GENERALIZATION OF THE EINSTEIN CONDITION FOR PSEUDO-RIEMANNIAN MANIFOLDS

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1 Introduction

1.1 Motivation and state of the art

In this thesis we treat $(\lambda, n+m)$-Einstein manifolds, $\lambda \in \mathbb{R}$, $m, n \in \mathbb{N}$. These are geodesically complete pseudo-Riemannian manifolds (M, g) possibly with boundary together with a smooth function f on M satisfying

$$\text{Hess} f = \frac{f}{m}(\text{Ric} - \lambda g) \tag{1}$$

where $f > 0$ on $\text{int}(M)$ and $f = 0$ on ∂M. If $m = 1$ we further assume that $\Delta f = -\lambda f$. A simple example of a $(\lambda, n+m)$-Einstein metric is when f is constant, which is then called a trivial space. As a classical example let $g = \epsilon dt^2 + u^2(t) g_N$, $u(t) = \cosh(t)$, on $M = [0, \infty) \times N$ where N is a geodesically complete pseudo-Riemannian Einstein metric with normalized scalar curvature $k = -\epsilon$ and further let $\epsilon\lambda = 1 - n - m$, $f(t) = \sinh(t)$. Then we obtain a non-trivial $(\lambda, n+m)$-Einstein manifold with the boundary the slice $\{t_0 = 0\} \times N$. We recall the Riemannian Schwarzschild metric on $E = \mathbb{R}^2 \times \mathbb{S}^2$, which can be considered as a Riemannian $(\lambda, 2+2)$-Einstein metric. In this particular example f is the unique positive function on $[0, \infty)$ satisfying the conditions

$$(f')^2 = 1 - f^{1-m} \;\; , f(0) = 1 \text{ and } f' \geq 0.$$

By [Bes08, Example 9.118(a)] the triple $(\mathbb{R}^2, g, f)$ with

$$g = dt^2 + (f'(t))^2 d\theta^2$$
$$f = f(t)$$

is a $(0, 2+m)$-Einstein metric which is not Einstein and its corresponding Einstein warped product (see Proposition 4.2) is complete and Ricci flat.

Our motivation to study $(\lambda, n+m)$-Einstein manifolds is due to [KK03, Proposition 5] which implies that corresponding to every Riemannian $(\lambda, n+m)$-Einstein manifold (M, g, f) there exists an $n+m$ dimensional warped product Einstein metric with the basis (M, g) and the warping function f. A $(\lambda, n+m)$-Einstein manifold (M, g, f) itself is not necessarily Einstein. More precisely,

Proposition 1.1. (*[HPW12], Proposition 1.1.*). *Suppose $\lambda \in \mathbb{R}$, $m > 1$ is an integer, (M, g) is a (geodesically) complete Riemannian manifold of dimension n and $f \in C^\infty(M)$ is non-negative. Then (M, g, f) is a $(\lambda, n+m)$-Einstein manifold if and only if there is a smooth $n+m$ dimensional Riemannian warped product Einstein metric g_E on $E = M^n \times F^m$ with Einstein constant λ of the form*

$$g_E = g + f^2 g_{F^m}$$

for a Riemannian Einstein manifold (F^m, g_F) *of dimension* m *satisfying* $Ric_F = \mu g_F$ *where* μ *satisfies* $\mu = f\Delta f + (m-1)|\nabla f|^2 + \lambda f^2$.

In the pseudo-Riemannian setting, we generalize the above statement to the case $m = 1$, see Corollary 4.31.

If we define $u \in C^\infty(M)$ via $e^{-\frac{u}{m}} = f$ on the interior of a $(\lambda, n+m)$-Einstein manifold (M, g, f), Equation (1) takes the form

$$Ric_u^m = Ric + Hessu - \frac{du \otimes du}{m} = \lambda g, \tag{2}$$

Ric_u^m is sometimes called the m-Bakry-Emery tensor. The so-called m-quasi-Einstein manifolds were introduced by J. Case, Y-J. Shu and G. Wei in [CSW11] as a triple (M, g, u) consisting of a Riemannian manifold (M, g) and a smooth function u as above which satisfies (2) for $0 < m \leq \infty$ and $\lambda \in \mathbb{R}$. By the results of [KK03], [CSW11, Theorem 2.2] characterizes m-quasi Einstein manifolds (M, g, u) as the base manifold of an Einstein warped product for which u is the warping function. Additionally they proved a rigidity property for scalar curvature through [CSW11, Proposition 3.6], by giving lower and upper bounds with respect to the sign of λ.

A bit later, Catino in [Cat12] considered the following extended form of the equation (2)

$$\mathrm{Ric} + \mathrm{Hess}u - \varsigma du \otimes du = \lambda g \tag{3}$$

for the so-called generalized quasi-Einstein manifolds. Here u, ς, λ are three smooth functions on a complete Riemannian manifold (M^n, g), $n \geq 3$. Equation (3) gives out Einstein condition when u and λ are constant, and a quasi-Einstein manifold when ς and λ are constant. Catino proves the following.

Proposition 1.2. (*[Cat12], Theorem 1.1.*). *Let* (M^n, g), $n \geq 3$, *be a generalized quasi-Einstein manifold with harmonic Weyl tensor and* $W(\nabla u, ., ., .) = 0$. *Then around any regular point of* u *the manifold* (M^n, g) *is locally a warped product with* $n-1$ *dimensional Einstein fibres.*

G. Catino, C. Mantegazza, L. Mazzieri and M. Rimoldi in [CMMR13] also consider Equation (3) where $\varsigma, \lambda \in \mathbb{R}$. They prove the following result for an arbitrary $\varsigma \in \mathbb{R}$.

Proposition 1.3. (*[CMMR13], Theorem 1.1.*). *Let* (M^n, g), $n \geq 3$, *be a complete locally conformally flat quasi Einstein manifold. Then the following hold:*

(i) If $\varsigma = \frac{1}{2-n}$, *then* (M^n, g) *is globally conformally equivalent to a space form.*

(ii) If $\varsigma \neq \frac{1}{2-n}$, *then around any regular point of* u, *the manifold* (M^n, g) *is locally a warped product with* $n-1$ *dimensional fibres of constant sectional curvature.*

In a series of papers [HPW12], [HPW14], [HPW15] C. He, P. Petersen and W. Wylie investigate warped product Einstein spaces using the concept of $(\lambda, m+n)$-Einstein metrics in the Riemannian case. These metrics can also be seen as generalizations of gradient Ricci solitons, which are invariant under the Ricci flow. Our work is mostly based on [HPW12] where the authors extend the works of [CSW11], [Cat12] and that of [KK03] on $(\lambda, n+m)$-Einstein manifolds with boundary. In [HPW12] setting new quantities $\rho := \frac{1}{m-1}((n-1)\lambda - Scal)$ and $P = Ric - \rho g$ the authors prove that given a $(\lambda, n+m)$-Einstein manifold (M, g, f) with suitable conditions on the Weyl tensor, at a point p where $\nabla f|_p \neq 0$, the tensor P (Schouten tensor and Hess f) has at most two eigenvalues, more precisely

Lemma 1.4. (*[HPW12], Lemma* 7.1). *Let (M, g, f) be a Riemannian $(\lambda, n+m)$-Einstein manifold with harmonic Weyl tensor and $W(\nabla f, Y, Z, \nabla f) = 0$. Then at a point p where $\nabla f \neq 0$, the tensor P (or Ricci tensor or Schouten tensor) has at most two eigenvalues. If it has two eigenvalues then one has multiplicity* 1 *with eigenvector ∇f, and the other one has multiplicity $n-1$ with vectors orthogonal to ∇f. If it has one eigenvalue then (M, g) is Einstein.*

In fact, in Lemma 1.4 the purpose of the conditions on the Weyl tensor is to get control on the number of eigenvalues of the tensor P (equivalently the Schouten tensor). Then the authors define $O = \{x \in M : df(x) \neq 0, \sigma_1(x) \neq \sigma_2(x)\}$ where σ_1 and σ_2 are the eigenvalues of the Schouten tensor. Using Lemma 1.4 they decompose the metric g in a neighborhood of a point $p \in O$ into a warped product $g = dt^2 + u^2(t)g_N$, and they further show that $f = f(t)$, see the following.

Theorem 1.5. (*[HPW12], Theorem* 7.1). *Suppose $m > 1$ and (M, g, f) is a $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ in an open set containing $p \in O$. Then*

$$g = dt^2 + u^2(t)g_N$$

$$f = f(t)$$

in a neighborhood of p, where g_N is an Einstein metric. Moreover if the metric is locally conformally flat around p, then N is a space of constant sectional curvature.

Then the authors globally characterize $(\lambda, n+m)$-Einstein metrics in the Riemannian case as their main result:

Theorem 1.6. (*[HPW12], Theorem* 7.2). *Let $m > 1$ and suppose that (M, g) is a complete, simply connected Riemannian manifold and has harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$, then (M, g, f) is a non-trivial $(\lambda, n+m)$-Einstein metric if and only if it is of the form*

$$g = dt^2 + u^2(t)g_N$$

$$f = f(t)$$

where g_N is an Einstein metric. Moreover, if $\lambda \geq 0$ then (N, g_N) has non-negative Ricci curvature, and if it is Ricci flat, then u is a constant, i.e, (M, g) is a Riemannian product.

There is a little observation in connecting Theorem 1.5 with Theorem 1.6 which is missing in [HPW12]. In fact, Theorem 1.5 is a classification around points $p \in O$, while making a global characterization it may happen that in a neighborhood of a regular point there are some points at which the Schouten tensor has only one eigenvalue, cf. Lemma 1.4. We go through this observation in the proof of Theorem 1.8.

1.2 Statement of results in the Riemannian case

We start our main works in this thesis by presenting the following statement which is essential to characterize $(\lambda, n+m)$-Einstein manifolds.

Proposition 1.7. *Let $\lambda \in \mathbb{R}$, $m \geq 1$, $n > 1$ integers and g_N a pseudo-Riemannian Einstein metric say with normalized scalar curvature $\varrho_N = k \in \mathbb{R}$, i.e. $\mathrm{Ric}_N = k(n-2)g_N$, on an $(n-1)$-dimensional manifold N and $g = \epsilon dt^2 + u^2(t)g_N$ a warped product metric on $M = I \times N$ with an interval $I \subset \mathbb{R}$. In addition suppose $f = f(t)$ is a smooth non-negative function on I. Then $(M = I \times N, g, f)$ satisfies Equation (1) of a $(\lambda, n+m)$-Einstein manifold if and only if the following conditions hold*

1. *On $int(M)$*

$$f'm\frac{u'}{u} + \{\epsilon\lambda - \frac{\epsilon(n-2)k - (n-2)u'^2 - uu''}{u^2}\}f = 0 \tag{4}$$

$$\begin{aligned} &\lambda^2u^4 - 2(n-2)k\lambda u^2 + (m+2(n-2))\lambda u^2u'^2 + (2+m)\lambda u^3u'' \\ &+(n-2)^2k^2 - (2(n-2)+m)(n-2)ku'^2 - (2+m)(n-2)kuu'' \\ &+(n-2)(m+n-2)u'^4 + (2(n-2)+m)uu'^2u'' + (1+m)u^2u''^2 \\ &-mu^2u'u''' = 0 \end{aligned} \tag{5}$$

2. *On ∂M*

$$f''(t) = u'(t) = 0. \tag{6}$$

Using Proposition 1.7 we then provide a characterization around regular points of f on a Riemannian manifold:

Theorem 1.8. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g) be a Riemannian manifold with a smooth function f defined on M. Then the following conditions are equivalent:*

1) (M, g, f) satisfies Equation (1) of a non-trivial $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ in a neighborhood of $p \in M$

with $\nabla f|_p \neq 0$.

2) (a) Case $p \in int(M)$*: There exist local coordinates* $(t, t_1, ..., t_{n-1})$ *with* $t \in (-\varepsilon, \varepsilon)$ *in a neighborhood of* $p \in M$ *and an Einstein Riemannian hypersurface* $(N, g_N = g_N(t_1, ..., t_{n-1}))$ *of* (M, g) *with normalized scalar curvature* $\varrho_N = k$ *and a function* $u = u(t) > 0$*, in addition* $f = f(t) > 0$ *satisfying* (4) *and* (5) *in Proposition 1.7 such that*

I) $g(\partial_t, \partial_t) = 1$
II) $g(\partial_t, \partial_{t_i}) = 0$, *for* $i = 1, ..., n-1$
III) $g(\partial_{t_i}, \partial_{t_j}) = u^2(t) g_N(\partial_{t_i}, \partial_{t_j})(t_1, ..., t_{n-1}) \quad i, j = 1, ..., n-1$.

(b) Case $p \in \partial M$*: There exist local coordinates* $(t, t_1, ..., t_{n-1})$ *with* $t \in [0, \varepsilon)$ *in a neighborhood of* p *and an Einstein Riemannian hypersurface* $(N, g_N = g_N(t_1, ..., t_{n-1}))$ *of* (M, g) *with normalized scalar curvature* $\varrho_N = k$ *and a function* $u = u(t) > 0$*, in addition* $f(t) > 0$ *for all* $t \in (0, \varepsilon)$ *satisfying* (4) & (5) *as well as* $f(0) = 0$ *satisfying* (6) *at* $t = 0$ *such that the conditions I, II, III in (a) hold.*

Any case of 2) implies that $g = dt^2 + u^2(t) g_N$ *around* p*. If the metric is locally conformally flat in a neighborhood of* p *then* N *must be a space of constant curvature.*

For a global characterization in the Riemannian case, in addition to Proposition 1.7 and Theorem 1.8, we need a characterization around critical points of f as well. To this end, we first show that critical points of f in a Riemannian $(\lambda, n+m)$-Einstein manifold with harmonic Weyl tensor satisfying $W(\nabla f, ., ., \nabla f) = 0$ are isolated:

Lemma 1.9. *Let* $m > 1$, $\lambda \in \mathbb{R}$ *and* (M, g) *be a connected Riemannian manifold with a smooth function* f *defined on* M*. Assume that* (M, g, f) *satisfies Equation* (1) *of a non-trivial* $(\lambda, n+m)$*-Einstein metric with harmonic Weyl tensor satisfying* $W(\nabla f, ., ., \nabla f) = 0$ *in a neighborhood of* $p \in M$ *with* $\nabla f|_p = 0$*. Then there exists a neighborhood* $\mathscr{U}$ *of* p *such that*

(i) p *is the only critical point of* f *in* $\mathscr{U}$.

