholds in dimension 3 (Theorem 1.2.11). Otherwise, we show the following key fact: there exists a divisorial contraction $Y_i \to X_i$ from a terminal variety Y_i which extracts a prime divisor E_i' over $X_i \ni x_i$ such that $a(E_i', X_i, \bar{B}_i) = \mathrm{mld}(X_i \ni x_i, \bar{B}_i)$. Note that when $\alpha = 1$, we may show the fact by standard tie breaking trick even in higher dimensions, see Lemma 3.1.4. The case when $\alpha > 1$ and either $x_i \in X_i$ is smooth or of type cA/n, which is one of our key observations, depends on the proofs of the classification of divisorial contractions for terminal threefolds [Kaw01, Kaw02, Kaw03, Kaw05, Yam18], see Lemmas 3.2.4, 3.2.5. Finally, Theorem 1.2.2 follows from the key fact and Lemma 3.2.1. We remark that Lemma 3.2.1 implies the ACC for threefold canonical thresholds (Theorem 1.2.10). We provide a flowchart of the structure of the paper (Table 2.2).

Table 2.2: Flowchart of the structure of the paper



($\star$): 3-fold divisorial contraction classification [Kaw01, Kaw02, Kaw03, Kaw05, Yam18]. ($\dagger$): The boundedness of lc complements [Bir19, HLS19]. ($\ddagger$): The theory of uniform rational polytopes [HLS19, CH21]. (II): Singular Riemann-Roch formula.

Chapter 3

Canonical Thresholds and Its Accumulation Points

3.1 Terminal Blow-ups

Definition 3.1.1. Let $(X \ni x, B)$ be an lc pair. We say that x is a *canonical center* of (X, B) if $\mathrm{mld}(X \ni x, B) = 1$ and $\dim x \leq \dim X - 2$. A prime divisor (resp. An analytic prime divisor) E that is exceptional over X is called a *canonical place* of (X, B) if $a(E, X, B) = 1$. Moreover, if $\mathrm{center}_X E = \bar{x}$, then E is called a *canonical place of* $(X \ni x, B)$.

Lemma 3.1.2. *Let* $(X \ni x, B)$ *be a germ such that* X *is terminal and* $\mathrm{mld}(X \ni x, B) = 1$. *Then there exists a pair* (X, B') *that is klt near* x*, such that*

- x *is the only canonical center of* (X, B')*,*
- *there exists exactly one canonical place* E *of* $(X \ni x, B')$*, and*
- $a(E, X, B) = 1$.

Proof. Possibly shrinking (X, B) to a neighborhood of x, we may assume that (X, B) is an lc pair. We play the so-called "tie-breaking trick" and follow the proof of [Kol07, Proposition 8.7.1].

Step 1. Let $f: W \to X$ be a log resolution of (X, B). We may write

$$K_W = f^*K_X + \sum_{i\in\mathfrak{I}} a_iE_i, \text{ and } f^*B = B_W + \sum_{i\in\mathfrak{I}} b_iE_i,$$

where $B_W := f_*^{-1}B$ is smooth, and $\{E_i\}_{i\in\mathfrak{I}}$ is the set of f-exceptional divisors. Let

$$\mathfrak{I}_x := \{i \in \mathfrak{I} \mid \operatorname{center}_X E_i = x\}, \text{ and } \mathfrak{I}_{x,0} := \{i \in \mathfrak{I}_x \mid a_i = b_i\} \neq \emptyset.$$

Then $a_i > 0$ for any $i \in \mathfrak{I}$, and $b_i > 0$, $a_i \geq b_i$ for each $i \in \mathfrak{I}_x$.

Let C be a very ample Cartier divisor such that $x \in \operatorname{Supp} C$. Possibly replacing C with $C' \in H^0(\mathcal{O}_X(kC) \otimes \mathfrak{m}_x)$ for some $k \gg 1$ and some irreducible C' which is sufficiently general, we may assume that C is a prime divisor whose support does not contain any $\operatorname{center}_X E_i$ $i \in \mathfrak{I} \setminus \mathfrak{I}_x$, or any canonical center of (X, B) except x. We may write $f^*C = C_W + \sum_{i\in\mathfrak{I}_x} c_iE_i$, where C_W is the strict transform of C on W, and $c_i > 0$ for each $i \in \mathfrak{I}_x$. Now we may choose a real number $0 < \epsilon \ll 1$, such that

$$t := \min_{i\in\mathfrak{I}_{x,0}} \frac{a_i - (1-\epsilon)b_i}{c_i} = \epsilon \min_{i\in\mathfrak{I}_{x,0}} \frac{a_i}{c_i} < \min_{i\in\mathfrak{I}_x\setminus\mathfrak{I}_{x,0}} \frac{a_i - b_i}{c_i} < \min_{i\in\mathfrak{I}_x\setminus\mathfrak{I}_{x,0}} \frac{a_i - (1-\epsilon)b_i}{c_i},$$

and $(1-\epsilon)B + tC \in [0, 1)$. Let $K_W + D_\epsilon := f^*(K_X + (1-\epsilon)B + tC)$ and $K_W + D := f^*(K_X + B)$. We have

$$D_\epsilon - D = \epsilon(\min_{i\in\mathfrak{I}_{x,0}} \frac{a_i}{c_i} f^*C - f^*B).$$

Consider the finite sets

$$\mathfrak{I}_{1,\epsilon} := \{\operatorname{codim} y - \operatorname{mult}_y D_\epsilon \mid y \in W, f(y) \neq x, \operatorname{codim} f(y) \geq 2\}, \text{ and}$$

$$\mathfrak{I}_{2,\epsilon} := \{\operatorname{codim} y - \operatorname{mult}_y D_\epsilon \mid y \in W, f(y) = x\}.$$

If F is an f-exceptional divisor over $X \ni x'$ for some $x' \neq x$ such that $a(F, X, B) = 1$, then $a(F, X, (1-\epsilon)B + tC) > 1$. By taking ϵ sufficiently small, we may assume that $1 \notin \mathfrak{I}_{1,\epsilon}$. Since B_W is smooth and

$\mathrm{mult}_{E_i} D_\epsilon \le 0$ for any $i \in \mathfrak{I}_x$, $1 \in \mathfrak{J}_{2,\epsilon} \subset [1,+\infty)$. By [CH21, Lemma 3.3], x is the only canonical center of $(X,(1-\epsilon)B+tC)$. Note that $1 \in \mathfrak{J}_{2,\epsilon}$.

Possibly replacing B with $(1-\epsilon)B+tC$, we may assume that x is the only canonical center of (X,B) and (X,B) is klt.

Step 2. We may assume that f is a composition of blow-ups of centers of codimension at least 2, hence there exists an f-ample $\mathbb{Q}$-divisor $-\sum_{i\in\mathfrak{I}} e_iE_i$. Note that $e_i>0$ for each i. Then there exists $i_0 \in \mathfrak{I}_{x,0}$, such that possibly replacing e_{i_0} with a bigger positive rational number, we may assume that $\lambda := \frac{a_{i_0}}{e_{i_0}} < \frac{a_i}{e_i}$ for any $i \in \mathfrak{I}_{x,0}\backslash\{i_0\}$. Moreover, there exists a positive real number $\epsilon_0' < 1$, such that $\lambda' := \epsilon_0'\lambda < \frac{a_i-b_i}{e_i} < \frac{a_i-(1-\epsilon_0')b_i}{e_i}$ for any $i \in \mathfrak{I}_x\backslash\mathfrak{I}_{x,0}$. Let $E := E_{i_0}$. Let H be an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $f^*H - \sum_{i\in\mathfrak{I}} e_iE_i$ is ample. Now $a_i-(1-\epsilon_0')b_i-\lambda'e_i>0$ for any $i\in\mathfrak{I}_x\setminus\{i_0\}$, and $a_{i_0}-(1-\epsilon_0')b_{i_0}-\lambda'e_{i_0}=0$. We have

$$K_W+(1-\epsilon_0')B_W+\lambda'(f^*H-\sum_{i\in\mathfrak{I}}e_iE_i)=f^*(K_X+(1-\epsilon_0')B+\lambda'H)+\sum_{i\in\mathfrak{I}}(a_i-(1-\epsilon_0')b_i-\lambda'e_i)E_i.$$

Let $A_W\ge 0$ be a $\mathbb{Q}$-divisor such that $A_W\sim_{\mathbb{Q}} f^*H-\sum_{i\in\mathfrak{I}}e_iE_i$, the coefficients of the prime components of A_W are sufficiently small, and $\mathrm{Supp}\,A_W\cup\mathrm{Supp}\,B_W\cup_{i\in\mathfrak{I}}\mathrm{Supp}\,E_i$ has simple normal crossings. Consider the pair $(X\ni x, B':=(1-\epsilon_0')B+\lambda'A)$, where A is the strict transform of A_W on X. We may write

$$K_W+D'_{\epsilon_0'} := f^*(K_X+(1-\epsilon')B+\lambda'A).$$

For any $\epsilon'\ge 0$, let

$$\mathfrak{J}'_{1,\epsilon'} := \{\mathrm{codim}\,y-\mathrm{mult}_y D'_{\epsilon'} \mid y\in W, f(y)\neq x, \mathrm{codim}\,f(y)\ge 2\},\ \text{and}$$

$$\mathfrak{J}'_{2,\epsilon',\ge 2} := \{\mathrm{codim}\,y-\mathrm{mult}_y D'_{\epsilon'} \mid y\in W, f(y)=x, \mathrm{codim}\,y\ge 2\}.$$

Since $1\notin\mathfrak{J}'_{1,0}$, by taking ϵ' small enough, we may assume that $1\notin\mathfrak{J}'_{1,\epsilon'}$. Thus x is the only canonical center of (X,B'). By our choices of ϵ_0', λ, and A_W, $\mathcal{J}'_{2,\epsilon_0',\ge 2}\subset(1,+\infty)$. Hence by [CH21, Lemma 3.3], E is the only canonical place of $(X \ni x, B')$. The pair $(X \ni x, B')$ satisfies all the requirements. $\square$

Definition 3.1.3. Let $(X \ni x, B)$ be an lc germ. A *terminal blow-up* of $(X \ni x, B)$ is a birational morphism $f : Y \to X$ which extracts a prime divisor E over $X \ni x$, such that Y is terminal, $a(E, X, B) = \mathrm{mld}(X \ni x, B)$, and $-E$ is f-ample.

Lemma 3.1.4. *Let $(X \ni x, B)$ be a germ such that X is terminal and $\mathrm{mld}(X \ni x, B) = 1$. Then there exists a terminal blow-up $f : Y \to X$ of $(X \ni x, B)$. Moreover, if X is $\mathbb{Q}$-factorial, then Y is $\mathbb{Q}$-factorial.*

Proof. By Lemma 3.1.2, possibly shrinking X to a neighborhood of x, we may assume that (X, B) is klt, and there exists exactly one canonical place E_W of $(X \ni x, B)$. By [BCHM10, Corollary 1.4.3], there exists a birational morphism $g : W \to X$ of (X, B) such that E_W is the only g-exceptional divisor. We may write $K_W + B_W := g^*(K_X + B)$, where B_W is the strict transform of B on W. Since (W, B_W) is klt, $(W, (1+\epsilon)B_W)$ is klt for some positive real number ϵ. Let $\phi : W \dashrightarrow Y$ be the lc model of $(W, (1+\epsilon)B_W)$ over X. Since X is terminal and E_W is a canonical place of $(X \ni x, B)$,

$$K_W + (1+\epsilon)B_W = g^*(K_X + (1+\epsilon)B) - eE_W$$

for some positive real number e. It follows that $-E$ is ample over X, where E is the strict transform of E_W on Y. Thus $f : Y \to X$ is an isomorphism over $X \backslash \{x\}$, and $\mathrm{Exc}(f) = \mathrm{Supp}\, E$.

It suffices to show that Y is terminal. Let F be any prime divisor that is exceptional over Y. If $\mathrm{center}_X\, F = x$, then $a(F, Y, 0) \geq a(F, X, B) > 1$ as E is the only canonical place of $(X \ni x, B)$. If $\mathrm{center}_X\, F \neq x$, then $a(F, Y, 0) = a(F, X, 0) > 1$ as f is an isomorphism over $X \backslash \{x\}$ and X is terminal.

Suppose that X is $\mathbb{Q}$-factorial. Then for any prime divisor $D_Y \neq E$ on Y, $f^* f_* D_Y - D_Y = (\mathrm{mult}_{E_Y} f_* D_Y) E$. It follows that D_Y is $\mathbb{Q}$-Cartier, and Y is $\mathbb{Q}$-factorial. $\square$

3.2 Weak Uniform Boundedness of Divisors Computing MLDs

Lemma 3.2.1. *Let I be a positive integer, $\alpha \geq 1$ a real number, and $\Gamma \subset [0,1]$ a DCC set. Then there exists a positive integer l depending only on I, α and Γ satisfying the following. Assume that*

1. *$(X \ni x, B := \sum_i b_i B_i')$ is a threefold germ,*
2. *X is terminal,*
3. *each $b_i \in \Gamma$ and each $B_i' \geq 0$ is a $\mathbb{Q}$-Cartier Weil divisor,*
4. *$\mathrm{mld}(X \ni x, B) = \alpha$,*
5. *IK_X is Cartier near x, and*
6. *there exists a terminal blow-up (see Definition 3.1.3) $f : Y \to X$ of $(X \ni x, B)$.*

Then there exists a prime divisor E over $X \ni x$, such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$ and $a(E, X, 0) = 1 + \frac{a}{I}$ for some positive integer $a \leq l$.

Proof. Suppose that the lemma does not hold. Then there exists a sequence of threefold germs $\{(X_i \ni x_i, B_i := \sum_{j=1}^{p_i} b_{i,j} B'_{i,j}))\}_{i=1}^{\infty}$ and terminal blow-ups $f_i : Y_i \to X_i$ corresponding to $(X \ni x, B := \sum_i b_i B_i')$ and $f : Y \to X$ as in (1)-(6), such that

- f_i extracts a prime divisor E_i,
- $K_{Y_i} = f_i^* K_{X_i} + \frac{a_i}{I} E_i$ for some positive real number a_i, and
- the following sequence of non-negative integers

$$A_i := \inf\{a_i' \mid F_i \text{ is over } X_i \ni x_i, a(F_i, X_i, B_i) = \alpha, a(F_i, X_i, 0) = 1 + \frac{a_i'}{I}\}$$

  is strictly increasing, and in particular, $\lim_{i \to +\infty} A_i = +\infty$.

Since f_i is a terminal blow-up, $a(E_i, Y_i, B_i) = \alpha$, hence $a_i \geq A_i$. Thus $\lim_{i\to+\infty} a_i = +\infty$. Possibly passing to a subsequence, we may assume that a_i is strictly increasing and $a_i > 5I$ for each i. By [Kaw01, Theorem 1.1] and Theorem 2.1.35, analytically locally, we have an embedding

$$(X_i \ni x_i) \hookrightarrow (\mathbb{C}^{m_i} \ni o)/\frac{1}{n_i}(\alpha_{1,i}, \dots, \alpha_{m,i})$$

for each i, where n_i is the index of $X_i \ni x_i$, $n_i \mid I$, $m_i \in \{3, 4, 5\}$, $\alpha_{1,i}, \dots, \alpha_{m,i} \in \mathbb{Z} \cap [1, I]$, and f_i is an admissible weighted blow-up with the weight $w_i \in \frac{1}{n_i}\mathbb{Z}^{m_i}_{>0}$. Moreover, for each i, j, we may assume that $B'_{i,j}$ is defined by $(h_{i,j} = 0)$ for some semi-invariant analytic power series $h_{i,j}$ near x_i.

By Theorems 2.1.13 and 2.1.33, possibly passing to a subsequence, we may assume that

- there exist positive integers $n, m, \alpha_1, \dots, \alpha_m$ and a non-negative integer p, such that $n_i = n$, $m_i = m$, $(\alpha_{1,i}, \dots, \alpha_{m,i}) = (\alpha_1, \dots, \alpha_m)$, and $p_i = p$ for each i,
- $b_{i,j}$ is increasing for any fixed j, and
- $\mathcal{N}(h_{i,j}) \subset \mathcal{N}(h_{i',j})$ for any $i > i'$ and any j.

By Lemma 2.1.29,

$$\frac{a_i}{I} = w_i(X_i \ni x_i) = w_i(B_i) + \alpha - 1 = \sum_{j=1}^{p} b_{i,j} w_i(h_{i,j}) + \alpha - 1.$$

We will show the following claim:

Claim 3.2.2. *Possibly passing to a subsequence, we may assume that $(X_1 \ni x_1, B_1)$ and $(X_2 \ni x_2, B_2)$ satisfy the following:*

1. *$A_2 > a_1$, and*
2. *there exists an admissible weighted blow-up f' of $(X_2 \ni x_2) \subset (\mathbb{C}^m \ni o)/\frac{1}{n}(\alpha_1, \dots, \alpha_m)$ with the*

weight $w' \in \frac{1}{n}\mathbb{Z}^m_{\geq 1}$, such that $w_1(B_2) \leq w'(B_2)$, $w'(X_2 \ni x_2) = w_1(X_1 \ni x_1)$, and the exceptional divisor E' of f' is an analytic prime divisor.

We proceed the proof assuming Claim 3.2.2. By Lemma **??**, we may assume that E' is a prime divisor over X_2. Since

$$w_1(X_1 \ni x_1) = w_1(B_1) + \alpha - 1 \leq w_1(B_2) + \alpha - 1 \leq w'(B_2) + \alpha - 1 \leq w'(X_2 \ni x_2) = w_1(X_1 \ni x_1),$$

$w'(X_2 \ni x_2) = w'(B_2) + \alpha - 1$ and $a(E', X_2, B_2) = \alpha = \mathrm{mld}(X_2 \ni x_2, B_2)$. Since $a(E', X_2, 0) = 1 + w'(X_2 \ni x_2) = 1 + w_1(X_1 \ni x_1) = 1 + \frac{a_1}{I}$, it follows that $a_1 < A_2 \leq a_1$, a contradiction. This finishes the proof. $\square$

Proof of Claim 3.2.2. Claim 3.2.2(1) follows from the fact that the sequence $\{A_i\}_{i=1}^{\infty}$ is strictly increasing. We now prove Claim 3.2.2(2) case by case. By [Kaw01, Theorem 1.1] and Theorem 2.1.35, we only need to consider the following cases.

Case 1. $X_i \ni x_i$ $(i = 1, 2)$ are all smooth. Then $m = 3$, and analytically locally, $(X_i \ni x_i) \cong (\mathbb{C}^3 \ni o)$. We may take $w' = w_1$ in this case.

Case 2. $X_i \ni x_i$ $(i = 1, 2)$ are all of type cD/n for $n = 1$ or 2. By Theorem 2.1.35(2-3), we only need to consider the following two subcases:

Case 2.1. $m = 4$. $f_i : Y_i \to X_i$ are divisorial contractions as in Theorem 2.1.35(2.1) when $n = 1$ and as in Theorem 2.1.35(3.1) when $n = 2$. In particular, there exist positive integers d_i and r_i, such that $2r_i + n = na_id_i$, and analytically locally,

$$(X_i \ni x_i) \cong (\phi_i = 0) \subset (\mathbb{C}^4 \ni o)/\frac{1}{n}(1,1,1,0)$$

for some semi-invariant analytic power series ϕ_i, and each f_i is a weighted blow-up with the weight $w_i :=$

$\frac{1}{n}(r_i+n, r_i, a_i, n)$. Possibly passing to a subsequence, we may assume that $d_1 \le d_2$. Let $s_2 := \frac{n}{2}(a_1d_2-1)$. Since $\frac{1}{n}(2s_2+n) = a_1d_2$ and $5 \le a_1 < a_2$, by [HLL22, Lemma C.8(1)](1) and [HLL22, Lemma C.10], the weighted blow-up at $x_2 \in X_2$ with the weight $w' := \frac{1}{n}(s_2+n, s_2, a_1, n)$ extracts an analytic prime divisor over $X_2 \ni x_2$, and

$$w'(X_2 \ni x_2) = \frac{a_1}{n} = w_1(X_1 \ni x_1).$$

Since $d_2 \ge d_1$ and $2r_1 + n = na_1d_1$, $s_2 = \frac{n}{2}(a_1d_2-1) \ge \frac{n}{2}(a_1d_1-1) = r_1$, hence $w_1(B_2) \le w'(B_2)$.

Case 2.2. $m = 5$. $f_i : Y_i \to X_i$ are divisorial contractions as in Theorem 2.1.35(2.2) when $n = 1$ and as in Theorem 2.1.35(3.2) when $n = 2$. In particular, there exist positive integers d_i and r_i, such that $r_i + n = a_id_i$, and analytically locally,

$$(X_i \ni x_i) \cong (\phi_{i,1} = \phi_{i,2} = 0) \subset (\mathbb{C}^5 \ni o)/\frac{1}{n}(1,1,1,0,1)$$

for some semi-invariant analytic power series $\phi_{i,1}, \phi_{i,2}$, and each f_i is a weighted blow-up with the weight $w_i := \frac{1}{n}(r_i+n, r_i, a_i, n, r_i+2n)$. Possibly passing to a subsequence, we may assume that $d_1 \le d_2$. Let $s_2 := a_1d_2 - n$. Since $s_2 + n = a_1d_2$ and $5 \le a_1 < a_2$, by [HLL22, Lemma C.9(1)](1) and [HLL22, Lemma C.11], the weighted blow-up at $x_2 \in X_2$ with the weight $w' := \frac{1}{n}(s_2+n, s_2, a_1, n, s_2+2n)$ extracts an analytic prime divisor over $X_2 \ni x_2$, and

$$w'(X_2 \ni x_2) = \frac{a_1}{n} = w_1(X_1 \ni x_1).$$

Since $d_2 \ge d_1$ and $r_1 + n = a_1d_1$, $s_2 = a_1d_2 - n \ge a_1d_1 - n = r_1$, hence $w_1(B_2) \le w'(B_2)$.

Case 3. $m = 4$, $X_i \ni x_i$ $(i = 1, 2)$ are of type cA/n and $f_i : Y_i \to X_i$ are divisorial contractions as in Theorem 2.1.35(1). In particular, possibly passing to a subsequence, there exist positive integers $d_i, r_{1,i}, r_{2,i}, b$, such that $r_{1,i} + r_{2,i} = a_i d_i n$, $b \in [1, n-1]$, $\gcd(b, n) = 1$, and analytically locally,

$$(X_i \ni x_i) \cong (\phi_i = 0) \subset (\mathbb{C}^4 \ni o)/\frac{1}{n}(1, -1, b, 0)$$

for some semi-invariant analytic power series ϕ_i, and each f_i is a weighted blow-up with the weight $w_i := \frac{1}{n}(r_{i,1}, r_{i,2}, a_i, n)$. Possibly passing to a subsequence, we may assume that $d_1 \leq d_2$. Let $s_{2,1} := r_{1,1}$ and $s_{2,2} := a_1 d_2 n - r_{1,1}$. Since $s_{2,1} + s_{2,2} = a_1 d_2 n$ and $5 \leq a_1 < a_2$, by [HLL22, Lemma C.7], the weighted blow-up at $x_2 \in X_2$ with the weight $w' := \frac{1}{n}(s_{2,1}, s_{2,2}, a_1, n)$ extracts an analytic prime divisor over $X_2 \ni x_2$, and

$$w'(X_2 \ni x_2) = \frac{a_1}{n} = w_1(X_1 \ni x_1).$$

Since $d_1 \leq d_2$ and $r_{1,1} + r_{1,2} = a_1 d_1 n$, $s_{2,2} = a_1 d_2 n - r_{1,1} \geq a_1 d_1 n - r_{1,1} = r_{1,2}$, hence $w_1(B_2) \leq w'(B_2)$. $\square$

Now we prove some boundedness results on divisors computing mlds when the germ is either smooth or a terminal singularity of type cA/n.

Lemma 3.2.3. *Let X be a smooth variety of dimension n for some $n \in \mathbb{Z}_{\geq 2}$ and $x \in X$ a closed point. Let $\pi : \tilde{X} \to X$ be the blow-up of X at x with the exceptional divisor E. For any hyperplane section $\tilde{H} \in |\mathcal{O}_E(1)|$ on E, there exists a Cartier divisor H on X, such that $x \in \operatorname{Supp} H$, $\operatorname{mult}_x H = 1$ and $\pi_*^{-1} H|_E = \tilde{H}$.*

Proof. Let $\mathfrak{m}_x$ be the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Then we have a canonical isomorphism (cf. [Har77, §2, Theorem 8.24(b)])

$$E \cong \mathbf{P}^{n-1} = \operatorname{Proj}_{\mathbb{C}} \oplus_{i=0}^{+\infty} \mathfrak{m}_x^i/\mathfrak{m}_x^{i+1},$$

where $\mathfrak{m}_x^0 := \mathcal{O}_{X,x}$. Thus there exists a nonzero element $\tilde{h} \in \mathfrak{m}_x/\mathfrak{m}_x^2$ such that $\tilde{H}$ is defined by $(\tilde{h} = 0)$ on

E. Let $h \in \mathfrak{m}_x$ be a preimage of $\tilde{h}$ under the morphism $\mathfrak{m}_x \to \mathfrak{m}_x/\mathfrak{m}_x^2$, and H the Cartier divisor locally defined by $(h = 0)$ near x. We have $\operatorname{mult}_x H = 1$ and $\pi_*^{-1}H|_E = \tilde{H}$ (cf. [EH00, Exercise III-29] and [EH00, Exercise IV-24]). □

Lemma 3.2.4. *Let $(X \ni x, B)$ be a threefold germ such that X is smooth and $\operatorname{mld}(X \ni x, B) \geq 1$. Then there exists a terminal blow-up (see Definition 3.1.3) $f : Y \to X$ of $(X \ni x, B)$, such that Y is $\mathbb{Q}$-factorial.*

Proof. Let $g_1 : X_1 \to X_0 := X$ be the blow-up at $x \in X$ and F_1 the g_1-exceptional divisor. When $\operatorname{mult}_x B \leq 1$, by [Kaw17, Proposition 6(i)], $a(F_1, X, B) = \operatorname{mld}(X \ni x, B)$, we may take $Y = X_1$, and $f = g_1$ in this case. From now on, we may assume that $\operatorname{mult}_x B > 1$.

Let $g : W \to X_1$ be a birational morphism which consists of a sequence of blow-ups at points with codimension at least two, such that the induced morphism $h : W \to X$ is a log resolution of (X, B). By Lemma 3.2.3, there exists a Cartier divisor H on X passing through x, such that $\operatorname{mult}_x H = 1$, $h^*(H + B)$ is an snc divisor on W, and $H_{X_1} := (g_1^{-1})_* H$ does not contain the center of any g-exceptional divisor on X_1.

Let $t := \operatorname{ct}(X \ni x, B; H)$. Since $\operatorname{mult}_x B > 1$ and $1 \leq a(E_1, X, B+tH) = 3 - \operatorname{mult}_x B - t$, $t < 1$. By Lemma 2.1.12(1), $\operatorname{mld}(X \ni x, B + tH) = 1$. By Lemma 3.1.4, there exists a terminal blow-up $f : Y \to X$ of $(X \ni x, B + tH)$ which extracts a prime divisor E over $X \ni x$ such that Y is $\mathbb{Q}$-factorial. In particular, $a(E, X, B + tH) = \operatorname{mld}(X \ni x, B + tH)$.

It suffices to show that $a(E, X, B) = \operatorname{mld}(X \ni x, B)$. By [Kaw01, Theorem 1.1], under suitable analytic local coordinates (x_1, x_2, x_3), f is the weighted blow-up of X with the weight $(1, a, b)$ for some coprime positive integers a and b. By [Kaw01, Proof of Proposition 3.6, line 6], $\operatorname{mult}_E F_1 = 1$[*]. By construction, $\operatorname{mult}_E H_X = \operatorname{mult}_E(H_{X_1} + F_1) = \operatorname{mult}_E F_1 = 1$. Let F be any prime divisor over $X \ni x$. We have

$$a(F, X, B) - t \operatorname{mult}_F H = a(F, X, B + tH) \geq a(E, X, B + tH) = a(E, X, B) - t \operatorname{mult}_E H.$$

[*] We recall that F_n and E in [Kaw01, Proposition 3.6] are the same divisorial valuation, see [Kaw01, Remark 3.3], and we use the same notion of F_1 and E as in [Kaw01, §3], see [Kaw01, Construnction 3.1].

Since $\mathrm{mult}_F H \geq 1 = \mathrm{mult}_E H$, $a(F, X, B) \geq a(E, X, B)$. It follows that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$, and $f : Y \to X$ is the desired terminal blow-up of $(X \ni x, B)$. $\square$

Lemma 3.2.5. *Let $x \in X$ be a threefold terminal singularity of type cA/n. Then there exists a Cartier divisor C near x satisfying the following.*

Let $(X \ni x, B)$ be a pair such that $\mathrm{mld}(X \ni x, B) \geq 1$. Then there exists a terminal blow-up of $(X \ni x, B)$ (see Definition 3.1.3) which extracts a prime divisor E over $X \ni x$, such that $a(E, X, B + tC) = 1$ and $\mathrm{mult}_E C = 1$, where $t := \mathrm{ct}(X \ni x, B; C)$.

Proof. By [Rei87, (6.1) Theorem] (cf. [Mor85, Theorems 12,23,25]), analytically locally,

$$(X \ni x) \cong (\phi := x_1x_2 + g(x_3^n, x_4) = 0) \subset (\mathbb{C}^4 \ni o)/\frac{1}{n}(1, -1, b, 0),$$

such that $b \in [1, n-1] \cap \mathbb{Z}$, $\gcd(b, n) = 1$, ϕ is a semi-invariant analytic power series, and $x_3^{dn} \in g(x_3^n, x_4)$ for some positive integer d. When $n = 1$, $x \in X$ is a terminal singularity of type cA_m (see [Kaw03, Page 333, Line 11]) for some positive integer m.

Claim 3.2.6. *There exist a Cartier divisor C on X and an integer $k \in \{1, 4\}$ depending only on $x \in X$ that satisfy the following.*

1. *x_k is invariant under the ξ_n-action on $\mathbb{C}^4$, and C is a Cartier divisor locally defined by an invariant analytic power series $(x_k + h = 0)$, where $h \in (\mathfrak{m}_o^{\mathrm{an}})^2$.*
2. *Let $t := \mathrm{ct}(X \ni x, B; C)$. Then $\mathrm{mld}(X \ni x, B + tC) = 1$, and there exists a terminal blow-up f of $(X \ni x, B + tC)$ which extracts a prime divisor E over $X \ni x$.*
3. *Possibly choosing a new local analytic coordinates x_1', x_2', x_3', x_4', where $x_k = x_k' + p_k'$ for some $p_k' \in \mathfrak{m}_o^{\mathrm{an}}$ such that $\lambda x_k' \notin p_k'$ for any $\lambda \in \mathbb{C}^*$, analytically locally, f is a weighted blow-up with the weight $w = (w(x_1'), w(x_2'), w(x_3'), w(x_4'))$ such that $w(x_k') = 1$.*

We proceed the proof assuming Claim 3.2.6. By Claim 3.2.6(1,3), C is locally defined by $(x'_k + h'_k = 0)$ for some $h'_k \in \mathfrak{m}_o^{\mathrm{an}}$ such that $\lambda x'_k \notin h'_k$ for any $\lambda \in \mathbb{C}^*$ under the new coordinates x'_1, x'_2, x'_3, x'_4, hence $1 \leq \operatorname{mult}_E C = w(x'_k + h') \leq w(x'_k) = 1$, and $\operatorname{mult}_E C = 1$. Let $F \neq E$ be any prime divisor over $X \ni x$. We have

$$a(F, X, B) - t \operatorname{mult}_F C = a(F, X, B + tC) \geq a(E, X, B + tC) = a(E, X, B) - t \operatorname{mult}_E C.$$

Since $\operatorname{mult}_F C \geq 1 = \operatorname{mult}_E C$, $a(F, X, B) \geq a(E, X, B)$. It follows that $a(E, X, B) = \operatorname{mld}(X \ni x, B)$, hence f is a terminal blow-up of $(X \ni x, B)$. □

Proof of Claim 3.2.6. We have an analytic isomorphism

$$\psi : \tilde{X} \ni \tilde{x} \to \tilde{Y} := (\phi : x_1 x_2 + g(x_3^n, x_4) = 0) \subset (\mathbb{C}^4 \ni o),$$

where $\pi : \tilde{X} \ni \tilde{x} \to X \ni x$ is the index one cover (cf. [KM98, Definition 5.19]). Under the analytic isomorphism ψ, the ξ_n-action on $\tilde{Y}$ induces the cyclic group action on $\tilde{X} \ni \tilde{x}$ which corresponds to π. By [HLL22, Lemma B.7], we can find a Cartier divisor $\tilde{C}$ on $\tilde{X}$ whose image under ψ is locally defined by $(x_k + h = 0)$ for some $h \in (\mathfrak{m}_o^{\mathrm{an}})^2$, and $x_k + h$ is invariant under ξ_n-action. Set $C := \pi(\tilde{C})$, we finish the proof of Claim 3.2.6(1). For Claim 3.2.6(2), since C is a prime divisor that is Cartier, by Lemma 2.1.12(3), $\operatorname{mld}(X \ni x, B + tC) = 1$. By Lemma 3.1.4, there exists a terminal blow-up of $(X \ni x, B + tC)$ which extracts a prime divisor E over $X \ni x$. Now we prove Claim 3.2.6(3) case by case.

Case 1. $n \geq 2$. By [Kaw05, Theorem 1.3], f is a divisorial contraction of ordinary type. By Theorem 2.1.35(1) and [Kaw05, Lemmas 6.1, 6.2 and 6.5], there exist analytic local coordinates x'_1, x'_2, x'_3, x'_4, such that analytically locally, f is a weighted blow-up with the weight $w := \frac{1}{n}(r'_1, r'_2, a, n)$, where r'_1, r'_2, a are positive integers such that $an \mid r'_1 + r'_2$. Moreover, by [Kaw05, Proof of Lemma 6.3, Line 7], $x'_4 = x_4 + x_1 p$ for some $p \in \mathfrak{m}_o^{\mathrm{an}}$. In this case, we take $k = 4$.