(ii) The level hypersurfaces of f *in* $\mathscr{U}$ *coincide with the geodesic distance spheres around* p.

Then we characterize locally conformally flat Riemannian $(\lambda, n+m)$-Einstein manifolds around critical points of f as follows.

Theorem 1.10. *Let* $m > 1$, $\lambda \in \mathbb{R}$ *and* (M, g) *be a Riemannian manifold with a smooth function* f *defined on* M*. Then the following conditions are equivalent:*

1) (M, g, f) is conformally flat and satisfies Equation (1) *of a non-trivial $(\lambda, n+m)$-Einstein metric in a neighborhood of $p \in M$ with $\nabla f|_p = 0$.*

2) There exist polar coordinates $(t, t_1, ..., t_{n-1}) \in I \times S^{n-1}(1)$, $I \subseteq \mathbb{R}$ being an open interval, in a neighborhood of p and an odd function $u = u(t)$, i.e. $u(0) = u^{(even)}(0) = 0$, with $u(t) > 0$ on $t \in I - \{0\}$ and $0 \neq (u')^2(0) = k$, such that in these coordinates $f = f(t)$ and

$$g = dt^2 + \frac{u^2(t)}{k} g_{S^{n-1}(1)} \tag{7}$$

where $g_{S^{n-1}(1)}$ denotes the line element of the standard unit sphere $S^{n-1}(1)$; In addition, the conditions (4) *and* (5) *in Proposition 1.7 hold.*

More details on the prerequisites for the restatement of the global statement 1.6 are listed right after Theorem 6.11. For example, under the same assumptions on the Weyl tensor W as in Theorem 1.8 part 1) the number of critical points of f is at most two and the warping function must be odd on the critical points of f. We restate Theorem 1.6 as follows $\big($ [(means either [or (, similarly does)]$\big)$.

Theorem 1.11. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g) be a connected Riemannian manifold with a smooth function f on M. Then the following conditions are equivalent:*

1) (M, g, f) is a non-trivial $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$.

2) If C denotes the set of critical points of f then $N' := |C| \leq 2$, and $(M \setminus C, g)$ is isometric with a warped product metric

$$g = dt^2 + u^2(t) g_N \tag{8}$$

$$f = f(t) \tag{9}$$

on $I \times N$ where (N, g_N) is a complete Einstein Riemannian hypersurface of (M, g) with normalized scalar curvature $k = \varrho_N$ and $I = [(\alpha_0, \beta_0)] \subset \mathbb{R}$ which is unlimited in both sides, i.e. $I = (-\infty, \infty)$ if there is niether a critical point for f nor a boundary point of M. Otherwise, it is closed in the left i.e. $I = [\alpha_0, \beta_0)]$ with $\alpha_0 \in \mathbb{R}$ if there exists a point $q_0 \in \partial M$ with $f(q_0) = f(\alpha_0) = 0$ (or similarly $I = [(\alpha_0, \beta_0]$ with $\beta_0 \in \mathbb{R}$ for a boundary point q_0 with $f(q_0) = f(\beta_0) = 0$). Or, $I = (\alpha_0, \beta_0)]$ has finite α_0 with open left side (or $I = [(\alpha_0, \beta_0)$ has finite β_0 with open right side) only if it corresponds to a minimum (or maximum) point q_0 of f with $f(q_0) = f(\alpha_0)$ (or $f(q_0) = f(\beta_0)$). In addition, in the latter case where $\gamma_0 = \alpha_0$ (or $\gamma_0 = \beta_0$) is finite and corresponds to a critical point q_0, $u = u(t)$ is odd at γ_0, i.e. $u^{(even)}(\gamma_0) = 0$, with $u'(\gamma_0) \neq 0$. In all cases $f((\alpha_0, \beta_0)) > 0$, and, $f(\alpha_0) = 0$ if $\{\alpha_0\} \times N \in \partial M$ (or $f(\beta_0) = 0$ if $\{\beta_0\} \times N \in \partial M$).
The product $I \times N$ becomes complete if we add the set C of critical points to it. In addition, $f = f(t)$ and $u = u(t)$ satisfy the equations (4), (5) *and* (6) *in Proposition 1.7.*

1.3 Statement of results in the pseudo-Riemannian case

Now we turn attention to our results in the pseudo-Riemannian setting. Behavior of f and g in the indefinite setting is much more complicated than the definite case. Using Proposition 1.7 we generalize Theorem 1.8 from the Riemannian to the pseudo-Riemannian setting by means of the assumption $|\nabla f| \neq 0$:

Theorem 1.12. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g) be a pseudo-Riemannian manifold and f a smooth function on M. Then the following conditions are equivalent:*

1) (M, g, f) satisfies Equation (1) of a non-trivial $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ in a neighborhood of p with $g(\nabla f, \nabla f)|_p \neq 0$.

2) (a) Case $p \in int(M)$: There exist local coordinates $(t, t_1, ..., t_{n-1})$ with $t \in (-\varepsilon, \varepsilon)$ in a neighborhood of $p \in M$ and an Einstein Riemannian hypersurface $(N, g_N = g_N(t_1, ..., t_{n-1}))$ of (M, g) with normalized scalar curvature $\varrho_N = k$ and a function $u = u(t) > 0$, in addition $f = f(t) > 0$ satisfying (4) and (5) in Proposition 1.7 such that

I) $g(\partial_t, \partial_t) = \epsilon, \quad \epsilon := sign\ g(\nabla f(p), \nabla f(p)) \in \{\pm 1\}$
II) $g(\partial_t, \partial_{t_i}) = 0, \quad for\ i = 1, ..., n-1$
III) $g(\partial_{t_i}, \partial_{t_j}) = u^2(t) g_N(\partial_{t_i}, \partial_{t_j})(t_1, ..., t_{n-1}) \quad i, j = 1, ..., n-1.$

(b) Case $p \in \partial M$: There exist local coordinates $(t, t_1, ..., t_{n-1})$ with $t \in [0, \varepsilon)$ in a neighborhood of p and an Einstein Riemannian hypersurface $(N, g_N = g_N(t_1, ..., t_{n-1}))$ of (M, g) with normalized scalar curvature $\varrho_N = k$ and a function $u = u(t) > 0$, in addition $f(t) > 0$ for all $t \in (0, \varepsilon)$ satisfying (4) & (5) as well as $f(0) = 0$ satisfying (6) at $t = 0$ such that the conditions I, II, III in (a) hold.

Any case of 2) implies that $g = \epsilon dt^2 + u^2(t) g_N$ around p. If the metric is locally conformally flat in a neighborhood of p then N is necessarily a space of constant curvature.

Considering a function f with isolated critical points as an additional assumption, we classify $(\lambda, n+m)$-Einstein manifolds (M, g, f) with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ around critical points of f in the pseudo-Riemannian setting:

Proposition 1.13. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g) be a pseudo-Riemannian manifold with a smooth non-constant f on M whose critical points are isolated. In addition suppose that (M, g, f) satisfies Equation (1) of a $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ in a neighborhood of $p \in M$ with $\nabla f|_p = 0$. Then there are functions $u_\pm \in \mathscr{F}$ such that the metric in geodesic polar coordinates $(t, x) \in A_u \subset \mathbb{R} \times \sum$ in a neighborhood $\mathscr{U}$ of p has* *the form*

$$g(t,x) = g_u(t,x) = \epsilon dt^2 + \frac{u_\epsilon(t)^2}{u'_\epsilon(0)^2} g_1(x); \epsilon = g(x,x) \in \{\pm 1\} \tag{10}$$

where $u(t,x) = u_\epsilon(t)$, $\epsilon = g(x,x)$. If all geodesics through p are defined on the whole real line $\mathbb{R}$, then the metric g is of the form (10) *for all $(t,x) \in A_u$, i.e. as long as u_ϵ does not vanish. Also, the conditions 1.) and 2.) of Proposition 1.7 are satisfied.*

To generalize our global result in the Riemannian case, namely Theorem 1.11, to the pseudo-Riemannian setting we have some obstacles. One of them is that the set of critical points of f may be in natural bijection with either the set $J = \{1, ..., m\}$ or $J = \mathbb{N}$ or $J = \mathbb{Z}$, cf. [KR97a, Theorem 4.3]. In addition we have points at which ∇f is null. Hence we may not expect a global characterization with such a nice behavior as in the Riemannian setting. Therefore we confine ourselves to considering the Brinkmann case where a pseudo-Riemannian non-trivial $(\lambda, n+m)$-Einstein manifold (M, g, f) is Einstein, and in addition ∇f is a non-vanishing and isotropic (i.e. null) vector field on an open subset of M. Then in particular the metric tensor can be converted in to the form $g = 2dt_1 dt_2 + g_*(t_1)$ where $\nabla f = \partial_{t_2} = \nabla t_1$ and where the $(n-2)$-dimensional metric $g_*(t_1)$ does not depend on t_2.

Our main results in this thesis are Theorem 1.8, Lemma 1.9, Theorem 1.10, Theorem 1.11 in the Riemannian case, and Theorem 1.12 in the pseudo-Riemannian setting.

1.4 Thesis organization at a glance

The different sections of this thesis are organized in the following way. In the first section we shortly remind some definitions through which we fix some notations. Section 2 gives a short investigation on conformally Einstein product spaces. In particular we recall two examples from [KR16] where the conformal factor depends only on one side of the product.

The third section of this thesis is devoted to $(\lambda, n+m)$-Einstein manifolds and the idea behind this notion. We also investigate the relation between the Weyl tensors of a local one dimensional basis warped product metric and its fibre (Lemma 4.27). Based on [KR16, Proposition 4.17] we present characterizations for a $(\lambda, n+m)$-Einstein structure when g is given as a warped product with one dimensional basis and Einstein fibre (Corollary 4.33, Corollary 4.34 and Proposition 4.35). Proposition 4.35 is the main result of this section and serves as an essential component to our characterizations of $(\lambda, n+m)$-Einstein manifolds. Then the case where the manifold itself is also Einstein is discussed (Corollary 4.42). We finish this section with some examples where the last one (Example 4.49) shows that the critical points of f in a $(\lambda, n+m)$-Einstein manifold in general are not isolated.

Section 4 contains step by step calculations from [HPW12] discussing the cases of ∇f being an eigenvector for the Schouten tensor, $|\nabla f|$ being constant on the connected components of the level sets of f, and the case that the Schouten tensor has at most two eigenvalues from which the first one corresponds to eigenvector ∇f (see Lemma 5.11, Remark 5.12 and Lemma 5.14 respectively). We also show that $g = \epsilon dt^2 + u^2(t) g_N$, $\epsilon \in \{\pm 1\}$, where g_N is Einstein has harmonic Weyl tensor and satisfies $W(\nabla f, ., ., \nabla f) = 0$ (Lemma 5.9).

Section 5 contains the main results of this thesis in the Riemannian case. In subsection 5.2 we extend Theorem 7.1 in [HPW12] to a characterization around regular points of f (Theorem 6.5). In subsection 5.3 we show that critical points of f in a triple (M, g, f) satisfying Equation (1) of a non-trivial $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ are isolated (Lemma 6.8). Then we characterize a triple (M, g, f) which is conformally flat and satisfies Equation (1) of a non-trivial $(\lambda, n+m)$-Einstein metric around a critical point of f (Theorem 6.10). In subsection 5.4 we explain the unnecessary and missing properties in the formulation of the global result [HPW12, Theorem 7.2] and restate it (Theorem 6.12). We finish this section by a short discussion on $(\lambda, n+m)$-Einstein metrics of constant scalar curvature.

Section 6 contains the main result of this thesis in the pseudo-Riemannian setting. In subsection 6.1 we generalize Theorem 6.5 to the pseudo-Riemannian setting for the neighborhoods of points satisfying $|\nabla f| \neq 0$ (Theorem 7.2). We then discuss a specific case for dimension 4 related to Theorem 7.2 in subsection 6.2 (Corollary 7.7). In subsection 6.3, assuming $f \in C^\infty(M)$ is a function with isolated critical points we classify a pseudo-Riemannian (M, g) where the triple (M, g, f) satisfies Equation (1) of $(\lambda, n+m)$-Einstein manifolds with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ around critical points of f (Theorem 7.9). Subsection 6.5 describes a big difference of the behavior of f between the Riemannian and the indefinite cases. We close with the Brinkmann case where the metric can be written in the form of $g = 2dt_1 dt_2 + g_*(t_1)$ where the $(n-2)$-dimensional metric $g_*(t_1)$ does not depend on t_2 (Proposition 7.11).

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2 Preliminaries and Notations

Let V be a finite-dimensional real vector space. A bilinear form on V is an $\mathbb{R}$-bilinear function $b : V \times V \to \mathbb{R}$. It is symmetric if $b(v, w) = b(w, v)$ for all $v, w \in V$. A symmetric bilinear form is nondegenerate if and only if its matrix relative to one (hence every) basis is invertible.