Case 2. $n = 1$ and $m \geq 2$. By [Kaw03, Theorem 1.13] and [Yam18, Theorem 2.6], there exist analytic local coordinates x'_1, x'_2, x'_3, x'_4, such that analytically locally, f is a weighted blow-up with the weight $w = (r_1, r_2, a, 1)$ for some positive integers r_1, r_2, a. Moreover, by [Kaw03, Proof of Lemma 6.1, Page 309, Line 5], the coordinates change relation for x_4 is given by $x_4 = x'_4 + c'x'_i$ for some $1 \leq i \leq 3$ and $c' \in \mathbb{C}$. Thus we may take $k = 4$.

Case 3. $n = 1$ and $m = 1$. By [Kaw02, Theorem 1.1], there exist analytic local coordinates x'_1, x'_2, x'_3, x'_4, such that analytically locally, f is a weighted blow-up with the weight w, where either $w = (s, 2a - s, a, 1)$ for some positive integers s and a, or $w = (1, 5, 3, 2)$. Moreover, by [Kaw02, Claim 6.13], the change of coordinates relations for x_i is given by $x_i = x'_i + p'_i(x'_4)$ and $x_4 = x'_4$ for $p'_i \in \mathfrak{m}_o^{\mathrm{an}}$ and $1 \leq i \leq 3$. Thus we may take $k = 1$ or 4. $\square$

Lemma 3.2.7. *Let $\Gamma \subset [0, 1]$ be a set such that $\gamma_0 := \inf\{b \mid b \in \Gamma \setminus \{0\}\} > 0$. Let $(X \ni x, B)$ be a threefold germ, such that*

- *$x \in X$ is a terminal singularity of type cA/n for some $n > N := \lceil \frac{3}{\gamma_0} \rceil$,*
- *$B := \sum_i b_i B_i$ for some $b_i \in \Gamma$, where $B_i \geq 0$ are $\mathbb{Q}$-Cartier Weil divisors,*
- *$\mathrm{mld}(X \ni x, B) \geq 1$, and*
- *there exists a terminal blow-up (see Definition 3.1.3) $f : Y \to X$ of $(X \ni x, B)$ which extracts a prime divisor E over $X \ni x$, such that $a(E, X, 0) = 1 + \frac{a}{n}$ for some positive integer $a \geq 3$.*

Then there exists a prime divisor $\bar{E}$ over $X \ni x$ such that $a(\bar{E}, X, B) = \mathrm{mld}(X \ni x, B)$ and $a(\bar{E}, X, 0) = 1 + \frac{3}{n}$. Moreover,

1. *if $\mathrm{mld}(X \ni x, B) > 1$, then $a = 3$, and*

2. *if* $\mathrm{mld}(X \ni x, B) = 1$ *and* Γ *is either a DCC set or an ACC set, then* $b_i \in \Gamma_0$ *for some finite set* Γ_0 *depending only on* Γ.

Proof. Since $n > 1$, by [Kaw05, Theorem 1.3], $f : Y \to X$ is a divisorial contraction of ordinary type as in Theorem 2.1.35(1). In particular, under suitable analytic local coordinates x_1, x_2, x_3, x_4, there exist positive integers r_1, r_2, b, d such that $r_1 + r_2 = adn$, $b \in \mathbb{Z} \cap [1, n-1]$, $\gcd(b, n) = 1$, $a \equiv br_1 \mod n$, and analytically locally,

$$(X \ni x) \cong (\phi(x_1, x_2, x_3, x_4) = 0) \subset (\mathbb{C}^4 \ni o)/\frac{1}{n}(1, -1, b, 0)$$

for some invariant analytic power series ϕ, and $f : Y \to X$ is a weighted blow-up with the weight $w := \frac{1}{n}(r_1, r_2, a, n)$. Assume that each B_i is locally defined by $(h_i = 0)$ for some semi-invariant analytic power series h_i.

Since $n > N \geq \frac{3}{\gamma_0}$, we can pick positive integers s_1, s_2, such that

- $s_1 + s_2 = 3dn$,
- $3 \equiv bs_1 \mod n$, and
- $s_1, s_2 > n$.

Let $\bar{w} := \frac{1}{n}(s_1, s_2, 3, n)$. Since $a \geq 3$, by [HLL22, Lemma C.7], the weighted blow-up with the weight $\bar{w}$ extracts an analytic prime divisor $\bar{E}$ over $X \ni x$, such that $a(\bar{E}, X, 0) = 1 + \bar{w}(X \ni x) = 1 + \frac{3}{n}$. By [HLL22, Lemma C.6], we may assume that $\bar{E}$ is a prime divisor over $X \ni x$. Since $a(\bar{E}, X, B) = 1 + \bar{w}(X \ni x) - \bar{w}(B) \geq \mathrm{mld}(X \ni x, B) \geq 1$,

$$\gamma_0 > \frac{3}{n} = \bar{w}(X \ni x) \geq \bar{w}(B) = \sum_i b_i \bar{w}(B_i) \geq \gamma_0 \sum_i \bar{w}(B_i),$$

which implies that $\bar{w}(h_i) = \bar{w}(B_i) < 1$ for each i. Since $\bar{w}(x_1) = \frac{s_1}{n} > 1$, $\bar{w}(x_2) = \frac{s_2}{n} > 1$, and $\bar{w}(x_4) = 1$, for each i, there exists a positive integer l_i, such that up to a scaling of h_i, $x_3^{l_i} \in h_i$ for each i, and $\bar{w}(B_i) = \bar{w}(h_i) = \bar{w}(x_3^{l_i})$. In particular,

$$\operatorname{mult}_{\bar{E}} B = \bar{w}(B) = \sum_i b_i l_i \bar{w}(x_3) = \frac{3}{n}\sum_i b_i l_i,$$

and $1 + \frac{3}{n} \geq \bar{w}(B) + \operatorname{mld}(X \ni x, B) = \frac{3}{n}\sum_i b_i l_i + \operatorname{mld}(X \ni x, B)$. This implies that

$$\operatorname{mld}(X \ni x, B) - 1 \leq \frac{3}{n}(1 - \sum_i b_i l_i). \tag{3.2.1}$$

On the other hand,

$$\operatorname{mult}_E B = w(B) = \sum_i b_i w(B_i) \leq \sum_i b_i w(x_3^{l_i}) = \frac{a}{n}\sum_i b_i l_i,$$

and

$$\frac{a}{n} - \operatorname{mld}(X \ni x, B) + 1 = w(X \ni x) - \operatorname{mld}(X \ni x, B) + 1 = w(B) \leq \frac{a}{n}\sum_i b_i l_i.$$

Combining with (3.2.1), we have

$$\frac{a}{n}(1 - \sum_i b_i l_i) \leq \operatorname{mld}(X \ni x, B) - 1 \leq \frac{3}{n}(1 - \sum_i b_i l_i). \tag{3.2.2}$$

If $\operatorname{mld}(X \ni x, B) > 1$, then by (3.2.2), $a \leq 3$, hence $a = 3$. It follows that $a(\bar{E}, X, B) = 1 + \frac{3}{n} - \bar{w}(B) = \operatorname{mld}(X \ni x, B)$ in this case. If $\operatorname{mld}(X \ni x, B) = 1$, then by (3.2.2), $\sum_i b_i l_i = 1$. In particular, $\bar{w}(B) = \frac{3}{n} = \bar{w}(X \ni x)$, hence $\operatorname{mld}(X \ni x, B) = a(\bar{E}, X, B) = 1$.

When $\operatorname{mld}(X \ni x, B) = 1$ and Γ is a DCC set or an ACC set, the equality $\sum_i b_i l_i = 1$ implies that B belongs to a finite subset $\Gamma_0 \subset \Gamma$. $\square$

Lemma 3.2.8. *Let γ_0 be a positive real number. Let $(X \ni x, B := \sum_i b_i B_i)$ be a threefold germ, where $x \in X$ is a terminal singularity of type cA/n for some $n > \lceil \frac{3}{\gamma_0} \rceil$, $b_i \geq \gamma_0$ and $\mathbb{Q}$-Cartier Weil divisors*

$B_i \geq 0$, *such that* $\mathrm{mld}(X \ni x, B) \geq 1$. *Then there exists a prime divisor* E *over* $X \ni x$, *such that* $a(E, X, B) = \mathrm{mld}(X \ni x, B)$ *and* $a(E, X, 0) \leq 1 + \frac{3}{n}$.

Proof. This follows from Lemmas 3.2.5 and 3.2.7. □

Theorem 3.2.9. *Let* $\Gamma \subset [0, 1]$ *be a DCC set. Then there exists a positive integer* l *depending only on* Γ *satisfying the following.*

Let $(X \ni x, B)$ *be a threefold pair such that* X *is terminal,* $B \in \Gamma$, *and* $\mathrm{mld}(X \ni x, B) = 1$. *Then there exists a prime divisor* E *over* $X \ni x$, *such that* $a(E, X, B) = 1$ *and* $a(E, X, 0) \leq 1 + \frac{l}{I}$, *where* I *is the index of* $X \ni x$. *In particular,* $a(E, X, 0) \leq 1 + l$.

Proof. Let Y be a small $\mathbb{Q}$-factorialization of X, and let $K_Y + B_Y := f^*(K_X + B)$. There exists a point $y \in Y$ such that $f(y) = x$ and $\mathrm{mld}(Y \ni y, B_Y) = \mathrm{mld}(X \ni x, B) = 1$. Moreover, the index of $Y \ni y$ divides the index of $X \ni x$. Possibly replacing $(X \ni x, B)$ with $(Y \ni y, B_Y)$, we may assume that X is $\mathbb{Q}$-factorial.

If $\dim x = 2$, then the theorem is trivial as we can take $l = 0$. If $\dim x = 1$, then X is smooth near x and $I = 1$. By Lemma 2.1.6, if E is the exceptional divisor of the blow-up at $x \in X$, then $a(E, X, B) = \mathrm{mld}(X \ni x, B)$. Since $a(E, X, 0) = 2$, we may take $l = 1$ in this case.

Now we may assume that $\dim x = 0$. By Lemma 3.1.4, there exists a terminal blow-up (see Definition 3.1.3) $f : Y \to X$ of $(X \ni x, B)$ which exactly extracts a prime divisor E over $X \ni x$. We may write

$$K_Y - \frac{a}{I} E = f^* K_X$$

for some positive integer a. Moreover, we may assume that $a \geq 5$.

By Theorem 2.1.35, f is a divisorial contraction of ordinary type. If $x \in X$ is a terminal singularity of type other than cA/n, then $I \leq 2$, and the theorem follows from Lemma 3.2.1. If $x \in X$ is a terminal singularity of type cA/n, then by Lemma 3.2.7, there exists an integer N' depending only on Γ', such that if $n = I \geq N'$, then there exists a prime divisor $\bar{E}$ over $X \ni x$ with $a(\bar{E}, X, B) = 1$ and $a(\bar{E}, X, 0) = 1 + \frac{3}{I}$. Hence when $I \geq N'$, we may take $l = 3$, and when $I < N'$, the theorem follows from Lemma 3.2.1. $\square$

3.3 ACC for Threefold Canonical Thresholds

The main goal for this section is to show Theorem 1.2.10.

Theorem 3.3.1. *Let $\Gamma \subset [0,1]$ be a DCC set. Then there exists a finite set $\Gamma_0 \subset \Gamma$ depending only on Γ satisfying the following. Assume that*

- *$(X \ni x, B := \sum_i b_i B_i)$ is a threefold pair,*
- *X is terminal,*
- *$b_i \in \Gamma$ and $B_i \geq 0$ are $\mathbb{Q}$-Cartier Weil divisors, and*
- *$\mathrm{mld}(X \ni x, B) = 1$.*

Then $b_i \in \Gamma_0$ for all i.

Proof. We may assume that $\dim x \leq 1$. If $\dim x = 1$, then X is smooth near x. By Lemma 2.1.6, $\mathrm{mld}(X \ni x, B) = 2 - \sum_i b_i \operatorname{mult}_x B_i = 1$ and $\operatorname{mult}_x B_i \in \mathbb{Z}_{>0}$ for each i, hence b_i belongs to a finite set $\Gamma_0 \subset \Gamma$ depending only on Γ. If $\dim x = 0$, then we let n be the index of $X \ni x$. By Theorem 3.2.9, there exists a prime divisor E over $X \ni x$ such that $a(E, X, B) = a(E, X, 0) - \operatorname{mult}_E B = 1$ and $a(E, X, 0) = 1 + \frac{a}{n}$ for some $a \leq l$, where l is a positive integer depending only on Γ. By [Kaw88, Lemma 5.1], $\operatorname{mult}_E B_i = \frac{1}{n} c_i$ for some positive integers c_i. It follows that $\frac{a}{n} = \frac{1}{n} \sum_{i=1}^{m} c_i b_i$. Thus $b_i \in \Gamma_0 \subset \Gamma$ for all i for some finite set $\Gamma_0 \subset \Gamma$ depending only on Γ. $\square$

As a consequence of Theorem 3.3.1, we show the ACC for $\mathrm{ct}(X \ni x, B; D)$ for terminal threefold singularities $x \in X$.

Theorem 3.3.2. *Let $\Gamma \subset [0,1],\ \Gamma' \subset [0,+\infty)$ be two DCC sets. Then the set*

$$\{\mathrm{ct}(X \ni x, B; D) \mid \dim X = 3,\ X \textit{ is terminal},\ B \in \Gamma,\ D \in \Gamma'\}$$

satisfies the ACC.

Proof. Pick $t \in \{\mathrm{ct}(X \ni x, B; D) \mid \dim X = 3,\ X \text{ is terminal},\ B \in \Gamma,\ D \in \Gamma'\}$. Then there exists a threefold pair $(X \ni x, B)$ and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D \in \Gamma' \backslash \{0\}$ on X, such that X is terminal, $B \in \Gamma$, and $t = \mathrm{ct}(X \ni x, B; D)$.

We only need to show that t belongs to an ACC set depending only on Γ and Γ'. By [HMX14, Theorem 1.1], we may assume that $\mathrm{mld}(X \ni x, B + tD) = 1$, and there exists a prime divisor E over $X \ni x$ such that $a(E, X, B + tD) = 1$. Possibly replacing X with a small $\mathbb{Q}$-factorialization X' and replacing x with the generic point of $\mathrm{center}_{X'} E$, we may assume that X is $\mathbb{Q}$-factorial. By Theorem 3.3.1, $B + tD$ belongs to a finite set depending only on Γ and Γ', hence t belongs to an ACC set depending only on Γ and Γ'. $\square$

Proof of Theorem 1.2.10. Let (X, B) be a canonical threefold pair and $D \geq 0$ a non-zero $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, such that $B \in \Gamma$ and $D \in \Gamma'$. Let $t := \mathrm{ct}(X, B; D)$. We only need to show that t belongs to an ACC set.

We may assume that $t > 0$. In particular, (X, B) is canonical. By [BCHM10, Corollary 1.4.3], there exists a birational morphism $f : Y \to X$ that exactly extracts all the exceptional divisors E over X such that $a(E, X, 0) = 1$. Since (X, B) is canonical, X is canonical, hence Y is terminal and $a(E, X, B + tD) = 1$ for any f-exceptional divisor E such that $a(E, X, 0) = 1$. We have $K_Y + B_Y + tD_Y = f^*(K_X + B + tD)$, where B_Y, D_Y are the strict transforms of B, D on Y respectively. Possibly replacing (X, B) and D with

(Y, B_Y) and D_Y respectively, we may assume that X is terminal.

Now there exists a point x on X such that $t = \mathrm{ct}(X \ni x, B; D)$. Theorem 1.2.10 follows from Theorem 3.3.2. $\square$

3.4 Uniform Canonical Rational Polytopes

[HLS19] established a general theory to show the boundedness of complements for DCC coefficients from the boundedness of complements for finite rational coefficients. We will follow this theory in our paper. As the key step, we need to show the existence of uniform canonical rational polytopes in this section. Recall that the proof of the uniform lc rational polytopes [HLS19] is based on some ideas in the proof of accumulation points of lc thresholds [HMX14], which relies on applying the adjunction formula to the lc places. Our proof is quite different from [HLS19] as we could not apply the adjunction formula to canonical places.

Lemma 3.4.1. *Let I, c, m be three non-negative integers, $r_1, \ldots, r_c$ real numbers such that $1, r_1, \ldots, r_c$ are linearly independent over $\mathbb{Q}$, and $s_1, \ldots, s_m : \mathbb{R}^{c+1} \to \mathbb{R}$ $\mathbb{Q}$-linear functions. Let $\boldsymbol{r} := (r_1, \ldots, r_c)$. Then there exists an open subset $U \subset \mathbb{R}^c$ depending only on $I, \boldsymbol{r}$ and $s_1, \ldots, s_m$, such that $U \ni \boldsymbol{r}$ satisfies the following.*

Let $x \in X$ be a terminal threefold singularity such that IK_X is Cartier near x, $B_1, \ldots, B_m \geq 0$ Weil divisors on X such that $(X \ni x, B := B(\boldsymbol{r}))$ is lc and $\mathrm{mld}(X \ni x, B) \geq 1$, where $B(\boldsymbol{v}) := \sum_{j=1}^{m} s_j(1, \boldsymbol{v}) B_j$ for any $\boldsymbol{v} \in \mathbb{R}^c$. Then $(X \ni x, B(\boldsymbol{v}))$ is lc and $\mathrm{mld}(X \ni x, B(\boldsymbol{v})) \geq 1$ for any $\boldsymbol{v} \in U$.

Proof. By [HLS19, Theorem 5.6], we may pick an open subset $U_0 \subset \mathbb{R}^c$ such that $\boldsymbol{r} \in U_0$ and $(X \ni x, B(\boldsymbol{v}))$ is lc for any $\boldsymbol{v} \in U_0$. By [Kaw88, Lemma 5.1], IB_j is Cartier near x for $1 \leq j \leq m$, we may write $B = \sum_{j=1}^{m} \frac{s_j(1,\boldsymbol{r})}{I} IB_j$. By [Nak16, Theorem 1.2], $\{a(E, X, B) \mid \mathrm{center}_X E = x\}$ belongs to a discrete set depending only on $I, \boldsymbol{r}$ and $s_1, \dots, s_m$. In particular, we may let

$$\alpha := \min\{a(E, X, B) \mid \operatorname{center}_X E = x,\ a(E, X, B) > 1\}.$$

Now we let

$$U := \{\frac{1}{\alpha}\boldsymbol{r} + \frac{\alpha - 1}{\alpha}\boldsymbol{v}_0 \mid \boldsymbol{v}_0 \in U_0\}.$$

We show that U satisfies our requirements. For any prime divisor E over $X \ni x$, if $a(E, X, B) = 1$, then $a(E, X, B(\boldsymbol{v})) = 1$ for any $\boldsymbol{v} \in U$ as $r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$. If $a(E, X, B) > 1$, then $a(E, X, B) \geq \alpha$. By the construction of U, for any $\boldsymbol{v} \in U$, there exists $\boldsymbol{v}_0 \in U_0$ such that $\boldsymbol{v} = \frac{1}{\alpha}\boldsymbol{r} + \frac{\alpha-1}{\alpha}\boldsymbol{v}_0$. Hence

$$a(E, X, B(\boldsymbol{v})) = \frac{1}{\alpha}a(E, X, B) + \frac{\alpha - 1}{\alpha}a(E, X, B(\boldsymbol{v}_0)) \geq 1.$$

It follows that $\operatorname{mld}(X \ni x, B(\boldsymbol{v})) \geq 1$ for any $\boldsymbol{v} \in U$. $\square$

Lemma 3.4.2. *Let $(X \ni x, B)$ be a threefold pair such that X is $\mathbb{Q}$-factorial and $x \in X$ is a terminal singularity of type cA/n. Assume that $\operatorname{mld}(X \ni x, B) \geq 1$ and $\lfloor B \rfloor \neq \emptyset$. Then $B = \lfloor B \rfloor$ is a prime divisor, $K_X + B$ is Cartier near x, and $\operatorname{mld}(X \ni x, B) = 1$.*

Proof. Let $S \subset \lfloor B \rfloor$ be a prime divisor. By Theorem 2.1.13(2), $B = \lfloor B \rfloor = S$ and $\operatorname{mld}(X \ni x, B) = 1$. If $n = 1$, by [Kaw88, Lemma 5.1], $K_X + B$ is Cartier near x. We may assume that $n \geq 2$. By [Kaw05, Theorem 1.3], Theorem 2.1.35(2), and [HLL22, Lemmas C.6 and C.7], there exist analytic local coordinates x_1, x_2, x_3, x_4 and a positive integer d, such that analytically locally, $x \in X$ is a hyperquotient singularity of the form

$$(X \ni x) \cong (\phi := x_1x_2 + g(x_3^n, x_4) = 0) \subset (\mathbb{C}^4 \ni o)/\frac{1}{n}(1, -1, b, 0),$$

where $b \in [1, n-1] \cap \mathbb{Z}$ and $\gcd(b, n) = 1$. Moreover, the weighted blow-up with the weight $w := \frac{1}{n}(s_1, dn - s_1, 1, n)$ extracts a prime divisor E such that $a(E, X, 0) = 1 + \frac{1}{n}$, where $s_1 \in [1, n-1] \cap \mathbb{Z}$

and $bs_1 \equiv 1 \mod n$. Let h be a semi-invariant analytic power series which defines S. Since $a(E, X, S) = a(E, X, 0) - \operatorname{mult}_E S = 1 + \frac{1}{n} - \operatorname{mult}_E S \geq 1$, $w(h) = \operatorname{mult}_E S = \frac{1}{n}$, and if $s_1 = 1$ (resp. $dn - s_1 = 1$), then either $x_3 \in h$ or $x_1 \in h$ (resp. $x_2 \in h$) up to a scaling of h. If $s_1 = 1$ (resp. $dn - s_1 = 1$), then $b = 1$ (resp. $b = -1$), and the analytic Cartier divisor $(x_3 = 0)$ is linearly equivalent to $(x_1 = 0)$ (resp. $(x_2 = 0)$), hence the $\mathbb{Q}$-Cartier divisor $(h = 0)$ is linear equivalent to the $\mathbb{Q}$-Cartier divisor $(x_3 = 0)$. By [Rei87, (6.4)(B.1)] and Lemma 2.1.10, $K_X + S$ is Cartier near x. □

Theorem 3.4.3. *Let c, m be two non-negative integers, $r_1, \dots, r_c$ real numbers such that $1, r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$, and $s_1, \dots, s_m : \mathbb{R}^{c+1} \to \mathbb{R}$ $\mathbb{Q}$-linear functions. Let $\boldsymbol{r} := (r_1, \dots, r_c)$. Then there exists an open subset $U \subset \mathbb{R}^c$ depending only on $\boldsymbol{r}$ and $s_1, \dots, s_m$, such that $U \ni \boldsymbol{r}$ satisfies the following.*

Let X be a terminal threefold, $x \in X$ a point, $B_1, \dots, B_m \geq 0$ distinct Weil divisors on X, and $B(\boldsymbol{v}) := \sum_{j=1}^m s_j(1, \boldsymbol{v})B_j$ for any $\boldsymbol{v} \in \mathbb{R}^c$. Assume that $(X \ni x, B := B(\boldsymbol{r}))$ is lc and $\operatorname{mld}(X \ni x, B) \geq 1$. Then $(X \ni x, B(\boldsymbol{v}))$ is lc and $\operatorname{mld}(X \ni x, B(\boldsymbol{v})) \geq 1$ for any $\boldsymbol{v} \in U$. Moreover, if $\operatorname{mld}(X \ni x, B) > 1$, then we may choose U so that $\operatorname{mld}(X \ni x, B(\boldsymbol{v})) > 1$ for any $\boldsymbol{v} \in U$.

Proof. Possibly replacing X with a small $\mathbb{Q}$-factorialization, we may assume that X is $\mathbb{Q}$-factorial.

By construction, we may assume that $s_j(1, \boldsymbol{r}) > 0$ for each j. If $\dim x = 2$, then the theorem is trivial. If $\dim x = 1$, then X is smooth near x. By Lemma 2.1.6, $\operatorname{mld}(X \ni x, B) = 2 - \operatorname{mult}_x B > 1$, thus $\operatorname{mult}_x B = \sum_{j=1}^m s_j(1, \boldsymbol{r}) \operatorname{mult}_x B_j < 1$, where $\operatorname{mult}_x B_j$ are non-negative integers. Hence $\sum_{j=1}^m s_j(1, \boldsymbol{r}) \operatorname{mult}_x B_j \leq 1 - \epsilon_0$ for some $\epsilon_0 \in (0, 1)$ depending only on $\boldsymbol{r}$ and $s_1, \dots, s_m$. By Lemma 2.1.6, $\operatorname{mld}(X \ni x, \frac{1}{1-\epsilon_0} B) \geq 1$, hence we can take $U := \{\boldsymbol{v} \mid 0 < s_j(1, \boldsymbol{v}) < \frac{1}{1-\epsilon_0} s_j(1, \boldsymbol{r}) \text{ for each } j\}$ in this case. Hence we may assume that $\dim x = 0$.

If $\lfloor B \rfloor \neq 0$, by [Rei87, (6.1) Theorem] and Lemma 3.4.2, we may assume that $12K_X$ is Cartier near x, and the theorem follows from Lemma 3.4.1. Thus we may assume that $\lfloor B \rfloor = 0$. By [BCHM10, Corollary 1.4.3], there exists a birational morphism $f : Y \to X$ from a $\mathbb{Q}$-factorial variety Y that exactly extracts all the exceptional divisors F over $X \ni x$ such that $a(F, X, B) = 1$. In particular, $a(F, X, B(\boldsymbol{v})) = 1$ for all $\boldsymbol{v} \in \mathbb{R}^c$. It follows that $f^*(K_X + B(\boldsymbol{v})) = K_Y + f_*^{-1}B(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{R}^c$. Hence it suffices to prove the theorem for all pairs $(Y \ni y, f_*^{-1}B)$, where $y \in f^{-1}(x)$ is a closed point. From now on, we may assume that $\mathrm{mld}(X \ni x, B) > 1$.

By [Rei87, (6.1) Theorem], if $x \in X$ is a terminal singularity of types other than cA/n, then the index of $X \ni x$ is ≤ 4, and the theorem holds by Lemma 3.4.1. From now on, we may assume that $x \in X$ is of type cA/n.

Claim 3.4.4. *There exist a positive integer N and a positive real number ϵ depending only on $\boldsymbol{r}$ and $s_1, \ldots, s_m$ satisfying the following.*

For any terminal threefold singularity $x \in X$ of type cA/n and $B := \sum_{j=1}^m s_j(1, \mathbf{r})B_j$, where $B_j \geq 0$ are $\mathbb{Q}$-Cartier Weil divisors on X and $\lfloor B \rfloor = 0$, if $\mathrm{mld}(X \ni x, B) > 1$ and $n > N$, then $t := \mathrm{ct}(X \ni x, 0; B) > 1 + \epsilon$.

We proceed the proof assuming Claim 3.4.4. By Lemma 3.4.1, we may assume that $n > N$. By Claim 3.4.4, $t > 1 + \epsilon$, hence we can take $U := \{\boldsymbol{v} \mid 0 < s_j(1, \boldsymbol{v}) < (1 + \epsilon)s_j(1, \boldsymbol{r}) \text{ for each } j\}$ in this case. Moreover, if $\mathrm{mld}(X \ni x, B) > 1$, then possibly replacing U with $\{\frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{r} \mid \boldsymbol{v} \in U\}$, we have $\mathrm{mld}(X \ni x, B(\boldsymbol{v})) > 1$ for all $\boldsymbol{v} \in U$. □

Proof of Claim 3.4.4. Since $\mathrm{mld}(X \ni x, B) > 1$, $t > 1$. Since $\lfloor B \rfloor = 0$, if $\lfloor tB \rfloor \neq 0$, then $t > 1 + \epsilon$ for some $\epsilon > 0$ depending only on $\boldsymbol{r}$ and $s_1, \ldots, s_m$. Thus we may assume that $\lfloor tB \rfloor = 0$. By Lemma 2.1.12(1), $\mathrm{mld}(X \ni x, tB) = 1$. Since $t > 1$, by Lemma 3.2.8, there exists a positive integer N depending only on $\boldsymbol{r}$ and $s_1, \ldots, s_m$, such that if $n > N$, then there exists a prime divisor $\bar{E}$ over $X \ni x$, such that

$a(\bar{E}, X, tB) = 1$ and $a(\bar{E}, X, 0) = 1 + \frac{a}{n}$ for some positive integer $a \leq 3$. Since

$$a(\bar{E}, X, tB) = a(\bar{E}, X, 0) - \operatorname{mult}_{\bar{E}} tB = 1 + \frac{a}{n} - \frac{t}{n} \sum_{j=1}^{m} l_j s_j(1, \boldsymbol{r}) = 1,$$

where $l_j := n \operatorname{mult}_{\bar{E}} B_j \in \mathbb{Z}_{>0}$ for each j, we have $t \sum_{j=1}^{m} l_j s_j(1, \boldsymbol{r}) = a$. Since $a, s_j(1, \boldsymbol{r})$ belong to a finite set of positive real numbers for any j, and l_j belongs to a discrete set of positive real numbers, t belongs to a set whose only accumulation point is 0. Since $t > 1$, there exists a positive real number ϵ depending only on $\boldsymbol{r}$ and $s_1, \ldots, s_m$, such that $t > 1 + \epsilon$. □

3.5 Accumulation Points of Canonical Thresholds

In this section, we prove Theorem 1.2.11.

Lemma 3.5.1 ([Che19, Lemma 2.1])**.** *Let $(X \ni x) \cong (\phi_1 = \cdots = \phi_m = 0) \subset (\mathbb{C}^d \ni o)/\frac{1}{n}(b_1, \ldots, b_d)$ be a germ, where $\phi_1, \ldots, \phi_m$ are semi-invariant analytic power series. Let $w, w' \in \frac{1}{n}\mathbb{Z}^d_{>0}$ be two weights and $f : Y \to X, f' : Y' \to X$ weighted blow-ups with the weights w, w' at $x \in X$ respectively, such that f extracts an analytic prime divisor E and f' extracts an analytic prime divisor E' respectively.*

Let $B \geq 0$ be a $\mathbb{Q}$-Cartier Weil divisor on X such that $1 = a(E, X, \operatorname{ct}(X \ni x, 0; D)D)$, $m := n \operatorname{mult}_E D$, and $m' := n \operatorname{mult}_{E'} D$. Then for any real number $\mu \geq 0$ such that $w' \succeq \mu w$ (see Definition 2.1.26),

$$\lceil \mu m \rceil \leq m' \leq \lfloor \frac{w'(X \ni x)}{w(X \ni x)} m \rfloor.$$

Lemma 3.5.2. *Let $\mathcal{TS}$ be a set of terminal threefold singularities, and*

$$\mathcal{TS}_1 := \{(\tilde{x} \in \tilde{X}) \mid \tilde{x} \in \tilde{X} \text{ is an index one cover of } (x \in X) \in \mathcal{TS}\}.$$

Then the set of accumulation points of

$$\{\operatorname{ct}(X \ni x, 0; D) \mid (x \in X) \in \mathcal{TS}, D \in \mathbb{Z}_{>0}\}$$

is a subset of the set of accumulation points of

$$\{\mathrm{ct}(X \ni x, 0; D) \mid (x \in X) \in \mathcal{TS}_1, D \in \mathbb{Z}_{>0}\}.$$

Proof. Let $\{(X_i \ni x_i, D_i)\}_{i=1}^{\infty}$ be a sequence of germs, such that $(x_i \in X_i) \in \mathcal{TS}$ and $D_i \geq 0$ are non-zero $\mathbb{Q}$-Cartier Weil divisors. Let n_i be the index of the terminal singularity $x_i \in X_i$ for each i. By Theorem 3.3.2, we may assume that the sequence $\{c_i := \mathrm{ct}(X_i \ni x_i, 0; D_i)\}$ is strictly decreasing with the limit point $c \geq 0$. It suffices to show that c is an accumulation point of $\{\mathrm{ct}(X \ni x, 0; D) \mid (x \in X) \in \mathcal{TS}_1, D \in \mathbb{Z}_{>0}\}$. We may assume that $c > 0$.

We may assume that $1 > c_i$ for each i, and by Lemma 2.1.12(1), $\mathrm{mld}(X_i \ni x_i, c_i D_i) = 1$. For each i, consider the pair $(X_i \ni x_i, c_i D_i)$, by Lemma 3.1.4, there exists a terminal blow-up (see Definition 3.1.3) of $(X_i \ni x_i, c_i D_i)$ which extracts a prime divisor E_i over $X_i \ni x_i$. We may write $a(E_i, X_i, 0) = 1 + \frac{a_i}{n_i}$ and $\mathrm{mult}_{E_i} D_i = \frac{m_i}{n_i}$ for some positive integers a_i, m_i. Set $t_i := \mathrm{lct}(X_i, 0; D_i)$ for each i, then we have $c_i = \frac{a_i}{m_i}$ and $t_i \leq \frac{a_i + n_i}{m_i}$. Since $\{c_i\}_{i=1}^{\infty}$ is strictly decreasing and $c > 0$, $\lim_{i \to +\infty} m_i = +\infty$ and $\lim_{i \to +\infty} a_i = +\infty$. In particular, possibly passing to a subsequence, we may assume that $a_i \geq 3$ for all i. Since $c_i > c$, by Lemma 3.2.7(2) and [Rei87, (6.1) Theorem], $n_i \leq \max\{4, \frac{3}{c}\}$ for all i.

Possibly shrinking X_i to a neighborhood of x_i, we may assume that $(X_i, c_i D_i)$ is lc for each i. It follows that

$$c = \lim_{i \to +\infty} \frac{a_i}{m_i} \leq \lim_{i \to +\infty} t_i \leq \lim_{i \to +\infty} \frac{a_i + n_i}{m_i} = c,$$

hence $c = \lim_{i \to +\infty} t_i$.

For each i, let $\pi_i : (\widetilde{X}_i \ni \tilde{x}_i) \to (X_i \ni x_i)$ be the index one cover of $x_i \in X_i$. Set $\widetilde{D}_i := \pi_i^{-1} D_i$, $\widetilde{c}_i := \mathrm{ct}(\widetilde{X}_i \ni \widetilde{x}_i, 0; \widetilde{D}_i)$ and $\widetilde{t}_i := \mathrm{lct}(\widetilde{X}_i, 0; \widetilde{D}_i)$. Possibly shrinking X_i to a neighborhood of x_i again, we may assume that $(\tilde{X}_i, \tilde{c}_i\tilde{D}_i)$ is lc. By [KM98, Proposition 5.20], $\widetilde{t}_i = t_i$ and $\widetilde{c}_i \geq c_i$. Now

$$c = \lim_{i\to+\infty} t_i = \lim_{i\to+\infty} \widetilde{t}_i \geq \lim_{i\to+\infty} \widetilde{c}_i \geq \lim_{i\to+\infty} c_i = c,$$

which implies that $c = \lim_{i\to+\infty} \widetilde{c}_i$. □

Theorem 3.5.3. *Let $\mathfrak{T}$ be the set of all terminal threefold singularities. Then the set of accumulation points of*

$$\mathcal{CT}_t := \{\mathrm{ct}(X \ni x, 0; D) \mid (x \in X) \in \mathfrak{T},\ D \in \mathbb{Z}_{>0}\}$$

is $\{0\} \cup \{\frac{1}{k} \mid k \in \mathbb{Z}_{\geq 2}\}$. Moreover, 0 is the only accumulation point of

$$\mathcal{CT}_{t,\neq sm,cA/n} := \left\{\mathrm{ct}(X \ni x, 0; D) \,\middle|\, \begin{array}{l} (x \in X) \in \mathfrak{T},\ (x \in X) \textit{ is neither smooth nor} \\ \textit{of type } cA/n \textit{ for any } n \in \mathbb{Z}_{>0}, D \in \mathbb{Z}_{>0}, \end{array} \right\}.$$

Proof. **Step 0**. By [Ste11, Theorem 3.6], $\{0\} \cup \{\frac{1}{k} \mid k \in \mathbb{Z}_{\geq 2}\}$ is a subset of the set of accumulation points of $\mathcal{CT}_t$. For any $(x \in X) \in \mathfrak{T}$, let $D \geq 0$ be a non-zero $\mathbb{Q}$-Cartier Weil divisor on of X. Then $\mathrm{ct}(X \ni x, 0, kD) = \frac{1}{k}\,\mathrm{ct}(X \ni x, 0; D)$ for any positive integer k, hence 0 is an accumulation point of $\mathcal{CT}_{t,\neq sm,cA/n}$.

It suffices to show the corresponding reverse inclusions. Let $c > 0$ be an accumulation point of $\mathcal{CT}_t$. We will finish the proof by showing that c is not an accumulation point of $\mathcal{CT}_{t,\neq sm,cA/n}$ in **Step 2** and $c = \frac{1}{k+1}$ for some positive integer k in **Step 3**.

Step 1. By Theorem 3.3.2, $c < 1$. Let k be a positive integer such that $\frac{1}{k+1} \leq c < \frac{1}{k}$. Consider the set

$$I_k := \{\frac{p}{q} \mid p, q \in \mathbb{Z}_{>0}, p \leq 16(k+1)^2\},$$

which is discrete away from 0. By Theorem 3.3.2, there exists a positive real number ϵ, such that for any $c' \in \mathcal{CT}_t$, if $0 < |c - c'| < \epsilon$, then $c < c' < \frac{1}{k}$, and $c' \notin I_k$.

By Lemma 3.5.2, there exists a Gorenstein terminal threefold singularity $x \in X$ and a non-zero $\mathbb{Q}$-Cartier

Weil divisor $D \geq 0$ on X, such that $0 < |c - \mathrm{ct}(X \ni x, 0; D)| < \epsilon$. We have $c < \mathrm{ct}(X \ni x, 0; D) < \frac{1}{k}$ and $\mathrm{ct}(X \ni x, 0; D) \notin I_k$. Let $D_0 := \mathrm{ct}(X \ni x, 0; D)D$. By Lemma 2.1.12(1), $\mathrm{mld}(X \ni x, D_0) = 1$. By Lemma 3.1.4, there exists a terminal blow-up $f : Y \to X$ of $(X \ni x, D_0)$ which extracts a prime divisor E over $X \ni x$, and $K_Y = f^*K_X + aE$, $f^*D = f^{-1}_*D + mE$ for some positive integers a, m. We have $\mathrm{ct}(X \ni x, 0; D) = \frac{a}{m}$. If $a \leq 4$ or $a \mid m$, then $\frac{a}{m} \in I_k$, a contradiction. Hence $a \geq 5$ and $a \nmid m$. By Theorem 2.1.35, f is a divisorial contraction of ordinary type as in Theorem 2.1.35(1-3) when $x \in X$ is not smooth or as in [Kaw01, Theorem 1.1] when $x \in X$ is smooth.

Step 2. We show that $x \in X$ is either smooth or of type cA in this step. In particular, by Theorem 3.5.2, c is not an accumulation point of $\mathcal{CT}_{t,\neq sm,cA/n}$.

Otherwise, by Theorem 2.1.35, X is of type cD, and there are two cases:

Case 2.1. f is a divisorial contraction as in Theorem 2.1.35(2.1). In particular, under suitable analytic local coordinates x_1, x_2, x_3, x_4, we have

$$(X \ni x) \cong (\phi := x_1^2 + x_1 q(x_3, x_4) + x_2^2 x_4 + \lambda x_2 x_3^2 + \mu x_3^3 + p(x_2, x_3, x_4) = 0) \subset (\mathbb{C}^4 \ni o)$$

for some analytic power series ϕ as in Theorem 2.1.35(2.1), and f is a weighted blow-up with the weight $w := (r + 1, r, a, 1)$, where r is a positive integer, $2r + 1 = ad$ for some integer $d \geq 3$, and a is an odd number. We have $w(X \ni x) = a$ and $w(D) = m$.

Since $a \nmid m$, $\mathrm{ct}(X \ni x, 0; D) \in (\frac{1}{k+1}, \frac{1}{k})$, and we have

$$\frac{2r+1}{md} = \frac{a}{m} = \mathrm{ct}(X \ni x, 0; D) \in (\frac{1}{k+1}, \frac{1}{k}),$$

hence $k(2r+1) < dm < (k+1)(2r+1)$. Consider the weighted blow-up $f' : Y' \to X$ (resp. $f'' : Y'' \to X$) at $x \in X$ with the weight $w' := (d, d, 2, 1)$ (resp. $w'' := (1 + r - d, r - d, a - 2, 1)$). Since $a - 2 \geq 3$, by [HLL22, Lemma C.8(1)], f'' extracts an analytic prime divisor, and $w''(X \ni x) = a - 2$. Since $a \geq 5$, by

[HLL22, Lemma C.8(2)], f' extracts an analytic prime divisor, and $w'(X\ni x)=2$. Since $w'\succeq \frac{d}{r+1}w$ and $w''\succeq \frac{r-d}{r}w$, by Lemma 3.5.1, $\lfloor \frac{2}{a}m\rfloor \geq \lceil \frac{d}{r+1}m\rceil$ and $\lfloor \frac{a-2}{a}m\rfloor \geq \lceil \frac{r-d}{r}m\rceil$. Thus

$$m-1=\lfloor \frac{2}{a}m\rfloor+\lfloor \frac{a-2}{a}m\rfloor \geq \lceil \frac{d}{r+1}m\rceil+\lceil \frac{r-d}{r}m\rceil \geq \lceil m-\frac{dm}{r(r+1)}\rceil,$$

where the first equality follows from $a\nmid 2m$ as $a\nmid m$ and a is an odd number. It follows that $\frac{dm}{r(r+1)}\geq 1$. Hence $(k+1)(2r+1)>dm\geq r(r+1)>r(r+\frac{1}{2})$, and $r<2(k+1)$. Since $2r+1=ad$, $a\leq 4k+4$. Therefore, $\mathrm{ct}(X\ni x,0;D)=\frac{a}{m}\in I_k$, a contradiction.

Case 2.2. f is a divisorial contraction as in Theorem 2.1.35(3.1). In particular, under suitable analytic local coordinates x_1,x_2,x_3,x_4,x_5, we have

$$(X\ni x)\cong \begin{pmatrix} \phi_1:=x_1^2+x_2x_5+p(x_2,x_3,x_4)=0 \\ \phi_2:=x_2x_4+x_3^d+q(x_3,x_4)x_4+x_5=0\end{pmatrix}\subset(\mathbb{C}^5\ni o)$$

for some analytic power series ϕ_1,ϕ_2 as in Theorem 2.1.35(3.1), and f is a weighted blow-up with the weight $w:=(r+1,r,a,1,r+2)$, where r is a positive integer such that $r+1=ad$ and $d\geq 2$ is an integer. We have $w(X\ni x)=a$ and $w(D)=m$.

Since $a\nmid m$, $\mathrm{ct}(X\ni x,0;D)\in(\frac{1}{k+1},\frac{1}{k})$, and we have

$$\frac{r+1}{dm}=\frac{a}{m}=\mathrm{ct}(X\ni x,0;D)\in(\frac{1}{k+1},\frac{1}{k}),$$

hence $k(r+1)<dm<(k+1)(r+1)$. Consider the weighted blow-up $f':Y'\to X$ (resp. $f'':Y''\to X$) at $x\in X$ with the weight $w':=(d,d,1,1,d)$ (resp. $w'':=(r+1-d,r-d,a-1,1,r+2-d)$). Since $a-1\geq 4$, by [HLL22, Lemma C.9(1)], f'' extracts an analytic prime divisor, and $w''(X\ni x)=a-1$. Since $a\geq 5$, by [HLL22, Lemma C.9(2)], f' extracts an analytic prime divisor, and $w'(X\ni x)=1$. Since $w'\succeq \frac{d}{r+2}w$ and $w''\succeq\frac{r-d}{r}w$, by Lemma 3.5.1, $\lfloor \frac{1}{a}m\rfloor\geq\lceil\frac{d}{r+2}m\rceil$ and $\lfloor\frac{a-1}{a}m\rfloor\geq\lceil\frac{r-d}{r}m\rceil$. Thus

$$m-1=\lfloor \frac{1}{a}m\rfloor+\lfloor \frac{a-1}{a}m\rfloor \geq \lceil \frac{d}{r+2}m\rceil+\lceil \frac{r-d}{r}m\rceil \geq \lceil m-\frac{2dm}{r(r+2)}\rceil,$$

where the first equality follows from $a \nmid m$. This implies that $\frac{2dm}{r(r+2)} \geq 1$. Hence $(k+1)(r+1) > dm \geq \frac{1}{2}r(r+2) > \frac{1}{2}r(r+1)$, and $r < 2(k+1)$. Since $r+1 = ad$, $a \leq 2k+2$. Therefore, $\mathrm{ct}(X \ni x, 0; D) = \frac{a}{m} \in I_k$, a contradiction.

Step 3. We show that $c = \frac{1}{k+1}$ in this step.

By **Step 1**, **Step 2**, [Kaw01, Theorem 1.1], and Theorem 2.1.35, there are two cases.

Case 3.1. $x \in X$ is smooth, and under suitable analytic local coordinates x_1, x_2, x_3, f is a weighted blow-up with the weight $w := (1, r_1, r_2)$ for some positive integers r_1, r_2, such that $\gcd(r_1, r_2) = 1$. Now $n = 1$, $a = r_1 + r_2$, and $\mathrm{ct}(X \ni x, 0; D) = \frac{r_1+r_2}{m}$, such that $r_1 + r_2 \nmid m$. Possibly switching x_2, x_3, we may assume that $r_1 \leq r_2$. By [Che19, Proposition 3.3(1)],

$$\mathrm{ct}(X \ni x, 0; D) = \frac{r_1 + r_2}{m} \leq \frac{1}{r_1} + \frac{1}{r_2}$$

when $r_1 \geq 2$. When $r_1 = 1$, we have

$$\mathrm{ct}(X \ni x, 0; D) < 1 < \frac{1}{r_1} + \frac{1}{r_2}.$$

Since $a \nmid m$, $\mathrm{ct}(X \ni x, 0; D) \in (\frac{1}{k+1}, \frac{1}{k})$, and we have $\frac{1}{r_1} + \frac{1}{r_2} > \frac{1}{k+1}$, and $r_1 < 2(k+1)$. If $k+2 \leq r_1$, then $r_2 < (k+1)(k+2)$ and $a = r_1 + r_2 \leq 16(k+1)^2$. It follows that $\mathrm{ct}(X \ni x, 0; D) = \frac{r_1+r_2}{m} \in I_k$, a contradiction. Hence $1 \leq r_1 \leq k+1$.

Consider the weighted blow-up with the weight $w' := (1, r_1, r_2 - 1)$. This weighted blow-up extracts an analytic prime divisor E' that is isomorphic to $\mathbf{P}(1, r_1, r_2 - 1)$, and $w'(X \ni x) = r_1 + r_2 - 1$. Since $w' \succeq \frac{r_2-1}{r_2} w$, by Lemma 3.5.1,

$$m - (k+1) = \lfloor \frac{r_1 + r_2 - 1}{r_1 + r_2} m \rfloor \geq \lceil \frac{r_2 - 1}{r_2} m \rceil,$$

where the equality follows from $\frac{m}{r_1+r_2} \in (k, k+1)$. It follows that $\frac{m}{r_2} \geq k+1$ and $(k+1)r_2 \leq m <$

$(k+1)(r_1+r_2)$. Thus $\mathrm{ct}(X \ni x, 0; D)$ belongs to the set

$$\{\frac{r_1+r_2}{m} \mid r_1, r_2, m \in \mathbb{Z}, 1 \leq r_1 \leq k+1, r_1 \leq r_2, (k+1)r_2 \leq m < (k+1)(r_1+r_2)\},$$

which has only one accumulation point $\frac{1}{k+1}$.

Case 3.2. $x \in X$ is of type cA and $f : Y \to X$ is a divisorial contraction of ordinary type as in Theorem 2.1.35(1). In particular, under suitable analytic local coordinates x_1, x_2, x_3, x_4, we have

$$(X \ni x) \cong (\phi := x_1x_2 + g(x_3, x_4) = 0) \subset (\mathbb{C}^4 \ni 0)$$

for some analytic power series ϕ as in Theorem 2.1.35(1), and f is a weighted blow-up with the weight $w := (r_1, r_2, a, 1)$, where r_1, r_2, d are positive integers such that $r_1 + r_2 = ad$. We have $w(X \ni x) = a$ and $w(D) = m$. By [Che19, Proposition 4.2],

$$\mathrm{ct}(X \ni x, 0; D) = \frac{a}{m} = \frac{r_1+r_2}{dm} \leq \frac{1}{r_1} + \frac{1}{r_2}.$$

Possibly switching x_1, x_2, we may assume that $r_1 \leq r_2$. Since $a \mid m$, $\mathrm{ct}(X \ni x, 0; D) \in (\frac{1}{k+1}, \frac{1}{k})$, and we have $\frac{1}{r_1} + \frac{1}{r_2} > \frac{1}{k+1}$, hence $r_1 < 2(k+1)$. If $k+2 \leq r_1$, then $r_2 < (k+1)(k+2)$, hence $a \leq r_1 + r_2 \leq 16(k+1)^2$, which implies that $\mathrm{ct}(X \ni x, 0; D) = \frac{a}{m} \in I_k$, a contradiction. Hence $1 \leq r_1 \leq k+1$.

Consider the weight $w' := (r_1, r_2 - d, a-1, 1)$. By [HLL22, Lemma C.7], the weighted blow-up with the weight w' extracts an analytic prime divisor, and $w'(X \ni x) = a - 1$. Since $w' \succeq \frac{r_2-d}{r_2}w$, by Lemma 3.5.1,

$$m - (k+1) = \lfloor \frac{a-1}{a} m \rfloor \geq \lceil \frac{r_2-d}{r_2} m \rceil,$$

where the equality follows from $\frac{m}{a} \in (k, k+1)$. It follows that $\frac{dm}{r_2} \geq k+1$ and $(k+1)r_2 \leq dm <$

$(k+1)(r_1+r_2)$. Thus $\mathrm{ct}(X \ni x, 0; D)$ belongs to the set

$$\{\frac{r_1+r_2}{dm} \mid r_1, r_2, d, m \in \mathbb{Z}_{>0}, 1 \le r_1 \le k+1, r_1 \le r_2, (k+1)r_2 \le dm < (k+1)(r_1+r_2)\},$$

which has only one accumulation point $\frac{1}{k+1}$. $\square$

Proof of Theorem 1.2.11. Let X be a canonical threefold and $D \ge 0$ a non-zero $\mathbb{Q}$-Cartier Weil divisor on X. Consider the pair $(X, D_0 := \mathrm{ct}(X, 0; D)D)$. For any exceptional prime divisor F over X such that $a(F, X, 0) = 1$, we have $a(F, X, D_0) = 1$. By [BCHM10, Corollary 1.4.3], there exists a $\mathbb{Q}$-factorial variety X' and a birational morphism $g : X' \to X$ that exactly extracts all exceptional divisors F such that $a(F, X, 0) = 1$. By construction, X' is terminal and $K_{X'} + g_*^{-1}D_0 = g^*(K_X + D_0)$, hence $\mathrm{ct}(X, 0; D) = \mathrm{ct}(X', 0; g_*^{-1}D)$. Possibly replacing (X, D) with $(X', g_*^{-1}D)$, we may assume that X is terminal.

Now either $\mathrm{ct}(X, 0; D) = 1$ or $t := \mathrm{ct}(X, 0; D) = \mathrm{ct}(X \ni x, 0; D) < 1$ for some point $x \in X$ of codimension ≥ 2. If $\dim x = 1$, then by Lemma 2.1.6, $\mathrm{mld}(X \ni x, tD) = 2 - \mathrm{mult}_x tD = 1$, hence $t = \frac{1}{\mathrm{mult}_x D} \in \{\frac{1}{m} \mid m \in \mathbb{Z}_{>0}\}$. If $\dim x = 0$, by Theorem 3.5.3, we are done. $\square$

3.6 ACC for Minimal Log Discrepancies on $[1, +\infty)$

In this section, we prove the following theorem:

Theorem 3.6.1. *Let $\Gamma \subset [0, 1]$ be a DCC set. Then the set*

$$\{\mathrm{mld}(X \ni x, B) \mid \dim X = 3, X \textit{ is terminal near } x, B \in \Gamma\} \cap [1, +\infty)$$

satisfies the ACC.

Proof of Theorem 3.6.1. **Step 1**. Suppose that Theorem 3.6.1 does not hold, then there exists a sequence of threefold pairs $\{(X_i \ni x_i, B_i)\}_{i=1}^{\infty}$, where X_i is terminal and $B_i \in \Gamma$ for each i, such that $\{\mathrm{mld}(X_i \ni$

$x_i, B_i)\}_{i=1}^{\infty} \subset (1, +\infty)$ is strictly increasing. Possibly replacing X_i with a small $\mathbb{Q}$-factorialization, we may assume that X_i is $\mathbb{Q}$-factorial. If $\dim x_i = 1$, then by Lemma 2.1.6, $\mathrm{mld}(X_i \ni x_i, B_i) = 2 - \mathrm{mult}_{x_i} B_i$, which belongs to an ACC set. Possibly passing to a subsequence, we may assume that $\dim x_i = 0$ for each i. [Amb99, Theorem 0.1], we may let $\beta := \lim_{i \to +\infty} \mathrm{mld}(X_i \ni x_i, B_i)$. By Theorem 2.1.13, possibly passing to a subsequence, there exists a non-negative integer p, such that $B_i := \sum_{j=1}^{p} b_{i,j} B_{i,j}$ for each i, where $B_{i,j}$ are distinct prime divisors. Set $b_j := \lim_{i \to +\infty} b_{i,j}$ for $1 \le j \le p$ and $\bar{B}_i := \sum_{j=1}^{p} b_j B_{i,j}$ for each i.

Let n_i be the index of $X_i \ni x_i$. By [Sho92, Appendix, Theorem], if $n_i \ge 2$, then there exists a prime divisor F_i over $X_i \ni x_i$, such that $a(F_i, X_i, 0) = 1 + \frac{1}{n_i}$. Thus

$$1 + \frac{1}{n_i} \ge a(F_i, X_i, B_i) \ge \mathrm{mld}(X_i \ni x_i, B_i) \ge \mathrm{mld}(X_1 \ni x_1, B_1) > 1,$$

and $n_i \le \frac{1}{\mathrm{mld}(X_1 \ni x_1, B_1) - 1}$. Hence, possibly passing to a subsequence, we may assume that there exists a positive integer n such that $n_i = n$ for all i. By [Kaw88, Lemma 5.1], nD_i is Cartier near x_i for any $\mathbb{Q}$-Cartier Weil divisor D_i on X_i and for each i.

By [Amb99, Theorem 0.1] and Theorem 3.3.2, $1 \le \mathrm{mld}(X_i \ni x_i) \le 3$. By [Nak16, Corollary 1.3], $\{\mathrm{mld}(X_i \ni x_i, \bar{B}_i) \mid i \in \mathbb{Z}_{>0}\} \subset [1, 3]$ is a finite set. Possibly passing to a subsequence, we may assume that there exists a positive real number $\alpha \ge 1$, such that $\mathrm{mld}(X_i \ni x_i, \bar{B}_i) = \alpha < \beta$ for all i.

Step 2. In this step, we show that for each i, there exists a prime divisor $\bar{E}_i$ over $X_i \ni x_i$, such that $a(\bar{E}_i, X_i, \bar{B}_i) = \mathrm{mld}(X_i \ni x_i, \bar{B}_i) = \alpha$, and $a(\bar{E}_i, X_i, 0) \le l$ for some positive real number l depending only on $\{b_j\}_{j=1}^{p}$.

By Lemmas 3.2.1, 3.2.4, 3.2.5, and Theorem 3.2.9, it suffices to show that if $\alpha > 1$, then either $x_i \in X_i$ is smooth or is of cA/n type for all but finitely many i.

Otherwise, $\alpha > 1$, and possibly passsing to a subsequence, we may assume that $x_i \in X_i$ is neither smooth nor of cA/n type for each i. Let E_i be a prime divisor over $X_i \ni x_i$, such that $a(E_i, X_i, B_i) =$

$\mathrm{mld}(X_i \ni x_i, B_i)$. By Theorem 3.4.3, there exist a positive integer m depending only on $\{b_j\}_{j=1}^p$, and $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors $\bar{B}_i' > 0$ on X_i, such that for each i,

1. $m\bar{B}_i'$ is a Weil divisor,
2. $\mathrm{mld}(X_i \ni x_i, \bar{B}_i') > 1$, and
3. $a(E_i, X_i, \bar{B}_i') \le a(E_i, X_i, \bar{B}_i) < \mathrm{mld}(X_i \ni x_i, B_i)$.

Since $\lim_{i\to+\infty} \mathrm{mld}(X_i \ni x_i, B_i) = \beta$, by [Nak16, Theorem 1.2], possibly passing to a subsequence, we may assume that there exists a positive real number γ, such that $\alpha \le a(E_i, X_i, \bar{B}_i) = \gamma < \mathrm{mld}(X_1 \ni x_1, B_1)$ for each i.

By Lemma 2.1.12(2), $t_i' := \mathrm{ct}(X_i \ni x_i, 0; \bar{B}_i') > 1$ for each i. We have

$$\gamma - (t_i' - 1)\,\mathrm{mult}_{E_i} \bar{B}_i' \ge a(E_i, X_i, \bar{B}_i') - (t_i' - 1)\,\mathrm{mult}_{E_i} \bar{B}_i' = a(E_i, X_i, t_i'\bar{B}_i') \ge 1,$$

which implies that $t_i' - 1 \le \frac{\gamma - 1}{\mathrm{mult}_{E_i} \bar{B}_i'}$. Since

$$\mathrm{mult}_{E_i}(\bar{B}_i - B_i) = a(E_i, X_i, B_i) - a(E_i, X_i, \bar{B}_i) \ge \mathrm{mld}(X_1 \ni x_1, B_1) - \gamma > 0,$$

$\lim_{i\to+\infty} \mathrm{mult}_{E_i} \bar{B}_i = +\infty$. Thus $\lim_{i\to+\infty} \mathrm{mult}_{E_i} \bar{B}_i' = +\infty$ as $a(E_i, X_i, \bar{B}_i') \le a(E_i, X_i, \bar{B}_i)$. It follows that $\lim_{i\to+\infty} t_i' = 1$. Hence $\mathrm{ct}(X_i \ni x_i, 0; m\bar{B}_i') > \frac{1}{m}$, and $\lim_{i\to+\infty} \mathrm{ct}(X_i \ni x_i, 0; m\bar{B}_i') = \frac{1}{m}$, which contradicts Theorem 3.5.3.

Step 3. By **Step 2**, $\mathrm{mult}_{\bar{E}_i} \bar{B}_i = a(\bar{E}_i, X_i, 0) - a(\bar{E}_i, X_i, B_i) \le l - \beta$. Thus $\lim_{i\to+\infty} \mathrm{mult}_{\bar{E}_i}(\bar{B}_i - B_i) = 0$. Hence

$$\begin{aligned}\alpha =& \mathrm{mld}(X_i \ni x_i, \bar{B}_i) + \lim_{i\to+\infty} \mathrm{mult}_{\bar{E}_i}(\bar{B}_i - B_i) = \lim_{i\to+\infty} a(\bar{E}_i, X_i, B_i)\\ \ge& \lim_{i\to+\infty} \mathrm{mld}(X_i \ni x_i, B_i) = \beta,\end{aligned}$$

a contradiction. $\square$

As a direct corollary, we have Theorem 1.2.9:

Proof. We follow the argument in [BS10, HLS19]. Suppose that the theorem does not hold. Then there exist a sequence of threefold a-lc germs $(X_i \ni x_i, B_i)$ such that X_i is terminal and $B_i \in \Gamma$, and a strictly increasing sequence of positive real numbers t_i, such that for every i, there exists an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D_i on X_i, such that $D_i \in \Gamma'$ and $t_i = a\text{-lct}(X_i \ni x_i, B_i; D_i)$. It is clear that $t := \lim_{i\to+\infty} t_i < +\infty$. Let $a_i := \mathrm{mld}(X_i \ni x_i, B_i + tD_i)$. By Theorem 3.3.2, possibly passing to a subsequence, we may assume that $a_i \geq 1$. Let $\{\epsilon_i\}_{i=1}^{\infty}$ be a strictly decreasing sequence which converges to 0, such that $0 < \epsilon_i < 1$ and $t_i' := t_i + \epsilon_i(t - t_i) \in (t_i, t_{i+1})$ for any i. Then all the coefficients of $B_i + t_i'D_i$ belong to a DCC set. By Theorem 3.6.1, the sequence $\{\mathrm{mld}(X_i \ni x_i, B_i + t_i'D_i)\}_{i=1}^{\infty}$ satisfies the ACC. By the convexity of minimal log discrepancies, we have

$$a > \mathrm{mld}(X_i \ni x_i, B_i + t_i'D_i)$$

$$= \mathrm{mld}(X_i \ni x_i, \frac{t_i' - t_i}{t - t_i}(B_i + tD_i) + \frac{t - t_i'}{t - t_i}(B_i + t_iD_i))$$

$$\geq \frac{t_i' - t_i}{t - t_i}\mathrm{mld}(X_i \ni x_i, B_i + tD_i) + \frac{t - t_i'}{t - t_i}\mathrm{mld}(X_i \ni x_i, B_i + t_iD_i)$$

$$\geq \frac{t_i' - t_i}{t - t_i}a_i + \frac{t - t_i'}{t - t_i}a = a - \frac{(t_i' - t_i)(a - a_i)}{t - t_i}$$

$$= a - \epsilon_i(a - a_i) \geq (1 - \epsilon_i)a.$$

Therefore, possibly passing to a subsequence, we may assume that $\mathrm{mld}(X_i \ni x_i, B_i + t_i'D_i)$ is strictly increasing and converges to a, which contradicts Theorem 3.6.1. $\square$

Chapter 4

Boundedness of Canonical Complements for Threefolds

4.1 Boundedness of Indices for Strictly Canonical Germs

Definition 4.1.1. Let $(X \ni x, B)$ be pair. We say that $(X \ni x, B)$ is *strictly canonical* if $\mathrm{mld}(X \ni x, B) = 1$. We say that (X, B) is *strictly canonical* if $\mathrm{mld}(X, B) = 1$.

Lemma 4.1.2. *Let $(X \ni x, B)$ be a strictly canonical germ such that B is a $\mathbb{Q}$-divisor. Let $f : Y \to X$ be a birational morphism which extracts a prime divisor E over $X \ni x$ such that $a(E, X, B) = 1$. Then the following holds.*

Let $\mathfrak{m}_x$ be the maximal ideal sheaf for $x \in X$, m the smallest positive integer such that mB is a Weil divisor, and r the smallest positive integer such that $rm(K_X + B)$ is Cartier near x. Then for any $i \in \mathbb{Z}$,

$$f_*\mathcal{O}_Y(im(K_Y + B_Y) - E) = \begin{cases} \mathfrak{m}_x\mathcal{O}_X(im(K_X + B)) & \text{if } r \mid i, \\ \mathcal{O}_X(im(K_X + B)) & \text{if } r \nmid i, \end{cases}$$

where $K_Y + B_Y := f^*(K_X + B)$.

Proof. If $r \mid i$, then by the projection formula,

$$f_*\mathcal{O}_Y(im(K_Y + B_Y) - E) = f_*\mathcal{O}_X(-E) \otimes \mathcal{O}_X(im(K_X + B)) = \mathfrak{m}_x\mathcal{O}_X(im(K_X + B)),$$

where the last equality follows from

$$f_*\mathcal{O}_Y(-E)(U) = \{u \in K(X) \mid ((u) - E)|_{f^{-1}(U)} \geq 0\} = \{u \in \mathcal{O}_X(U) \mid \operatorname{mult}_x(u) > 0\},$$

where U is an arbitrary open neighborhood of $x \in X$, $K(X)$ is the field of rational functions of X, and (u) is the Cartier divisor defined by the rational function u.

If $r \nmid i$, then $f_*\mathcal{O}_Y(im(K_Y + B_Y) - E)(U) \subset \mathcal{O}_X(im(K_X + B))(U)$ for any open set $U \subset X$ as

$$f_*\mathcal{O}_Y(im(K_Y + B_Y) - E)(U) = \{u \in K(X) \mid ((u) + imf^*(K_X + B) - E)|_{f^{-1}(U)} \geq 0\}, \text{ and}$$

$$\mathcal{O}_X(im(K_X + B))(U) = \{u \in K(X) \mid ((u) + im(K_X + B))|_U \geq 0\}.$$

Suppose that $u \in K(X)$ satisfies $((u) + im(K_X + B))|_U \geq 0$. Since $r \nmid i$, $(u) + im(K_X + B)$ is not Cartier at x, so there exists an effective $\mathbb{Q}$-Cartier Weil divisor D passing through x such that $((u) + im(K_X + B) - D)|_U \geq 0$, which implies that $((u) + im(K_Y + B_Y) - f^*D)|_{f^{-1}(U)} \geq 0$. Since $E \subset \operatorname{Supp}(f^*D)$, we obtain $((u) + im(K_Y + B_Y) - E)|_{f^{-1}(U)} \geq 0$. Thus $f_*\mathcal{O}_Y(im(K_Y + B_Y) - E) = \mathcal{O}_X(im(K_X + B))$ in this case. ☐

Notation $(\star)$. Let $f : Y \to X$ be a divisorial contraction of a prime divisor E over $X \ni x$ as in Theorem 2.1.35(1) (see also [Kaw05, Theorem 1.2(1)]). Recall that in this case, $x \in X$ is a terminal singularity of type cA/n. In particular, under suitable analytic local coordinates x_1, x_2, x_3, x_4,

$$(X \ni x) \cong (\phi := x_1x_2 + g(x_3^n, x_4) = 0) \subset (\mathbb{C}^4 \ni o)/\frac{1}{n}(1, -1, b, 0),$$

where $b \in [1, n-1] \cap \mathbb{Z}$ such that $\gcd(b,n) = 1$ and f is a weighted blow-up with the weight $w = \frac{1}{n}(r_1, r_2, a, n)$ for some positive integers a, r_1, r_2, such that $an \mid r_1 + r_2$ and $a \equiv br_1 \mod n$.

Let J' be the Reid basket for $f : Y \to X$ (see Definition 2.1.20). By [Kaw05, Theorem 1.2], we have three cases: $J' = \emptyset$, $J' = \{(r'_{Q'}, 1)_{Q'}\}$, or $J' = \{(r'_{Q'_1}, 1)_{Q'_1}, (r'_{Q'_2}, 1)_{Q'_2}\}$, where $r'_{Q'}, r'_{Q'_1}, r'_{Q'_2} \in \mathbb{Z}_{\geq 2}$, and Q', Q'_1, Q'_2 are fictitious singularities (see Definition-Lemma 2.1.17). In the case when $J' = \{(r'_{Q'_1}, 1)_{Q'_1}, (r'_{Q'_2}, 1)_{Q'_2}\}$, Q'_1, Q'_2 come from two different non-Gorenstein points on Y. In the following, we introduce the set $J := \{(r_{Q_1}, 1)_{Q_1}, (r_{Q_2}, 1)_{Q_2}\}$ for f, here Q_1, Q_2 may not be fictitious cyclic quotient singularities any more as they could be smooth points. We let Q_1, Q_2 be any smooth points on Y and $(r_{Q_1}, r_{Q_2}) := (1,1)$ when $J' = \emptyset$, Q_1 any smooth closed point on Y, $Q_2 = Q'$ and $(r_{Q_1}, r_{Q_2}) := (1, r'_{Q'})$ when $J' = \{(r_{Q'}, 1)_{Q'}\}$, and $Q_1 := Q'_1, Q_2 := Q'_2$, $(r_{Q_1}, r_{Q_2}) := (r'_{Q'_1}, r'_{Q'_2})$ when $J' = \{(r'_{Q'_1}, 1)_{Q'_1}, (r'_{Q'_2}, 1)_{Q'_2}\}$.