A pseudo-Riemannian metric g on a smooth n dimensional manifold M is a symmetric nondegenerate $(0, 2)$ smooth tensor field on M which everywhere has constant index $(j, n-j), 0 \leq j \leq n$. A pseudo-Riemannian manifold (M, g) is a smooth n dimensional manifold M possibly with boundary together with a pseudo-Riemannian metric. We assume all manifolds are connected.

Definition 2.1. *A pseudo-Riemannian manifold (M, g) which has no boundary point is said to be geodesically complete provided every maximal geodesic γ: $I \to M$ is defined on the whole $\mathbb{R}$, i.e. $I = \mathbb{R}$. Moreover a pseudo-Riemannian manifold with non-empty boundary is said to be geodesically complete provided every maximal geodesic $\gamma : I \to M$ maps each end point of I to a boundary point of ∂M when it is finite, i.e. every maximal geodesic $\gamma : I \to M$ satisfies in one of the following conditions:*
1). $\gamma : [a, \infty) \to M$ then $\gamma(a) \in \partial M$
2). $\gamma : (-\infty, b] \to M$ then $\gamma(b) \in \partial M$
3). $\gamma : [a, b] \to M$ then $\gamma(a)$, $\gamma(b) \in \partial M$
4). $\gamma : (-\infty, \infty) \to M$ then $\gamma(-\infty, \infty) \cap \partial M = \emptyset$

Definition 2.2. *A connection ∇ on a smooth pseudo-Riemannian manifold M is a function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying*

(1) $\nabla_V W$ *is $C^\infty(M)$-linear in V,*
(2) $\nabla_V W$ *is $\mathbb{R}$-linear in W,*
(3) $\nabla_V(fW) = (Vf)W + f\nabla_V W$ *for $f \in C^\infty(M)$.*

$\nabla_V W$ *is called the covariant derivative of W with respect to V.*

The following properties determine a unique connection ∇ on a pseudo-Riemannian manifold (M, g): If for any $X, V, W \in \mathfrak{X}(M)$

(4) $[V, W] = \nabla_V W - \nabla_W V$, (Torsion free)
(5) $Xg(V, W) = g(\nabla_X V, W) + g(V, \nabla_X W)$, (Metric compatibility)

then ∇ is called the Levi-Civita connection of M which is derived out of the Koszul formula

$$\begin{aligned} 2g(\nabla_V W, X) =& Vg(W, X) + Wg(X, V) - Xg(V, W) \\ & - g(V, [W, X]) + g(W, [X, V]) + g(X, [V, W]). \end{aligned}$$

Definition 2.3. *Let ∇ be the Levi-Civita connection on a pseudo-Riemannian manifold (M, g). The tensor field $R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ given by*

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

is called the curvature of M.

For a smooth function f the gradient vector field ∇f (or grad f) is defined to be metrically equivalent to $df \in \mathfrak{X}^*(M)$. More precisely

$$g(\nabla f, X) = df(X) = Xf \ \ for \ \ any \ \ X \ \ \in \mathfrak{X}(M),$$

in which the coordinate expression of $df = \sum(\frac{\partial f}{\partial x^i})dx^i$ takes the form

$$\nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \partial_j.$$

Next we recall the definition of divergence for a vector field and more generally for a tensor of arbitrary type (k, l): If V is a vector field, then in an orthonormal frame field $(E_i)_1^n$ with $\epsilon_i = g(E_i, E_i)$

$$divV := \sum \varepsilon_i g(\nabla_{E_i} V, E_i)$$

in which a coordinate system takes the form

$$div \ \ V = \sum_i \{\frac{\partial V^i}{\partial x^i} + \sum_j \Gamma^i_{ij} V^j\}.$$

Definition 2.4. *By the Levi-Civita connection on a pseudo-Riemannian manifold, the divergence of a (k, l) tensor field A is a $(k-1, l)$ tensor field defined by*

$$(divA)(X_1, ..., X_{k-1}, \theta_1, ..., \theta_l) := \sum_{j=1}^n \epsilon_j (\nabla_{E_j} A)(E_i, X_1, ..., X_{k-1}, \theta_1, ..., \theta_l)$$

where $(E_i)_1^n$ is an orthonormal basis with $\epsilon_i = g(E_i, E_i)$. The Hessian tensor of a smooth function f is a symmetric $(0, 2)$-tensor field defined by $Hessf = \nabla(\nabla f)$, in more details

$$Hessf(X, Y) = XYf - (\nabla_X Y)f = g(\nabla_X \nabla f, Y).$$

The Laplacian Δf of a function $f \in C^\infty(M)$ is defined to be the divergence of its gradient, i.e. $\Delta f = div(\nabla f)$, in which coordinates takes the form

$$\Delta f = \sum_{ij} g^{ij} \{\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma^k_{ij} \frac{\partial f}{\partial x^k}\}.$$

Let R denotes the curvature tensor of a pseudo-Riemannian manifold (M, g). Then we define the Ricci curvature tensor Ric of (M, g) to be the contraction of R with respect to the first and fourth components, i.e.

$$Ric(X, Y) = \sum_i \varepsilon_i g(R(E_i X)Y, E_i)$$

for an orthonormal frame field $(E_i)_1^n$ with $\varepsilon_i = g(E_i, E_i)$. It is a symmetric tensor of type-$(0, 2)$ and in coordinate expression becomes $R_{ij} := Ric(\partial_i, \partial_j) =$

$\sum R^k{}_{ijk}$. By symmetries the only nonzero contractions of R are $\pm Ric$. Moreover, if the Ricci tensor of a manifold vanishes identically it is called Ricci flat. Obviously a flat manifold is Ricci flat, but the converse does not hold in general.

The scalar curvature Scal of a manifold (M, g) is the contraction of the Ricci tensor, hence in coordinate systems

$$Scal = \sum g^{ij} R_{ij} = \sum g^{ij} R^k{}_{ijk}.$$

A pseudo-Riemannian manifold (M, g) is Einstein if its Ricci tensor is proportional to the metric, i.e.

$$Ric = \gamma g.$$

Then we necessarily have $\gamma = \frac{Scal}{n}$ which is constant for $n \geq 3$. A Ricci flat manifold is a special case of Einstein manifolds for which the Einstein function γ is zero. For more convenience, we consider the normalized scalar curvature $\varrho = \frac{Scal}{n(n-1)}$ which is relevant to the normalized Einstein constant by

$$\frac{\gamma}{(n-1)} := \frac{Scal}{n(n-1)} = \varrho,$$

hence we have $\varrho = 1$ on the unit sphere of dimension n. In particular, when $n = 2$ we have $\varrho = \gamma = \frac{Scal}{2} = K$ which is the Gaussian curvature. A conformal mapping between two pseudo-Riemannian manifolds $(M, g), (\tilde{M}, \tilde{g})$ is a smooth map $f : (M, g) \to (\tilde{M}, \tilde{g})$ such that $f^*\tilde{g} = \varphi^{-2} g$ where $\varphi : M \to \mathbb{R}^+$ is a smooth function. More explicitly

$$\tilde{g}_{f(x)}(df_x(X), df_x(Y)) = \varphi^{-2}(x) g_x(X, Y); \quad X, Y \in T_x M.$$

When φ is constant it is called a homothety, in addition, if $\varphi = 1$ it is called an isometry and if $\varphi = -1$ an anti-isometry. An isometry is the special type of mapping that expresses the notion of isomorphism for pseudo-Riemannian manifolds.

3 Conformally Einstein product spaces in the pseudo-Riemannian setting

Our work has a close connection with the search for Einstein metrics which are conformally equivalent to a product metric. These metrics in the Riemannian and pseudo-Riemannian setting are investigated by Kühnel and Rademacher in [KR16]. In particular we are interested in [KR16, Theorem 3.2.(1)] which results in an Einstein warped product metric by which Corollary 4.31 is equivalent to existence of a $(\lambda, n+m)$-Einstein manifold, where the basis is a boundaryless pseudo-Riemannian manifold. Moreover, if the resulted Einstein warped product is complete and Riemannian, by Proposition 4.2 it is equivalent to existence of a $(\lambda, n+m)$-Einstein manifold (M, g, f) possibly with boundary under some condition on f.

Investigation of conformal changes between two Einstein manifolds started with Brinkmann [Bri25]. More results on Einstein spaces which are conformally equivalent with a Riemannian product can be found in the references [Cor00], [MO08], [HPW12] and [Tas85]. The products of Riemannian manifolds of arbitrary dimensions is also stated in [Cle08], but the 4-dimensional case of a product of two surfaces having non-constant sectional curvature is not discussed in [Cle08].

In [KR16] the authors start with Lemma 2.1 for the change of the Ricci tensor under a conformal change of a metric by which conformal mappings between two Einstein spaces are classified. By this basic lemma they classify conformally Einstein pseudo-Riemannian metrics through [KR16, Corollary 2.2]. If in additon g is Einstein then [KR16, Corollary 2.4] gives a simpler classification for conformally Einstein metrics. If the conformal factor depends only on one factor of the pseudo-Riemannian product then the conformal change of the metric results in a warped product metric; as a case where the conformal factor depends on a real parameter:

Proposition 3.1. (*[KR16], Proposition 3.1.*). *Suppose f is a non-constant function that depends only on the real parameter t. Then the metric $\bar{g} = f^{-2}(\epsilon dt^2 + g_N)$ is Einstein if and only if (N, g_N) is an Einstein manifold of dimension n and f satisfies the differential equation $\varrho_N f^2 - \epsilon(f')^2 = \bar{\varrho}$, where $\bar{\varrho}$ is the normalized scalar curvature of $\bar{g}$.*

See the reference for a detailed proof including a list of solutions of f for different initial values in the Riemannian case $\epsilon = 1$.

Corollary 3.2. *Applying Proposition 3.1 above for a smooth positive function f together with Corollary 4.31 we see that $(\mathbb{R}, f^{-2}\epsilon dt^2, f^{-1})$ is a $(\lambda, 1+m)$-Einstein manifold with $m := n$ if and only if (N, g_N) is an Einstein manifold of dimension n and f satisfies $\varrho_N f^2 - \epsilon(f')^2 = \frac{\lambda}{n}$(normalized λ).*

As a case where the conformal factor of a pseudo-Riemannian product $\mathbb{R} \times N$ depends only on the $(n-1)$ dimensional factor N see [KR16, Proposition 4.1].

In case that the conformal factor depends on both factors of the pseudo-Riemannian product there is a complete classification of the solutions by [KR16, Theorem 3.2] which also covers the missed 4-dimenisional case in [Cle08]. As a specific case of [KR16, Theorem 3.2] where one factor of a pseudo-Riemannian product is a subset of $\mathbb{R}$ see [KR16, Corollary 3.9].

There are some classifications for warped products which are also Einstein. For a thorough classification for complete Riemannian Einstein warped products on $\mathbb{R}^2 \times F$ see [Bes08, Theorem 9.119]. Manifolds (M, g) admitting a nonconstant solution ψ of $\nabla^2\psi = \frac{\Delta\psi}{n} g$, in the Riemannian case by [Küh88] and in the pseudo-Riemannian setting by [KR97b] and [KR09], are classified into Einstein warped products. As classifications of the base manifold of Einstein warped products with additional assumptions on the Weyl tensor of the base manifold, in the Riemannian case we have theorems 6.5, 6.10 and 6.12 and in the pseudo-Riemannian setting Theorem 7.2 and Proposition 7.9 in this thesis.

4 Investigation of $(\lambda, n+m)$-Einstein metrics and their characterization for a local warped metric

Most statements throughout this section are in the pseudo-Riemannian setting. For every statement we have cleared whether the setting is Riemannian or pseudo-Riemannian. From now on when we talk about completeness, in general we mean geodesically completeness unless we specifically mention some other sense of completeness. In particular we are interested in a direct relation between completeness of a warped product metric and its basis and fibre. We will see this works very well in the Riemannian case but breaks down when the metric is indefinite, cf. the famous example by Beem and Buseman [O'N10, Example 7.41].

From now on we consider manifolds possibly with boundary unless we mention the opposite. The following is the main definition in this thesis which is adapted to the pseudo-Riemannian setting from the Riemannian one.

Definition 4.1. (*The pseudo-Riemannian version adopted from [HPW12]). A (λ,n+m)-Einstein manifold* (M, g, f) *is a geodesically complete pseudo-Riemannian manifold* (M^n, g) *which may have boundary together with a smooth function* f *(here smooth means of class* C^2*) on* M *satisfying*

$$\begin{aligned} Hessf &= \frac{f}{m}(Ric - \lambda g) \qquad (11) \\ f &> 0 \text{ on } \mathrm{int}(M) \\ f &= 0 \text{ on } \partial M. \end{aligned}$$

If $m = 1$ we additionally assume that $\Delta f = -\lambda f$. In [CSW11] (after a change of variable $e^{-\frac{u}{m}} = f$) the Riemannian metrics satisfying (11) are called m-quasi Einstein metrics. A trivial case of $(\lambda, n+m)$-Eintstein manifolds happens when f is constant. Consequently from Equation (11) one obtains $Ric = \lambda g$. Moreover as f is constant it can not be identically zero, hence $\partial M = \emptyset$. In [HPW12] this case is called a λ-Einstein manifold.

Reminder: For the case $m = 1$ see [Cor00]. In this thesis and in particular in the main results, i.e. theorems 6.5, 6.10, 6.12, 7.2 and 7.9 we focus on $m > 1$. For a Riemannian manifold M without boundary compare [KK03].

Our initial purpose to study Equation (11) is due to [KK03, Proposition 5] which implies that for an integer m, a boudaryless Riemannian $(\lambda, n + m)$-Einstein structure is the base of a complete Einstein Riemannian warped product of dimension $n + m$ having complete Einstein fibre of dimension m.