Lemma 4.1.3. *With* **Notation** $(\star)$. *Up to a permutation, we have $r_{Q_1} = r_1$ and $r_{Q_2} = r_2$. Moreover, Q_1, Q_2 are indeed singularities (possibly smooth) on Y.*

Proof. By [Kaw05, Theorem 6.5, Page 112, Line 12-14] and [CH11, Proposition 2.15, Page 9, Line 15], there are two cyclic quotient terminal singularities $P_1, P_2 \in Y$ of type $\frac{1}{r_1}(1, -1, b_1)$, $\frac{1}{r_2}(1, -1, b_2)$ respectively and possibly a cA/n type singularity $P_3 \in Y$. By [Kaw05, Theorem 4.3], possibly changing the order of the indices, Q_1, Q_2 are P_1, P_2 on Y respectively. It follows that $r_{Q_1} = r_1$, and $r_{Q_2} = r_2$. □

Lemma 4.1.4. *Let $(X \ni x, B)$ be a threefold germ and B a $\mathbb{Q}$-divisor, such that X is terminal and $\mathrm{mld}(X \ni x, B) = 1$. Let $f : Y \to X$ be a divisorial contraction of a prime divisor E over $X \ni x$ as in* **Notation** $(\star)$*, such that $a(E, X, B) = 1$. Let m be the smallest positive integer such that mB is a Weil divisor near x, and r the smallest positive integer such that $rm(K_X + B)$ is Cartier near x. Then $r \mid \gcd(r_1, r_2)$.*

Proof. Possibly shrinking and compactifying X, we may assume that X is projective and terminal. In particular, Y is also projective and terminal. For each $r \in \mathbb{Z}_{>0}$ and $i \in \mathbb{Z}$, we define

$$\delta_r(i) := \begin{cases} 1 & \text{if } r \mid i, \\ 0 & \text{if } r \nmid i. \end{cases}$$

Let $D_{i,m} := im(K_Y + B_Y) = imf^*(K_X + B)$ for each $i \in \mathbb{Z}$. Since E is $\mathbb{Q}$-Cartier, by [KM98, Proposition 5.26], we have the following short exact sequence

$$0 \to \mathcal{O}_Y(D_{i,m} - E) \to \mathcal{O}_Y(D_{i,m}) \to \mathcal{O}_E(D_{i,m}|_E) \to 0.$$

Since $D_{i,m} - E$ and $D_{i,m}$ are both f-big and f-nef, by the Kawamata-Viehweg vanishing theorem [KMM87, Theorem 1.2.5], $R^j f_* \mathcal{O}_Y(D_{i,m} - E) = R^j f_* \mathcal{O}_Y(D_{i,m}) = 0$ for all $j \in \mathbb{Z}_{>0}$. It follows that $h^j(\mathcal{O}_E(D_{i,m}|_E)) = 0$ for all $j \in \mathbb{Z}_{>0}$. By Lemma 4.1.2,

$$\delta_r(i) = h^0(\mathcal{O}_X(im(K_X + B))/f_*\mathcal{O}_Y(im(K_Y + B_Y)) - E))$$

$$= h^0(\mathcal{O}_E(D_{i,m}|_E)) = \chi(\mathcal{O}_E(D_{i,m}|_E))$$

$$= \chi(\mathcal{O}_Y(D_{i,m})) - \chi(\mathcal{O}_Y(D_{i,m} - E)).$$

For each fictitious singularity $Q \in Y_Q$ of some closed point on Y,

$$(D_{1,m})_Q \sim d_Q K_{Y_Q} \text{ and } E_Q \sim f_Q K_{Y_Q}$$

near $Q \in Y_Q$ for some integers $f_Q, d_Q \in [1, r_Q]$, where Y_Q is the deformed variety on which Q appears as a cyclic quotient terminal singularity, and $E_Q, (D_{i,m})_Q$ are the corresponding deformed divisors on Y_Q (see Definition-Lemma 2.1.17). By Theorem 2.1.19,

$$\delta_r(i) = \chi(\mathcal{O}_Y(D_{i,m})) - \chi(\mathcal{O}_Y(D_{i,m} - E)) = \Delta_1 + \Delta_2 + \frac{1}{12} E \cdot c_2(X)$$

with

$$\Delta_1 = T(D_{i,m}) - T(D_{i,m} - E), \Delta_2 = \sum_{y \in Y, \dim y = 0} (c_y(D_{i,m}) - c_y(D_{i,m} - E)),$$

and

$$T(D) = \frac{1}{12} D(D - K_Y)(2D - K_Y).$$

Since $D_{i,m} \cdot E^2 = D_{i,m}^2 \cdot E = D_{i,m} \cdot E \cdot K_Y = 0$, $\Delta_1 = \frac{1}{6}E^3 + \frac{1}{4}E^2 \cdot K_Y$. For any fictitious point Q, if E_Q is Cartier, then $c_Q((D_{i,m})_Q) = c_Q((D_{i,m} - E)_Q)$. By Definition-Lemma 2.1.17 and Definition 2.1.20,

$$\Delta_2 = \sum_{Q \in J} c_Q(D_{i,m}) - c_Q(D_{i,m} - E) = \sum_{Q \in J} (A_Q(id_Q) - A_Q(id_Q - f_Q)),$$

where J is defined as in **Notation** ($\star$) for $f : Y \to X$, and

$$A_Q(i) := -i \frac{r_Q^2 - 1}{12 r_Q} + \sum_{j=1}^{i-1} \frac{\overline{(jb_Q)}_{r_Q} (r_Q - \overline{(jb_Q)}_{r_Q})}{2 r_Q}.$$

By $Q \in J$, we mean Q is a fictitious singularity that contributes to J. Here we allow $i < 0$ if we adopt the notation of generalized summation (see Definition 2.1.16). It is worthwhile to mention that $A_Q(i) = A_Q(\overline{(i)}_{r_Q})$ as $\gcd(b_Q, r_Q) = 1$. Now

$$\delta_r(i+1) - \delta_r(i) = \sum_{Q \in J} \sum_{j = id_Q}^{(i+1)d_Q - 1} (B_Q(jb_Q) - B_Q(jb_Q - v_Q)). \tag{4.1.1}$$

Here, for each $Q \in J$, $B_Q(i)$ is an even periodic function with period r_Q defined by

$$B_Q(i) := \frac{\overline{(i)}_{r_Q} (r_Q - \overline{(i)}_{r_Q})}{2 r_Q}.$$

By Lemma 4.1.3, $J = \{(r_1, 1)_{Q_1}, (r_2, 1)_{Q_2}\}$, where Q_1, Q_2 are cyclic quotient terminal singularities (might be smooth) on Y. It follows that

$$\delta_r(i+1) - \delta_r(i) = \sum_{k=1,2} \sum_{j = id_{Q_k}}^{(i+1)d_{Q_k} - 1} (B_{Q_k}(jb_{Q_k}) - B_{Q_k}(jb_{Q_k} - 1)). \tag{4.1.2}$$

Claim 4.1.5. *We have the following equality:*

$$r = \mathrm{lcm}\{\frac{r_1}{\gcd(r_1, d_{Q_1})}, \frac{r_2}{\gcd(r_2, d_{Q_2})}\}.$$

We proceed the proof assuming Claim 4.1.5. Let $l_1 := \frac{r_1}{\gcd(r_1,r_2)}$ and $l_2 := \frac{r_2}{\gcd(r_1,r_2)}$. We have $\gcd(l_1 l_2, l_1 + l_2) = 1$ as $\gcd(l_1, l_2) = 1$. By Claim 4.1.5, $r \mid \mathrm{lcm}(r_1, r_2) = \gcd(r_1, r_2) l_1 l_2$. Since $r \mid n$ and $n \mid r_1 + r_2$, $r \mid \gcd(r_1, r_2)(l_1 + l_2)$. Hence $r \mid \gcd(r_1, r_2)$. $\square$

Proof of Claim 4.1.5. Let $\lambda \in \mathbb{Z}$ such that $r_{Q_k} \mid \lambda d_{Q_k}$ for $k = 1, 2$. By (4.1.2),

$$\delta_r(\lambda + 1) - \delta_r(\lambda) = \sum_{k=1,2} \sum_{j=\lambda d_{Q_k}}^{(\lambda+1)d_{Q_k}-1} (B_{Q_k}(jb_{Q_k}) - B_{Q_k}(jb_{Q_k} - 1))$$

$$= \sum_{k=1,2} \sum_{j=0}^{d_{Q_k}-1} (B_{Q_k}(jb_{Q_k}) - B_{Q_k}(jb_{Q_k} - 1)) = \delta_r(1) - \delta_r(0).$$

Thus $r \mid \lambda$. By Lemma 4.1.3, $r \mid \mathrm{lcm}\{\frac{r_1}{\gcd(r_1,d_{Q_1})}, \frac{r_2}{\gcd(r_2,d_{Q_2})}\}$.

Since $rD_{1,m} = rm(K_Y + B_Y)$ is Cartier, $r_k \mid rd_{Q_k}$ for $k = 1, 2$. Thus $\frac{r_k}{\gcd(r_k,d_{Q_k})} \mid r$ for $k = 1, 2$, and the claim is proved. $\square$

Remark 4.1.6. Let $n = r(4r^2 - 2r - 1)$, $a = r$, $b = 4r^2 + 2r - 1$, $r_1 = r_{Q_1} = (2r - 1)^2 r^2, d_{Q_1} = (2r - 1)^2 r^2, b_{Q_1} = 4r^3 - r + 1$, $r_2 = r_{Q_2} = 2r^2(r - 1)$, $d_{Q_2} = 2r(r - 1)$, and $b_{Q_2} = 2r^2 - 1$. Then $(n, a, b, r_{Q_1}, d_{Q_1}, b_{Q_1}, r_{Q_2}, d_{Q_2}, b_{Q_2})$ satisfies both (4.1.1) and Claim 4.1.5. Moreover, as $r \mid n$, $\gcd(b, n) = 1$, $n \mid a - br_{Q_1}$, $an \mid r_{Q_1} + r_{Q_2}$, and $\gcd(\frac{a-br_{Q_1}}{n}, r_{Q_1}) = 1$, $(n, a, b, r_{Q_1}, d_{Q_1}, b_{Q_1}, r_{Q_2}, d_{Q_2}, b_{Q_2})$ also satisfies the restrictions proved in [Kaw05, Theorem 1.2(1)]. Hence we could not show Theorem 4.1.7 by simply applying singular Riemann-Roch formula for terminal threefold as [Kaw15a] did for the case when $B = 0$, X is canonical and x is an isolated canonical center of X.

Theorem 4.1.7. *Let $\Gamma \subset [0, 1]$ be a set and m a positive integer such that $m\Gamma \subset \mathbb{Z}$. Then the positive integer*

$N := 12m^2$ *satisfies the following.*

Let $(X \ni x, B)$ *be a threefold germ such that* X *is terminal,* $B \in \Gamma$*, and* $\mathrm{mld}(X \ni x, B) = 1$*. Then* $I(K_X + B)$ *is Cartier near* x *for some positive integer* $I \leq N$*.*

Proof. Let r be the smallest positive integer such that $rm(K_X + B)$ is Cartier near x.

By [Rei87, (6.1) Theorem], if $x \in X$ is a terminal singularity of types other than cA/n, then the index of $x \in X$ divides 12. By [Kaw88, Lemma 5.1], $12m(K_X + B)$ is Cartier near x. From now on, we may assume that $x \in X$ is a terminal singularity of type cA/n.

By Lemma 3.1.4, there exists a terminal blow-up (see Definition 3.1.3) $f : Y \to X$ of $(X \ni x, B)$ which extracts a prime divisor E over $X \ni x$. By [Kaw05, Theorem 1.1], f is either of ordinary type or of exceptional type.

If f is of exceptional type, then by [Kaw05, Theorem 1.3], $x \in X$ is a terminal singularity of type cA. Hence by [Kaw88, Lemma 5.1], $m(K_X + B)$ is Cartier.

We may now assume that f is of ordinary type, and we write $f^*K_X + \frac{a}{n}E = K_Y$ for some positive integer $a \geq 1$. Now $f : Y \to X$ is a divisorial contraction of ordinary type as in Theorem 2.1.35(1). In particular, under suitable analytic local coordinates x_1, x_2, x_3, x_4,

$$(X \ni x) \cong (\phi := x_1x_2 + g(x_3^n, x_4) = 0) \subset (\mathbb{C}^4 \ni o)/\frac{1}{n}(1, -1, b, 0),$$

where $b \in [1, n-1] \cap \mathbb{Z}$, $\gcd(b, n) = 1$, and f is a weighted blow-up at $x \in X$ with the weight $w := \frac{1}{n}(r_1, r_2, a, n)$. Now mB is a Weil divisor locally defined by a semi-invariant analytic power series $(h(x_1, x_2, x_3, x_4) = 0)$.

Claim 4.1.8. *Either* $r \leq 3m$*, or* $x_3^m \in h$ *(up to a scaling of* h*).*

We proceed the proof assuming Claim 4.1.8. If $r \leq 3m$, then $Im(K_X + B)$ is Cartier for some $I \leq 3m$. Otherwise, $r > 3m$. By Claim 4.1.8, up to a scaling of h, we have $h = x_3^m + p$ for some analytic power series p such that $\lambda x_3^m \notin p$ for any $\lambda \in \mathbb{C}^*$. Recall that ξ_n is the primitive n-th root of unity. Since $h = x_3^m + p$ is semi-invariant with respect to the ξ_n-action: $(x_1, x_2, x_3, x_4) \to (\xi_n x_1, \xi_n^{-1} x_2, \xi_n^b x_3, x_4)$, $\xi_n(h)/h = \xi_n^{mb}$. Since $\xi_n(x_3^m)/x_3^m = \xi_n^{mb}$, $\xi_n(\frac{h}{x_3^m}) = \frac{h}{x_3^m}$, and $(\frac{h}{x_3^m})$ is ξ_n-invariant, hence $(\frac{h}{x_3^m})$ is a rational function on X which defines a principle divisor. Now $mB = (x_3^m + p = 0) \sim (x_3^m = 0)$ near x. Let S be the analytic Cartier divisor locally defined by $(x_3 = 0)$ on X. By [Rei87, (6.4)(B.1)] and Lemma 2.1.10, $K_X + S$ is Cartier near x. It follows that $m(K_X + B) \sim m(K_X + S)$ is Cartier near x. □

Proof of Claim 4.1.8. Assume that $r > 3m$. By Lemma 4.1.4, $r_1 > 3m$ and $r_2 > 3m$. Note also that $n \geq r > 3m$. When $a \leq 2$,

$$w(h) = mw(B) = mw(X \ni x) = \frac{am}{n} < \frac{3m}{n}.$$

Since $w(x_k) = \frac{r_k}{n} > \frac{3m}{n}$ for $k = 1, 2$ and $w(x_4) = 1 > \frac{3m}{n}$, up to a scaling of h, $x_3^l \in h$ for some $l \in \mathbb{Z}_{>0}$ and $w(h) = \frac{la}{n}$. Thus $l = m$, and the claim follows in this case.

When $a \geq 3$, we can pick positive integers s_1, s_2 such that

- $s_1 + s_2 = 3dn$,
- $3 \equiv bs_1 \mod n$, and
- $s_1, s_2 > n$.

Let $\bar{w} := \frac{1}{n}(s_1, s_2, 3, n)$. Since $a \geq 3$, by [HLL22, Lemma C.7], the weighted blow-up with the weight $\bar{w}$ extracts a prime analytic divisor $\bar{E}$ such that $\bar{w}(X \ni x) = \frac{3}{n}$. By [HLL22, Lemma C.6], we may assume that E is a prime divisor over $X \ni x$. Since $\mathrm{mld}(X \ni x, B) = 1$, $a(\bar{E}, X, B) = 1 + \bar{w}(X \ni x) - \bar{w}(B) \geq 1$, thus

$$1 > \frac{3m}{n} = m\bar{w}(X \ni x) \geq m\bar{w}(B) = \bar{w}(h).$$

Since $\bar{w}(x_1) = \frac{s_1}{n} > 1$, $\bar{w}(x_2) = \frac{s_2}{n} > 1$, and $\bar{w}(x_4) = 1$, there exists a positive integer l, such that up to a scaling of h, $x_3^l \in h$ and $\bar{w}(h) = \bar{w}(x_3^l) = \frac{3l}{n}$. Since $\frac{3m}{n} \geq \bar{w}(h) = \frac{3l}{n}$, $l \leq m$. On the other hand,

$$\frac{am}{n} = mw(X \ni x) = w(mB) = w(h) \leq w(x_3^l) = \frac{al}{n},$$

which implies that $l \geq m$. Thus $l = m$, and $x_3^m \in h$ up to a scaling of h. □

Proof of Theorem 1.2.8. If $\dim x = 2$, then the theorem is trivial. If $\dim x = 1$, then X is smooth near x. By Lemma 2.1.6, $\mathrm{mld}(X \ni x, B) = 2 - \mathrm{mult}_x B = \epsilon$, hence the coefficients of B belong to a finite set of rational numbers depending only on ϵ and Γ, and the theorem holds in this case. Thus we may assume that $\dim x = 0$ and X is not smooth at x.

Let B_1 be any component of B with coefficient b_1. Let $f : Y \to X$ be a small $\mathbb{Q}$-factorialization, and $B_{1,Y}$ the strict transform of B_1 on Y. We may write $K_Y + B_Y := f^*(K_X + B)$. Let $Y \dashrightarrow Z$ be the canonical model of $(Y, B_Y - b_1 B_{1,Y})$ over X. Let B_Z and $B_{1,Z}$ be the strict transforms of B_Y and $B_{1,Y}$ on Z respectively. Since $-B_{1,Z}$ is ample over X, $\mathrm{Supp}\, B_{1,Z}$ contains $g^{-1}(x)$, where $g : Z \to X$ is the natural induced morphism. Moreover, since g is small, $K_Z = g^* K_X$ and Z is terminal.

Since $\mathrm{mld}(X \ni x, B) = \epsilon$, there exists a point $z \in g^{-1}(x)$ such that $\mathrm{mld}(Z \ni z, B_Z) = \epsilon$. Then

$$b_1 = \epsilon\text{-}\mathrm{lct}(Z \ni z, B_Z - b_1 B_{1,Z}; B_{1,Z}) \in \Gamma.$$

By Theorem 1.2.9, b_1 belongs to a finite set depending only on Γ. Hence we may assume that the coefficients of B belong to a finite set Γ' depending only on Γ.

If $\epsilon = 1$, then Theorem 1.2.8 follows from Theorem 4.1.7. If $\epsilon > 1$, then we let n be the index of $X \ni x$.

By [Sho92, Appendix, Theorem] and [Mar96, Theorem 0.1], there exists a prime divisor E over $X \ni x$ such that $a(E, X, 0) = 1 + \frac{1}{n}$. Since $1 + \frac{1}{n} \geq a(E, X, B) \geq \epsilon$, $n \leq \frac{1}{\epsilon - 1}$, and the theorem follows from [Kaw88, Lemma 5.1]. □

4.2 Boundedness of Complements for Finite Rational Coefficients

We prove Theorem 4.2.4 in this section, and Theorem 1.2.6 follows as a direct corollary. Also, for any $\epsilon \geq 1$, we prove the existence of (ϵ, N)-complements for terminal pairs (see Theorem 4.2.5).

Lemma 4.2.1. *Let $x \in X$ be an isolated singularity such that X is affine. Let $D \geq 0$ be a Weil divisor on X and E a prime divisor over $X \ni x$. Then there exists a finite dimensional linear system $\mathfrak{d} \subset |D|$, such that*

1. *$\mathfrak{d}$ contains D,*
2. *the base locus of $\mathfrak{d}$ is x, and*
3. *if D is $\mathbb{Q}$-Cartier, then $\operatorname{mult}_E D' \geq \operatorname{mult}_E D$ for any $D' \in \mathfrak{d}$.*

In particular, if $x \in X$ is a terminal threefold singularity such that X is affine, then there exists a finite dimensional linear system $\mathfrak{d} \subset |-K_X|$ such that $\mathfrak{d}$ contains an elephant (cf. [Rei87, (6.4)(B)]) of $x \in X$, and the base locus of $\mathfrak{d}$ is x.

Proof. Let $\mathcal{I}_E$ be the ideal sheaf on X such that for any open set $x \in U \subset X$,

$$\mathcal{I}_E(U) = \begin{cases} \{u \in \mathcal{O}_X(U) \mid \operatorname{mult}_E(u) > 0\} & \text{if } D \text{ is not } \mathbb{Q}\text{-Cartier}, \\ \{u \in \mathcal{O}_X(U) \mid \operatorname{mult}_E(u) \geq \operatorname{mult}_E D\} & \text{if } D \text{ is } \mathbb{Q}\text{-Cartier}, \end{cases}$$

where (u) is the Cartier divisor defined by the rational function u, and $\mathcal{I}_E(U) = \mathcal{O}_X(U)$ for any open set $x \notin U \subset X$. Since X is affine, the coherent sheaf $\mathcal{O}_X(D) \otimes \mathcal{I}_E$ is globally generated. Since $\mathcal{O}_X(D) \otimes \mathcal{I}_E|_{X \setminus \{x\}} = \mathcal{O}_X(D)|_{X \setminus \{x\}}$ is an invertible sheaf, there exist finitely many sections $s_1, \dots, s_m \in$

$H^0(X, \mathcal{O}_X(D) \otimes \mathcal{I}_E)$, such that the linear system defined by $s_1, \ldots, s_m$ is base point free on $X \backslash \{x\}$. Pick $s_0 \in H^0(X, \mathcal{O}_X(D) \otimes \mathcal{I}_E)$ such that $(s_0) = 0$. Then the linear system $\mathfrak{d}$ defined by $s_0, \ldots, s_m$ satisfies our requirements. $\square$

Lemma 4.2.2. *Let $(X \ni x, B)$ be a canonical threefold germ such that X is affine terminal but not smooth, m a positive integer such that $mB \in \mathbb{Z}$, and $\mathfrak{d}$ a finite dimensional linear system whose base locus is x. Then for any integer $N > m$ and any general element $(D_1, \ldots, D_N) \in \mathfrak{d}^N$, the divisor $D := \sum_{i=1}^N D_i$ satisfies that* $\operatorname{ct}(X, B; D) = \operatorname{ct}(X \ni x, B; D)$.

Proof. Let $f : Y \to X$ be a log resolution of (X, B) such that

- $f^*|\mathfrak{d}| = F + |M|$, and
- $\operatorname{Supp} B_Y \cup \operatorname{Supp} F \cup \operatorname{Supp}(\operatorname{Exc}(f))$ is snc,

where B_Y is the strict transform of B on Y, F is the fixed part of $f^*|\mathfrak{d}|$, and M is a base point free Cartier divisor. Note that $\operatorname{Supp} F \subset f^{-1}(x)$ as $\mathfrak{d}$ is base point free on $X \setminus \{x\}$. Let $M' := \sum_{i=1}^N M_i$, where $M_1, \cdots, M_N$ are N general elements in $|M|$. Set $D := f_*(NF + M')$. Then $D = \sum_{i=1}^N D_i$, where $D_i := f_*(F + M_i)$ for each i, and $(D_1, \ldots, D_N) \in \mathfrak{d}^N$ is a general element. Since D has N distinct components, by Theorem 2.1.13(2), $t := \operatorname{ct}(X, B; D) \leq \frac{1}{N}$.

If $\lfloor B \rfloor \neq 0$, then $B = \lfloor B \rfloor$ is a $\mathbb{Q}$-Cartier prime divisor and $t = 0$. By Theorem 2.1.13(2), $\operatorname{mld}(X \ni x, B) = 1$. Hence $\operatorname{ct}(X \ni x, B; D) = \operatorname{ct}(X, B; D) = 0$ in this case.

If $\lfloor B \rfloor = 0$, then $B \in (0, 1 - \frac{1}{m}]$. Since $K_Y + B_Y + tM' + tNF = f^*(K_X + B + tD) + G$ for some $\mathbb{Q}$-divisor $G \geq 0$, $\operatorname{Supp} M' \cup \operatorname{Supp} B_Y$ is snc, and $t \leq \frac{1}{N} < \frac{1}{m}$, we have $(Y, B_Y + tM' + tNF)$ is terminal on $Y \setminus \operatorname{Supp} F$. Hence $\operatorname{ct}(X, B; D) = \operatorname{ct}(X \ni x, B; D)$ as $\operatorname{Supp} F$ contains at least one canonical place of $(X, B + tD)$. $\square$

Lemma 4.2.3. *Let I be a positive integer and $\Gamma\subset[0,1]\cap\mathbb{Q}$ a finite set. Then there exists a positive integer N depending only on I and Γ satisfying the following.*

Let $(X\ni x,B)$ be a threefold germ such that X is terminal, $B\in\Gamma$, $\mathrm{mld}(X\ni x,B)\geq 1$, and IK_X is Cartier. Then there exists a monotonic N-complement $(X\ni x,B^+)$ of $(X\ni x,B)$ such that $\mathrm{mld}(X\ni x,B^+)=1$, and if $(X\ni x,B)$ is canonical near x and $x\in X$ is not smooth, then $(X\ni x,B^+)$ is canonical near x.

Proof. Possibly shrinking X near x, we may assume that (X,B) is lc and X is affine. For any positive real number $\epsilon<1$, since $(X\ni x,(1-\epsilon)B)$ is a klt germ, by [HLS19, Lemmas 3.12 and 3.13], there exists a $\mathbb{Q}$-factorial weak plt blow-up $f_\epsilon:Y_\epsilon\to X$ of $(X\ni x,(1-\epsilon)B)$, such that $(Y_\epsilon,(1-\epsilon)B_{Y_\epsilon}+E_{Y_\epsilon})$ is $\mathbb{Q}$-factorial plt, where B_{Y_ϵ} is the strict transform of B on Y_ϵ, and E_{Y_ϵ} is the reduced exceptional divisor of f_ϵ. By [HMX14, Theorem 1.1], we may choose $\epsilon<1$ such that $(Y_\epsilon,B_{Y_\epsilon}+E_{Y_\epsilon})$ is lc. Let $Y:=Y_\epsilon, B_Y:=B_{Y_\epsilon}$, $E:=E_{Y_\epsilon}$, and $f:=f_\epsilon$. By [HLS19, Corollary 3.20], there exists a $\mathbb{Q}$-divisor $G_Y\geq 0$ on Y, such that $N'G_Y$ is a Weil divisor, and $(Y/X\ni x,B_Y+E+G_Y)$ is an N'-complement of $(Y/X\ni x,B_Y+E)$ for some positive integer N' depending only on Γ. Then $K_Y+B_Y+G_Y+E=f^*(K_X+B+G)$, $(X\ni x,B+G)$ is an N'-complement of $(X\ni x,B)$, and $\mathrm{mld}(X\ni x,B+G)=0$, where $G:=f_*G_Y$.

Let m be a positive integer such that $m\Gamma\subset\mathbb{Z}$ and $N'':=m(m+1)N'$. Since $mN'G\in|-mN'(K_X+B)|$ near x, by Lemma 4.2.1, there exists a finite dimensional linear system $\mathfrak{d}\subset|-mN'(K_X+B)|$ such that $\mathfrak{d}$ contains $mN'G$, the base locus of $\mathfrak{d}$ is x, and $\mathrm{mult}_E G'\geq\mathrm{mult}_E mN'G$ for any $G'\in\mathfrak{d}$. By Lemma 4.2.2, when (X,B) is canonical and $x\in X$ is not smooth, the divisor $G'':=G'_1+\cdots+G'_{m+1}\in|-N''(K_X+B)|$ satisfies that $t:=\mathrm{ct}(X,B;\frac{1}{N''}G'')=\mathrm{ct}(X\ni x,B;\frac{1}{N''}G'')$, where $(G'_1,\ldots,G'_{m+1})\in\mathfrak{d}^{m+1}$ is a general element. By construction, $(X\ni x,B+\frac{1}{N''}G'')$ is an N''-complement of $(X\ni x,B)$. Since $\mathrm{mult}_E\frac{1}{N''}G''\geq\mathrm{mult}_E G$, $\mathrm{mld}(X\ni x,B+\frac{1}{N''}G'')=0$. In particular, $t<1$.

By Theorem 4.1.7, it suffices to show that t belongs to a finite set of rational numbers depending only on I and Γ. By Lemma 2.1.12(1), there exists a prime divisor F over $X \ni x$ such that $a(F, X, B + \frac{t}{N''}G'') = 1$. Then $a(F, X, B + \frac{1}{N''}G'') = \frac{i}{N''}$ for some non-negative integer $i < N''$. We have

$$a(F, X, B + \frac{t}{N''}G'') = a(F, X, B + \frac{1}{N''}G'') + (1-t)\operatorname{mult}_F \frac{1}{N''}G'' = \frac{i}{N''} + (1-t)\operatorname{mult}_F \frac{1}{N''}G'',$$

and $\operatorname{mult}_F \frac{1}{N''}G'' = \frac{1}{1-t}(1 - \frac{i}{N''})$. By Theorem 3.3.1, $\delta \leq 1 - t$ for some positive real number δ depending only on Γ. Thus $\operatorname{mult}_F \frac{1}{N''}G'' \leq \frac{1}{\delta}(1 - \frac{i}{N''})$. By [Kaw88, Lemma 5.1], IG'' is Cartier near x. It follows that $\operatorname{mult}_F \frac{1}{N''}G''$ belongs to a finite set depending only on I and Γ. Hence $t = 1 - (1 - \frac{i}{N''})\frac{1}{\operatorname{mult}_F \frac{1}{N''}G''}$ belongs to a finite set of rational numbers depending only on I and Γ. $\square$

Theorem 4.2.4. *Let $\Gamma \subset [0, 1] \cap \mathbb{Q}$ be a finite set. Then there exists a positive integer N depending only on Γ satisfying the following.*

Let $(X \ni x, B)$ be a threefold germ such that X is a terminal, $B \in \Gamma$, and $\operatorname{mld}(X \ni x, B) \geq 1$. Then there exists a monotonic N-complement $(X \ni x, B^+)$ of $(X \ni x, B)$ such that $\operatorname{mld}(X \ni x, B^+) = 1$. Moreover, if (X, B) is canonical near x and $x \in X$ is not smooth, then (X, B^+) is canonical near x.

Proof. Let m be a positive integer such that $m\Gamma \subset \mathbb{Z}$.

By Theorem 4.1.7, we may assume that $\operatorname{mld}(X \ni x, B) > 1$. Let n be the index of $X \ni x$. By Lemma 4.2.3, we may assume that $n > 4m$. By [Rei87, (6.1) Theorem] (cf. [Mor85, Theorems 12,23,25]), $x \in X$ is of type cA/n for some $n > 4m$.

Possibly shrinking X near x, we may assume that X is affine. By Lemma 4.2.1, there exists a finite dimensional linear system $\mathfrak{d} \subset |-K_X|$ such that $\mathfrak{d}$ contains an elephant of $x \in X$ and the base locus of $\mathfrak{d}$ is x. By Lemma 4.2.2, if (X, B) is canonical and $x \in X$ is not smooth, then the divisor $D = D_1 + \cdots + D_{m+1}$ satisfies $t := \operatorname{ct}(X, B; \frac{1}{m+1}D) = \operatorname{ct}(X \ni x, B; \frac{1}{m+1}D)$, where $(D_1, \ldots, D_{m+1}) \in \mathfrak{d}^{m+1}$ is a general element. Since $(X, \frac{1}{m+1}D)$ is canonical near x, by Theorem 2.1.13(2), $\mathrm{mld}(X \ni x, \frac{1}{m+1}D) = 1$.

By Lemma 3.1.4, there exists a terminal blow-up (see Definition 3.1.3) $f : Y \to X$ of $(X \ni x, B + \frac{t}{m+1}D)$ which extracts a prime divisor E over $X \ni x$. Since $x \in X$ is a terminal singularity of type cA/n for some $n > 4m > 1$, by [Kaw05, Theorem 1.3] and Theorem 2.1.35, $f : Y \to X$ is a divisorial contraction of ordinary type as in Theorem 2.1.35(1). We may write $K_Y = f^*K_X + \frac{a}{n}E$ for some positive integer a.

Since $(X, \frac{1}{m+1}D)$ is canonical near x, $a(E, X, \frac{1}{m+1}D) = a(E, X, 0) - \mathrm{mult}_E \frac{1}{m+1}D = 1 + \frac{a}{n} - \mathrm{mult}_E \frac{1}{m+1}D \geq 1$. It follows that $\mathrm{mult}_E \frac{1}{m+1}D \leq \frac{a}{n}$. Since $B = \frac{1}{m}mB$, $n > 3m$, and $\mathrm{mld}(X \ni x, B) > 1$, by Lemma 3.2.7(1), $a \leq 3$, and $\mathrm{mult}_E \frac{1}{m+1}D \in \{\frac{i}{(m+1)n} \mid i \in \mathbb{Z} \cap [1, 3(m+1)]\}$. Since $nm(K_X + B)$ is Cartier, $a(E, X, B) = 1 + \frac{k}{nm}$ for some positive integer k. Since $a(E, X, B + \frac{t}{m+1}D) = 1$, $t\,\mathrm{mult}_E \frac{1}{m+1}D = a(E, X, B) - a(E, X, B + \frac{t}{m+1}D) = \frac{k}{nm}$, which implies that $t = \frac{k}{nm\,\mathrm{mult}_E \frac{1}{m+1}D} \in \frac{1}{(3m+3)!m}\mathbb{Z}_{>0}$. Now the coefficients of $B + \frac{t}{m+1}D$ belong to $\frac{1}{(3m+3)!(m+1)m}\mathbb{Z} \cap [0, 1]$. By Theorem 1.2.8, $(X \ni x, B + \frac{t}{m+1}D)$ is a monotonic N-complement of $(X \ni x, B)$ for some positive integer N depending only on Γ satisfying all the required properties. $\square$

Proof of Theorem 1.2.6. When X is smooth near x, in particular, when $\dim x \geq 1$, we may take $G = 0$, and (X, B) is an m-complement of itself, where m is a positive integer such that $m\Gamma \subset \mathbb{Z}$. When $x \in X$ is a closed point that is not smooth, by Theorem 4.2.4, we are done. $\square$

Theorem 4.2.5. *Let $\epsilon \geq 1$ be a rational number and $\Gamma \subset [0, 1] \cap \mathbb{Q}$ a finite set. Then there exists a positive integer N depending only on ϵ and Γ satisfying the following.*

Let $(X \ni x, B)$ be a threefold ϵ-lc pair such that X is terminal and $B \in \Gamma$. Then there exists a monotonic (ϵ, N)-complement $(X \ni x, B^+)$ of $(X \ni x, B)$.

Proof. If $\dim x = 2$, then the theorem is trivial. When X is smooth near x, in particular, when $\dim x = 1$, we may take $G = 0$, and (X, B) is a monotonic (ϵ, m)-complement of itself, where m is a positive integer such that $m\Gamma \subset \mathbb{Z}$. From now on, we may assume that $x \in X$ is a closed point that is not smooth.

When $\epsilon = 1$, the theorem follows from Theorem 4.2.4. When $\epsilon > 1$, let n be the index of the terminal singularity $X \ni x$. By [Sho92, Appendix, Theorem] and [Mar96, Theorem 0.1], there exists a prime divisor E over $X \ni x$ such that $a(E, X, 0) = 1 + \frac{1}{n}$, hence $1 + \frac{1}{n} \geq a(E, X, B) \geq \epsilon$, and $n \leq \lfloor \frac{1}{\epsilon - 1} \rfloor$. It follows that $(X \ni x, B)$ is a monotonic $(\epsilon, \lfloor \frac{1}{\epsilon-1} \rfloor ! m)$-complement of itself. $\square$

4.3 Boundedness of Complements for DCC Coefficients

Definition 4.3.1. For any $\boldsymbol{v} = (v_1, \dots, v_m) \in \mathbb{R}^m$, we defined $||\boldsymbol{v}|| := \max_i\{v_i\}$. For an $\mathbb{R}$-divisor $B = \sum b_i B_i$, where B_i are the distinct prime divisors of $\operatorname{Supp} B$, we define $||B|| := \max_i\{b_i\}$.

Theorem 4.3.2. *Let m be a positive integer, $\epsilon \geq 1$ a real number, and $\boldsymbol{v} = (v_1^0, \dots, v_m^0) \in \mathbb{R}^m$ a point. Then there exist a rational polytope $\boldsymbol{v} \in P \subset \mathbb{R}^m$ with vertices $\boldsymbol{v}_j = (v_1^j, \dots, v_m^j)$, positive real numbers a_j, and positive real numbers ϵ_j depending only on m, ϵ and $\boldsymbol{v}$ satisfying the following.*

1. *$\sum_j a_j = 1, \sum_j a_j \boldsymbol{v}_j = \boldsymbol{v}$, and $\sum_j a_j \epsilon_j \geq \epsilon$.*
2. *Assume that $(X \ni x, B := \sum_{i=1}^m v_i^0 B_i)$ is a threefold germ such that X is terminal, $(X \ni x, B)$ is ϵ-lc, and $B_1, \dots, B_m \geq 0$ are Weil divisors. Then for any j,*

$$\operatorname{mld}(X \ni x, \sum_{i=1}^m v_i^j B_i) \geq \epsilon_j.$$

Moreover, if $\epsilon > 1$, then the function $P \to \mathbb{R}$ defined by

$$(v_1, \dots, v_m) \mapsto \operatorname{mld}(X \ni x, \sum_{i=1}^m v_i B_i)$$

is a linear function; if $\epsilon \in \mathbb{Q}$, then we may pick $\epsilon_j = \epsilon$ for any j.

Proof. **Step 1**. There exist $\mathbb{Q}$-linearly independent real numbers $r_0 = 1, r_1, \dots, r_c$ for some $0 \leq c \leq m$, and

$\mathbb{Q}$-affine functions $s_i : \mathbb{R}^c \to \mathbb{R}$ such that $s_i(\boldsymbol{r}_0) = v_i^0$ for any $1 \le i \le m$, where $\boldsymbol{r}_0 := (r_1, \dots, r_c)$. Note that the map $\mathbb{R}^c \to V$ defined by

$$\boldsymbol{r} \mapsto (s_1(\boldsymbol{r}), \dots, s_m(\boldsymbol{r}))$$

is one-to-one, where $V \subset \mathbb{R}^m$ is the rational envelope of $\boldsymbol{v}$.

If $c = 0$, then $P = V = \{\boldsymbol{v}\}$, and there is nothing to prove. Suppose that $c \ge 1$. Let $B(\boldsymbol{r}) := \sum_{i=1}^m s_i(\boldsymbol{r})B_i$. Then $B(\boldsymbol{r}_0) = \sum_{i=1}^m v_i^0 B_i = B$. By [HLS19, Lemma 5.4(1)], $K_X + B(\boldsymbol{r})$ is Cartier near x for any $\boldsymbol{r} \in \mathbb{R}^c$.

Step 2. We will show that there exist a positive real number δ and a $\mathbb{Q}$-affine function $f(\boldsymbol{r})$ depending only on $m, \epsilon, c, \boldsymbol{r}_0, s_1, \dots, s_m$ such that $f(\boldsymbol{r}_0) \ge \epsilon$, and for any $\boldsymbol{r} \in \mathbb{R}^c$ satisfying $||\boldsymbol{r} - \boldsymbol{r}_0|| \le \delta$, $\mathrm{mld}(X \ni x, B(\boldsymbol{r})) \ge f(\boldsymbol{r})$, moreover, when $\epsilon > 1$,

$$\mathrm{mld}(X \ni x, B(\boldsymbol{r})) = a(E, X, B(\boldsymbol{r})) \ge f(\boldsymbol{r})$$

for some prime divisor E over $X \ni x$.

When $\epsilon = 1$, we may take $f(\boldsymbol{r}) = 1$, and the assertion follows from Theorem 3.4.3. When $\epsilon > 1$, by [Sho92, Appendix, Theorem], for all germs $(X \ni x, B)$ which is ϵ-lc, the index of $X \ni x$ is bounded from above by $I_0 := \lfloor \frac{1}{\epsilon - 1} \rfloor$. Note that by [Kaw88, Lemma 5.1], $I_0!D$ is Cartier for any Weil divisor D on X. Also note that $(X \ni x, B(\boldsymbol{r}_0))$ is ϵ-lc, and $\mathrm{mld}(X \ni x) \le 3$ (cf. [Amb99, Theorem 0.1]). The existence of δ and $f(\boldsymbol{r})$ in this case follows from [CH21, Lemma 4.7].

Step 3. We finish the proof in this step. It follows from the same line of the proof of [CH21, Theorem 7.15].

Note that if $\epsilon \in \mathbb{Q}$, then $f(\boldsymbol{r}) = \epsilon$ for any $\boldsymbol{r} \in \mathbb{R}^c$. We may find 2^c positive rational numbers $r_{i,1}, r_{i,2}$ such that $r_{i,1} < r_i < r_{i,2}$ and $\max\{r_i - r_{i,1}, r_{i,2} - r_i\} \le \delta$ for any $1 \le i \le c$. By our choice of δ, the function $\mathbb{R}^c \to \mathbb{R}$ defined by

$$\boldsymbol{r} \mapsto \operatorname{mld}(X \ni x, B(\boldsymbol{r}))$$

is a linear function on $\boldsymbol{r} \in U_c := [r_{1,1}, r_{1,2}] \times \cdots \times [r_{c,1}, r_{c,2}]$.

Let $\boldsymbol{r}_j$ be the vertices of U_c. Set $\epsilon_j := f(\boldsymbol{r}_j), v_i^j := s_i(\boldsymbol{r}_j)$ for any i, j. Note that if $\epsilon \in \mathbb{Q}$, then $\epsilon_j = f(\boldsymbol{r}_j) = \epsilon$ for any j. Let $P := \{(s_1(\boldsymbol{r}), \dots, s_m(\boldsymbol{r})) \mid \boldsymbol{r} \in U_c\} \subset V$. Then the function $P \to \mathbb{R}$: $(v_1, \dots, v_m) \mapsto \operatorname{mld}(X \ni x, \sum_{i=1}^m v_i B_i)$ is linear, $(v_1^j, \dots, v_m^j)$ are vertices of P, and

$$\operatorname{mld}(X \ni x, \sum_{i=1}^m v_i^j B_i) = \operatorname{mld}(X \ni x, B(\boldsymbol{r}_j)) \geq f(\boldsymbol{r}_j) \geq \epsilon_j$$

for any j.

Finally, we may find positive real numbers a_j such that $\sum_j a_j = 1$ and $\sum_j a_j \boldsymbol{r}_j = \boldsymbol{r}_0$. Then $\sum_j a_j \boldsymbol{v}_j = \boldsymbol{v}$ and $\sum_j a_j \epsilon_j \geq \epsilon$ as $\sum_j a_j v_i^j = \sum_j a_j s_i(\boldsymbol{r}_j) = s_i(\sum_j a_j \boldsymbol{r}_j) = s_i(\boldsymbol{r}_0) = v_i^0$ for any $1 \leq i \leq m$, and $\sum_j a_j \epsilon_j = \sum_j a_j f(\boldsymbol{r}_j) = f(\sum_j a_j \boldsymbol{r}_j) = f(\boldsymbol{r}_0) \geq \epsilon$. $\square$

Theorem 4.3.3. *Let p be a positive integer, $\epsilon \geq 1$ a real number, and $\Gamma \subset [0,1]$ a finite set. Then there exists a positive integer N depending only on ϵ, p and Γ, such that $p \mid N$ and N satisfies the following.*

Let $(X \ni x, B)$ be a pair such that X is a terminal threefold, $B \in \Gamma$, and $\operatorname{mld}(X \ni x, B) \geq \epsilon$. Then there exists an N-complement $(X \ni x, B^+)$ of $(X \ni x, B)$ such that $\operatorname{mld}(X \ni x, B^+) \geq \epsilon$. Moreover, if $\operatorname{Span}_{\mathbb{Q}_{\geq 0}}(\bar{\Gamma} \cup \{\epsilon\} \backslash \mathbb{Q}) \cap (\mathbb{Q} \backslash \{0\}) = \emptyset$, then we may pick $B^+ \geq B$.

Proof. By Theorem 4.3.2, there exist three finite sets $\Gamma_1 \subset (0,1]$, $\Gamma_2 \subset [0,1] \cap \mathbb{Q}$ and $\mathcal{M}$ of non-negative rational numbers depending only on ϵ, Γ, such that

- $\sum a_i \epsilon_i \geq \epsilon$,
- $K_X + B = \sum a_i (K_X + B^i)$, and

- $(X \ni x, B^i)$ is ϵ_i-lc at x for any i,

for some $a_i \in \Gamma_1, B^i \in \Gamma_2$ and $\epsilon_i \in \mathcal{M}$. By Theorem 4.2.5, there exists a positive integer n_0 which only depends on Γ_2 and $\mathcal{M}$, such that $(X \ni x, B^i)$ has an (ϵ_i, n_0)-complement $(X \ni x, B^i + G^i)$ for some $\mathbb{Q}$-Cartier divisor $G^i \geq 0$ for any i. Let $G := \sum a_i G^i$.

By [CH21, Lemma 6.2], there exists a positive integer n depending only on $\epsilon, p, n_0, \Gamma, \Gamma_1, \Gamma_2, \mathcal{M}$, such that there exist positive rational numbers a_i' with the following properties:

- $pn_0 | n$,
- $\sum a_i' = 1$,
- $\sum a_i' \epsilon_i \geq \epsilon$,
- $na_i' \in n_0\mathbb{Z}$ for any i, and
- $nB' \geq n\lfloor B \rfloor + \lfloor (n+1)\{B\} \rfloor$, where $B' := \sum a_i' B^i$.

Let $G' := \sum a_i' G^i$. Then

$$n(K_X + B' + G') = n \sum a_i'(K_X + B^i + G^i) = \sum \frac{a_i' n}{n_0} \cdot n_0(K_X + B^i + G^i) \sim_Z 0$$

and

$$a(E, X, B' + G') = \sum a_i'(E, X, B^i + G^i) \geq a_i'\epsilon_i \geq \epsilon$$

for any prime divisor E over $X \ni x$. Hence $(X \ni x, B' + G')$ is an (ϵ, n)-complement of $(X \ni x, B)$.

Moreover, if $\mathrm{Span}_{\mathbb{Q}_{\geq 0}}(\bar{\Gamma} \cup \{\epsilon\} \backslash \mathbb{Q}) \cap (\mathbb{Q} \backslash \{0\}) = \emptyset$, then $B' \geq B$ by [CH21, Lemmas 6.2, 6.4]. $\square$

Proposition 4.3.4 and Theorem 4.3.6 study the inversion of stability property for $\mathbb{R}$-Cartier divisors, and give a positive answer to [HL20, Conjecture 7.8] in some special cases.

Proposition 4.3.4. *Let I be a positive integer and $\Gamma \subset [0,1]$ a finite set. Then there exists a positive real number τ depending only on I and Γ satisfying the following.*

Let $x \in X$ be a terminal threefold singularity, and $B \geq 0, B' \geq 0$ two $\mathbb{R}$-divisors on X, such that

1. *IK_X is Cartier near x,*

2. *$B \geq B'$, $||B - B'|| < \tau, B \in \Gamma$,*

3. *$\mathrm{mld}(X \ni x, B') \geq 1$, and*

4. *$K_X + B'$ is $\mathbb{R}$-Cartier.*

Then $K_X + B$ is $\mathbb{R}$-Cartier.

Proof. Suppose that the proposition does not hold, then there exist $X_i \ni x_i, B_i, B_i', \tau_i$ corresponding to $X \ni x, B, B', \tau$ as in the assumptions, and a DCC set Γ', such that

- $\lim_{i \to +\infty} \tau_i = 0$,

- $B_i' \in \Gamma'$, and

- $K_{X_i} + B_i$ is not $\mathbb{R}$-Cartier.

Let $f_i : Y_i \to X_i$ be a small $\mathbb{Q}$-factorialization of X_i. Let B_{Y_i} be the strict transform of B_i on Y_i. Possibly replacing Y_i with a minimal model of (Y_i, B_{Y_i}) over X_i, we may assume that $K_{Y_i} + B_{Y_i}$ is big and nef over X. We may write $K_{Y_i} + B_{Y_i}' := f_i^*(K_{X_i} + B_i')$. Since $\mathrm{mld}(Y_i/X_i \ni x_i, B_{Y_i}') = \mathrm{mld}(X_i \ni x_i, B_i') \geq 1$, by Theorem 1.2.10, possibly passing to a subsequence, we may assume that $(Y_i/X_i \ni x_i, B_{Y_i})$ is 1-lc over x_i for any i. Since the Cartier index of any Weil divisor on Y_i is bounded from above by I, by [Nak16, Theorem 1.2] and [Amb99, Theorem 0.1], $\{\mathrm{mld}(Y_i/X_i \ni x_i, B_{Y_i})\}_{i=1}^{\infty}$ belongs to a finite set. Thus possibly passing to a subsequence, we may assume that there exists a real number $\epsilon \geq 1$, such that $\epsilon := \operatorname{mld}(Y_i/X_i \ni x_i, B_{Y_i})$ for any i. Since $K_{Y_i} + B'_{Y_i} \leq K_{Y_i} + B_{Y_i}$, $(Y_i/X_i \ni x_i, B'_{Y_i})$ is an $(\epsilon, \mathbb{R})$-complement of itself.

Note that Y_i is of Fano type over X_i. Let Y'_i be a minimal model of $-(K_{Y_i} + B_{Y_i})$ over X_i. Then $(Y'_i/X_i \ni x_i, B'_{Y'_i})$ is an $(\epsilon, \mathbb{R})$-complement of itself, where $B'_{Y'_i}$ is the strict transform of B'_{Y_i} on Y'_i. In particular, $\operatorname{mld}(Y'_i/X_i \ni x_i, B'_{Y'_i}) \geq \epsilon$. By Theorem 1.2.9, possibly passing to a subsequence, we may assume that $(Y'_i/X_i \ni x_i, B_{Y'_i})$ is ϵ-lc over x_i, where $B_{Y'_i}$ is the strict transform of B_{Y_i} on Y'_i. Thus $(Y'_i/X_i \ni x_i, B_{Y'_i})$ is $(\epsilon, \mathbb{R})$-complementary as $-(K_{Y'_i} + B_{Y'_i})$ is big and nef over X_i. By [CH21, Lemma 3.13], $(Y_i/X_i \ni x_i, B_{Y_i})$ has an $(\epsilon, \mathbb{R})$-complement $(Y_i/X_i \ni x_i, B_{Y_i} + G_{Y_i})$ for some $\mathbb{R}$-divisor $G_{Y_i} \geq 0$.

Let $Y_i \to Z_i$ be the canonical model of (Y_i, B_{Y_i}) over X and B_{Z_i} the strict transform of B_{Y_i} on Z_i. Then $-G_{Z_i}$ is ample over X, where G_{Z_i} is the strict transform of G_{Y_i} on Z_i. Since $K_{X_i} + B_i$ is not $\mathbb{R}$-Cartier, the natural induced morphism $g_i : Z_i \to X_i$ is not the identity, and $G_{Z_i} \neq 0$. It follows that $\operatorname{Supp} G_{Z_i}$ contains $g_i^{-1}(x_i)$. Thus

$$\epsilon = \operatorname{mld}(Y_i/X_i \ni x_i, B_{Y_i}) = \operatorname{mld}(Z_i/X_i \ni x_i, B_{Z_i}) > \operatorname{mld}(Z_i/X_i \ni x_i, B_{Z_i} + G_{Z_i}) \geq \epsilon,$$

a contradiction. □

Remark 4.3.5. Note that on any fixed potential klt variety X, the Cartier index of any Weil $\mathbb{Q}$-Cartier divisor is bounded from above (cf. [CH21, Lemma 7.14]). Thus the proof of Proposition 4.3.4 also works for any fixed potential klt variety X by assuming the ACC conjecture for minimal log discrepancies. It would be interesting to ask if it is necessary to assume X is fixed in higher dimensional cases.

Theorem 4.3.6. *Let $\Gamma \subset [0, 1]$ be a finite set. Then there exists a positive real number τ depending only on Γ satisfying the following.*

Let $x \in X$ be a terminal threefold singularity, and $B \geq 0, B' \geq 0$ two $\mathbb{R}$-divisors on X, such that

1. $B' \leq B$, $||B-B'||<\tau, B\in\Gamma$,

2. $\operatorname{mld}(X\ni x, B')\geq 1$, *and*

3. K_X+B' *is* $\mathbb{R}$*-Cartier.*

Then K_X+B *is* $\mathbb{R}$*-Cartier.*

Proof. Let τ be the positive real number constructed in Proposition 4.3.4 which only depends on Γ and $I:=1$.

Let $f: Y\to X$ be an index one cover of K_X. We may write $K_Y+B'_Y := f^*(K_X+B')$, and $K_Y+B_Y := f^*(K_X+B)$. Then K_Y is Cartier, $B'_Y\leq B_Y$, $||B_Y-B'_Y||<\tau$, and $B_Y\in\Gamma$. Moreover, by [KM98, Proposition 5.20], Y is terminal, and $\operatorname{mld}(Y\ni y, B'_Y)\geq \operatorname{mld}(X\ni x, B')\geq 1$ for any point $y\in f^{-1}(x)$. Thus by Proposition 4.3.4, K_Y+B_Y is $\mathbb{R}$-Cartier. We conclude that K_X+B is $\mathbb{R}$-Cartier as f is a finite morphism. $\square$

Theorem 4.3.7. *Let p be a positive integer, $\epsilon\geq 1$ a real number, and $\Gamma\subset[0,1]$ a finite set. Then there exists a positive integer N depending only on ϵ, p and Γ, such that $p\mid N$ and N satisfies the following.*

Let $(X\ni x, B)$ be a pair such that X is a terminal threefold, $B\in\Gamma$ and $\operatorname{mld}(X\ni x,B)\geq\epsilon$. Then there exists an N-complement $(X\ni x, B^+)$ of $(X\ni x, B)$ such that $\operatorname{mld}(X\ni x, B^+)\geq\epsilon$. Moreover, if $\operatorname{Span}_{\mathbb{Q}_{\geq 0}}(\bar{\Gamma}\cup\{\epsilon\}\backslash\mathbb{Q})\cap(\mathbb{Q}\backslash\{0\})=\emptyset$, then we may pick $B^+\geq B$.

Proof. By Theorems 4.3.6, 1.2.9, [HLS19, Lemma 5.17] (see also [CH21, Lemma 5.5]) and follow the same lines of the proof of [CH21, Theorem 5.6] (see also [HLS19, Theorem 5.18]), possibly replacing Γ by a finite subset of $\bar{\Gamma}$, we may assume that Γ is a finite set. Now the theorem follows from Theorem 4.3.3. $\square$

Proof of Theorem 1.2.7. This is a special case of Theorem 4.3.7. $\square$

Chapter 5

Proof of the Main Results

5.1 Proof of Theorem 1.2.2

In this subsection, we prove the following theorem:

Theorem 5.1.1. *Let $\Gamma \subset [0,1]$ be a DCC set. Then 1 is not an accumulation point of*

$$\{\mathrm{mld}(X,B) \mid \dim X = 3, B \in \Gamma\}$$

from below.

Definition 5.1.2. Let (X,B) be a pair. We say that (X,B) is *extremely non-canonical* if $\mathrm{mld}(X,B) < 1$, and the set

$$\{E \mid E \text{ is exceptional over } X, a(E,X,B) \leq 1\}$$

contains a unique element. In particular, any extremely non-canonical pair is klt.

A pair $(X \ni x, B)$ is called *extremely non-canonical* if (X,B) is extremely non-canonical near x and $\mathrm{mld}(X \ni x, B) = \mathrm{mld}(X,B) < 1$.

Lemma 5.1.3. *Let d be a positive integer and $\Gamma \subset [0,1]$ a set. Let (X,B) be a klt pair of dimension d such that $B \in \Gamma$ and $\mathrm{mld}(X,B) < 1$. Then there exists a $\mathbb{Q}$-factorial extremely non-canonical klt pair (Y,B_Y) of* *dimension d, such that $B_Y \in \Gamma$ and* $\mathrm{mld}(X,B) \leq \mathrm{mld}(Y,B_Y)$.

Proof. Since (X,B) is klt, by [BCHM10, Corollary 1.4.3], there exists a birational morphism $f: W \to X$ from a $\mathbb{Q}$-factorial variety W which extracts exactly all the exceptional divisors E over X such that $a(E,X,B)=1$. Let $K_W+B_W := f^*(K_X+B)$. Possibly replacing (X,B) with (W,B_W), we may assume that $a(E,X,B)\neq 1$ for any prime divisor E that is exceptional over X.

Since (X,B) is klt and $\mathrm{mld}(X,B)<1$, there exist prime divisors $E_1,\dots,E_k$ that are exceptional over X, such that

$$\{E_1,\dots,E_k\}=\{E \mid E \text{ is exceptional over } X,\ a(E,X,B)<1\}.$$

Let $\alpha_j := 1-a(E_j,X,B)$ for each j. By [Liu18, Lemma 5.3], there exists $i\in\{1,2,\dots,k\}$ and a birational morphism $h: Y\to X$ from a $\mathbb{Q}$-factorial variety Y, such that

- f exactly extracts $E_1,\dots,E_{i-1},E_{i+1},\dots,E_k$, and
- $\mathrm{mult}_{E_i}\sum_{j\neq i}\alpha_j E_{j,Y}<\alpha_i$, where $E_{j,Y}=\mathrm{center}_Y E_j$ for each $j\neq i$.

Let $B_Y := h^{-1}_*B$. Then

$$\begin{aligned}\mathrm{mld}(Y,B_Y)\leq a(E_i,Y,B_Y)&=a(E_i,Y,B_Y+\sum_{j\neq i}\alpha_j E_{j,Y})+\mathrm{mult}_{E_i}\sum_{j\neq i}\alpha_j E_{j,Y}\\ &<a(E_i,X,B)+\alpha_i=1,\end{aligned}$$

and for any prime divisor $F\neq E_i$ that is exceptional over Y,

$$a(F,Y,B_Y)\geq a(F,Y,B_Y+\sum_{j\neq i}\alpha_j E_{j,Y})=a(F,X,B)>1.$$

Thus (Y,B_Y) satisfies our requirements. $\square$

Lemma 5.1.4. *Let d be a positive integer and $\Gamma \subset [0,1]$ a DCC set. Then there exists a positive real number t depending only on d and Γ satisfying the following.*

Let (X,B) be a klt pair of dimension d, E a prime exceptional divisor over X such that $a(E,X,B) < 1$, and x the generic point of $\operatorname{center}_X E$. Let $f: Y \to X$ be a birational morphism which only extracts E. Then $(Y, B_Y + tE)$ is lc over a neighborhood of x.

Proof. By Theorem 2.1.8, there exist a positive integer n and a finite set $\Gamma_0 \subset (0,1]$, such that $(X \ni x, B)$ has an (n,Γ_0)-decomposable $\mathbb{R}$-complement $(X \ni x, B^+)$ of $(X \ni x, B)$. In particular, there exist real numbers $a_1,\dots,a_k \in \Gamma_0$ and lc pairs $(X \ni x, B_i^+)$, such that $\sum_{i=1}^k a_i = 1$, $\sum_{i=1}^k a_i B_i^+ = B^+$, and each $(X \ni x, B_i^+)$ is an n-complement of itself. Let

$$\Gamma_0' := \{\sum_{i=1}^k s_i a_i \mid ns_i \in \mathbb{Z}_{\geq 0}\}.$$

Then $\Gamma_0' \subset [0,+\infty)$ is a discrete set, and we may let

$$\gamma_0 := \max\{\gamma \in \Gamma_0' \mid \gamma < 1\}.$$

Since $na(E,X,B_i^+) \in \mathbb{Z}_{\geq 0}$ for every i,

$$1 > a(E,X,B) \geq a(E,X,B^+) = \sum_{i=1}^k a_i a(E,X,B_i^+) \in \Gamma_0',$$

so $a(E,X,B^+) \leq \gamma_0$. Thus $(Y, B_Y + (1-\gamma_0)E)$ is lc over a neighborhood of x. We may take $t := 1-\gamma_0$. $\square$

Lemma 5.1.5. *Let $\Gamma \subset [0,1]$ be a DCC set. Then there exists a positive real number ϵ depending only on Γ satisfying the following.*

Let (X,B) be a $\mathbb{Q}$-factorial extremely non-canonical threefold pair such that X is strictly canonical. Then $\operatorname{mld}(X,B) \leq 1-\epsilon$.

Proof. Since (X,B) is extremely non-canonical, (X,B) is klt. Let E be the unique prime divisor that is exceptional over X such that $a(E,X,B)\leq 1$. Then $a(E,X,B)<1$. Moreover, for any prime divisor $F\neq E$ that is exceptional over X, $a(F,X,0)\geq a(F,X,B)>1$. Since X is strictly canonical, $a(E,X,0)=1$. In particular, $\operatorname{mult}_E B>0$.

Let $\gamma_0:=\min\{\gamma\in\Gamma\mid\gamma>0\}$ and let $m:=\lceil\frac{1}{\gamma_0}\rceil$. Then $B\geq\frac{1}{m}\operatorname{Supp}B$,

$$a(E,X,B)\leq a(E,X,\frac{1}{m}\operatorname{Supp}B)<a(E,X,0)=1,$$

and

$$1<a(F,X,B)\leq a(F,X,\frac{1}{m}\operatorname{Supp}B)$$

for any prime divisor $F\neq E$ that is exceptional over X. Thus possibly replacing Γ with $\{0,\frac{1}{m}\}$ and B with $\frac{1}{m}\operatorname{Supp}B$, we may assume that Γ is a finite set of rational numbers.

Let $f:Y\to X$ be a birational morphism which extracts E, and let B_Y be the strict transform of B on Y. Then $K_Y=f^*K_X$, and

$$K_Y+B_Y+(1-a(E,X,B))E=f^*(K_X+B).$$

Let x be the generic point of $\operatorname{center}_X E$. If $\dim x=1$, by taking general hyperplane sections, the lemma follows from [Ale93, Theorem 3.8] (see also [Sho94b] and [HL20, Theorem 1.5]). Therefore, we may assume that x is a closed point.

By Theorem 2.1.11, $60K_Y$ is Cartier over a neighborhood of x. By our construction, Y is terminal. By [Kaw88, Lemma 5.1], $60D$ is Cartier over an neighborhood of x for any Weil divisor D on Y.

By Lemma 5.1.4, there exists a positive integer n depending only on Γ, such that $n\Gamma\subset\mathbb{Z}_{\geq 0}$ and $(Y,B_Y+\frac{1}{n}E)$ is lc over a neighborhood of x.

If $a(E,X,B)\leq 1-\frac{1}{2n}$, then we are done. Thus we may assume that $a(E,X,B)>1-\frac{1}{2n}$. In this case, by the boundedness of length of extremal rays (cf. [Fuj17, Theorem 4.5.2(5)]), there exists a $(K_Y + B_Y + \frac{1}{n}E)$-negative extremal ray R in $\overline{NE}(Y/X)$ which is generated by a rational curve C, such that

$$0 > (K_Y + B_Y + \frac{1}{n}E) \cdot C \geq -6.$$

Since $(K_Y + B_Y + (1 - a(E, X, B))E) \cdot C = 0$, we have

$$0 < (a(E, X, B) - 1 + \frac{1}{n})(-E \cdot C) \leq 6.$$

Hence $0 < (-E \cdot C) < 12n$ as $a(E, X, B) > 1 - \frac{1}{2n}$. Since $60n(K_Y + B_Y)$ is Cartier over a neighborhood of x, we have

$$a(E, X, B) = 1 - \frac{60n(K_Y + B_Y) \cdot C}{60n(-E \cdot C)},$$

so $a(E, X, B) < 1 - \frac{1}{720n^2}$ and we are done. □

Lemma 5.1.6. *Let $\Gamma \subset [0, 1]$ be a DCC set. Then there exists a positive real number ϵ depending only on Γ satisfying the following.*

Let (X, B) be a $\mathbb{Q}$-factorial extremely non-canonical threefold pair such that X is terminal. Then $\mathrm{mld}(X, B) \leq 1 - \epsilon$.

Proof. Let E be the unique divisor that is exceptional over X such that $a(E, X, B) \leq 1$. Then $0 < a(E, X, B) < 1$. Since X is terminal, $a(E, X, 0) > 1$. Thus $\mathrm{mult}_E B > 0$.

Let x be the generic point of $\mathrm{center}_X E$. If $\dim x = 1$, by taking general hyperplane sections, the lemma follows from [Ale93, Theorem 3.8] (see also [Sho94b] and [HL20, Theorem 1.5]). Therefore, we may assume that x is a closed point.

Let $t := \mathrm{ct}(X, 0; B)$. Since X is terminal and (X, B) is extremely non-canonical, we have $a(E, X, cB) = 1$ and $t = \mathrm{ct}(X \ni x, 0; B) < 1$. By Theorem 1.2.10, there exists a real number $\delta \in (0, 1)$ depending only on Γ such that $t \le 1-\delta$. Let $\gamma_0 := \min\{\gamma \in \Gamma \mid \gamma > 0\}$, $m := \lceil \frac{1}{\delta\gamma_0} \rceil$, and $B' := \frac{1}{m}\lfloor mB \rfloor$. Then $||B - B'|| < \delta\gamma_0$. Hence $B \ge B' \ge tB$ and $\operatorname{Supp}(B' - tB) = \operatorname{Supp} B$. Since $\operatorname{mult}_E B > 0$,

$$a(E, X, B) \le a(E, X, B') < a(E, X, tB) = 1,$$

and

$$1 < a(F, X, B) \le a(F, X, B')$$

for any prime divisor $F \neq E$ that is exceptional over X. Thus possibly replacing Γ with $\frac{1}{m}\mathbb{Z}_{\ge 0} \cap [0,1]$, B with B', and t with $\operatorname{ct}(X, 0; B')$ respectively, we may assume that $\Gamma \subset \frac{1}{m}\mathbb{Z} \cap [0,1]$.

If $x \in X$ is a terminal singularity of types other than cA/n or of type cA/n with $n \le 2$, then by [Rei87, (6.1) Theorem], the index of $X \ni x$ divides 12. By [Kaw88, Lemma 5.1], $12m(K_X + B)$ is Cartier, and we may take $\epsilon = \frac{1}{12m}$ in this case. Thus we may assume that $x \in X$ is a terminal singularity of type cA/n for some $n \ge 3$.

By construction, (X, tB) is extremely non-canonical. By Lemma 3.1.4, there exists a terminal blow-up $f: Y \to X$ of $(X \ni x, tB)$ which extracts E. Since $n \ge 3$, by [Kaw05, Theorem 1.3], f is of ordinary type. Let $a := a(E, X, B) + 1$. By Theorem 2.1.35(1), under suitable analytic local coordinates x_1, x_2, x_3, x_4, there exist positive integers r_1, r_2, b, d, where $\gcd(b, n) = 1$, $r_1 + r_2 = adn$ and $a \equiv br_1 \mod n$, such that analytically locally,

$$(X \ni x) \cong (\phi(x_1, x_2, x_3, x_4) = 0) \subset (\mathbb{C}^4 \ni o)/\frac{1}{n}(1, -1, b, 0)$$

for some invariant analytic power series ϕ, and $f: Y \to X$ is a weighted blow-up at $x \in X$ with the weight $w := \frac{1}{n}(r_1, r_2, a, n)$. Assume that mB is locally defined by $(h(x_1, x_2, x_3, x_4) = 0)$ for some semi-invariant analytic power series h.