Proposition 4.2. (*Restating of [HPW12, Proposition 1.1]). Suppose* $m > 1$ *is an integer and* (M, g) *is an* n *dimensional complete Riemannian manifold possibly with boundary. Then* (M, g, f) *is a* $(\lambda, n + m)$*-Einstein manifold if and* *only if either of the following occurs*

(a) CASE $\partial M = \emptyset$: There is a smooth $n+m$ dimensional complete Einstein Riemannian warped product metric

$$g_E = g + f^2 g_F \ \ on \ \ E = M \times F^m$$

with $Ric_E = \lambda g_E$ for a complete Einstein Riemannian manifold (F^m, g_F) of dimension m satisfying $Ric_F = \mu g_F$ where μ satisfies (17)*.*

(b) CASE $\partial M \neq \emptyset$: The boundary ∂M is totally geodesic and there is a smooth $n+m$ dimensional Einstein Riemannian warped product metric $g_E = g_M + f^2 g_{S^m(1)}$

$$on \ \ E = (M \times S^m(1))/\sim \ \ (where \ \ (x,p) \sim (x,p') \ \ for \ \ x \in \partial M)$$

with $Ric_E = \lambda g_E$ and $\begin{cases} f > 0 & ; \ on \ \ int(M) \\ f = 0 & ; \ on \ \ \partial M \end{cases}$, where $g_{S^m(1)}$ is the line element of the standard unit sphere $S^m(1)$ and where $\mu = m-1$ satisfies (17)*, i.e. $Ric_{S^m(1)} = (m-1) g_{S^m(1)}$.*

In the following we present a step by step proof for Proposition 4.2. Let's begin the first step by the following result which holds in the pseudo-Riemannian setting:

Proposition 4.3. (*[O'N10], Corollary 7.43). Let $E = M \times_f F$ be a warped product with $m = dimF > 1$, where $X, Y \in TM$ and $Z, V \in TF$. Then the Ricci tensor of E satisfies:*

$$Ric_E(X,Y) = Ric(X,Y) - \frac{m}{f} Hess f(X,Y), \tag{12}$$

$$Ric_E(X,Z) = 0 \tag{13}$$

$$Ric_E(Z,V) = Ric_F(Z,V) - g_E(Z,V) f^{\#}, f^{\#} = \frac{\Delta f}{f} + \frac{m-1}{f^2} g_E(\nabla f, \nabla f) \tag{14}$$

where Ric_E, Ric and Ric_F denote the Ricci tensor of (E, g_E), (M, g) and (F, g_F) respectively, and Δf=trace(Hess f) denotes the Laplacian of f on M.

This gives in turn the following characterization of: When is a warped product $E = M \times_f F$ Einstein?

Corollary 4.4. *A pseudo-Riemannian warped product $E = M \times_f F$, with $m = dimF > 1$, is Einstein with $Ric_E = \lambda g_E$ if and only if the following conditions hold*

$$Hess f = \frac{f}{m}(Ric - \lambda g), \tag{15}$$

$$(F, g_F) \ \ is \ Einstein \ with \ \ Ric_F = \mu g_F, \tag{16}$$

$$\mu = f \Delta f + (m-1)|\nabla f|^2 + \lambda f^2. \tag{17}$$

As the second step, we note the following lemma and proposition by Kim-Kim. We consider the pseudo-Riemannian versions of them (we note that the Riemannian curvature tensor in [KK03] has the opposite sign).

Lemma 4.5. (*The pseudo-Riemannian version of [KK03, Lemma 4]). Suppose f is a smooth function on a pseudo-Riemannian manifold M. Then for an arbitrary vector field X we have*

$$div(Hessf)(X) = Ric(\nabla f, X) + d(\Delta f)(X). \tag{18}$$

Proof. The same proof to the reference works for the pseudo-Riemannian case as well. □

Proposition 4.6. (*The pseudo-Riemannian version of [KK03, Proposition 5]). Suppose (M, g) is an $n(\geq 2)$ dimensional pseudo-Riemannian manifold. Let f be a non-constant smooth function on M satisfying* (11) *for a constant $\lambda \in \mathbb{R}$ and a natural number $m \in \mathbb{N}$. Then f satisfies Equation* (17) *for some constant $\mu \in \mathbb{R}$. Therefore for an Einstein space (F, g_F) of dimension m satisfying $Ric_F = \mu g_F$, we obtain an Einstein warped product space $E = M \times_f F$ satisfying $Ric_E = \lambda g_E$.*

Proof. We first consider Equatoin (11) through which tracing gives us

$$\Delta f = \frac{f}{m}(Scal - \lambda n) \tag{19}$$

and for its divergence

$$\begin{aligned} mdiv(Hessf)(X) &= div(fRic)(X) - \lambda div(fg)(X) \Rightarrow \\ &= Ric(\nabla f, X) + f divRic(X) - \lambda df(X) \end{aligned}$$

using Equation (18) as well as the second Bianchi identity we get

$$m(Ric(\nabla f, X) + d(\Delta f)(X)) = Ric(\nabla f, X) + \frac{f}{2} dScal(X) - \lambda df(X) \Rightarrow$$

$$(m-1)Ric(\nabla f, X) + md(\Delta f)(X) = \frac{f}{2} dScal(X) - \lambda df(X). \tag{20}$$

On the other hand from Equation (11) we have $mHessf(X, \nabla f) = fRic(X, \nabla f) - \lambda fg(X, \nabla f)$ which together with $Hessf(X, \nabla f) = \frac{1}{2} d(|\nabla|^2)(X)$ gives us

$$fRic(\nabla f, X) = \frac{m}{2} d(|\nabla f|^2)(X) + \lambda f df(X), \tag{21}$$

and by the differential of Equation (19) we get

$$fdScal = \lambda n df + md(\Delta f) - df Scal. \tag{22}$$

Now we multiply both sides of Equation (20) by f and then substitute the equations (21) and (22) for the appropriate terms and obtain

$$\begin{aligned} &(m-1).\frac{m}{2} d(|\nabla f|^2)(X) + (m-1)\lambda f df(X) + mfd(\Delta f)(X) \\ &= \frac{f}{2}(\lambda n df + md(\Delta f) - df Scal) - \lambda f df(X) \\ &= \frac{\lambda n}{2} f df(X) + \frac{m}{2} fd(\Delta f)(X) - \frac{df(X)}{2} fScal - \lambda f df(X). \end{aligned} \tag{23}$$

By Equation (19) we also get $fScal = \lambda nf + m\Delta f$ which in combination with (23) gives us

$$(m-1).\frac{m}{2}d(|\nabla f|^2)(X) + (m-1)\lambda f df(X) + mfd(\Delta f)(X)$$
$$= \frac{\lambda n}{2} f df(X) + \frac{m}{2} fd(\Delta f)(X) - \frac{df(X)}{2}\lambda nf - \frac{df(X)}{2} m\Delta f - \lambda f df(X).$$

By simplification this becomes

$$(m-1).\frac{m}{2}d(|\nabla f|^2) + (m-1)\lambda f df + mfd(\Delta f) = \frac{m}{2} fd(\Delta f) - \frac{df}{2} m\Delta f - \lambda f df$$

giving

$$d(f\Delta f + (m-1)|\nabla f|^2 + \lambda f^2) = 0.$$

Hence for a pseudo-Riemannian Einstein space (F, g_F) of dimension m satisfying $Ric_F = \mu g_F$, by Corollary 4.4 we obtain a pseudo-Riemannian Einstein warped product space $E = M \times_f F$ with $Ric_E = \lambda g_E$. □

Remark 4.7. *Kim Kim in [KK03, Proposition 5] assume compact Riemannian manifolds. But they did not use compactness in their proof. Hence, for an Einstein manifold (F^m, g_F) whose Einstein constant μ satisfies* (17) *and is not necessarily compact, one obtains an Einstein warped product $E = M \times_f F$ through their proof. Accordingly we implemented this point in Proposition 4.6.*

Lemma 4.8. (*[O'N10], Lemma 7.40.). Suppose (M, g) and (F, g_F) are complete Riemannian manifolds. Then for every warping function f the warped product $E = M \times_f F$ is complete.*

Now suppose (M, g, f) is a $(\lambda, n+m)$-Einstein metric for $m > 1$. Through Proposition 4.6 together with Corollary 4.4 we then have an $n+m$ dimensional Einstein Riemannian warped product metric $g_E = g + f^2 g_{F^m}$ for an Einstein Riemannian manifold (F^m, g_F) of dimension m where $Ric_F = \mu g_F$ and where μ satisfies (17).

In CASE $\partial M = \emptyset$: g_E is smooth on the topological product $M \times F$, furthermore, by Lemma 4.8 g_E would be complete for a complete g_F. Conversely, if $g_E = g + f^2 g_{F^m}$ is a smooth complete Einstein Riemannian metric with $Ric_E = \lambda g_E$ then (M, g) would be complete, hence using Corollary 4.4 (M, g, f) is a Riemannian $(\lambda, n+m)$-Einstein manifold.

Now in order to proceed our discussion into the case with boundary, we recall the following statements describing some behaviors of $|\nabla f|$ and μ with respect to ∂M.

Proposition 4.9. (*[HPW12], Proposition 2.2.). On the boundary ∂M of a Riemannian $(\lambda, n+m)$-Einstein manifold (M, g, f) we always have $|\nabla f| \neq 0$.*

Proof. For the proof see the reference. □

This in turn gives us the following result.

Proposition 4.10. (*[HPW12], Proposition 2.3.*). *The boundary ∂M of a Riemannian $(\lambda, n+m)$-Einstein metric (M, g, f) is totally geodesic, and further, $|\nabla f|$ is constant on the connected components of ∂M.*

Proof. Consider Equation (11) of a $(\lambda, n+m)$-Einstein manifold,

$$Hessf = \frac{f}{m}(Ric - \lambda g)$$

from which it follows that Hess $f|_{\partial M} = 0$ due to $\partial M = f^{-1}(0)$. By Proposition 4.9 $|\nabla f| \neq 0$ hence the second fundamental form, cf. [Pet16, Proposition 3.2.1],

$$\Pi(X, Y) = \frac{1}{|\nabla f|} Hessf(X, Y) \text{ for all } X, Y \in T\partial M \tag{24}$$

vanishes on ∂M saying it is totally geodesic. Furthermore $|\nabla f|$ is locally constant along ∂M because

$$D_X |\nabla f|^2 = 2Hessf(X, \nabla f) = 2\frac{f}{m}(Ric(X, \nabla f) - \lambda g(X, \nabla f)) = 0, \quad X \in TM$$

□

Corollary 4.11. (*[HPW12], Corollary 2.1.*). *Let $m > 1$ and (M, g, f) be a Riemannian $(\lambda, n+m)$-Einstein metric. Then $|\nabla f|^2$ has the same value on all connected components of ∂M. Moreover, if $\partial M \neq \emptyset$ then we have $\mu > 0$.*

Proof. On the boundary ∂M, Equation (17) takes the simple form

$$\mu = (m-1)|\nabla f|^2. \tag{25}$$

Since μ is constant the first part is proved. If $\partial M \neq \emptyset$ then using Proposition 4.9 on ∂M and noting $m > 1$ it follows that $\mu > 0$. □

In CASE $\partial M \neq \emptyset$: by Proposition 4.10 the boundary ∂M would be totally geodesic. We consider g_E on

$$E = (M \times F)/\sim$$

where $(x, p) \sim (x, p')$ if $x \in \partial M$, implying that ∂M collapses to an element in E with the quotient topology. Hence $f^{-1}(0)$ is isolated in this topology. Near ∂M the topology of the space is $\partial M \times F^m$. Since f vanishes on ∂M we investigate the conditions under which we may smoothly extend g_E onto the boundary. Since $|\nabla f| \neq 0$ on ∂M we may write

$$g_E = \frac{df^2}{|\nabla f|^2} + g_f + f^2 g_F \tag{26}$$

near $\partial M \times F$. Also, due to $|\nabla f|_{\partial M} \neq 0$ we may choose a coordinate system $(t_1, ..., t_{n-1})$ on the hypersurface $f^{-1}(0) = \partial M$, furthermore, there is a smooth function t from a neighborhood of ∂M to $\mathbb{R}$ with $dt = \frac{df}{|\nabla f|}$. Following that t is a smooth distance function, i.e. $|\nabla t| = 1$, the trajectories of $\frac{\nabla f}{|\nabla f|}$ are normal geodesics to ∂M and to level sets of f close to it, compare proof of Theorem 6.5. Therefore we may extend $(t_1, ..., t_{n-1})$ on ∂M to geodesic parallel coordinates $(t_1, ..., t_{n-1}, t)$ in a neighborhood of a point $q \in f^{-1}(0)$. This implies that the different t-levels are parallel to each other. In addition, since 0 is not a critical value of f (see Proposition 4.9) and since M is a manifold with boundary we have a local coordinate system $(t_1, ..., t_{n-1}, f)$ around any $q \in f^{-1}(0)$. These are also parallel to each other. It follows that t-levels coincide with the f-levels, hence f can be regarded as a function of t: $f(t, m) = f(t)$, $m \in \partial M$, as well as $\nabla f(t, m) = f'(t)\partial_t$. Therefore in these coordinates around each $q \in \partial M$ we obtain

$$\begin{aligned} g_E &= \frac{df^2}{|\nabla f|^2} + f^2 g_F + g_f \\ &= dt^2 + f^2(t) g_F + g_{\partial M}(t_1, ..., t_{n-1}), \ \ t \in [0, a). \end{aligned} \tag{27}$$

For (F, g_{F^m}) we may write $F = F(s_1, ..., s_m)$. As $f(t)$ vanishes at $t_0 = t(q) = 0$, similar to the proof of Theorem 6.10, we let X, Y be two orthonormal vectors in $[0, a) \times F$ which are tangent to F. By Equation (36) we then relate the sectional curvatures Sec resp. Sec_F of the (X, Y)-plane in $([0, a) \times F)$ resp. (F, g_F)

$$\begin{aligned} Sec &= g(R(X, Y)Y, X) \\ &= g(R_F(X, Y)Y, X) - \frac{(f'(t_0))^2}{(f(t_0))^2} \\ &= \frac{1}{(f^2(t_0))^2}(Sec_F - (f'(t_0))^2). \end{aligned}$$

We note that g_F is independent of t when it tends to zero. Following $f(t_0) = 0$ we obtain

$$0 = \lim_{t \to 0}(Sec_F - (f'(t))^2) = Sec_F - (f'(0))^2.$$

It implies that (F, g_F) is a space of constant curvature $Sec_F = (f'(0))^2$. Since $\mu > 0$, cf. Corollary 4.11, it follows that $Sec_F > 0$ and furthermore $F^m = S^m(1)$ and $g_F = \frac{1}{(f'(0))^2} S^m(1)$. Moreover (27) takes the form

$$g_E = dt^2 + \frac{f^2(t)}{(f'(t_0))^2} g_{S^m(1)} + g_{\partial M}(t_1, ..., t_n), \ \ (f'(t_0))^2 = \varrho_{S^m(1)} = 1 \tag{28}$$

Since $f(0)' \neq 0$ (by Proposition 4.9) and $f(0)'' = 0$ (which is true by Equation (11) where f vanishes on the boundary ∂M and where $\nabla^2 f(\partial_t, \partial_t) = f''(t) g(\partial_t, \partial_t)$), through the same discussion as in [Pet16, 1.4.4] we see that Equation (28) extends smoothly onto $f^{-1}(0) = \partial M$ (we note that smoothness here means of class C^2).