Claim 5.1.7. *If either $d \geq 4$ or $a \geq 4$, then $n \leq 3m$.*

We proceed the proof assuming Claim 5.1.7. If either $a \geq 4$ or $d \geq 4$, then by Claim 5.1.7, $n \leq 3m \leq \frac{3m}{t}$. Since $a(E, X, tB) = a(E, X, 0) - t\,\mathrm{mult}_E\, B = 1 + \frac{a}{n} - t\,\mathrm{mult}_E\, B$,

$$\frac{a}{n} = t\,\mathrm{mult}_E\, B. \tag{5.1.1}$$

It follows that

$$\mathrm{mult}_E\, B = \frac{a}{tn} \geq \frac{a}{3m} \geq \frac{1}{3m}.$$

Thus $a(E, X, B) = a(E, X, tB) - (1-t)\,\mathrm{mult}_E\, B \leq 1 - \frac{\delta}{3m}$. We can take $\epsilon = \frac{\delta}{3m}$ in this case.

We may now assume that $a \leq 3$ and $d \leq 3$. Since $a \equiv br_1 \mod n$, $\gcd(r_1, n) \mid 6$. Since $r_1 + r_2 = adn$, $\gcd(r_2, n) \mid 6$, and $\gcd(r_1, r_2) \mid adn$. This implies that $\gcd(r_1, r_2) \mid 216$. Let m' be the smallest positive integer such that $m'tB$ is an integral divisor and r the smallest positive integer such that $rm'(K_X + tB)$ is Cartier. By Lemma 4.1.4, $r \mid \gcd(r_1, r_2)$. Thus $r \mid 216$. By (5.1.1), $t = \frac{a}{n\,\mathrm{mult}_E\, B} = \frac{a}{N}$, where $N = n\,\mathrm{mult}_E\, B$ is a positive integer. We may write $tB = \frac{a}{mN} mB$, then $216mN(K_X + tB)$ is Cartier.

By [Sho94a, 4.8 Corollary], there exists a prime divisor $E_1 \neq E$ over $X \ni x$ such that $a(E_1, X, 0) = 1 + \frac{a_1}{n}$ for some positive integer $a_1 \leq 2$.

Since $a(E_1, X, tB) > 1$,

$$1 + \frac{a_1}{n} = a(E_1, X, 0) \geq a(E_1, X, tB) \geq 1 + \frac{1}{216mN},$$

hence $n \leq 432mN = \frac{432am}{t}$. It follows that $tn \leq 432am \leq 1296m$. By (5.1.1),

$$\mathrm{mult}_E\, B = \frac{a}{tn} \geq \frac{1}{1296m},$$

and $a(E, X, B) = a(E, X, tB) - (1-t)\,\mathrm{mult}_E\, B \leq 1 - \frac{\delta}{1296m}$. We can take $\epsilon = \frac{\delta}{1296m}$ in this case. $\square$

Proof of Claim 5.1.7. Suppose that $n > 3m$. If either $d \geq 4$ or $a \geq 4$, then we can pick positive integers s_1, s_2 such that

- $s_1 + s_2 = a'dn$ for some $a' \leq \min\{a, 3\}$,
- $a' \equiv bs_1 \mod n$,
- $s_1 > n, s_2 > n$, and
- $\frac{1}{n}(s_1, s_2, a', n) \neq \frac{1}{n}(r_1, r_2, a, n)$.

In fact, when $a \geq 4$, we may take $a' = 3$. When $d \geq 4$, we may take $a' = 1$ and $(s_1, s_2) \neq (r_1, r_2)$. Let $w' := \frac{1}{n}(s_1, s_2, a', n)$.

Since $a \geq a'$, by [HLL22, Lemma C.7], the weighted blow-up with the weight w' at $x \in X$ extracts an analytic prime divisor $E' \neq E$ such that $w'(X \ni x) = \frac{a'}{n}$. By [HLL22, Lemma C.6], we may assume that E' is a prime divisor over $X \ni x$. By our assumption, $a(E', X, B) = 1 + w'(X \ni x) - w'(B) > 1$, thus

$$\frac{1}{m} > \frac{3}{n} \geq \frac{a'}{n} = w'(X \ni x) > w'(B) = \frac{1}{m}w'(mB),$$

which implies that $w'(h) = w'(mB) < 1$. Since $w'(x_1) = \frac{s_1}{n} > 1$, $w'(x_2) = \frac{s_2}{n} > 1$ and $w'(x_4) = 1$, up to a scaling of h, there exists a positive integer l, such that $x_3^l \in h$ and $w'(mB) = w'(h) = w'(x_3^l)$. In particular, $w'(h) = lw'(x_3) = \frac{a'l}{n}$ and

$$w'(X \ni x) = \frac{a'}{n} > w'(B) = \frac{a'}{n}\frac{l}{m},$$

this implies that $\frac{l}{m} < 1$. On the other hand,

$$1 > a(E, X, B) = 1 + w(X \ni x) - w(B) \geq 1 + \frac{a}{n} - \frac{1}{m}w(x_3^l) = 1 + \frac{a}{n}(1 - \frac{l}{m}) > 1,$$

a contradiction. □

Proof of Theorem 5.1.1. Let (X,B) be a threefold pair such that $B\in\Gamma$. Possibly replacing (X,B) with a $\mathbb{Q}$-factorialization and replacing Γ with $\Gamma\cup\{1\}$, we may assume that (X,B) is $\mathbb{Q}$-factorial dlt.

If (X,B) is klt, then by Lemma 5.1.3, we may assume that (X,B) is extremely non-canonical. Then either X is not canonical, or X is strictly canonical, or X is terminal. If X is not canonical, then by [LX21, Theorem 1.4] (see also [Jia21, Theorem 1.3]), $\mathrm{mld}(X,B)\leq\mathrm{mld}(X)\leq\frac{12}{13}$. If X is strictly canonical, then the theorem follows from Lemma 5.1.5. If X is terminal, then the theorem follows from Lemma 5.1.6.

If (X,B) is not klt, then we let E be a prime divisor that is exceptional over X such that $a(E,X,B)=\mathrm{mld}(X,B)$. If $\operatorname{center}_X E\not\subset\lfloor B\rfloor$, then

$$\mathrm{mld}(X,B)=a(E,X,B)=a(E,X,\{B\})\geq\mathrm{mld}(X,\{B\})\geq\mathrm{mld}(X,B)$$

and the theorem follows from the klt case. If $\operatorname{center}_X E\subset\lfloor B\rfloor$, then there exists a prime divisor $S\subset\lfloor B\rfloor$ such that $\operatorname{center}_X E\subset S$. We let $K_S+B_S:=(K_X+B)|_S$. By [BCHM10, Corollary 1.4.5], $\mathrm{mld}(X,B)=a(E,X,B)$ is equal to the total minimal log discrepancy of (S,B_S). Since $B_S\in D(\Gamma)$ which satisfies the DCC, the theorem follows from [Ale93, Theorem 3.8] (see also [Sho04b] and [HL20, Theorem 1.5]). $\square$

Proof of Theorem 1.2.2. This follows from Theorems 3.6.1 and 5.1.1. $\square$

5.2 Proof of Theorem 1.2.5

Lemma 5.2.1. *Let I be a positive integer, and $\Gamma\subset[0,1]$ a finite set. Then there exists a positive real number l depending only on I and Γ satisfying the following. Assume that*

1. *$(X\ni x,B)$ is a threefold pair,*
2. *X is terminal,*
3. *$B\in\Gamma$,*

4. $\mathrm{mld}(X \ni x, B) \geq 1$*, and*

5. IK_X *is Cartier near* x*.*

Then there exists a prime divisor E *over* $X \ni x$*, such that* $a(E, X, B) = \mathrm{mld}(X \ni x, B)$ *and* $a(E, X, 0) = 1 + \frac{a}{I}$ *for some non-negative integer* $a \leq l$*.*

Proof. We will use some ideas of Kawakita [Kaw21, Theorem 4.6]. Possibly replacing X with a small $\mathbb{Q}$-factorialization, we may assume that X is $\mathbb{Q}$-factorial.

If $\dim x = 2$, then the lemma is trivial as we can take $l = 0$.

If $\dim x = 1$, then X is smooth near x. By Lemma 2.1.6, $\mathrm{mld}(X \ni x, B) = a(E, X, B)$, where E is the exceptional divisor obtained by blowing up $x \in X$. We have $a(E, X, 0) = 2$, and the theorem holds in this case.

If $\dim x = 0$ and suppose that the theorem does not hold, then there exists a sequence of threefold germs $(X_i \ni x_i, B_i)$ satisfying (1-5), and a strictly increasing sequence of positive integers l_i, such that for each i,

$$\min\{a(E, X_i, 0) \mid \mathrm{center}_{X_i} E = x_i, a(E, X_i, B_i) = \mathrm{mld}(X_i \ni x_i, B_i)\} = 1 + \frac{l_i}{I}.$$

By [Kaw88, Lemma 5.1], ID is Cartier near x_i for any Weil divisor D on X_i. By [Amb99, Theorem 0.1], $1 \leq \mathrm{mld}(X_i \ni x_i, B_i) \leq 3$ for any i. By [Nak16, Corollary 1.3], possibly passing to a subsequence, we may assume that there exists a real number $\alpha \geq 1$, such that $\mathrm{mld}(X_i \ni x_i, B_i) = \alpha$ for any i. By [Nak16, Theorem 1.2], there exists a real number $\alpha' > \alpha$, such that for any i and any prime divisor F_i over $X_i \ni x_i$, if $a(F_i, X_i, B_i) > \alpha$, then $a(F_i, X_i, B_i) > \alpha'$. Therefore,

$$\alpha'\text{-}\mathrm{lct}(X_i \ni x_i, 0; B_i) = \frac{1 + \frac{l_i}{I} - \alpha'}{1 + \frac{l_i}{I} - \alpha} = 1 - \frac{\alpha' - \alpha}{1 + \frac{l_i}{I} - \alpha} < 1$$

is strictly increasing, which contradicts Theorem 1.2.9. $\square$

The following theorem answers a question of [HL20, Conjecture 7.2] for terminal threefold pairs. We will use it to prove Theorem 1.2.5.

Theorem 5.2.2. *Let I be a positive integer, and $\Gamma \subset [0,1]$ a finite set. Then there exists a positive real number τ depending only on I and Γ satisfying the following. Assume that $(X \ni x, B)$ and $(X \ni x, B')$ are two threefold lc pairs and E' is a prime divisor over $X \ni x$, such that*

1. *X is terminal,*

2. *$B \geq B', ||B - B'|| < \tau$, and $B \in \Gamma$,*

3. *$a(E', X, B') = \mathrm{mld}(X \ni x, B') \geq 1$, and*

4. *IK_X is Cartier near x.*

Then $a(E', X, B) = \mathrm{mld}(X \ni x, B)$.

Proof. We may assume that $\{0\} \subsetneq \Gamma$, otherwise $B = B' = 0$ and the theorem is obvious.

Since IK_X is Cartier near x and X is terminal, by [Kaw88, Lemma 5.1], ID is Cartier for any $\mathbb{Q}$-Cartier Weil divisor on X. By [HLS19, Theorem 5.6], there exist positive real numbers $a_1, \ldots, a_k \in (0,1]$ depending only on Γ, a positive integer I' depending only on I and Γ, and $\mathbb{Q}$-divisors $B_1, \ldots, B_k \geq 0$ on X, such that

- $\sum_{i=1}^{k} a_i = 1$,

- $\sum_{i=1}^{k} a_i B_i = B$,

- $(X \ni x, B_i)$ is lc for any i. In particular, $K_X + B_i$ is $\mathbb{Q}$-Cartier for any i, and

- $I'(K_X + B_i)$ is Cartier near x for each i.

Thus there exists a positive real number δ depending only on I and Γ, such that for any prime divisor F over $X \ni x$ and $a(F, X, B) > \mathrm{mld}(X \ni x, B)$, we have $a(F, X, B) > \mathrm{mld}(X \ni x, B) + \delta$.

By Lemma 5.2.1, there exists a prime divisor E over $X \ni x$, such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$ and $a(E, X, 0) \leq l$ for some positive integer l depending only on I and Γ. In particular, $\mathrm{mult}_E B = a(E, X, 0) - a(E, X, B) \leq l$.

We show that we may take $\tau := \frac{\delta}{2l} \cdot \min\{\gamma \in \Gamma \mid \gamma > 0\}$. In this case, $B' \geq (1 - \frac{\delta}{2l})B$. Since

$$a(E, X, 0) - \mathrm{mult}_E B' = a(E, X, B') \geq a(E', X, B') \geq a(E', X, B)$$

$$=(a(E', X, B) - a(E, X, B)) + a(E, X, 0) - \mathrm{mult}_E B,$$

we have

$$0 \leq a(E', X, B) - a(E, X, B) \leq \mathrm{mult}_E(B - B') \leq \frac{\delta}{2l} \mathrm{mult}_E B \leq \frac{\delta}{2},$$

which implies that $a(E', X, B) = a(E, X, B) = \mathrm{mld}(X \ni x, B)$. □

Lemma 5.2.3. *Let m_0 be a positive integer and let $\{a_{i,1}\}_{i=1}^{\infty}, \{a_{i,2}\}_{i=1}^{\infty}, \dots, \{a_{i,m_0}\}_{i=1}^{\infty}$ be m_0 sequences of positive real numbers. Then there exists an integer $1 \leq k \leq m_0$, such that possibly passing to a subsequence, $\{\frac{a_{i,j}}{a_{i,k}}\}_{i=1}^{\infty}$ are decreasing (resp. increasing) for all $1 \leq j \leq m_0$.*

Proof. Possibly passing to a subsequence, we may assume that for any k, j, $\{\frac{a_{i,j}}{a_{i,k}}\}_{i=1}^{\infty}$ is either decreasing or strictly increasing (resp. either increasing or strictly decreasing). Suppose that the lemma does not hold. Then there exists a function $\pi : \{1, 2, \dots, m_0\} \to \{1, 2, \dots, m_0\}$, such that $\{\frac{a_{i,\pi(j)}}{a_{i,j}}\}_{i=1}^{\infty}$ is strictly increasing (resp. strictly decreasing) for any j. We may pick $1 \leq j_0 \leq m$ such that $\pi^{(l)}(j_0) = j_0$ for some positive integer l. Then

$$\{1\}_{i=1}^{\infty} = \{\frac{a_{i,\pi(j_0)}}{a_{i,j_0}} \cdot \frac{a_{i,\pi(\pi(j_0))}}{a_{i,\pi(j_0)}} \cdot \dots \cdot \frac{a_{i,\pi^{(l)}(j_0)}}{a_{i,\pi^{(l-1)}(j_0)}}\}_{i=1}^{\infty}$$

is strictly increasing (resp. strictly decreasing), which is absurd. $\square$

Lemma 5.2.4. *Let $m_0 \geq 0, I > 0$ be integers, $\Gamma_0 \subset [0,1]$ a finite set, and $\Gamma \subset [0,1]$ a DCC set. Then there exists a positive integer l depending only on m_0, I, Γ_0, and Γ satisfying the following.*

Assume that $\{(X_i \ni x_i, B_i := \sum_{j=1}^{m_0} b_{i,j} B_{i,j} + B_{i,0})\}_{i=0}^{\infty}$ is a sequence of $\mathbb{Q}$-factorial threefold germs, such that

1. *X_i is terminal for each i,*
2. *$\{b_{i,j}\}_{i=1}^{\infty}$ is strictly increasing for any fixed j,*
3. *$b_{i,j} \in \Gamma$ and $B_{i,0} \in \Gamma_0$ for each i and j,*
4. *$B_{i,j} \geq 0$ is a Weil divisor on X_i for each i and j,*
5. *$\operatorname{mld}(X_i \ni x_i, B_i) > 1$ for each i,*
6. *IK_{X_i} is Cartier near x_i for each i, and*
7. *$1 + \frac{l_i}{I} := \min\{a(E_i, X_i, 0) \mid \operatorname{center}_{X_i} E_i = x_i, a(E_i, X_i, B_i) = \operatorname{mld}(X_i \ni x_i, B_i)\}$.*

Then possibly passing to a subsequence, we have $l_i \leq l$ for each i.

Proof. **Step 1**. We prove the lemma by induction on m_0. When $m_0 = 0$, the lemma follows from Lemma 5.2.1. Thus we may assume that $m_0 \geq 1$.

Let $\gamma_0 := \min\{1, \gamma \mid \gamma \in \Gamma, \gamma > 0\}$. Let $b_j := \lim_{i \to +\infty} b_{i,j}$, $\bar{B}_i := \sum_{j=1}^{m_0} b_j B_{i,j} + B_{i,0}$ for any i, and $\Gamma_0' := \Gamma_0 \cup \{b_1, \ldots, b_{m_0}\}$. By Theorem 1.2.9, possibly passing to a subsequence, we may assume that $\operatorname{mld}(X_i \ni x_i, \bar{B}_i) \geq 1$ for each i.

By [Kaw88, Lemma 5.1], for each i, ID_i is Cartier near x_i for any Weil divisor D_i on X_i. By [Amb99, Theorem 0.1], $1 < \mathrm{mld}(X_i \ni x_i, \bar{B}_i) \leq 3$. By [Nak16, Theorem 1.2], possibly passing to a subsequence, we may assume that there exist two real numbers $\alpha \geq 1$ and $\delta > 0$, such that for any i,

- $\mathrm{mld}(X_i \ni x_i, \bar{B}_i) = \alpha$, and
- for any prime divisor F_i over $X_i \ni x_i$ such that $a(F_i, X_i, \bar{B}_i) > \mathrm{mld}(X_i \ni x_i, \bar{B}_i)$, we have $a(F_i, X_i, \bar{B}_i) > \alpha + \delta$.

For each i, let E_i be a prime divisor over $X_i \ni x_i$ such that $a(E_i, X_i, B_i) = \mathrm{mld}(X_i \ni x_i, B_i)$ and $a(E_i, X_i, 0) = 1 + \frac{l_i}{I}$. By Theorem 5.2.2, possibly passing to a subsequence, we may assume that $a(E_i, X_i, \bar{B}_i) = \mathrm{mld}(X_i \ni x_i, \bar{B}_i) = \alpha$.

Step 2. For any i and any $1 \leq j \leq m_0$, we define $B'_{i,j} := \sum_{k \neq j} b_{i,k} B_{i,k} + b_j B_{i,j} + B_{i,0}$.

By the induction for $m_0 - 1, I, \Gamma'_0$ and Γ, possibly passing to a subsequence, we may assume that there exists a positive integer l' depending only on m_0, I, Γ_0 and Γ, such that for any $1 \leq j \leq m_0$, there exists a prime divisor $E_{i,j}$ over $X_i \ni x_i$, such that

- $a(E_{i,j}, X_i, B'_{i,j}) = \mathrm{mld}(X_i \ni x_i, B'_{i,j})$, and
- $a(E_{i,j}, X_i, 0) \leq 1 + \frac{l'}{I} \leq 1 + l'$.

Since

$$a(E_{i,j}, X_i, B_i) \geq \mathrm{mld}(X_i \ni x_i, B_i) = a(E_i, X_i, B_i)$$

and

$$a(E_{i,j}, X_i, B'_{i,j}) = \mathrm{mld}(X_i \ni x_i, B'_{i,j}) \leq a(E_i, X_i, B'_{i,j}),$$

we have $\mathrm{mult}_{E_{i,j}}(B'_{i,j}-B_i)\geq \mathrm{mult}_{E_i}(B'_{i,j}-B_i)$. By the construction of $B'_{i,j}$, we have

$$\mathrm{mult}_{E_{i,j}} B_{i,j} \geq \mathrm{mult}_{E_i} B_{i,j}$$

for any i and $1\leq j\leq m_0$. Since

$$\begin{aligned}1 &\leq \mathrm{mld}(X_i\ni x_i,\bar{B}_i)\leq \mathrm{mld}(X_i\ni x_i,B'_{i,j})\\ &= a(E_{i,j},X_i,B'_{i,j})\leq a(E_{i,j},X_i,b_{i,j}B_{i,j})\leq a(E_{i,j},X_i,\gamma_0 B_{i,j})\\ &= a(E_{i,j},X_i,0)-\gamma_0\,\mathrm{mult}_{E_{i,j}} B_{i,j}\leq 1+l'-\gamma_0\,\mathrm{mult}_{E_{i,j}} B_{i,j},\end{aligned}$$

for any i and $1\leq j\leq m_0$, we have

$$\mathrm{mult}_{E_i} B_{i,j}\leq \mathrm{mult}_{E_{i,j}} B_{i,j}\leq \frac{l'}{\gamma_0}.$$

Step 3. Let $a_{i,j}:=b_j-b_{i,j}$ for any i and any $1\leq j\leq m_0$. By Lemma 5.2.3, possibly re-odering indices and passing to a subsequence, we may assume that $\{\frac{a_{i,j}}{a_{i,1}}\}_{i=1}^{\infty}$ is decreasing for any $1\leq j\leq m_0$. Let $M:=\max\{\frac{a_{1,j}}{a_{1,1}}\mid 1\leq j\leq m_0\}$, $t:=\frac{\delta\gamma_0}{m_0Ml'}$, $\tilde{b}_{i,j}:=b_j-t\frac{a_{i,j}}{a_{i,1}}$ for any i,j, and $\tilde{B}_i:=\sum_{j=1}^{m_0}\tilde{b}_{i,j}B_{i,j}+B_{i,0}$. Possibly passing to a subsequence, we may assume that $a_{i,1}<t$ for any i as $\lim_{i\to+\infty}a_{i,1}=0$.

There exist a positive integer k and a finite set $\Lambda\subset\{1,2,\ldots,m_0\}$, such that $|\Lambda|=k$, and $\tilde{b}_{i,j}=\tilde{b}_{1,j}$ for any i and $j\in\Lambda$ as $\tilde{b}_{i,1}=b_1-t$. By the induction for $m_0-k,I,\Gamma_0\cup\{\tilde{b}_{1,j}\mid j\in\Lambda\}$, and $\{\tilde{b}_{i,j}\}_{i\geq 1,1\leq j\leq m_0}$, possibly passing to a subsequence, for any i, there exists a prime divisor $\tilde{E}_i$ over $X_i\ni x_i$, such that

- $a(\tilde{E}_i,X_i,\tilde{B}_i)=\mathrm{mld}(X_i\ni x_i,\tilde{B}_i)$, and
- $a(\tilde{E}_i,X_i,0)\leq 1+\frac{l}{I}$ for some positive integer l depending only on m_0,I,Γ_0 and Γ.

Since

$$\tilde{b}_{i,j} \geq b_j - \frac{\delta\gamma_0}{m_0 M l'} \cdot M = b_j - \frac{\delta\gamma_0}{m_0 l'},$$

we have

$$\operatorname{mld}(X_i \ni x_i, \tilde{B}_i) \leq a(E_i, X_i, \tilde{B}_i) = a(E_i, X_i, \bar{B}_i) + \operatorname{mult}_{E_i}(\bar{B}_i - \tilde{B}_i)$$

$$= \alpha + \sum_{j=1}^{m_0}(b_j - \tilde{b}_{i,j}) \operatorname{mult}_{E_i} B_{i,j} \leq \alpha + \sum_{j=1}^{m_0} \frac{\delta\gamma_0}{m_0 l'} \cdot \frac{l'}{\gamma_0} = \alpha + \delta.$$

Therefore, $a(\tilde{E}_i, X_i, \bar{B}_i) \leq a(\tilde{E}_i, X_i, \tilde{B}_i) = \operatorname{mld}(X_i \ni x_i, \tilde{B}_i) \leq \alpha + \delta$, and by our choice δ, we have $a(\tilde{E}_i, X_i, \bar{B}_i) = \operatorname{mld}(X_i \ni x_i, \bar{B}_i) = \alpha$. By the construction of $\tilde{B}_i$,

$$B_i = \frac{a_{i,1}}{t}\tilde{B}_i + (1 - \frac{a_{i,1}}{t})\bar{B}_i.$$

It follows that $a(\tilde{E}_i, X_i, B_i) = \operatorname{mld}(X_i \ni x_i, B_i)$. Thus $a(E_i, X_i, 0) = 1 + \frac{l_i}{I} \leq a(\tilde{E}_i, X_i, 0) \leq 1 + \frac{l}{I}$. □

Theorem 5.2.5. *Let $\Gamma \subset [0, 1]$ be a DCC set. Then there exists a positive integer l depending only on Γ satisfying the following.*

Assume that $(X \ni x, B)$ is a threefold pair such that X is terminal, $B \in \Gamma$, and $\operatorname{mld}(X \ni x, B) \geq 1$. Then there exists a prime divisor E over $X \ni x$, such that $a(E, X, B) = \operatorname{mld}(X \ni x, B)$ and $a(E, X, 0) \leq 1 + \frac{l}{I}$, where I is the index of $X \ni x$. In particular, $a(E, X, 0) \leq 1 + l$.

Proof. Possibly replacing X with a small $\mathbb{Q}$-factorialization, we may assume that X is $\mathbb{Q}$-factorial.

Let $\gamma_0 := \min\{\gamma \in \Gamma, 1 \mid \gamma > 0\}$. Suppose that the theorem does not hold. Then by Lemmas 2.1.6 and 3.2.8, Theorems 2.1.13 and 3.2.9, there exist a positive integer I, an integer $0 \leq m \leq \frac{2}{\gamma_0}$, a strictly increasing sequence of positive integers l_i, and a sequence of threefold germs $(X_i \ni x_i, B_i = \sum_{j=1}^{m} b_{i,j} B_{i,j})$, such that

- X_i is $\mathbb{Q}$-factorial terminal for each i,

- $b_{i,j} \in \Gamma$, and $B_{i,j} \geq 0$ is a Weil divisor for any i, j,

- $\mathrm{mld}(X_i \ni x_i, B_i) > 1$ for each i,

- IK_{X_i} is Cartier near x_i for each i, and

- $1 + \frac{l_i}{I} = \min\{a(E_i, X_i, 0) \mid \mathrm{center}_{X_i} E_i = x_i, a(E_i, X_i, B_i) = \mathrm{mld}(X_i \ni x_i, B_i)\}$.

Possibly passing to a subsequence, we may assume that $\{b_{i,j}\}_{i=1}^{\infty}$ is increasing for any fixed j. We let $b_j := \lim_{i \to +\infty} b_{i,j}$ for any j, and $\Gamma_0 := \{b_1, \ldots, b_m\}$. Possibly reordering indices and passing to a subsequence, we may assume that there exists an integer $0 \leq m_0 \leq m$, such that

- $b_{i,j} \neq b_j$ for any i when $j \leq m_0$, and

- $b_{i,j} = b_j$ for every i when $j > m_0$.

Let $B_{i,0} := \sum_{j=m_0+1}^{m} b_j B_{i,j}$. Then $B_i = \sum_{j=1}^{m_0} b_{i,j} B_{i,j} + B_{i,0}$. By Lemma 5.2.4, possibly passing to a subsequence, $l_i \leq l$ for some positive integer l depending only on Γ, a contradiction. $\square$

Proof of Theorem 1.2.5. This follows from Theorem 5.2.5. $\square$

Theorem 5.2.6. *Let $\Gamma_0 = \{b_1, \ldots, b_m\} \subset [0,1]$ be a finite set. Then there exist a positive integer l and a positive real number ϵ depending only on Γ_0 satisfying the following.*

Assume that $(X \ni x, B' = \sum_i b_i' B_i')$ is a threefold pair such that

1. *X is terminal,*

2. *$b_i - \epsilon < b_i' < b_i$ for each i and $B_i' \geq 0$ are Weil divisors on X,*

3. *$\mathrm{mld}(X \ni x, B') \geq 1$, and*

4. *IK_X is Cartier near x for some positive integer I.*

Then for any prime divisor E over $X \ni x$ such that $a(E,X,B) = \mathrm{mld}(X \ni x, B)$, we have $a(E,X,0) \leq 1 + \frac{l}{I}$. In particular, $a(E,X,0) \leq 1 + l$.

Proof. Suppose that the theorem does not hold. Then there exist a strictly increasing sequence of positive integers l_i and a sequence of threefold pairs $(X_i \ni x_i, B_i = \sum_{j=1}^m b_{i,j}B_{i,j})$, such that

- X_i is terminal,
- $b_{i,j}$ is strictly increasing with $\lim_{i\to+\infty} b_{i,j} = b_j$ for each j, and $B_{i,j} \geq 0$ is a Weil divisor for any i, j,
- $\mathrm{mld}(X_i \ni x_i, B_i) \geq 1$ for any i,
- $I_i K_{X_i}$ is Cartier near x_i for some positive integer I_i, and
- there exists a prime divisor E_i over $X_i \ni x_i$ such that $a(E_i, X_i, B_i) = \mathrm{mld}(X_i \ni x_i, B_i)$, and $a(E_i, X_i, 0) \geq \frac{l_i}{I_i}$.

Possibly replacing each X_i with a small $\mathbb{Q}$-factorialization, we may assume that X_i is $\mathbb{Q}$-factorial for each i.

Let $\bar{B}_i := \sum_{j=1}^m b_j B_{i,j}$ for any i. By Theorem 1.2.9, possibly passing to a subsequence, we may assume that $\mathrm{mld}(X_i \ni x_i, \bar{B}_i) \geq 1$ for any i.

By Lemma 5.2.3, possibly reordering the indices and passing to a subsequence, we may assume $\{\frac{b_{i,j}}{b_{i,1}}\}_{i=1}^\infty$ is an increasing sequence for each j. In particular, $\frac{b_{i,j}}{b_{i,1}} \leq \frac{b_j}{b_1}$ as $\lim_{i\to+\infty} \frac{b_{i,j}}{b_{i,1}} = \frac{b_j}{b_1}$. For each i, j, let

$$b'_{i,j} := b_{i,j} + (b_1 - b_{i,1})\frac{b_{i,j}}{b_{i,1}} = \frac{b_1 b_{i,j}}{b_{i,1}} \leq b_j,$$

and $\bar{B}'_i := \sum_{j=1}^m b'_{i,j}B_{i,j} \leq \bar{B}_i$. Then $\mathrm{mld}(X_i \ni x_i, \bar{B}'_i) \geq 1$. Note that $\Gamma' := \{\frac{b_1 b_{i,j}}{b_{i,1}}\}_{i\in\mathbb{Z}_{\geq 1}, 1\leq j\leq m}$ satisfies the DCC. By Theorem 5.2.5, there exists a positive integer l depending only on Γ' and a prime divisor E'_i

over $X_i \ni x_i$ for each i, such that $a(E_i', X_i, \bar{B}_i') = \mathrm{mld}(X_i \ni x_i, \bar{B}_i')$, and $a(E_i', X_i, 0) \le 1 + \frac{l}{I_i}$. Since

$$a(E_i', X_i, \bar{B}_i') + \mathrm{mult}_{E_i}(\bar{B}_i' - B_i) = \mathrm{mld}(X_i \ni x_i, \bar{B}_i') + \mathrm{mult}_{E_i}(\bar{B}_i' - B_i)$$

$$\le a(E_i, X_i, \bar{B}_i') + \mathrm{mult}_{E_i}(\bar{B}_i' - B_i) = a(E_i, X_i, B_i) = \mathrm{mld}(X_i \ni x_i, B_i)$$

$$\le a(E_i', X_i, B_i) = a(E_i', X_i, \bar{B}_i') + \mathrm{mult}_{E_i'}(\bar{B}_i' - B_i),$$

we have $\mathrm{mult}_{E_i}(\bar{B}_i' - B_i) \le \mathrm{mult}_{E_i'}(\bar{B}_i' - B_i)$. By the construction of $\bar{B}_i'$,

$$\bar{B}_i' - B_i = \sum_{j=1}^{m}(b_{i,j}' - b_{i,j})B_{i,j} = \sum_{j=1}^{m}(b_1 - b_{i,1})\frac{b_{i,j}}{b_{i,1}}B_{i,j} = \frac{b_1 - b_{i,1}}{b_{i,1}}B_i.$$

It follows that $\mathrm{mult}_{E_i} B_i \le \mathrm{mult}_{E_i'} B_i$. Hence

$$a(E_i, X_i, 0) = a(E_i, X_i, B_i) + \mathrm{mult}_{E_i} B_i \le a(E_i', X_i, B_i) + \mathrm{mult}_{E_i'} B_i$$

$$= a(E_i', X_i, 0) \le 1 + \frac{l}{I_i},$$

a contradiction. $\square$

5.2.1 Proof of Theorem 1.2.12

Definition 5.2.7 (Log Calabi–Yau pairs)**.** A log pair (X, B) is called a *log Calabi–Yau pair* if $K_X + B \sim_{\mathbb{R}} 0$.

Definition 5.2.8 (Bounded pairs)**.** A collection of varieties $\mathcal{D}$ is said to be *bounded* (resp. *birationally bounded*, *bounded in codimension one*) if there exists a projective morphism $h\colon \mathcal{Z} \to S$ of schemes of finite type such that each $X \in \mathcal{D}$ is isomorphic (resp. birational, isomorphic in codimension one) to $\mathcal{Z}_s$ for some closed point $s \in S$.

We say that a collection of log pairs $\mathcal{D}$ is *log birationally bounded* (resp. *log bounded*, *log bounded in codimension one*) if there exist a quasi-projective scheme $\mathcal{Z}$, a reduced divisor $\mathcal{E}$ on $\mathcal{Z}$, and a projective morphism $h\colon \mathcal{Z} \to S$, where S is of finite type and $\mathcal{E}$ does not contain any fiber, such that for every $(X, B) \in \mathcal{D}$, there exist a closed point $s \in S$ and a birational map (resp. isomorphism, isomorphism in codimension one) $f\colon \mathcal{Z}_s \dashrightarrow X$ such that $\mathcal{E}_s$ contains the support of $f_*^{-1}B$ and any f-exceptional divisor (resp. $\mathcal{E}_s$ coincides with the support of $f_*^{-1}B$, $\mathcal{E}_s$ coincides with the support of $f_*^{-1}B$).

Moreover, if $\mathcal{D}$ is a set of klt Calabi–Yau varieties (resp. klt log Calabi–Yau pairs), then it is said to be *bounded modulo flops* (resp. *log bounded modulo flops*) if it is bounded (resp. log bounded) in codimension one, each fiber $\mathcal{Z}_s$ corresponding to a member in $\mathcal{D}$ is normal projective, and $K_{\mathcal{Z}_s}$ is $\mathbb{Q}$-Cartier (resp. $K_{\mathcal{Z}_s} + f_*^{-1}B$ is $\mathbb{R}$-Cartier).

Proof of Theorem 1.2.12. We follow the proof of [Jia21, Theorem 6.1] and [CDHJS21, Theorem 5.1]. By Theorem 1.2.2, there exists a positive real number $\delta < 1$ depending only on Γ, such that $\operatorname{mld}(X, B) \leq 1 - \delta$. By [BCHM10, Corollary 1.4.3], there exists a birational morphism $f : Y \to X$ which extracts exactly one exceptional divisor E with $a := a(E, X, B) \leq 1 - \delta$. By [HLS19, Lemma 3.21], Y is of Fano type over X. Possibly replacing Y with the canonical model of $-E$ over X, we may assume that $-E$ is ample over X, and $\operatorname{Exc}(f) = \operatorname{Supp} E$. We may write

$$K_Y + B_Y + (1 - a)E = f^*(K_X + B) \equiv 0,$$

where B_Y is the strict transform of B on Y. By [HMX14, Theorem 1.5], there exists a finite subset $\Gamma_0 \subset \Gamma$ depending only on Γ, such that $B \in \Gamma_0$. Possibly replacing Γ with Γ_0, we may assume that Γ is finite. By [HMX14, Theorem 1.5] again (see also the proof of [CDHJS21, Lemma 3.12]), there exists a positive real number $\epsilon < \frac{1}{2}$ depending only on Γ such that (X, B) is (2ϵ)-lc. Thus $(Y, B_Y + (1 - a)E)$ is a (2ϵ)-lc log Calabi-Yau pair with $1 - a \geq \delta > 0$. By [HM07, Corollary 1.4], each fiber of f is rationally chain connected. Since $\operatorname{Exc}(f) = \operatorname{Supp} E$, E is uniruled. Now by [Jia21, Proposition 6.4], the pairs $(Y, B_Y + (1 - a)E)$ are log bounded modulo flops. That is, there are finitely many normal varieties $\mathcal{W}_i$, an $\mathbb{R}$-divisor $\mathcal{B}_i$ and a reduced divisor $\mathcal{E}_i$ on $\mathcal{W}_i$, and a projective morphism $\mathcal{W}_i \to S_i$, where S_i is a normal variety of finite type, and $\mathcal{B}_i, \mathcal{E}_i$ do not contain any fiber of $\mathcal{W}_i \to S_i$, such that for every $(Y, B_Y + (1-a)E)$, there is an index i, a closed point $s \in S_i$, and a small birational map $g : \mathcal{W}_{i,s} \dashrightarrow Y$ such that $\mathcal{B}_{i,s} = g_*^{-1}B_Y$ and $\mathcal{E}_{i,s} = g_*^{-1}E$. We may assume that the set of points s corresponding to such Y is dense in each S_i. We may just consider a fixed index i and ignore the index in the following argument.

For the point s corresponding to $(Y, B_Y + (1-a)E)$,

$$K_{\mathcal{W}_s} + g_*^{-1}B_Y + (1-a)g_*^{-1}E \equiv f_*^{-1}(K_Y + B_Y + (1-a)E) \equiv 0$$

and therefore $(\mathcal{W}_s, g_*^{-1}B_Y + (1-a)g_*^{-1}E)$ is a (2ϵ)-lc log Calabi–Yau pair.

Let $h : \mathcal{W}' \to \mathcal{W}$ be a log resolution of $(\mathcal{W}, \mathcal{B} + \mathcal{E})$, $\mathcal{B}'$ the strict transforms of $\mathcal{B}$ on $\mathcal{W}'$, and $\mathcal{E}'$ the sum of all h-exceptional reduced divisors and the strict transform of $\mathcal{E}$ on $\mathcal{W}'$. Then there exists an open dense subset $U \subset S$ such that for the point $s \in U$ corresponding to $(Y, B_Y + (1-a)E)$, $h_s : \mathcal{W}'_s \to \mathcal{W}_s$ is a log resolution of $(Y, B_Y + (1-a)E)$. Since $(\mathcal{W}_s, g_*^{-1}B_Y + (1-a)g_*^{-1}E)$ is (2ϵ)-lc,

$$K_{\mathcal{W}'_s} + \mathcal{B}'_s + (1-\epsilon)\mathcal{E}'_s - h_s^*(K_{\mathcal{W}_s} + g_*^{-1}B_Y + (1-a)g_*^{-1}E)$$

is an h_s-exceptional $\mathbb{R}$-divisor whose support coincides with $\operatorname{Supp} \mathcal{E}'_s$. Note that $\dim \mathcal{W}_s = 3$. By [HH20, Lemma 2.10, Theorem 1.1], we may run a $(K_{\mathcal{W}'} + \mathcal{B}' + (1-\epsilon)\mathcal{E}')$-MMP with scaling of an ample divisor over S and reach a relative minimal model $\tilde{\mathcal{W}}$ over S. For the point $s \in U$ corresponding to $(Y, B_Y + (1-a)E)$, $\mathcal{E}'_s$ is contracted, and hence $\tilde{\mathcal{W}}_s$ is isomorphic to X in codimension one. This gives a bounded family modulo flops over U. Applying Noetherian induction on S, the family of all such X is bounded modulo flops. $\square$

Remark 5.2.9. It is possible to replace Theorem 1.2.2 with the uniform lc rational polytopes [HLS19, Theorem 5.6] and the boundedness of indices of log Calabi-Yau threefolds [Xu19, Theorem 1.13] to conclude $\operatorname{mld}(X, B) \leq 1 - \delta$ in the beginning of the proof of Theorem 1.2.12. We briefly describe the idea here. By

[HMX14, Theorem 1.5], we may assume that Γ is a finite set. By [HLS19, Theorem 5.6], we may reduce the theorem to the case $\Gamma \subset \mathbb{Q}$. By [Xu19, Theorem 1.13], $I(K_X + B)$ is Cartier for some positive integer I which only depends on Γ. In particular, $\mathrm{mld}(X, B) \leq 1 - \delta$.

Chapter 6

Questions and Open Problems

It would be interesting to ask if Lemmas 3.2.4 3.2.5 hold for all terminal threefolds.

Conjecture 6.0.1. *Let $(X \ni x, B)$ be an lc threefold pair, such that X is terminal and $\mathrm{mld}(X \ni x, B) \geq 1$. Then there exists a prime divisor E over $X \ni x$, and a divisorial contraction $f : Y \to X$ of E, such that Y is terminal, and $a(E, X, B) = \mathrm{mld}(X \ni x, B)$.*

We remark that the assumption "$\mathrm{mld}(X \ni x, B) \geq 1$" is necessary in Conjecture 6.0.1. Indeed, [KSC04, Excerise 6.45], and [Kaw17, Example 5] show that there exists a $\mathbb{Q}$-divisor B on $X := \mathbb{C}^3$, such that $\mathrm{mld}(X \ni x, B) = 0$, there is exactly one prime divisor E over $X \ni x$ with $a(E, X, B) = \mathrm{mld}(X \ni x, B)$, and E is not obtained by a weighted blow-up. Recall that any divisorial contraction from a terminal threefold to a smooth variety is always a weighted blow-up.

Conjecture 6.0.2 (cf. [HL20, Introduction])**.** *Let d be a positive integer and $\Gamma \subset [0, 1]$ a DCC set. Then there exists a positive real number l depending only on d and Γ satisfying the following.*

Assume that $(X \ni x, B)$ is an lc pair of dimension d such that X is $\mathbb{Q}$-Gorenstein and $B \in \Gamma$. Then there exists a prime divisor E over $X \ni x$, such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$ and $a(E, X, 0) \leq l$.

[MN18, Conjecture 1.1] and [CH21, Problem 7.17] are exactly Conjecture 6.0.2 for the case when $X \ni x$

is a fixed germ and Γ is a finite set, and when $X \ni x$ is a fixed germ respectively. Conjecture 6.0.2 holds when $\dim X = 2$ [HL20, Theorem 1.2]. In this paper, we give a positive answer for terminal threefolds. A much ambitious problem is the following.

Question 6.0.3. *Let $\Gamma \subset [0,1]$ be a DCC set. Assume that $(X \ni x, B)$ is an lc pair such that X is klt near x and $\operatorname{mld}(X \ni x, B) > 0$.*

1. *Will there exist a divisorial contraction $f : Y \to X$ of a prime divisor E over $X \ni x$, such that $a(E, X, B) = \operatorname{mld}(X \ni x, B)$?*
2. *Moreover, if $B \in \Gamma$, and $\operatorname{mld}(X \ni x, B) \geq 1$, will $a(E, X, B) = \operatorname{mld}(X \ni x, B)$ and $a(E, X, 0) \leq l$ for some real number l depending only on $\dim X$ and Γ?*

It was shown in [HLQ21, Theorem 1.1] that the lc threshold polytopes satisfy the ACC, and a conjecture due to the first author asks whether the volumes of lc threshold polytopes satisfy the ACC. In the same fashion, we ask the following.

Question 6.0.4 (ACC for CT-polytopes)**.** *Let d, s be positive integers, and $\Gamma \subseteq \mathbb{R}_{\geq 0}$ a DCC set. Let $\mathcal{S}$ be the set of all $(X, \Delta; D_1, \ldots, D_s)$, where*

1. *$\dim X = d$, (X, Δ) is canonical, and $\Delta \in \Gamma$, and*
2. *$D_1, \ldots, D_s$ are $\mathbb{R}$-Cartier divisors, and $D_1, \ldots, D_s \in \Gamma$.*

Then

1. *$\{P(X, \Delta; D_1, \ldots, D_s) \mid (X, \Delta; D_1, \ldots, D_s) \in \mathcal{S}\}$ satisfies the ACC (under the inclusion), and*
2. *$\{\operatorname{Vol}(P(X, \Delta; D_1, \ldots, D_s)) \mid (X, \Delta; D_1, \ldots, D_s) \in \mathcal{S}\}$ satisfies the ACC,*

where

$$P(X, \Delta; D_1, \ldots, D_s) \coloneqq \{(t_1, \ldots, t_s) \in \mathbb{R}^s_{\geq 0} \mid (X, \Delta + t_1 D_1 + \ldots + t_s D_s) \text{ is canonical}\}.$$

Chapter 7

Appendix—Boundedness of Divisors Computing Minimal Log Discrepancies for Surfaces

We prove Theorem 1.2.4 in this chapter. We work over an algebraic closed field of arbitrary characteristic in this chapter.

7.1 Preliminary Results

7.1.1 Arithmetic of sets

Lemma 7.1.1. *Let $\Gamma \subseteq [0,1]$ be a set which satisfies the DCC, and n a non-negative integer. There exists a positive real number γ which only depends on n and Γ, such that*

$$\{\sum_i n_i b_i - n > 0 \mid b_i \in \Gamma, n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\gamma, +\infty).$$

Proof. The existence of γ follows from that the set $\{\sum_i n_i b_i - n \mid b_i \in \Gamma, n_i \in \mathbb{Z}_{\geq 0}\}$ satisfies the DCC. $\square$

Definition 7.1.2. Let $\epsilon \in \mathbb{R}$, $I \in \mathbb{R}\backslash\{0\}$, and $\Gamma \subseteq \mathbb{R}$ a set of real numbers. We define $\Gamma_\epsilon := \cup_{b\in\Gamma}[b-\epsilon, b]$, and $\frac{1}{I}\Gamma := \{\frac{b}{I} \mid b \in \Gamma\}$.

Lemma 7.1.3. *Let $\Gamma \subseteq [0,1]$ be a set which satisfies the DCC. Then there exist positive real numbers $\epsilon, \delta \leq 1$, such that*

$$\{\sum_i n_i b_i' - 1 > 0 \mid b_i' \in \Gamma_\epsilon \cap [0,1], n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\delta, +\infty).$$

Proof. We may assume that $\Gamma \setminus \{0\} \neq \emptyset$, otherwise we may take $\epsilon = \delta = 1$.

Since Γ satisfies the DCC, by Lemma 7.1.1, there exists a real number $\gamma \in (0,1]$ such that $\Gamma \setminus \{0\} \subseteq (\gamma, 1]$, and $\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}, b_i \in \Gamma\} \subseteq [\gamma, +\infty)$. It suffices to prove that there exist $0 < \epsilon, \delta < \frac{\gamma}{2}$, such that the set $\{\sum_i n_i b_i' - 1 \in (0,1] \mid b_i' \in \Gamma_\epsilon \cap [0,1], n_i \in \mathbb{Z}_{\geq 0}\}$ is bounded from below by δ, or equivalently

$$\{\sum_i n_i b_i' - 1 > 0 \mid b_i' \in \Gamma_\epsilon \cap [0,1], n_i \in \mathbb{Z}_{\geq 0}, \sum_i n_i \leq \frac{4}{\gamma}\} \subseteq [\delta, +\infty).$$

We claim that $\epsilon = \frac{\gamma^2}{8}, \delta = \frac{\gamma}{2}$ have the desired property. Let $b_i' \in \Gamma_\epsilon \cap [0,1]$ and $n_i \in \mathbb{Z}_{\geq 0}$, such that $\sum_i n_i b_i' - 1 > 0$ and $\sum_i n_i \leq \frac{4}{\gamma}$. We may find $b_i \in \Gamma$, such that $0 \leq b_i - b_i' \leq \epsilon$ for any i. In particular, $\sum_i n_i b_i - 1 > 0$. By the choice of γ, $\sum_i n_i b_i - 1 \geq \gamma$. Thus

$$\sum_i n_i b_i' - 1 = (\sum_i n_i b_i - 1) - \sum_i n_i (b_i - b_i') \geq \gamma - \frac{4}{\gamma}\epsilon = \frac{\gamma}{2},$$

and we are done. $\square$

We will use the following lemma frequently without citing it in this article.

Lemma 7.1.4. *Let $\Gamma \subseteq [0,1]$ be a set, and $\gamma \in (0,1]$ be a real number. If $\{\sum_i n_i b_i - 1 > 0 \mid b_i \in \Gamma, n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\gamma, +\infty)$, then $\Gamma \setminus \{0\} \subseteq [\gamma, 1]$.*

Proof. Otherwise, we may find $b \in \Gamma$, such that $0 < b < \gamma$. Then $0 < (\lfloor \frac{1}{b} \rfloor + 1) \cdot b - 1 \leq b < \gamma$, a contradiction. $\square$

7.1.2 Minimal resolution

Let X be a normal quasi-projective surface and $x \in X$ a closed point. Then a birational morphism $f: Y \to X$ (respectively $f: Y \to X \ni x$) is called a *minimal resolution* of X (respectively $X \ni x$) if Y is smooth (respectively smooth over a neighborhood of $x \in X$) and there is no (-1)-curve on Y (respectively over a neighborhood of $x \in X$).

Note that the existence of resolutions of singularities for surfaces (see [Lip18]) and the minimal model program for surfaces (see [Tan14] and [Tan18]) are all known in positive characteristic. In particular, for any surface X (respectively surface germ $X \ni x$), we can construct a minimal resolution $\tilde{f}: \widetilde{X} \to X$ (respectively $\widetilde{f}: \widetilde{X} \to X \ni x$).

Definition 7.1.5. Let $\widetilde{f}: \widetilde{X} \to X \ni x$ be the minimal resolution of $X \ni x$, and we may write $K_{\widetilde{X}} + B_{\widetilde{X}} + \sum_i (1-a_i)E_i = \widetilde{f}^*(K_X + B)$, where $B_{\widetilde{X}}$ is the strict transform of B, E_i are $\widetilde{f}$-exceptional prime divisors and $a_i := a(E_i, X, B)$ for all i. The *partial log discrepancy* of $(X \ni x, B)$, $\mathrm{pld}(X \ni x, B)$, is defined as follows.

$$\mathrm{pld}(X \ni x, B) := \begin{cases} \min_i\{a_i\} & \text{if } x \in X \text{ is a singular point,} \\ +\infty & \text{if } x \in X \text{ is a smooth point.} \end{cases}$$

7.1.3 Dual graphs

Definition 7.1.6 (c.f. [KM98, Definition 4.6])**.** Let $C = \cup_i C_i$ be a collection of proper curves on a smooth surface U. We define the *dual graph* $\mathcal{DG}$ of C as follows.

1. The vertices of $\mathcal{DG}$ are the curves C_j.
2. Each vertex is labelled by the negative self intersection of the corresponding curve on U, we call it the *weight* of the vertex (curve).
3. The vertices C_i, C_j are connected with $C_i \cdot C_j$ edges.

Let $f : Y \to X \ni x$ be a projective birational morphism with exceptional divisors $\{E_i\}_{1\leq i\leq m}$, such that Y is smooth. Then the dual graph $\mathcal{DG}$ of f is defined as the dual graph of $E = \cup_{1\leq i\leq m} E_i$. In particular, $\mathcal{DG}$ is a connected graph.

Definition 7.1.7. A *cycle* is a graph whose vertices and edges can be ordered $v_1, \ldots, v_m$ and $e_1, \ldots, e_m$ $(m \geq 2)$, such that e_i connects v_i and v_{i+1} for $1 \leq i \leq m$, where $v_{m+1} = v_1$.

Let $\mathcal{DG}$ be a dual graph with vertices $\{C_i\}_{1\leq i\leq m}$. We call $\mathcal{DG}$ a *tree* if

1. $\mathcal{DG}$ does not contain a subgraph which is a cycle, and

2. $C_i \cdot C_j \leq 1$ for all $1 \leq i \neq j \leq m$.

Moreover, if C is a vertex of $\mathcal{DG}$ that is adjacent to more than three vertices, then we call C a *fork* of $\mathcal{DG}$. If $\mathcal{DG}$ contains no fork, then we call it a *chain*.

Lemma 7.1.8. *Let $X \ni x$ be a surface germ. Let Y, Y' be smooth surfaces, and let $f : Y \to X \ni x$ and $f' : Y' \to X \ni x$ be two projective birational morphisms, such that f' factors through f.*

$$\begin{tikzcd}
Y' \arrow[r, "g"] \arrow[dr, "f'"'] & Y \arrow[d, "f"] \\
 & X \ni x
\end{tikzcd}$$

If the dual graph of f is a tree whose vertices are all smooth rational curves, then the dual graph of f' is a tree whose vertices are all smooth rational curves.

Proof. Let $g : Y' \to Y$ be the projective birational morphism such that $f \circ g = f'$. Since g is a composition of blow-ups at smooth closed points, by inducion on the number of blow-ups, we may assume that g is a single blow-up of Y at a smooth closed point $y \in Y$.

Let E' be the g-exceptional divisor on Y', $\{E_i\}_{1\leq i\leq m}$ the set of distinct exceptional curves of f on Y, and $\{E_i'\}_{1\leq i\leq m}$ their strict transforms on Y'. By assumption $E_i'\cdot E_j'\leq g^*E_i\cdot E_j'=E_i\cdot E_j\leq 1$ for $1\leq i\neq j\leq m$. Since E_i is smooth, $0=g^*E_i\cdot E'\geq (E_i'+E')\cdot E'$. It follows that $E'\cdot E_i'\leq 1$ for $1\leq i\leq m$.

If the dual graph of f' contains a cycle, then E' must be a vertex of this cycle. Let $E',E_{i_1}',\dots,E_{i_k}'$ be the vertices of this cycle, $1\leq k\leq m$. Then the vertex-induced subgraph by $E_{i_1},\dots,E_{i_k}$ of the dual graph of f is a cycle, a contradiction. $\square$

The following lemma maybe well-known to experts. For the reader's convenience, we include the proof here.

Lemma 7.1.9. *Let $\epsilon_0\in(0,1]$ be two real numbers. Let $(X\ni x,B)$ be an lc surface germ, Y a smooth surface, and $f:Y\to X\ni x$ a projective birational morphism with the dual graph $\mathcal{DG}$. Let $\{E_k\}_{1\leq k\leq m}$ be the set of vertices of $\mathcal{DG}$, and $w_k:=-E_k\cdot E_k$, $a_k:=a(E_k,X,B)$ for each k. Suppose that $a_k\leq 1$ for any $1\leq k\leq m$, then we have the following:*

1. *$w_k\leq\frac{2}{a_k}$ if $a_k>0$, and in particular, $w_k\leq\frac{2}{\epsilon_0}$ for $1\leq k\leq m$ if $\mathrm{mld}(X\ni x,B)\geq\epsilon_0$.*
2. *If $w_k\geq 2$ for some k, then for any E_{k_1},E_{k_2} which are adjacent to E_k, we have $2a_k\leq a_{k_1}+a_{k_2}$. Moreover, if the equality holds, then $f_*^{-1}B\cdot E_k=0$, and either $w_k=2$ or $a_k=a_{k_1}=a_{k_2}=0$.*
3. *If E_{k_0} is a fork, then for any E_{k_1},E_{k_2},E_{k_3} which are adjacent to E_{k_0} with $w_{k_i}\geq 2$ for $0\leq i\leq 2$, $a_{k_3}\geq a_{k_0}$. Moreover, if the equality holds, then $w_{k_i}=2$ and $f_*^{-1}B\cdot E_{k_i}=0$ for $0\leq i\leq 2$.*
4. *Let E_{k_0},E_{k_1},E_{k_2} be three vertices, such that E_{k_1},E_{k_2} are adjacent to E_{k_0}. Assume that $a_{k_1}\geq a_{k_2}$, $a_{k_1}\geq\epsilon_0$, and $w_{k_0}\geq 3$, then $a_{k_1}-a_{k_0}\geq\frac{\epsilon_0}{3}$.*

5. *If E_{k_0} is a fork, and there exist three vertices $E_{k_1}, E_{k_2}, E_{k_3}$ which are adjacent to E_{k_0} with $w_{k_i} \geq 2$ for $0 \leq i \leq 3$, then $a(E, X, B) \geq a_{k_0}$ for any vertex E of $\mathcal{DG}$.*

6. *Let $\{E_{k_i}\}_{0 \leq i \leq m'}$ be a set of distinct vertices such that E_{k_i} is adjacent to $E_{k_{i+1}}$ for $0 \leq i \leq m' - 1$, where $m' \geq 2$. If $a_{k_0} = a_{k_{m'}} = \mathrm{mld}(X \ni x, B) > 0$ and $w_{k_i} \geq 2$ for $1 \leq i \leq m' - 1$, then $a_{k_0} = a_{k_1} = \cdots = a_{k_{m'}}$ and $w_{k_i} = 2$ for $1 \leq i \leq m' - 1$.*

Proof. For (1), we may write

$$K_Y + f_*^{-1}B + \sum_{1 \leq i \leq m} (1 - a_i)E_i = f^*(K_X + B).$$

For each $1 \leq k \leq m$, we have

$$0 = (K_Y + f_*^{-1}B + \sum_{1 \leq i \leq m} (1 - a_i)E_i) \cdot E_k,$$

or equivalently,

$$a_k w_k = 2 - 2p_a(E_k) - \sum_{i \neq k} (1 - a_i)E_i \cdot E_k - f_*^{-1}B \cdot E_k. \tag{7.1.1}$$

So $a_k w_k \leq 2$, and $w_k \leq \frac{2}{a_k}$.

For (2), by (7.1.1),

$$2a_k \leq a_k w_k \leq a_{k_1} + a_{k_2} - f_*^{-1}B \cdot E_k \leq a_{k_1} + a_{k_2}.$$

If $2a_k = a_{k_1} + a_{k_2}$, then $f_*^{-1}B \cdot E_k = 0$, and either $w_k = 2$ or $a_k = a_{k_1} = a_{k_2} = 0$.

For (3), let $k = k_i$ in (7.1.1) for $i = 1, 2$,

$$a_{k_i} w_{k_i} \leq 1 + a_{k_0} - (\sum_{j \neq k_0, k_i} (1 - a_j)E_j \cdot E_{k_i} + f_*^{-1}B \cdot E_{k_i}) \leq 1 + a_{k_0},$$

or $a_{k_i} \leq \frac{1+a_{k_0}}{w_{k_i}}$. Thus let $k = k_0$ in (7.1.1), we have

$$a_{k_3} \geq a_{k_0} w_{k_0} + 1 - a_{k_1} - a_{k_2} + f_*^{-1}B \cdot E_{k_0}$$

$$\geq a_{k_0}(w_{k_0} - \frac{1}{w_{k_1}} - \frac{1}{w_{k_2}}) + (1 - \frac{1}{w_{k_1}} - \frac{1}{w_{k_2}}) \geq a_{k_0}.$$

If the equality holds, then $w_{k_i} = 2$ and $f_*^{-1}B \cdot E_i = 0$ for $0 \leq i \leq 2$.

For (4), by (7.1.1), we have $a_{k_0} w_{k_0} \leq a_{k_1} + a_{k_2} - d$, where $d := f_*^{-1}B \cdot E_{k_0} + \sum_{j \neq k_1, k_2, k_0} (1 - a_j) E_j \cdot E_{k_0}$. Hence

$$a_{k_1} - a_{k_0} \geq \frac{(w_{k_0} - 1)a_{k_1} - a_{k_2} + d}{w_{k_0}} \geq \frac{(w_{k_0} - 2)a_{k_1}}{w_{k_0}} \geq \frac{\epsilon_0}{3}.$$

For (5), we may assume that $E \neq E_{k_0}$. There exist $m' + 1$ distinct vertices $\{F_i\}_{0 \leq i \leq m'}$ of $\mathcal{DG}$, such that

- $F_0 = E_{k_0}$, $F_{m'} = E$, and
- F_i is adjacent to F_{i+1} for $0 \leq i \leq m' - 1$.

Denote $a_i' := a(F_i, X, B)$ for $0 \leq i \leq m'$. By (3), we have $a_1' \geq a_0'$, and by (2), $a_{i+1}' - a_i' \geq a_i' - a_{i-1}'$ for $1 \leq i \leq m' - 1$. Thus $a_m' - a_0' \geq 0$.

For (6), by (2) $a_{k_0} \leq a_{k_1} \leq \ldots \leq a_{k_{m'-1}} \leq a_{k_{m'}}$. Thus $a_{k_0} = \ldots = a_{k_{m'}}$. By (2) again, $w_{k_i} = 2$ for $1 \leq i \leq m' - 1$.

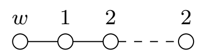
$\square$

Lemma 7.1.10. *Let $(X \ni x, B)$ be an lc surface germ. Let Y be a smooth surface and $f : Y \to X \ni x$ a birational morphism with the dual graph $\mathcal{DG}$. If $\mathcal{DG}$ contains a (-1)-curve E_0, then*

1. *E_0 can not be adjacent to two (-2)-curves in $\mathcal{DG}$,*

2. *if either* $\mathrm{mld}(X \ni x, B) \neq \mathrm{pld}(X \ni x, B)$ *or* $\mathrm{mld}(X \ni x, B) > 0$*, then* E_0 *is not a fork in* $\mathcal{DG}$*, and*

3. *if* $E, E_0, \ldots, E_m$ *are distinct vertices of* $\mathcal{DG}$ *such that* E *is adjacent to* E_0*,* E_i *is adjacent to* E_{i+1} *for* $0 \leq i \leq m-1$*, and* $-E_i \cdot E_i = 2$ *for* $1 \leq i \leq m$*, then* $m+1 < -E \cdot E = w$*.*



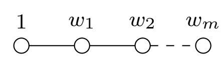
Proof. For (1), if E_0 is adjacent to two (-2)-curves E_{k_1} and E_{k_2} in $\mathcal{DG}$, then we may contract E_0 and get a smooth model $f' : Y' \to X \ni x$ over X, whose dual graph contains two adjacent (-1)-curves, this contradicts the negativity lemma.

By [KM98, Theorem 4.7] and the assumptions in (2), the dual graph of the minimal resoltion of $X \ni x$ is a tree. If E_0 is a fork, we may contract E_0 and get a smooth model $f' : Y' \to X \ni x$, whose dual graph contains a cycle, this contradicts Lemma 7.1.8.

For (3), we will construct a sequence of contractions of (-1)-curve $X_0 := X \to X_1 \to \ldots X_m \to X_{m+1}$ inductively. Let E_{X_k} be the strict transform of E on X_k, and $w_{X_k} := -E_{X_k} \cdot E_{X_k}$. For simplicity, we will always denote the strict transform of E_k on X_j by E_k for all k, j. Let $f_1 : X_0 \to X_1$ be the contraction of E_0 on X_0, then $w_{X_1} = w - 1$, and $E_1 \cdot E_1 = -1$ on X_1. Let $f_2 : X_1 \to X_2$ be the contraction of E_1 on X_1, then $w_{X_2} = w_{X_1} - 1 = w - 2$, and $E_2 \cdot E_2 = -1$ on X_2. Repeating this procedure, we have $f_k : X_{k-1} \to X_k$ the contraction of E_{k-1}, and $w_{X_k} = w - k$, $E_k \cdot E_k = -1$ on X_k for $1 \leq k \leq m+1$. By the negativity lemma, $w_{X_{m+1}} = w - (m+1) > 0$, and we are done. $\square$

Lemma 7.1.11. *Let* $\gamma \in (0,1]$ *be a real number. Let* $(X \ni x, B := \sum_i b_i B_i)$ *be an lc surface germ, where* B_i *are distinct prime divisors. Let* Y *be a smooth surface and* $f : Y \to X \ni x$ *a birational morphism with the dual graph* $\mathcal{DG}$*. Let* $\{E_k\}_{0 \leq k \leq m}$ *be a vertex-induced sub-chain of* $\mathcal{DG}$*, such that* E_k *is adjacent to* E_{k+1} *for* $0 \leq k \leq m-1$*, and let* $w_k := -E_k \cdot E_k$*,* $a_k := a(E_k, X, B)$ *for all* k*. Suppose that* $w_0 = 1$*, and* E_0 *is* *adjacent to only one vertex E_1 of $\mathcal{DG}$, $a_k \le 1$, and $w_k \ge 2$ for each $k \ge 1$, then*

1. *if $\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\ge 0}\} \subseteq [\gamma, +\infty)$, and $a_0 < a_1$, then $m \le \frac{1}{\gamma}$,*
2. *if $\sum_i n_i b_i - 1 \neq 0$ for all $n_i \in \mathbb{Z}_{\ge 0}$, and $a_0 \le a_1$, then $a_0 < a_1$, and*
3. *if $\{\sum_i n_i b_i - 1 \ge 0 \mid n_i \in \mathbb{Z}_{\ge 0}\} \subseteq [\gamma, +\infty)$, and $a_0 \le a_1$, then $m \le \frac{1}{\gamma}$.*



Proof. We may write $K_Y + f_*^{-1}B + \sum_i (1 - a_i)E_i = f^*(K_X + B)$, then

$$-2 + f_*^{-1}B \cdot E_0 + w_0 a_0 + \sum_{i \neq 0} (1 - a_i)E_i \cdot E_0 = 0. \tag{7.1.2}$$

Since E_0 is adjacent to only one vertex E_1 of $\mathcal{DG}$, by (7.1.2) we have

$$a_1 - a_0 = f_*^{-1}B \cdot E_0 - 1.$$

For (1), since $1 \ge a_1 > a_0$, it follows that $f_*^{-1}B \cdot E_0 - 1 \in \{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\ge 0}\} \subseteq [\gamma, +\infty)$, Thus $a_1 - a_0 \ge \gamma$. By Lemma 7.1.9(2), we have $a_{i+1} - a_i \ge a_1 - a_0 \ge \gamma$ for any $0 \le i \le m - 1$, and $1 \ge a_m \ge \gamma m$. So $m \le \frac{1}{\gamma}$.

For (2), since $f_*^{-1}B \cdot E_0 - 1 = \sum_i n_i b_i - 1 \neq 0$ for some $n_i \in \mathbb{Z}_{\ge 0}$, $a_0 < a_1$.

(3) follows immediately from (1) and (2). $\square$

7.1.4 Extracting divisors computing mlds

We first simplify our notions.

Definition 7.1.12. Let $X \ni x$ be a smooth surface germ. We say $X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 := X$ is a *sequence of blow-ups with the data* $(f_i, F_i, x_i \in X_i)$ if

- $f_i : X_i \to X_{i-1}$ is the blow-up of X_{i-1} at a closed point $x_{i-1} \in X_{i-1}$ with the exceptional divisor F_i for any $1 \leq i \leq n$, where $x_0 := x$, and
- $x_i \in F_i$ for any $1 \leq i \leq n-1$.

In particular, F_n is the only exceptional (-1)-curve over X.

For convenience, we will always denote the strict transform of F_i on X_j by F_i for any $n \geq j \geq i$.

The following lemma is well known. For a proof, see for example, [HL20, Lemma 3.15].

Lemma 7.1.13. *Let $(X \ni x, B)$ be an lc surface germ such that $\operatorname{mld}(X \ni x, B) > 1$, then $\operatorname{mld}(X \ni x, B) = 2 - \operatorname{mult}_x B$, and there is exactly one prime divisor E over $X \ni x$ such that $a(E, X, B) = \operatorname{mld}(X \ni x, B)$.*

Lemma 7.1.14 ([MN18, Lemma 4.2])**.** *Let $X \ni x$ be a smooth surface germ, and $X_{l_0} \to \cdots \to X_1 \to X_0 := X$ a sequence of blow-ups with the data $(f_i, E_i, x_i \in X_i)$, then $a(E_{l_0}, X, 0) \leq 2^{l_0}$.*

Now we will construct some birational models for surfaces.