Up to now we have the necessary conditions in Proposition 4.2 in CASE $\partial M \neq \emptyset$. For the inverse, if ∂M is totally geodesic then using Equation (24) and $|\nabla f| \neq 0$ on ∂M we obtain

$$Hessf(X,Y) = |\nabla f|.\Pi(X,Y) = 0, \text{ for all } X,Y \in T\partial M. \tag{29}$$

Also by assumption f vanishes on ∂M, i.e. $f|_{\partial M} = 0$. Hence Equation (11) of the $(\lambda, n+m)$-Einstein structure holds on the boundary ∂M.

Additionally, since $g_E = g + f^2 g_{S^m(1)}$ extends smoothly on ∂M, by Corollary 4.4 it follows that Equation (11) holds on int(M) as well. Therefore (M, g, f) becomes a $(\lambda, n+m)$-Einstein manifold. Now the proof of Proposition 4.2 is complete.

Unfortunately, it is not in general clear how one can generalize Proposition 4.2 to the pseudo-Riemannian manifolds with boundary. In fact, the main key to obtain Proposition 4.2 is Proposition 4.9 by which $|\nabla f| \neq 0$ on ∂M in the Riemannian setting. The rest of discussion in Proposition 4.2 is also based on this property.

Any way we can partially extend Proposition 4.2 in case $\partial M = \emptyset$ to pseudo-Riemannian manifolds without boundary. Although for indefinite metrics we have no similar result to Lemma 4.8, cf. the famous example by Beem and Busemann [Pet16, Example 7.41], but we can take into consideration the following definition which is based on [CS08, Lemma 3.36] and relevant to the case where the boundary set is empty.

Definition 4.12. (*[CS08], Definition 3.38.*). *A triple (M, g_M, f) with a positive $f \in C^\infty(M)$ is said to be (resp. timelike, lightlike or spacelike) warped complete if for any complete fibre (F, g_F) the warped product $M \times_f F$ is (resp. timelike, lightlike or spacelike) complete.*

Then using Proposition 4.6, Corollary 4.4 and Defintion 4.12 we may obtain the following result corresponding to the pseudo-Riemannian setting which is weaker than Proposition 4.2.

Proposition 4.13. *Let $m > 1$ be an integer, $\lambda \in \mathbb{R}$ and (M^n, g) be a complete pseudo-Riemannian manifold without boundary. Then (M, g, f) is a warped complete $(\lambda, n+m)$-Einstein manifold if and only if there is a smooth $n+m$ dimensional complete Einstein pseudo-Riemannian warped product metric g_E on $E = M \times F^m$ of the form*

$$g_E = g + f^2 g_{F^m}$$

for a complete Einstein pseudo-Riemannian manifold (F^m, g_F) with $Ric_F = \mu g_F$ where μ satisfies (17).

We note that in Proposition 4.13 the assumption $m > 1$ is due to Corollary 4.4.

Remark 4.14. *The main idea that we define $(\lambda, n+m)$-Einstein manifolds to be complete, is the nice behavior of boundaryless Riemannian manifolds with respect to Proposition 4.2.CASE $\partial M = \emptyset$. This means that the completeness issue between the corresponding warped product g_E and its fibre g_F works well for boundaryless Riemannian manifolds, cf. Lemma 4.8. More explicitly, The Einstein Riemannian metric $g_E = g + f^2 g_F$ on $E = M \times F^m$ corresponding to a $(\lambda, n+m)$-Einstein manifold (M, g, f) without boundary, would be complete for any complete Einstein Riemannian manifold (F^m, g_F) of dimension m satisfying $Ric_F = \mu g_F$ where μ satisfies* (17).

Now as more results in the pseudo-Riemannian setting we have the following which is weaker than Proposition 4.9 as ∇f may be a null vector field on ∂M.

Proposition 4.15. *On the boundary ∂M of a pseudo-Riemannian $(\lambda, n+m)$-Einstein manifold (M, g, f) we have $\nabla f \neq 0$.*

Proof. Let $x_0 \in \partial M$ and let $\gamma(t)$ be a spacelike or timelike (or null) geodesic starting from x_0 such that $\gamma'(0)$ points inward the manifold, i.e. into $\text{int}(M)$. Also let $h(t) = f(\gamma(t))$ and $\Theta(t) = Ric(\gamma'(t), \gamma'(t)) - \lambda g(\gamma'(t), \gamma'(t))$. Hence the equation for f becomes a linear second-order ordinary differential equation for h along $\gamma(t)$ and we further have

$$\begin{aligned} h''(t) &= Hess f(\gamma'(t), \gamma(t)) \\ &= \frac{1}{m}\Theta(t)h(t) \\ h(0) &= 0, \\ h'(0) &= g(\nabla f, \gamma')_{x_0}. \end{aligned}$$

Hence if $\nabla f(x_0) = 0$, then $h'(0) = 0$. Therefore $h = 0$ along all of γ by the initial conditions. On the other hand, as $\gamma'(0)$ points inward the manifold it follows that $\gamma(t) \in int(M)$ for $0 < t < \epsilon$, and hence $h(t) > 0$ when $0 < t < \epsilon$. This is in contradiction with the fact that $h|_\gamma = 0$. □

The statements 4.10 and 4.11 are not completely true in the pseudo-Riemannian setting, because $|\nabla f| = 0$ may occur. But we can formulate some parts of them as following:

Corollary 4.16. *Let (M, g, f) be a pseudo-Riemannian $(\lambda, n+m)$-Einstein metric. Then $|\nabla f|$ is constant on the connected components of ∂M. If $m > 1$ then $|\nabla f|^2$ has the same value on the connected components of ∂M, in addition, if $\partial M \neq \emptyset$ and $|\nabla f| \neq 0$ then $\mu > 0$ (or $\mu < 0$) and ∂M is totally geodesic.*

The proof is a combination of similar proofs to those of Proposition 4.10 and Corollary 4.11.

Remark 4.17. *When $m = 1$ the additional condition $\Delta f = -\lambda f$ is equivalent to $\mu = 0$. This is necessary for the existence of F because in one dimensional manifolds we must have $Ric \equiv 0$. Moreover, combining this additional condition* *with the trace of Equation* (11) *we obtain* $Scal = (n-1)\lambda$*. In addition, if* g *is a pseudo-Riemannian Einstein manifold then via derivative of the equation in Corollary 4.24 we get*

$$(n-1)\lambda = Scal = -\epsilon(n-1)n\frac{u''}{u}. \tag{30}$$

Theorem 4.18. *([HPW12], Theorem 4.1.). Suppose* (M, g, f) *is a non-trivial Riemannian* $(\lambda, n+m)$*-Einstein manifold. Then* M *is compact if and only if* $\lambda > 0$*.*

Corollary 4.19. *([HPW12], Corollary 4.2.). Suppose* (M, g, f) *is a Riemannian* $(\lambda, n+m)$*-Einstein metric with* $m > 1$*,* $\lambda \geq 0$ *and* $\mu \leq 0$*, then it is trivial and satisfies* $\lambda = \mu = 0$*.*

Definition 4.20. *. If* $n > 2$ *the Weyl tensor* W *on a pseudo-Riemannian manifold with arbitrary signature* $0 \leq j \leq n$ *is defined as*

$$R = W + \frac{2}{n-2}Ric \odot g - \frac{Scal}{(n-1)(n-2)}g \odot g \tag{31}$$

in which the Kulkarni-Nomizu product $s \odot r$ *of two symmetric* $(0, 2)$ *tensors* s *and* r *is a* $(0, 4)$*-tensor defined by*

$$\begin{aligned}(s \odot r)(X, Y, Z, V) =& \frac{1}{2}(r(X, V)s(Y, Z) + r(Y, Z)s(X, V) \\ &- r(X, Z)s(Y, V) - r(Y, V)s(X, Z)).\end{aligned}$$

Definition 4.21. *A pseudo-Riemannian manifold is conformally flat if each point has a neighborhood that can be mapped conformally to a flat space.*

Lemma 4.22. *[Sán95]. Consider a pseudo-Riemannian warped product over* $M = I \times N$*,* $I \subset \mathbb{R}$*, of the form*

$$g = \epsilon dt^2 + u^2(t)g_N, \;\; \epsilon := sign \;\; g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) \in \{\pm 1\} \tag{32}$$

where $u > 0$ *is a smooth function on* I*. If* (M, g) *is geodesically complete then* (N, g_N) *is also geodesically complete.*

In the sequel, an investigation of some geometric properties of warped product metrics of type (32) is presented.

Lemma 4.23. *([KR97a], Lemma 2.5). Let* $\nabla_N, R_N, Ric_N, \varrho_N$ *denote the Levi–Civita covariant derivative, the Riemannian curvature tensor, the Ricci tensor and the normalized scalar curvature of* (N, g_N)*. Also let* ∂_t *denotes the unit tangent vector in direction of the first side of the product* $I \times N$ *and* X, Y, Z *denote the lifts of vector fields on* N*. "I" is an open interval in* $\mathbb{R}$*. Then for the warped product metric*

$$(I, \epsilon dt^2) \times_u (N, g_N) = \Big((I \times N) \;\; , \;\; (g = \epsilon dt^2 + u^2(t)g_N)\Big)$$

we have the following formulae:

1).

$$\nabla_{\partial_t}\partial_t = 0 \tag{33}$$

$$\nabla_{\partial_t}X = \nabla_X\partial_t = \frac{u'}{u}X \tag{34}$$

$$\nabla_X Y = -\frac{g(X,Y)}{u}\epsilon u'\partial_t + \nabla_{N X}Y \tag{35}$$

2).

$$R(X,Y)Z = R_N(X,Y)Z - \epsilon\frac{u'^2}{u^2}\{g(Y,Z)X - g(X,Z)Y\} \tag{36}$$

$$R(X,Y)\partial_t = 0 \tag{37}$$

$$R(X,\partial_t)\partial_t = -\frac{u''}{u}X \tag{38}$$

3).

$$Ric(Y,Z) = Ric_N(Y,Z) - \frac{\epsilon}{u^2}\{(n-2)u'^2 + u''u\}g(Y,Z) \tag{39}$$

$$Ric(Y,\partial_t) = 0 \tag{40}$$

$$Ric(\partial_t,\partial_t) = -(n-1)\frac{u''}{u} \tag{41}$$

4).

$$u^2\varrho = \frac{n-2}{n}\varrho_N - \frac{n-2}{n}u'^2\epsilon - \frac{2}{n}\epsilon u''u \tag{42}$$

These relations follow from the corresponding formulae for warped products, compare [O'N10, chapter 7] in which the Riemannian curvature tensor has the opposite sign. Following from

$$\nabla u = \epsilon u'\partial_t, \;\; \nabla^2_{\partial_t,\partial t}u = g(\nabla_{\partial_t}\nabla u, \partial_t) = u''$$

the formulae above in the Riemannian and the pseudo-Riemannian cases coincide , if for $\epsilon = -1$ *we let the warped product* $\tilde{g} = -dt^2 + u^2(t)\tilde{g}_N$ *satisfies* $\tilde{g}_N = -g_N$ *which is anti-isometric to* g *(hence* $\tilde{\varrho} = -\varrho$, $\tilde{\varrho}_N = -\varrho_N$*,...). Also similar to the Riemannian case we obtain:*

Corollary 4.24. (*[KR97a], Corollary* 2.6). $g = (I, \epsilon dt^2)\times_u (N, g_N)$ *is an Einstein metric (has constant sectional curvature) if and only if* g_N *is an Einstein metric (has constant sectional curvature) and* $u'^2 + \varrho\epsilon u^2 = \epsilon\varrho_N$.

Remark 4.25. *In Lemma 4.23 above if the hypersurface* (N^{n-1}, g_N) *is considered to be a pseudo-Riemannian manifold then all the formulae in particular those including the Ricci tensor Ric and the normalized scalar curvature* ϱ *are still valid. Here* $Ric(X,Y) = \sum_m \varepsilon_m g(R(E_m X)Y, E_m)$ *for an orthonormal basis* $(E_i)_1^n$ *with* $\epsilon_i = g(E_i, E_i)$.

In the Riemannian case, Lemma 4.23 parts 2), 3), 4) are also stated in [Küh88, Lemma 13]. We just recall the last part of the Riemannian result:

Lemma 4.26. (*[Küh88], Lemma* 13). *Consider the Riemannian warped product $ds^2 = dt^2 + u^2(t)g_N$. Then*

$$(iv) \text{ g is Einstein (has constant sectional curvature)} \Leftrightarrow g_N \text{ is Einstein (has constant sectional curvature) and } \varrho = -\frac{u''}{u}. \tag{43}$$

Equation $\varrho = -\frac{u''}{u}$ in (43) corresponds to Equation $u'^2 + \varrho\epsilon u^2 = \epsilon\varrho_N$ in Corollary 4.24 via a derivative step.

Via the following result we see how the Weyl tensors of g and g_N are related to each other in the pseudo-Riemannian setting.

Lemma 4.27. *Consider the pseudo-Riemannian warped product $g = \epsilon dt^2 + u^2(t)g_N$, $\epsilon \in \{\pm 1\}$, then we have the following equations for vector fields X, Y, Z, V orthogonal to ∂_t:*

$$(i)\ W(X,Y,Z,V) = u^2 W_N(X,Y,Z,V) + \frac{2}{(n-2)(n-3)} u^2 Ric_N \odot g_N - \frac{2}{(n-3)} \varrho_N u^2 g_N \odot g_N$$

$$(ii)\ W(X,\partial_t,\partial_t,Y) = \epsilon\varrho_N g_N(X,Y) - \frac{\epsilon}{(n-2)} Ric_N(X,Y)$$

$$(iii)\ W(X,Y,\partial_t,Z) = 0.$$

If g_N is Einstein then Equation (ii) vanishes for vector fields $X, Y \perp \partial_t$, and also, the last two terms on the right-hand side of Equation (i) vanish, hence W becomes conformal to the Weyl tensor W_N of g_N i.e. $W(X,Y,Z,V) = u^2 W_N(X,Y,Z,V)$.

Proof. (i): From the definition of Weyl tensor, see (31), one obtains

$$W(X,Y,Z,V) = R(X,Y,Z,V) - \frac{2}{n-2} Ric \odot g(X,Y,Z,V) + \frac{Scal}{(n-1)(n-2)} g \odot g(X,Y,Z,V)$$

which in combination with the equations (36) and (39) gives us

$$W(X,Y,Z,V) = g(R_N(X,Y)Z,V) - \epsilon(\frac{u'}{u})^2 g \odot g(X,Y,Z,V) - \frac{2}{(n-2)}\{Ric_N \odot g(X,Y,Z,V) - \epsilon(n-2)(\frac{u'}{u})^2 g \odot g(X,Y,Z,V) - \epsilon(\frac{u''}{u}) g \odot g(X,Y,Z,V)\} + \frac{n}{n-2}\varrho g \odot g(X,Y,Z,V).$$

Using (31) for the hypersurface N and $g(X,Y) = u^2 g_N(X,Y)$ for $X, Y \in TN$ we obtain

$$W(X,Y,Z,V) = u^2 W_N(X,Y,Z,V) + \frac{2}{(n-2)(n-3)} u^2 Ric_N \odot g_N(X,Y,Z,V)$$
$$- \frac{2}{(n-3)} \varrho_N u^2 g_N \odot g_N(X,Y,Z,V)$$

(ii): By (31) we have

$$W(X,\partial_t,\partial_t,Y) = R(X,\partial_t,\partial_t,Y) - \frac{2}{(n-2)} Ric \odot g(X,\partial_t,\partial_t,Y)$$
$$+ \frac{Scal}{(n-1)(n-2)} g \odot g(X,\partial_t,\partial_t,Y).$$

Using Equation (38) for $R(X,\partial_t,\partial_t,Y)$ and Equation (39) for $Ric(X,Y)$, where $X, Y \perp \partial_t$, together with $\frac{\varrho_N}{u^2} g(X,Y) = \varrho_N g_N(X,Y)$ we obtain

$$W(X,\partial_t,\partial_t,Y) = \epsilon \varrho_N g_N(X,Y) - \frac{\epsilon}{(n-2)} Ric_N(X,Y)$$

(iii): Using (31) we have

$$W(X,Y,\partial_t,Z) = R(X,Y,\partial_t,Z) - \frac{2}{(n-2)} Ric \odot g(X,Y,\frac{\partial}{\partial t},Z)$$
$$+ \frac{Scal}{(n-1)(n-2)} g \odot g(X,Y,\partial_t,Z)$$

which by (37), i.e. $R(X,Y,\partial_t,Z) = 0$, becomes

$$W(X,Y,\partial_t,Z) = 0 - \frac{1}{(n-2)} \{Ric(X,Z)g(Y,\partial_t) + Ric(Y,\partial_t)g(X,Z) - Ric(X,\partial_t)$$
$$g(Y,Z) - Ric(Y,Z)g(X,\partial_t)\} + \frac{Scal}{(n-1)(n-2)} \{g(X,Z)g(Y,\partial_t) - g(X,\partial_t)g(Y,Z)\}.$$

Furthermore using (40) we conclude

$$W(X,Y,\partial_t,Z) = 0.$$

□

Remark 4.28. *Since the dimension of a hypersurface N of an n dimensional pseudo-Riemannian manifold M is $n-1$, by definition*

$$R_N = W_N + \frac{2}{n-3} Ric_N \odot g_N - \frac{Scal}{(n-2)(n-3)} g_N \odot g_N$$

the Weyl tensor W_N of (N, g_N) is not defined when $n = 3$, hence one should pay attention the restriction of dim $M = n \geq 4$ when connecting the Weyl tensors W and W_N.

Lemma 4.29. (*[Che17], Theorem* 3.6). *Let $M^n = I \times_u N$ be a pseudo-Riemannian warped product $g = \epsilon dt^2 + u^2(t) g_N$, $\epsilon \in \{\pm 1\}$. Then (M, g) is locally conformally flat if and only if (N, g_N) is a space of constant curvature.*

Proof. For a complete proof see the reference. But, as an application of Lemma 4.27 we give a proof for manifolds of dimension $M^{n\geq 4}$ (here we note Remark 4.28):
We know from the literature that a manifold $(M^n, g), n \geqslant 4$ is locally conformally flat if and only if its Weyl tensor vanishes. Then Lemma 4.27.(ii) implies that (N, g_N) is Einstein, respectively Equation (i) in Lemma 4.27 turns into

$$W(X, Y, Z, V) = u^2 W_N(X, Y, Z, V).$$

Since the Weyl tensor W vanishes it follows that $W_N(X, Y, Z, V) = 0$ implying g_N is also locally conformally flat. On the other hand, from the literature we also know that a manifold is both Einstein and locally conformally flat if and only if it has constant curvature, cf. [Pet16, page.110]. Therefore (N, g_N) has constant sectional curvature.

Conversely if (N, g_N) has constant curvature, then it is both Einstein and locally conformally flat. To see that g is locally conformally flat it suffices to prove the Weyl tensor W vanishes for all vector fields.

For that purpose, since g_N is Einstein the tensor W vanishes for vector fields in the style of case (ii). Since g_N is locally conformally flat it follows that $W_N(X, Y, Z, V) = 0$; $X, Y, Z, V \in TN$ (if $n = 3$, then $W \equiv 0$) from which together with the fact that g_N is Einstein one obtains $W(X, Y, Z, V) = 0$, i.e. case (i) holds as well. Thus the Weyl tensor identically vanishes, respectively g is locally conformally flat. □

4.1 Local characterization of $(\lambda, n+m)$-Einstein structure for a local warped product metric

One dimensional Riemannian $(\lambda, 1+m)$-Einstein manifolds are classified in [HPW12, Example 3.1]. We consider $dim(M) = n > 1$ unless the opposite is stated. In the following we investigate $(\lambda, n+m)$-Einstein manifolds provided with a metric of the form $g = \epsilon dt^2 + u^2(t) g_N$, $\epsilon \in \{\pm 1\}$ where g_N is Einstein. We begin with the case $m = 1$ which is already solved. We present then Proposition 4.35 for the general case $m \geq 1$.

Proposition 4.30. (*[KR16, Corollary 4.3], [Cor00, Proposition 2.7]*). *Let (M, g) be a pseudo-Riemannian manifold on which a function f is defined. Then the metric $\bar{g} = \pm f^2 ds^2 + g$ on $\mathbb{R} \times M$ is Einstein if and only if the equation*

$$f.Ric - \nabla^2 f + (\Delta f) g = 0 \tag{44}$$

holds.

Proposition 4.13 can be extended to the case $m = 1$ via Proposition 4.30.

Corollary 4.31. *Let $m \geq 1$ be an integer, (M^n, g) be a complete pseudo-Riemannian manifold without boundary, and f be a smooth non-negative function on M. Then (M, g, f) is a warped complete $(\lambda, n+m)$-Einstein manifold if and only if there is an $n+m$ dimensional complete Einstein pseudo-Riemannian warped product metric g_E on $E = M \times F^m$ of the form*

$$g_E = g + f^2 g_{F^m}$$

for a complete Einstein pseudo-Riemannian manifold (F^m, g_F) with $Ric_F = \mu g_F$ where μ satisfies (17).

Proof. For $m > 1$ we refer to the proof of Proposition 4.13. If $m = 1$ by the additional assumption corresponding to $m = 1$, i.e. $\Delta f = -\lambda f$, Equation (11) of a $(\lambda, n+1)$-Einstein manifold (M, g, f)

$$\nabla^2 f = \frac{f}{1}(Ric - \lambda g)$$

changes into the form

$$f Ric - \nabla^2 f + \Delta f.g = 0$$

by which Proposition 4.30 is equivalent to say that $g_E = g + f^2 g_{F^1}$ is Einstein. For the completeness issue we refer to Definition 4.12. □

As another application of Proposition 4.30 we note the following statement which is actually adapted to pseudo-Riemannian manifolds of arbitrary signatures.

Proposition 4.32. (*[KR16], Proposition 4.7*). *Suppose g_N is a pseudo-Riemannian Einstein metric with $Ric_N = k(n-2)g_N, k \in \mathbb{R}$ on a manifold N of dimension $n-1$. Let $g = \epsilon dt^2 + u^2(t) g_N$, $\epsilon \in \{\pm 1\}$ be a warped product metric on $M = I \times N$ with an interval $I \subseteq \mathbb{R}$. Also let $f = f(t)$ be a smooth function on I. Then Equation* (44) *holds if and only if $f(t) = a u'(t)$ for some constant $a \neq 0$ and*

$$u^2 u''' + (n-3) u u' u'' - (n-2) u'^3 + \epsilon k (n-2) u' = 0. \tag{45}$$

Therefore for a positive solution u of Equation (45) *the warped product*

$$\bar{g} = \pm u'^2(t) ds^2 + \epsilon dt^2 + u^2(t) g_N$$

is an Einstein metric on $I \times I \times N$. The interval I is chosen such that u and u' are positive on it. We may consider $\bar{g}$ as a warped product $g \pm u'^2(t) ds^2$ with a basis of dimension n and a fibre of dimension 1 or as a warped product $(\pm u'^2(t) ds^2 + \epsilon dt^2) + u^2(t) g_N$ with a basis of dimension 2 in the (s, t)-plane.

Proof. Let (E_i) be a pseudo-orthonormal frame, i.e. $g(E_i, E_j) = \epsilon_i \delta_{ij}$ with $\epsilon_i \in \{\pm 1\}$, and let $E_1 = \partial_t$. Using equations (33) resp. (34) one obtains

$$\nabla^2 f(\partial_t, \partial_t) = \epsilon f'' g(\partial_t, \partial_t) = f''$$

$$\nabla^2 f(E_j, E_j) = \epsilon \frac{f'u'}{u} g(E_j, E_j) = \epsilon\epsilon_j \frac{f'u'}{u}, \;\; j \in \{2, ...n\}.$$

Hence,

$$\Delta f = \text{trace}(\nabla^2 f) = \sum_{i=1}^{n} \epsilon_i g(\nabla_{E_i} \nabla f, E_i) = \epsilon f'' + \epsilon(n-1)\frac{f'u'}{u}$$

which gives us the following equations

$$-\nabla^2 f(\partial_t, \partial_t) + \Delta f.g(\partial_t, \partial_t) = (n-1)\frac{f'u'}{u}$$

$$-\nabla^2 f(E_j, E_j) + \Delta f.g(E_j, E_j) = \epsilon\epsilon_j f'' + \epsilon\epsilon_j (n-2)\frac{f'u'}{u}.$$

Now using Equation (41) one gets

$$\begin{aligned} & fRic(\partial_t, \partial_t) - \nabla^2 f(\partial_t, \partial_t) + \Delta f.g(\partial_t, \partial_t) \\ & = \frac{n-1}{u}(f'u' - fu'') \end{aligned} \tag{46}$$

and via (39) one has

$$\begin{aligned} & fRic(E_j, E_j) - \nabla^2 f(E_j, E_j) + \Delta f.g(E_j, E_j) \\ = & \frac{f}{u^2}\Big(k(n-2) - \epsilon\{(n-2)u'^2 + uu''\}\Big)\epsilon_j + \epsilon\epsilon_j f'' + \epsilon\epsilon_j (n-2)\frac{f'u'}{u}. \end{aligned} \tag{47}$$

By Equation (44) the right hand sides of (46) and (47) vanish. From Equation (46) it follows that $f'u' = fu''$ giving $f = au'$ for some constant a. Without loss of generality we may assume $a = 1$. Similarly from Equation (47) we conclude Equation (45).

Conversely if $f(t) = au'(t)$ as well as Equation (45) hold then (46) and (47) equal zero. This implies that Equation (44) holds. □

As a $(\lambda, n+1)$-Einstein manifold may have boundary, based on Proposition 4.32 we present the following local characterizations in either case of the boundary status under the hypotheses that g is a warped product with one dimensional basis and f is a one-variable function.