Lemma 7.1.15. *Let $(X \ni x, B)$ be an lc surface germ. Let $h : W \to (X, B)$ be a log resolution, and $S = \{E_j\}$ a finite set of valuations of h-exceptional prime divisors over $X \ni x$ such that $a(E_j, X, B) \leq 1$ for all j. Then there exist a smooth surface Y and a projective birational morphism $f : Y \to X \ni x$ with the following properties.*

1. *$K_Y + B_Y = f^*(K_X + B)$ for some $\mathbb{R}$-divisor $B_Y \geq 0$ on Y,*
2. *each valuation in S corresponds to some f-exceptional divisor on Y, and*
3. *each f-exceptional (-1)-curve corresponds to some valuation in S.*

Proof. We may write

$$K_W + B_W = h^*(K_X + B) + F_W,$$

where $B_W \geq 0$ and $F_W \geq 0$ are $\mathbb{R}$-divisors with no common components. We construct a sequence of $(K_W + B_W)$-MMP over X as follows. Each time we will contract a (-1)-curve whose support is contained in F_W. Suppose that $K_W + B_W$ is not nef over X, then $F_W \neq 0$. By the negativity lemma, there exists a h-exceptional irreducible curve $C \subseteq \operatorname{Supp} F_W$, such that $F_W \cdot C = (K_W + B_W) \cdot C < 0$. Since $B_W \cdot C \geq 0$, $K_W \cdot C < 0$. Thus C is a h-exceptional (-1)-curve. We may contract C, and get a smooth surface $Y_0 := W \to Y_1$ over X. We may continue this process, and finally reach a smooth model Y_k on which $K_{Y_k} + B_{Y_k}$ is nef over X, where B_{Y_k} is the strict transform of B_W on Y_k. By the negativity lemma, F_W is contracted in the MMP, thus $K_{Y_k} + B_{Y_k} = h_k^*(K_X + B)$, where $h_k : Y_k \to X$. Since $a(E_j, X, B) \leq 1$, E_j is not contracted in the MMP for any $E_j \in S$.

We now construct a sequence of smooth models over X, $Y_k \to Y_{k+1} \to \cdots$, by contracting a curve C' satisfying the following conditions in each step.

- C' is an exceptional (-1)-curve over X, and
- $C' \notin S$.

Since each time the Picard number of the variety will drop by one, after finitely many steps, we will reach a smooth model Y over X, such that $f : Y \to X$ and (Y, B_Y) satisfy (1)–(3), where B_Y is the strict transform of B_{Y_k} on Y. □

We will need Lemma 7.1.16 to prove our main results. It maybe well known to experts. Lemma 7.1.16(1)–(4) could be proved by constructing a sequence of blow-ups (c.f. [CH21, Lemma 4.3]). We give another proof here.

We remark that Lemma 7.1.16(5) will only be applied to prove Theorem 1.2.4.

Lemma 7.1.16. *Let $(X \ni x, B)$ be an lc surface germ such that $1 \geq \mathrm{mld}(X \ni x, B) \neq \mathrm{pld}(X \ni x, B)$. There exist a smooth surface Y and a projective birational morphism $f : Y \to X$ with the dual graph $\mathcal{DG}$, such that*

1. *$K_Y + B_Y = f^*(K_X + B)$ for some $\mathbb{R}$-divisor $B_Y \geq 0$ on Y,*
2. *there is only one f-exceptional divisor E_0 such that $a(E_0, X, B) = \mathrm{mld}(X \ni x, B)$,*
3. *E_0 is the only (-1)-curve of $\mathcal{DG}$, and*
4. *$\mathcal{DG}$ is a chain.*

Moreover, if $X \ni x$ is not smooth, let $\widetilde{f} : \widetilde{X} \to X \ni x$ be the minimal resolution of $X \ni x$, and let $g : Y \to \widetilde{X}$ be the morphism such that $\widetilde{f} \circ g = f$, then

(5) there exist a $\widetilde{f}$-exceptional prime divisor $\widetilde{E}$ on $\widetilde{X}$ and a closed point $\widetilde{x} \in \widetilde{E}$, such that $a(\widetilde{E}, X, B) = \mathrm{pld}(X \ni x, B)$, and $\mathrm{center}_{\widetilde{X}} E = \widetilde{x}$ for all g-exceptional divisors E.

Proof. By Lemma 7.1.15, we can find a smooth surface Y_0 and a birational morphism $h : Y_0 \to X \ni x$, such that $a(E_0', X, B) = \mathrm{mld}(X \ni x, B)$ for some h-exceptional divisor E_0', and $K_{Y_0} + B_{Y_0} = h^*(K_X + B)$ for some $B_{Y_0} \geq 0$ on Y_0.

We now construct a sequence of smooth models over X, $Y_0 \to Y_1 \to \cdots$, by contracting a curve C' satisfying the following conditions in each step.

- C' is an exceptional (-1)-curve over X, and
- there exists $C'' \neq C'$ over X, such that $a(C'', X, B) = \mathrm{mld}(X \ni x, B)$.

Since each time the Picard number of the variety will drop by one, after finitely many steps, we will reach a smooth model Y over X, such that $f : Y \to X$ and (Y, B_Y) satisfy (1), where B_Y is the strict transform of

B_{Y_0} on Y. Since $\operatorname{mld}(X \ni x, B) \neq \operatorname{pld}(X \ni x, B)$, by the construction of Y, there exists a curve E_0 on Y satisfying (2)–(3).

For (4), by [KM98, Theorem 4.7], the dual graph of the minimal resolution $\widetilde{f} : \widetilde{X} \to X \ni x$ is a tree whose vertices are smooth rational curves. Since Y is smooth, f factors through $\widetilde{f}$. By Lemma 7.1.8, the dual graph $\mathcal{DG}$ of f is a tree whose vertices are smooth rational curves. It suffices to show that there is no fork in $\mathcal{DG}$. By Lemma 7.1.10(2), E_0 is not a fork. Suppose that $\mathcal{DG}$ contains a fork $E' \neq E_0$, by (3) and (5) of Lemma 7.1.9, we have $a(E', X, B) \leq a(E_0, X, B)$, this contradicts (2). Thus $\mathcal{DG}$ is a chain.

For (5), since there exists only one f-exceptional (-1)-curve, there is at most one closed point $\widetilde{x} \in \widetilde{X}$, such that $\operatorname{center}_{\widetilde{X}} E = \widetilde{x}$ for all g-exceptional divisors E. Thus the dual graph of g, which is denoted by $\mathcal{DG}'$, is a vertex-induced connected sub-chain of $\mathcal{DG}$ by all g-exceptional divisors. Since $\operatorname{mld}(X \ni x, B) \neq \operatorname{pld}(X \ni x, B)$, we have $\mathcal{DG}' \subsetneq \mathcal{DG}$.

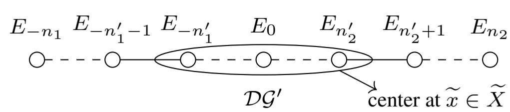


Figure 7.1: The dual graph of f

We may index the vertices of $\mathcal{DG}$ as $\{E_i\}_{-n_1 \leq i \leq n_2}$ for $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, such that E_i is adjacent to E_{i+1}, and $a_i := a(E_i, X, B)$ for all possible i. We may assume that the set of vertices of $\mathcal{DG}'$ is $\{E_j\}_{-n_1' \leq j \leq n_2'}$, where $0 \leq n_1' \leq n_1$ and $0 \leq n_2' \leq n_2$ (see Figure 7.1). If $n_1 > n_1'$, then by Lemma 7.1.9(2), $a_k - a_{-n_1'-1} \geq \min\{0, a_{-1} - a_0\} \geq 0$ for all $-n_1 \leq k < -n_1'$. If $n_2 > n_2'$, then again by Lemma 7.1.9(2), $a_{k'} - a_{n_2'+1} \geq \min\{0, a_1 - a_0\} \geq 0$ for all $n_2' < k' \leq n_2$. Set $a_{-n_1-1} = 1, E_{-n_1-1} = E_{-n_1}$ if $n_1 = n_1'$, and set $a_{n_2+1} = 1, E_{n_2+1} = E_{n_2}$ if $n_2 = n_2'$. Then $\min\{a_{n_2+1}, a_{-n_1-1}\} = \operatorname{pld}(X \ni x, B)$, and $\widetilde{x} = g(E_{-n_1'-1}) \cap g(E_{n_2+1}) \in \widetilde{E}$, where $a(\widetilde{E}, X, B) = \operatorname{pld}(X \ni x, B)$. $\square$

The following lemma gives an upper bound for number of vertices of certain kind of $\mathcal{DG}$ constructed in

Lemma 7.1.16, with the additional assumption that $\mathrm{mld}(X \ni x, B)$ is bounded from below by a positive real number.

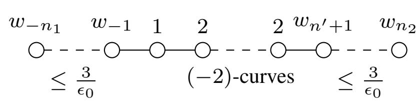
Lemma 7.1.17. *Let $\epsilon_0 \in (0,1]$ be a real number. Then $N_0' := \lfloor \frac{8}{\epsilon_0} \rfloor$ satisfies the following properties.*

Let $(X \ni x, B := \sum b_i B_i)$ be an lc surface germ such that $\mathrm{mld}(X \ni x, B) \geq \epsilon_0$, where B_i are distinct prime divisors. Let Y be a smooth surface, and $f : Y \to X \ni x$ a birational morphism with the dual graph $\mathcal{DG}$, such that

- *$K_Y + B_Y = f^*(K_X + B)$ for some $\mathbb{R}$-divisor $B_Y \geq 0$ on Y,*
- *$\mathcal{DG}$ is a chain with only one (-1)-curve E_0,*
- *$a(E_0, X, B) = \mathrm{mld}(X \ni x, B)$, and*
- *E_0 is adjacent to two vertices of $\mathcal{DG}$.*

Then the number of vertices of $\mathcal{DG}$ is bounded from above by N_0'.

Proof. Let $\{E_i\}_{-n_1 \leq i \leq n_2}$ be the vertices of $\mathcal{DG}$, such that E_i is adjacent to E_{i+1} for $-n_1 \leq i \leq n_2 - 1$, and $w_i := -(E_i \cdot E_i), a_i := a(E_i, X, B)$ for all i. We may assume that E_0 is adjacent to two vertices E_{-1}, E_1 of $\mathcal{DG}$.



By Lemma 7.1.10(1), we may assume that $w_{-1} > 2$. By (2) and (4) of Lemma 7.1.9, $a_{i-1} - a_i \geq \frac{\epsilon_0}{3}$ for any $-n_1 + 1 \leq i \leq -1$, and $a_{-1} \geq \frac{\epsilon_0}{3}$. Since $a_{-n_1} \leq 1$, $n_1 \leq \frac{3}{\epsilon_0}$. Similarly, $n_2 - n' \leq \frac{3}{\epsilon_0}$, where n' is the largest non-negative integer such that $w_i = 2$ for any $1 \leq i \leq n'$. By Lemma 7.1.9(1), $w_{-1} \leq \frac{2}{\epsilon_0}$, and by Lemma 7.1.10(3), $n' < \frac{2}{\epsilon_0} - 1$. Hence $n_1 + n_2 + 1$, the number of vertices of $\mathcal{DG}$, is bounded from above by $\frac{8}{\epsilon_0}$. $\square$

7.2 Proof of Theorem 1.2.4

7.2.1 Smooth case

The goal in this subsection is to prove a modified version of Theorem 1.2.4 with an additional assumption that the germ $x \in X$ is smooth while the coefficient set is larger.

Theorem 7.2.1. *Let $\gamma \in (0, 1]$ be a real number. Then $N_0 := \lfloor 1 + \frac{32}{\gamma^2} + \frac{1}{\gamma} \rfloor$ satisfies the following.*

Let $(X \ni x, B := \sum_i b_i B_i)$ be an lc surface germ, where $X \ni x$ is smooth, and B_i are distinct prime divisors. Suppose that $\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\gamma, +\infty)$. Then there exists a prime divisor E over X such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$, and $a(E, X, 0) \leq 2^{N_0}$.

Lemma 7.2.2 is crucial in the proof of Theorem 7.2.1. Before providing the proof, we introduce some notations first.

Notation $(\star)$. Let $X \ni x$ be a smooth surface germ, and let $g : X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 := X$ be a sequence of blow-ups with the data $(f_i, F_i, x_i \in X_i)$. Let $\mathcal{DG}$ be the dual graph of g, and assume that $\mathcal{DG}$ is a chain.

Let $n_3 \geq 2$ be the largest integer, such that $x_i \in F_i \setminus F_{i-1}$ for any $1 \leq i \leq n_3 - 1$, where we set $F_0 := \emptyset$. Let $\{E_j\}_{-n_1 \leq j \leq n_2}$ be the vertices of $\mathcal{DG}$, such that $E_0 := F_n$ is the only g-exceptional (-1)-curve on X_n, $E_{n_2} := F_1$, and E_i is adjacent to E_{i+1} for any $-n_1 \leq i \leq n_2 - 1$(see Figure 7.2).

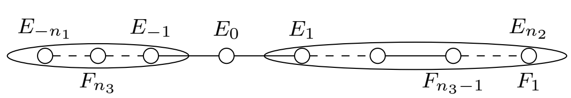


Figure 7.2: The dual graph of g

We define $n_i(g) := n_i$ for $1 \leq i \leq 3$, $n(g) = n$, $w_j(g) := -E_j \cdot E_j$ for all j, and $W_1(g) := \sum_{j<0} w_j(g)$ and $W_2(g) := \sum_{j>0} w_j(g)$.

Lemma 7.2.2. *With* **Notation** $(\star)$. *Then*

$$(W_1(g) - n_1(g)) + n_3(g) - 1 = W_2(g) - n_2(g). \tag{7.2.1}$$

In particular, $n(g) = n_1(g) + n_2(g) + 1 \leq n_3(g) + \min\{W_1(g), W_2(g)\}$.

Proof. For simplicity, let $n := n(g)$, $n_i := n_i(g)$ for $1 \leq i \leq 3$, $w_j := w_j(g) = -E_j \cdot E_j$ for all j, and $W_j := W_j(g)$ for $j = 1, 2$.

We prove (7.2.1) by induction on the non-negative integer $n - n_3$.

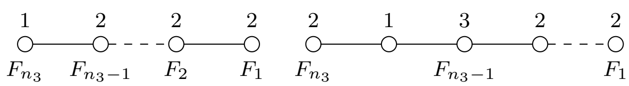


Figure 7.3: The dual graph for the case $n = n_3$ and $n = n_3 + 1$.

If $n = n_3$, then $n_1 = W_1 = 0$, $n_2 = n_3 - 1$, and $W_2 = 2n_3 - 2$, thus (7.2.1) holds (see Figure 7.3). If $n = n_3 + 1$, then $x_{n_3} \in F_{n_3} \cap F_{n_3-1}$. In this case, $n_1 = 1$, $W_1 = 2$, $n_2 = n_3 - 1$, and $W_2 = 2n_3 - 1$, thus (7.2.1) holds (see Figure 7.3).

In general, suppose (7.2.1) holds for any sequence of blow-ups g as in **Notation** $(\star)$ with positive integers n, n_3 satisfying $1 \leq n - n_3 \leq k$. For the case when $n - n_3 = k + 1$, we may contract the (-1)-curve on X_n, and consider $g' : X_{n-1} \to \cdots \to X_0 := X$, a subsequence of blow-ups of g with the data $(f_i, F_i, x_i \in X_i)$ for $0 \leq i \leq n - 1$. Denote $n'_i := n_i(g')$ for any $1 \leq i \leq 3$, and $W'_j := W_j(g')$ for any $1 \leq j \leq 2$. By Lemma 7.1.10(1), either $w_{-1} = 2$ or w_1=2. In the former case, $W'_1 = W_1 - 2, W'_2 = W_2 - 1, n'_1 = n_1 - 1$, $n'_2 = n_2$, and $n'_3 = n_3$. In the latter case, $W'_1 = W_1 - 1, W'_2 = W_2 - 2, n'_1 = n_1, n'_2 = n_2 - 1$, and $n'_3 = n_3$. In both cases, by induction,

$$W'_2 - n'_2 - (W'_1 - n'_1) = (W_2 - n_2) - (W_1 - n_1) = n_3 - 1.$$

Hence we finish the induction, and (7.2.1) is proved.

Since $w_j \geq 2$ for $j \neq 0$, we have $W_1 = \sum_{-n_1 \leq j \leq -1} w_j \geq 2n_1$ and $W_2 = \sum_{1 \leq j \leq n_2} w_j \geq 2n_2$. By (7.2.1),

$$n_1 + n_2 + 1 \leq n_1 + W_2 - n_2 + 1 = W_1 + n_3,$$

and

$$n_1 + n_2 + 1 \leq W_1 - n_1 + n_2 + n_3 - 1 = W_2,$$

which imply that $n = n_1 + n_2 + 1 \leq n_3 + \min\{W_1, W_2\}$. $\square$

We will need Lemma 7.2.3 to prove Theorems 7.2.1.

Lemma 7.2.3. *Let $\gamma \in (0, 1]$ be a real number. Let $N_0 := \lfloor 1 + \frac{32}{\gamma^2} + \frac{1}{\gamma} \rfloor$, then we have the following.*

Let $(X \ni x, B := \sum_i b_i B_i)$ be an lc surface germ, such that $X \ni x$ is smooth, and B_i are distinct prime divisors. Suppose that $\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\gamma, +\infty)$. Let Y be a smooth surface and $f : Y \to X \ni x$ be a birational morphism with the dual graph $\mathcal{DG}$, such that

- *$K_Y + B_Y = f^*(K_X + B)$ for some $B_Y \geq 0$ on Y,*
- *$\mathcal{DG}$ is a chain that contains only one (-1)-curve E_0,*
- *E_0 is adjacent to two vertices of $\mathcal{DG}$, and*
- *either E_0 is the only vertex of $\mathcal{DG}$ such that $a(E_0, X, B) = \mathrm{mld}(X \ni x, B)$, or $a(E_0, X, B) = \mathrm{mld}(X \ni x, B) > 0$ and $\sum_i n_i b_i \neq 1$ for all $n_i \in \mathbb{Z}_{\geq 0}$.*

Then the number of vertices of $\mathcal{DG}$ is bounded from above by N_0.

Proof. By Lemma 7.1.4, $b_i \geq \gamma$ for all i.

If $\mathrm{mld}(X \ni x, B) \geq \frac{\gamma}{2}$, then by Lemma 7.1.17 with $\epsilon_0 = \frac{\gamma}{2}$, the number of vertices of $\mathcal{DG}$ is bounded from above by $\frac{16}{\gamma}$.

Thus we may assume that $0 \leq \mathrm{mld}(X \ni x, B) \leq \frac{\gamma}{2}$. We may index the vertices of $\mathcal{DG}$ as $\{E_j\}_{-n_1 \leq j \leq n_2}$ for some positive integer n_1, n_2, where E_j is adjacent to E_{j+1} for $-n_1 \leq j \leq n_2 - 1$. Let $w_j := -E_j \cdot E_j$ and $a_j := a(E_j, X \ni x, B)$ for all j.

For all $-n_1 \leq k \leq n_2$, we have

$$(K_Y + f_*^{-1}B + \sum_j (1 - a_j)E_j) \cdot E_k = f^*(K_X + B) \cdot E_k = 0. \tag{7.2.2}$$

Let $k = 0$, (7.2.2) becomes $0 = -2 + f_*^{-1}B \cdot E_0 + (1 - a_{-1}) + (1 - a_1) + w_0 a_0$, thus

$$(a_1 - a_0) + (a_{-1} - a_0) = f_*^{-1}B \cdot E_0 - a_0.$$

By the last assumption in the lemma, either $(a_{-1} - a_0) + (a_1 - a_0) > 0$ or $a_0 > 0$, thus $f_*^{-1}B \cdot E_0 > 0$ in both cases. Hence $f_*^{-1}B \cdot E_0 - a_0 \geq \gamma - \frac{\gamma}{2} = \frac{\gamma}{2}$. Possibly switching E_j $(j < 0)$ with E_j $(j > 0)$, we may assume that $a_{-1} - a_0 \geq \frac{\gamma}{4}$.

By Lemma 7.1.9(2), $a_{-j} - a_{-j+1} \geq a_{-1} - a_0 \geq \frac{\gamma}{4}$ for $1 \leq j \leq n_1$, thus $n_1 \cdot \frac{\gamma}{4} \leq a_{-n_1} \leq 1$, and $n_1 \leq \frac{4}{\gamma}$. Since $a_j \geq \frac{\gamma}{4}$ for all $-n_1 \leq j \leq -1$, by Lemma 7.1.9(1), $w_j \leq \frac{8}{\gamma}$ for all $-n_1 \leq j \leq -1$. Thus $\sum_{j=-1}^{-n_1} w_j \leq n_1 \cdot \frac{8}{\gamma} \leq \frac{32}{\gamma^2}$. Note that $X \ni x$ is smooth and $\mathcal{DG}$ has only one (-1)-curve, thus $f : Y \to X$ is a sequence of blow-ups as in Definition 7.1.12. Moreover, $\mathcal{DG}$ is a chain, thus by Lemma 7.2.2, $1 + n_1 + n_2 \leq n_3 + \frac{32}{\gamma^2}$, where $n_3 = n_3(f)$ is defined as in **Notation** $(\star)$.

It suffices to show that n_3 is bounded, we may assume that $n_3 > 2$. By the definition of n_3, there exists a sequence of blow-ups $X_{n_3} \to \ldots X_1 \to X_0 := X$ with the data $(f_i, F_i, x_i \in X_i)$, such that $x_i \in F_i \setminus F_{i-1}$ for any $1 \leq i \leq n_3 - 1$. Here $F_0 := \emptyset$.

Let B_{X_i} be the strict transform of B on X_i for $0 \leq i \leq n_3$, and let $a'_i := a(F_i, X, B)$ for $1 \leq i \leq n_3$, and $a'_0 := 1$. Since $x_i \in F_i \setminus F_{i-1}$, $a'_i - a'_{i+1} = \mathrm{mult}_{x_i} B_{X_i} - 1$ for any $n_3 - 1 \geq i \geq 0$. By Lemma 7.1.9(2), $a'_i - a'_{i+1} \geq \min\{a_1 - a_0, a_{-1} - a_0\} \geq 0$ for $1 \leq i \leq n_3 - 2$ (see Figure 7.2). Thus by the last assumption in the lemma, either $\min\{a_1 - a_0, a_{-1} - a_0\} > 0$, or $\mathrm{mult}_{x_i} B_{X_i} - 1 > 0$, in both cases we have $a'_i - a'_{i+1} = \mathrm{mult}_{x_i} B_{X_i} - 1 > 0$. Hence $a'_i - a'_{i+1} = \mathrm{mult}_{x_i} B_{X_i} - 1 \geq \gamma$ for any $1 \leq i \leq n_3 - 2$ as $\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\gamma, +\infty)$. Therefore,

$$0 \leq a'_{n_3-1} = a'_0 + \sum_{i=0}^{n_3-2} (a'_{i+1} - a'_i) \leq 1 - (n_3 - 1)\gamma,$$

and $n_3 \leq 1 + \frac{1}{\gamma}$.

To sum up, the number of vertices of $\mathcal{DG}$ is bounded from above by $\lfloor 1 + \frac{32}{\gamma^2} + \frac{1}{\gamma} \rfloor$. $\square$

Now we are ready to prove Theorem 7.2.1.

Proof of Theorem 7.2.1. By Lemma 7.1.13, we may assume that $\mathrm{mld}(X \ni x, B) \leq 1$.

Let $f : Y \to X \ni x$ be the birational morphism constructed in Lemma 7.1.16 with the dual graph $\mathcal{DG}$. We claim that the number of vertices of $\mathcal{DG}$ is bounded from above by $N_0 := \lfloor 1 + \frac{32}{\gamma^2} + \frac{1}{\gamma} \rfloor$.

Assume the claim holds, then by Lemma 7.1.14, $a(E, X, 0) \leq 2^{N_0}$ for some exceptional divisor E such that $a(E, X, B) = \mathrm{mld}(X \ni X, B)$, we are done. It suffices to show the claim.

If the f-exceptional (-1)-curve is adjacent to only one vertex of $\mathcal{DG}$, then by Lemma 7.1.11(1), the number of vertices of $\mathcal{DG}$ is bounded from above by $1 + \frac{1}{\gamma}$.

If the f-exceptional (-1)-curve is adjacent to two vertices of $\mathcal{DG}$, then by Lemma 7.2.3, the number of vertices of $\mathcal{DG}$ is bounded from above by $\lfloor 1 + \frac{32}{\gamma^2} + \frac{1}{\gamma} \rfloor$. Thus we finish the proof. $\square$

7.2.2 General case

The following result is known as the ACC for PLDs (for surfaces), and it plays an important role in the proof of Theorem 1.2.4.

Theorem 7.2.4 ([Ale93, Theorem 3.2],[HL20, Theorem 2.2])**.** *Let $\Gamma\subseteq[0,1]$ be a set which satisfies the DCC. Then*

$$\mathrm{Pld}(2,\Gamma):=\{\mathrm{pld}(X\ni x,B)\mid (X\ni x,B) \textit{ is lc}, \dim X=2, B\in\Gamma\},$$

satisfies the ACC.

Proof of Theorem 1.2.4. We may assume that $\Gamma\setminus\{0\}\neq\emptyset$.

Let $(X\ni x,B)$ be an lc surface germ with $B\in\Gamma$. By Lemma 7.1.13, we may assume that $\mathrm{mld}(X\ni x,B)\leq 1$. By Theorem 7.2.1, it suffices to show the case when $X\ni x$ is not smooth.

If $\mathrm{mld}(X\ni x,B)=\mathrm{pld}(X\ni x,B)$, then $a(E,X,0)\leq 1$ for some prime divisor E over $X\ni x$ such that $a(E,X,B)=\mathrm{mld}(X\ni x,B)$. So we may assume that $\mathrm{mld}(X\ni x,B)\neq\mathrm{pld}(X\ni x,B)$.

By Lemma 7.1.16, there exists a birational morphism $f:Y\to X\ni x$ which satisfies Lemma 7.1.16(1)–(5). Let $\widetilde{f}:\widetilde{X}\to X$ be the minimal resolution of $X\ni x$, $g:Y\to\widetilde{X}\ni\widetilde{x}$ the birational morphism such that $\widetilde{f}\circ g=f$, where $\widetilde{x}\in\widetilde{X}$ is chosen as in Lemma 7.1.16(5), and there exists a $\widetilde{f}$-exceptional prime divisor $\widetilde{E}$ over $X\ni x$ such that $a(\widetilde{E},X,B)=\mathrm{pld}(X\ni x,B)$ and $\widetilde{x}\in\widetilde{E}$. Moreover, there is at most one other vertex $\widetilde{E'}$ of $\widetilde{\mathcal{DG}}$ such that $\widetilde{x}\in\widetilde{E'}$.

Let $\widetilde{\mathcal{DG}}$ be the dual graph of $\widetilde{f}$, and $\{F_i\}_{-n_1\leq i\leq n_2}$ the vertices of $\widetilde{\mathcal{DG}}$, such that $n_1,n_2\in\mathbb{Z}_{\geq 0}$, F_i is adjacent to F_{i+1}, $w_i:=-F_i\cdot F_i, a_i:=a(F_i,X,B)$ for all i, and $F_0:=\widetilde{E},F_1:=\widetilde{E'}$ (see Figure 7.4). We may write $K_{\widetilde{X}}+B_{\widetilde{X}}=\widetilde{f}^*(K_X+B)$, where $B_{\widetilde{X}}:=\widetilde{f}^{-1}_*B+\sum_i(1-a_i)F_i$, and we define $\widetilde{B}:=\widetilde{f}^{-1}_*B+\sum_{\widetilde{x}\in F_i}(1-a_i)F_i$.

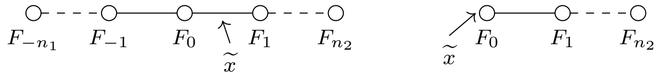


Figure 7.4: Cases when $\widetilde{x} \in F_0 \cap F_1$ and when $\widetilde{x} \notin F_i$ for $i \neq 0$.

If $\widetilde{x} \notin F_i$ for all $i \neq 0$, then we consider the surface germ $(\widetilde{X} \ni \widetilde{x}, \widetilde{B} = \widetilde{f}_*^{-1}B + (1-a_0)F_0)$, where $\widetilde{B} \in \Gamma' := \Gamma \cup \{1 - a \mid a \in \mathrm{Pld}(2, \Gamma)\}$. By [HL20, Theorem 2.9], Γ' satisfies the DCC. Thus by Theorem 7.2.1, we may find a positive integer N_1 which only depends on Γ, and a prime divisor E over $\widetilde{X} \ni \widetilde{x}$, such that $a(E, \widetilde{X}, \widetilde{B}) = a(E, X, B) = \mathrm{mld}(X \ni x, B)$, and $a(E, X, 0) \leq a(E, \widetilde{X}, 0) \leq N_1$.

So we may assume that $\widetilde{x} = F_0 \cap F_1$. By Lemma 7.1.3, there exist positive real numbers $\epsilon, \delta \leq 1$ depending only on Γ, such that $\{\sum_i n_i b_i - 1 > 0 \mid b_i \in \Gamma'_\epsilon \cap [0,1], n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\delta, +\infty)$. Recall that $\Gamma'_\epsilon = \cup_{b' \in \Gamma'}[b' - \epsilon, b']$.

If $a_1 - a_0 \leq \epsilon$, then we consider the surface germ $(\widetilde{X} \ni \widetilde{x}, \widetilde{B} = \widetilde{f}_*^{-1}B + (1-a_0)F_0 + (1-a_1)F_1)$, where $\widetilde{B} \in \Gamma'_\epsilon \cap [0,1]$. By Theorem 7.2.1, there exist a positive integer N_2 which only depends on Γ, and a prime divisor E over $\widetilde{X} \ni \widetilde{x}$, such that $a(E, \widetilde{X}, \widetilde{B}) = a(E, X, B) = \mathrm{mld}(X \ni x, B)$ and $a(E, X, 0) \leq a(E, \widetilde{X}, 0) \leq N_2$.

If $a_1 - a_0 \geq \epsilon$, then we claim that there exists a DCC set Γ'' depending only on Γ, such that $1 - a_1 \in \Gamma''$.

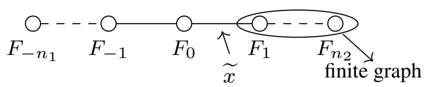


Figure 7.5: Cases when $a_1 - a_0 \geq \epsilon$.

Assume the claim holds, then we consider the surface germ $(\widetilde{X} \ni \widetilde{x}, \widetilde{B} = \widetilde{f}_*^{-1}B + (1-a_0)F_0 + (1-a_1)F_1)$, where $\widetilde{B} \in \Gamma'' \cup \Gamma'$. By Theorem 7.2.1, we may find a positive integer N_3 which only depends on Γ, and a prime divisor E over $\widetilde{X} \ni \widetilde{x}$, such that $a(E, \widetilde{X}, \widetilde{B}) = a(E, X, B) = \mathrm{mld}(X \ni x, B)$ and $a(E, X, 0) \leq a(E, \widetilde{X}, 0) \leq N_3$. Let $N := \max\{N_1, N_2, N_3\}$, and we are done.

It suffices to show the claim. By Lemma 7.1.9(1), $w_i \leq \frac{2}{\epsilon}$ for any $0 < i \leq n_2$. Since $1 \geq a_{n_2} = a_0 + \sum_{i=0}^{n_2-1}(a_{i+1} - a_i) \geq n_2\epsilon$, $n_2 \leq \frac{1}{\epsilon}$. We may write

$$K_{\widetilde{X}} + \widetilde{f}_*^{-1}B + \sum_{-n_1 \leq i \leq n_2} (1 - a_i)F_i = \widetilde{f}^*(K_X + B),$$

For each $1 \leq j \leq n_2$, we have

$$(K_{\widetilde{X}} + \widetilde{f}_*^{-1}B + \sum_{-n_1 \leq i \leq n_2} (1 - a_i)F_i) \cdot F_j = 0,$$

which implies $\sum_{-n_1 \leq i \leq n_2}(a_i - 1)F_i \cdot F_j = -F_j{}^2 - 2 + \widetilde{f}_*^{-1}B \cdot F_j$, or equivalently,

$$\begin{pmatrix} F_1 \cdot F_1 & \cdots & F_{n_2} \cdot F_1 \\ \vdots & \ddots & \vdots \\ F_1 \cdot F_{n_2} & \cdots & F_{n_2} \cdot F_{n_2} \end{pmatrix} \begin{pmatrix} a_1 - 1 \\ \vdots \\ a_{n_2} - 1 \end{pmatrix} = \begin{pmatrix} w_1 - 2 + \widetilde{f}_*^{-1}B \cdot F_1 + (1 - a_0) \\ \vdots \\ w_{n_2} - 2 + \widetilde{f}_*^{-1}B \cdot F_{n_2} \end{pmatrix}.$$

By assumption, $w_j - 2 + \widetilde{f}_*^{-1}B \cdot F_j$ belongs to a DCC set, and by Lemma 7.2.4, $1 - a_0$ belongs to the DCC set $\{1 - a \mid a \in \mathrm{Pld}(2, \Gamma)\}$.

By [KM98, Lemma 3.40], $(F_i \cdot F_j)_{1 \leq i,j \leq n_2}$ is a negative definite matrix. Let $(s_{ij})_{n_2 \times n_2}$ be the inverse matrix of $(F_i \cdot F_j)_{1 \leq i,j \leq n_2}$. By [KM98, Lemma 3.41], $s_{ij} < 0$ for any $1 \leq i, j \leq n_2$, thus

$$1 - a_1 = -s_{11}(w_1 - 2 + \widetilde{f}_*^{-1}B \cdot F_1 + (1 - a_0)) - \sum_{j=2}^{n_2} s_{1j}(w_j - 2 + \widetilde{f}_*^{-1}B \cdot F_j)$$

belongs to a DCC set. $\square$

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