Corollary 4.33. *Let $\lambda \in \mathbb{R}$, $m \geq 1$, $n > 1$ integers and g_N a pseudo-Riemannian Einstein metric say with normalized scalar curvature $\varrho_N = k \in \mathbb{R}$, i.e. $Ric_N = k(n-2)g_N$, on an $(n-1)$-dimensional manifold N and $g = \epsilon dt^2 + u^2(t)g_N$ a warped product metric on $M = I \times N$ with an interval $I \subset \mathbb{R}$. In addition suppose $\partial M = \emptyset$ and $f = f(t)$ is a smooth non-negative function on I. Then (M, g, f) satisfies Equation (11) of a $(\lambda, n+1)$-Einstein manifold on $M = I \times N$ if and only if $f(t) = au'(t)$ for some constant $a \neq 0$ and*

$$u^2 u''' + (n-3)uu'u'' - (n-2)u'^3 + \epsilon k(n-2)u' = 0 \tag{48}$$

where I is an interval on which u and au' are positive.

Proof. Since $m = 1$ by the assumption $\Delta f = -\lambda f$ Equation (11) of a $(\lambda, n+1)$-Einstein manifold (M, g, f) i.e.

$$\nabla^2 f = \frac{f}{1}(Ric - \lambda g) \tag{49}$$

turns into

$$fRic - \nabla^2 f + \Delta f g = 0 \tag{50}$$

which is Equation (44). Also by assumption, $g = \epsilon dt^2 + u^2(t)g_N$ on M and $f = f(t)$ on I. By Proposition 4.32 Equation (50) holds if and only if $f(t) = au'(t)$ for some constant $a \neq 0$ and $u^2u''' + (n-3)uu'u'' - (n-2)u'^3 + \epsilon k(n-2)u' = 0$. □

The following characterization is in accordance with the situation where a pseudo-Riemannian manifold has boundary.

Corollary 4.34. *Let $\lambda \in \mathbb{R}$, $m \geq 1$, $n > 1$ integers and g_N a pseudo-Riemannian Einstein metric say with normalized scalar curvature $\varrho_N = k \in \mathbb{R}$, i.e. $Ric_N = k(n-2)g_N$, on an $(n-1)$-dimensional manifold N and $g = \epsilon dt^2 + u^2(t)g_N$ a warped product metric on $M = I \times N$ with an interval $I \subset \mathbb{R}$. In addition suppose $f = f(t)$ is a smooth non-negative function on I. Then (M, g, f) satisfies Equation* (11) *of a $(\lambda, n+1)$-Einstein manifold on $M = I \times N$ if and only if*

1. *On $int(M)$*

$$f(t) = au'(t), \quad a \neq 0 \tag{51}$$

$$u^2u''' + (n-3)uu'u'' - (n-2)u'^3 + \epsilon k(n-2)u' = 0 \tag{52}$$

2. *On ∂M*

$$f''(t) = u'(t) = 0. \tag{53}$$

Proof. Through a similar argument as in the proof of Corollary 4.33 we see that Equation (11) of a $(\lambda, n+1)$-Einstein manifold (for which $m = 1$)

$$\nabla^2 f = \frac{f}{1}(Ric - \lambda g), \tag{54}$$

holds on $I \times N$ if and only if $f(t) = au'(t)$ for some constant $a \neq 0$ and $u^2u''' + (n-3)uu'u'' - (n-2)u'^3 + \epsilon k(n-2)u' = 0$. On ∂M the condition $f(t) = au'(t)$ reduces to $u'(t) = 0$ as $f(t)$ vanishes on ∂M. Consequently, $u^2u''' + (n-3)uu'u'' - (n-2)u'^3 + \epsilon k(n-2)u' = 0$ simplifies to $f''(t) = 0$ because $f(t) = au'(t) \Rightarrow f''(t) = au'''(t)$. □

Now we generalize corollaries 4.33 and 4.34 to pseudo-Riemannian $(\lambda, n+m)$-Einstein manifolds with $m \geq 1$.

Proposition 4.35. *Let $\lambda \in \mathbb{R}$, $m \geq 1$, $n > 1$ integers and g_N a pseudo-Riemannian Einstein metric say with normalized scalar curvature $\varrho_N = k \in \mathbb{R}$, i.e. $Ric_N = k(n-2)g_N$, on an $(n-1)$-dimensional manifold N and $g = \epsilon dt^2 + u^2(t)g_N$ a warped product metric on $M = I \times N$ with an interval $I \subset \mathbb{R}$. In addition suppose $f = f(t)$ is a smooth non-negative function on I. Then $(M = I \times N, g, f)$ satisfies Equation (11) of a $(\lambda, n+m)$-Einstein manifold if and only if the following conditions hold*

1. *On $int(M)$*

$$f'm\frac{u'}{u} + \{\epsilon\lambda - \frac{\epsilon(n-2)k - (n-2)u'^2 - uu''}{u^2}\}f = 0 \tag{55}$$

$$\begin{aligned} &\lambda^2u^4 - 2(n-2)k\lambda u^2 + (m+2(n-2))\lambda u^2u'^2 + (2+m)\lambda u^3u'' \\ &+(n-2)^2k^2 - (2(n-2)+m)(n-2)ku'^2 - (2+m)(n-2)kuu'' \\ &+(n-2)(m+n-2)u'^4 + (2(n-2)+m)uu'^2u'' + (1+m)u^2u''^2 \\ &-mu^2u'u''' = 0 \end{aligned} \tag{56}$$

2. *On ∂M*

$$f''(t) = u'(t) = 0. \tag{57}$$

Proof. As (M, g) may have boundary, first we consider $g = \epsilon dt^2 + u^2(t)g_N$ on $int(M = I \times N)$ on which $f > 0$. Using the relations $\nabla_X \partial_t = \frac{u'}{u}X$ and $\nabla_{\partial_t}\partial_t = 0$ one obtains

$$\nabla^2 f(\partial_t, \partial_t) = \epsilon f'' g(\partial_t, \partial_t) = f'' \tag{58}$$

$$\nabla^2 f(X, X) = \epsilon\frac{f'u'}{u}g(X, X). \tag{59}$$

Also, from Lemma 4.23 we have the following equations where k signifies the normalized scalar curvature of g_N

$$Ric(\partial_t, \partial_t) = -\epsilon(n-1)\frac{u''}{u}g(\partial_t, \partial_t) = -(n-1)\frac{u''}{u} \tag{60}$$

and

$$\begin{aligned} Ric(X, X) &= Ric_N(X, X) - \frac{\epsilon}{u^2}[(n-2)u'^2 + uu'']g(X, X) \\ &= k(n-2)g_N(X, X) - \frac{\epsilon}{u^2}[(n-2)u'^2 + uu'']g(X, X) \end{aligned} \tag{61}$$

where X denotes any unit tangent vector orthogonal to ∂_t. Let (M, g, f) satisfies Equation (11) of a $(\lambda, n+m)$-Einstein manifold on $int(M)$. Tangent vectors to M can be divided into the category of those tangent in the direction of the first factor of $I \times N$, i.e. scalar multiplications of ∂_t, and into the category of tangent vectors to N. Tensorial Equation (11) on vectors of the first category,

$$\nabla^2 f(\partial_t, \partial_t) = \frac{f}{m}(Ric(\partial_t, \partial_t) - \lambda g(\partial_t, \partial_t)), \tag{62}$$

via (58) & (60) becomes

$$f'' + (\frac{(n-1)}{m}\frac{u''}{u} + \epsilon\frac{\lambda}{m})f = 0, \tag{63}$$

the so-called first necessary condition. Now let's pay attention to evaluation on the second category of vector fields

$$\nabla^2 f(X, Y) = \frac{f}{m}(Ric(X, Y) - \lambda g(X, Y)) \qquad X, Y \perp \frac{\partial}{\partial t}, \tag{64}$$

which by (59) & (61) forms the so-called second necessary condition,

$$f' m\frac{u'}{u} + \{\epsilon\lambda - \frac{\epsilon(n-2)k - (n-2)u'^2 - uu''}{u^2}\}f = 0. \tag{65}$$

Note that (55) is actually the second necessary condition (65). Furthermore, through the functions

$$\begin{aligned} a(t) &= \frac{(n-1)}{m}\frac{u''}{u} + \epsilon\frac{\lambda}{m} \\ b(t) &= m\frac{u'}{u} \\ c(t) &= \epsilon\lambda - \frac{\epsilon(n-2)k - (n-2)u'^2 - uu'')}{u^2} \end{aligned}$$

we can denote the necessary conditions (63) and (65) respectively by

$$f'' = -af \tag{66}$$

$$f' = -\frac{c}{b}f. \tag{67}$$

We still need to show that the second claim (56) is satisfied: We differentiate (67) and then compare it to (66) which gives us

$$f'' = -af = (-\frac{c}{b}f)' = -(\frac{c}{b})'f - (\frac{c}{b})f'. \tag{68}$$

By further application of (67) we obtain

$$-af = -(\frac{c}{b})'f + (\frac{c}{b})^2 f \tag{69}$$

which reduces to

$$-a = -(\frac{c}{b})' + (\frac{c}{b})^2. \tag{70}$$

We rewrite (70) in terms of $u(t)$, $u'(t)$ and $u''(t)$ as

$$-\Big(\frac{(n-1)}{m}\frac{u''}{u}+\epsilon\frac{\lambda}{m}\Big)=-\Big(\{\epsilon\lambda-\frac{\epsilon(n-2)k-(n-2)u'^2-uu''}{u^2}\}\frac{u}{mu'}\Big)'$$
$$+\Big(\epsilon\lambda-\frac{\epsilon(n-2)k-(n-2)u'^2-uu''}{u^2}\Big)^2\frac{u^2}{m^2u'^2} \tag{71}$$

which after simplification gives us (56).

On $int(M)$ for the converse, suppose that we have $f=f(t)$ and $g=\epsilon dt^2+u^2(t)g_N$ where g_N is Einstein with normalized scalar curvature $\varrho_N=k$. Additionally, assume Equation (55) for some $\lambda\in\mathbb{R}$ holds. This says that the so-called second necessary condition (65) holds.

It remains to show that tensorial Equation (11) is also satisfied with vectors tangent on the first factor of $I\times N$ to prove the triple (M,g,f) satisfies Equation (11) of $(\lambda,n+m)$-Einstein manifolds. To this end, we utilize Equation (56) which is the simplification of (71). On the other hand by the labels $a(t)$, $b(t)$ and $c(t)$ Equation (71) can be written in the form of (70) which after multiplication by f gives out (69) (note that $f>0$ on $int(M)$). From Equation (69) together with Equation (67), which is actually (55) using the labels $a(t)$, $b(t)$ and $c(t)$, we see that Equation (68) holds. On the other hand, the derivative of (68) comparing with (67) gives out Equation (66) which is equivalent to the so-called first necessary condition (63). So, for all vector fields tangent to $int(M)$ the tensorial Equation (11) of a $(\lambda,n+m)$-Einstein manifold holds.

Secondly let's investigate the characterization on ∂M: Since the Hessian tensor $\nabla^2 f$ vanishes on the boundary ∂M, equations (58) and (59) imply

$$f''(t)=0 \tag{72}$$

and

$$f'(t)u'(t)=0 \tag{73}$$

from which the latter together with Proposition 4.9 (which says $f'(t)\neq 0$ on ∂M) implies $u'(t)=0$. Therefore on ∂M one obtains Equation (57).

Conversely, as f vanishes on ∂M the right-hand side of Equation (11) becomes zero. Thus the left-hand side of Equation (11) must also vanish on it. On the other hand by Equation (57) the Hessian tensor $\nabla^2 f$ identically vanishes on ∂M, hence Equation (11) holds in this case as well.

Therefore in each case the tensorial Equation (11) of a $(\lambda,n+m)$-Einstein manifold holds on M . □

Remark 4.36. *In Proposition 4.35 for a positive solution $u(t)$ of* (56) *on $int(M)$, Equation* (55) *expresses $f(t)$ in terms of $u(t)$ by $f(t)=e^{-\int_0^t\frac{c(s)}{b(s)}ds}f(0)$ as far as $u'\neq 0$. On ∂M the functions $f(t)$ and $u(t)$ satisfy Equation* (57)*.*

Remark 4.37. *The procedure through equations* (66) *to* (70) *in the proof of Proposition 4.35 shows that if f satisfies* (55)*, i.e. the so-called second necessary* *condition , then it also satisfies the so-called first necessary condition* (63) *of a* $(\lambda, n+m)$*-Einstein manifold under condition* (56) *which is equivalent to* (70) *and hence, the domain of solution of the differential equations system of the* $(\lambda, n+m)$*-Einstein manifold consisting of* (63) *and* (65) *is not empty for* f*.*

Remark 4.38. *Classical solutions like* $u(t) = t, sin(t), e^t$*,* $cosh(t)$ *give us one dimensional basis warped product* $(\lambda, n+m)$*-Einstein metrics for appropriate choices of* $f(t)$*,* k*,* λ *and* m*. In particular* $u(t) = e^t$ *with* $\epsilon\lambda = -n-m+1, \ \ k = 0$ *and* $f(t) = ae^t, \ \ a \in \mathbb{R}^+$ *provides us with an example where* (M, g) *is Einstein with normalized scalar curvature* $k_g = -\epsilon$ *(by Equation* (42)*). Here the manifold* (M, g) *is without boundary as the function* f *is always positive. This example is not interesting by Proposition 4.45 and [KR09, page 434].*

Next we bring a classical example with non-empty boundary.

Example 4.39. *Let* $g = \epsilon dt^2 + u^2(t)g_N$ *on* $M = [0, \infty) \times N$ *where* $u(t) = cosh(t)$ *and where* g_N *is a complete pseudo-Riemannian Einstein metric with* $\varrho_N = k = -\epsilon$*. Also let* $f(t) = sinh(t)$ *and* $\epsilon\lambda = 1 - n - m$*. The boundary would be the slice* $\{t_0 = 0\} \times N$*. Then one can check that the conditions* (55)*,* (56) *and* (57)*, e.g.* $f''(0) = u'(0) = 0$*, in Proposition 4.35 are satisfied for this example. Therefore we obtain a non-trivial* $(\lambda, n+m)$*-Einstein manifold.*

As Proposition 4.35 is formulated for a $(\lambda, n+m)$-Einstein structure in the general case $m \geq 1$ we expect that it be compatible with the preceding results obtained by conformal change of the metric. More precise statement is the following remark:

Remark 4.40. *If* $m = 1$*, then the characterization in Proposition 4.35 is compatible with the corresponding previous results, i.e. with corollaries 4.33 and 4.34 which are based on Proposition 4.32.*

Proof. We need to show that when $m = 1$ the necessary and sufficient conditions (55) & (56) in Proposition 4.35 match with the corresponding ones in corollaries 4.33 & 4.34. We begin with (55): Recall the additional assumption $\Delta f = -\lambda f$ when $m = 1$, changing Equation (11) of a $(\lambda, n+1)$-Einstein manifold into $fRic - \nabla^2 f + \Delta f g = 0$ through which taking the trace gives out $Scal = (n-1)\lambda$. Thus, $\lambda = \frac{Scal}{n-1} = n\varrho$. Hence, through Equation (42) and $\varrho_N = k$ we obtain

$$\begin{aligned} \epsilon\lambda u^2 = \epsilon n u^2 \varrho &= \epsilon(n-2)\varrho_N - (n-2)u'^2 - 2uu'' \\ &= \epsilon(n-2)k - (n-2)u'^2 - 2uu''. \end{aligned} \tag{74}$$

Then, as in the proof of Proposition 4.2 we use the auxiliary functions $b(t)$ and

$c(t)$:

$$
\begin{aligned}
b(t) &= \frac{u'}{u} && (75)\\
c(t) &= \epsilon\lambda - \frac{\epsilon(n-2)k-(n-2)u'^2-uu''}{u^2}\\
&= \frac{\epsilon\lambda u^2-\epsilon(n-2)k+(n-2)u'^2+uu''}{u^2}\\
&= -\frac{u''}{u} && (76)
\end{aligned}
$$

in which we used (74). We rewrite Equation (55) in the form $f' = -\frac{c(t)}{b(t)}f$ by which (75) and (76) becomes $f' = \frac{u''}{u'}f$ with the solution $f = au'$ for a constant $a \neq 0$. This coincides with the first necessary condition in corollaries 4.33 and 4.34.

Now we consider the second condition (56) which is equivalent to (71). Dividing (74) by 2, and applying it in the left hand side of (71) gives us

$$
-(n-1)\frac{u''}{u} - \epsilon\lambda = -(n-1)\frac{u''}{u} - \frac{1}{u^2}\Big(\epsilon(n-2)k-(n-2)u'^2-2uu''\Big) \quad (77)
$$

in the right hand side of (71) gives out

$$
\begin{aligned}
&-\Big(\{\epsilon\lambda - \frac{\epsilon(n-2)k-(n-2)u'^2-uu''}{u^2}\}\frac{u}{mu'}\Big)'\\
&+\Big(\epsilon\lambda - \frac{\epsilon(n-2)k-(n-2)u'^2-uu''}{u^2}\Big)^2\frac{u^2}{m^2u'^2} \qquad (78)\\
&= \frac{u'''}{u'}.
\end{aligned}
$$

Now the equality (77)=(78), i.e.

$$
-(n-1)\frac{u''}{u} - \frac{1}{u^2}\Big(\epsilon(n-2)k-(n-2)u'^2-2uu''\Big) = \frac{u'''}{u'},
$$

after simplification becomes

$$
u^2u''' + (n-3)uu'u'' - (n-2)u'^3 + \epsilon k(n-2)u' = 0
$$

which coincides with (48) and (52) as the second necessary condition in corollaries 4.33 and 4.34. ∎

Proposition 4.41. (*The pseudo-Riemannian version of [CSW11, Proposition 3.4]). Suppose $m \neq 1$. Then a $(\lambda, n+m)$-Einstein metric has constant scalar curvature if and only if the following equation holds*

$$
Ric(\nabla f) = \frac{1}{m-1}((n-1)\lambda - Scal)\nabla f. \quad (79)
$$

Proof. The proof is the same as the reference. ∎

4.2 Local characterization of a $(\lambda, n+m)$-Einstein metric which is also Einstein, for a local warped product structure

In the following we see that when a $(\lambda, n+m)$-Einstein manifold is also Einstein, under the same assumptions as in Proposition 4.35, the necessary and sufficient conditions reduce to simple formulae.

Corollary 4.42. *Let $\lambda \in \mathbb{R}$, $m \geq 1$, $n > 1$ integers and g_N a pseudo-Riemannian Einstein metric say with normalized scalar curvature $\varrho_N = k$, i.e. $Ric_N = k(n-2)g_N$, $k \in \mathbb{R}$ on an $(n-1)$-dimensional manifold N and $g = \epsilon dt^2 + u^2(t)g_N$ a warped product metric on $M = I \times N$ with an interval $I \subset \mathbb{R}$. In addition suppose $f = f(t)$ is a smooth non-negative function on I. Then (M, g, f) satisfies Equation (11) of a $(\lambda, n+m)$-Einstein manifold on $I \times N$ if and only if the following conditions hold*

1. On $int(M)$

$$f' = au(t), \;\; a \in \mathbb{R}^+ \tag{80}$$

$$\frac{u''}{u} = \frac{u'''}{u'} \tag{81}$$

2. On ∂M

$$f''(t) = u'(t) = 0. \tag{82}$$

Hence in Proposition 4.45.TABLE 2 each warping function u satisfies (81)*, also for the entries with non-empty boundary the functions f and u must satisfy* (82) *on ∂M.*

Proof. Here, besides all the assumptions in Proposition 4.35 we additionally assume that g is Einstein. Accordingly, we just need to justify the way Einstein property of g affects equations (55) & (56): Let (M^n, g, f) satisfies Equation (11) of a $(\lambda, n+m)$-Einstein manifold, i.e.

$$\nabla^2 f = \frac{f}{m}(Ric - \lambda g),$$

taking trace then implies

$$\Delta f = \frac{f}{m}(Scal - \lambda n). \tag{83}$$

Furthermore using $\nabla_X \frac{\partial}{\partial t}$, as in the proof of Proposition 4.32, we get

$$\Delta f = \epsilon f'' + \epsilon(n-1)f'\frac{u'}{u}. \tag{84}$$

Since g is Einstein, via a derivative step from the equation in Corollary 4.24 we obtain

$$\varrho = -\epsilon\frac{u''}{u} \Leftrightarrow$$

$$Scal = -\epsilon(n-1)n\frac{u''}{u} \tag{85}$$

as well as via equations (39) & (41) it follows that

$$\epsilon u'^2 - \epsilon u u'' = k. \tag{86}$$

By substitution of equations (84) and (85) in (83) we obtain

$$\epsilon\lambda f = -(n-1) f \frac{u''}{u} - \frac{m}{n} f'' - \frac{m(n-1)}{n} f' \frac{u'}{u}. \tag{87}$$

Replacing equations (86) and (87) in the first necessary condition (55),

$$f' m \frac{u'}{u} + \{\epsilon\lambda - \frac{\epsilon(n-2)k - (n-2)u'^2 - uu''}{u^2}\} f = 0, \tag{88}$$

gives us

$$f' \frac{u'}{u} = f'' \tag{89}$$

from which by an integration step it follows that

$$f' = au, \ \ a \in \mathbb{R}. \tag{90}$$

Now consider the so-called second necessary condition (56). For facility we instead study Equation (71) with which it is equivalent: When $m > 1$ one may use Equation (79) in Proposition 4.41 which in combination with (85) gives

$$\lambda = \epsilon(1 - n - m)\frac{u''}{u}. \tag{91}$$

We note that g has constant scalar curvature as it is Einstein. On the other hand for the case $m = 1$ by Remark 4.17.(30) one has

$$\lambda = -\epsilon n \frac{u''}{u} \tag{92}$$

which is the extension of Equation (91) to $m = 1$. Hence using the formula (91) for $m > 1$ and (92) for $m = 1$, the left side of Equation (71) becomes

$$-\Big(\frac{(n-1)}{m}\frac{u''}{u} + \epsilon\frac{\lambda}{m}\Big) = \frac{u''}{u}. \tag{93}$$

The right hand side of (71) using equations (86), (91) for $m > 1$ and (92) for $m = 1$ becomes

$$\begin{aligned} &-\Big(\{\epsilon\lambda - \frac{\epsilon(n-2)k - (n-2)u'^2 - uu''}{u^2}\}\frac{u}{mu'}\Big)' \\ &+\Big(\epsilon\lambda - \frac{\epsilon(n-2)k - (n-2)u'^2 - uu''}{u^2}\Big)^2 \frac{u^2}{m^2 u'^2} \\ &= (\frac{u''}{u'})' + \frac{u''^2}{u'^2} = \frac{u'''}{u'}. \end{aligned} \tag{94}$$

By Equation (71) the right hand sides of equations (93) and (94) are equal. This implies

$$\frac{u''}{u} = \frac{u'''}{u'} \tag{95}$$

which is (81). □

4.3 Some examples of $(\lambda, n+m)$-Einstein manifolds

We already have some classical examples via Remark 4.38 and Example 4.39. We may recall [HPW12, Example 3.1] which classifies Riemannian non-trivial $(\lambda, 1+m)$-Einstein manifolds. Examples 4.45 and 4.48 give classifications of non-trivial $(\lambda, n+m)$-Einstein manifolds which are in addition Einstein. The last example itself contains two examples both showing that critical points of f in a $(\lambda, n+m)$-Einstein manifold (M, g, f) in general are not isolated.

Example 4.43. *([Bes08], Example 9.118(a)). Suppose f is the unique positive function on $[0, \infty)$ satisfying the conditions*

$$(f')^2 = 1 - f^{1-m} \quad , f(0) = 1 \;\; \text{and} \;\; f' \geq 0. \tag{96}$$

Then by [Bes08, Example 9.118(a)] the triple $(\mathbb{R}^2, g, f)$ with

$$\begin{aligned} g &= dt^2 + \frac{4(f'(t))^2}{(m-1)^2} d\theta^2 \\ f &= f(t) \end{aligned} \tag{97}$$

is a $(0, 2+m)$-Einstein metric where its corresponding Einstein warped product (see Proposition 4.2) is complete and Ricci flat. To observe the conditions in Proposition 4.35 are satisfied here, first we note that comparing (97) with the corresponding metric in Proposition 4.35 one has $u = \frac{2f'}{m-1}$ and hence $u'' = \frac{2f'''}{m-1}$. Moreover from (96) derivating f implies

$$f'' = \frac{(m-1)}{2} f^{-m} \tag{98}$$

hence

$$f''' = -\frac{m(m-1)}{2} f' f^{-m-1} \tag{99}$$

so one obtains

$$\frac{u''}{u} = \frac{f'''}{f'} = -\frac{m(m-1)}{2} f^{-m-1}. \tag{100}$$

At this moment consider the first necessary condition (55) of Proposition 4.35, i.e.

$$f'm\frac{u'}{u} + \{\epsilon\lambda - \frac{\epsilon(n-2)k - (n-2)u'^2 - uu''}{u^2}\} f = 0,$$

which due to the assumptions $n = 2$ and $\lambda = 0$ reduces to

$$f'm\frac{u'}{u} + \frac{u''}{u} f = 0$$

and via $u = f'$ it further changes into

$$m.u' + \frac{u''}{u} f = 0. \tag{101}$$

Through equations (98) *and* (100) *one sees Equation* (101) *holds.*

Next we need to verify the second condition (56) *of Proposition 4.35 holds as well. As $n = 2$ and $\lambda = 0$ in this case,* (56) *reduces to*

$$muu'^2u'' + (1+m)u^2u''^2 - mu^2u'u''' = 0$$

or equivalently

$$m\frac{u''}{u} = m\frac{u'''}{u'} - (1+m)\frac{u''^2}{u'^2}$$

which can be reformulated in the more appropriate form

$$m\frac{u''}{u} = m(\frac{u''}{u'})' - (\frac{u''}{u'})^2. \tag{102}$$

In order to confirm Equation (102) *we use* (98) & (99) *giving us*

$$\frac{u''}{u'} = \frac{f'''}{f''} = \frac{-\frac{m(m-1)}{2}f'f^{-m-1}}{\frac{(m-1)}{2}f^{-m}} = -m\frac{f'}{f}. \tag{103}$$

Thus the right hand side of (102) *becomes*

$$m(\frac{u''}{u'})' - (\frac{u''}{u'})^2 = m(-m\frac{f'}{f})' - (-m\frac{f'}{f})^2 = -m^2\frac{f''}{f} \tag{104}$$

which equals the left hand side through division (98) *by f and then using* (100)*. By Equations* (96),(97) *and* (98) *we obtain*

$$u'(0) = \frac{2f''(0)}{m-1} = \frac{2}{m-1}\frac{m-1}{2}f^{-m}(0) = 1$$

which implies that Equation (97) *is smooth as a metric of a surface of revolution (warped product).*

Next we recall a local Riemannian $(\lambda, 3+m)$-Einstein structure which is not locally conformally flat.

Example 4.44. (*[HPW12], Example 3.5.*)*. Let the metric be a doubly warped product of the form*

$$g = dt^2 + \phi^2 d\theta_1^2 + \psi^2 d\theta_2^2$$

then applying Equation (11) *of $(\lambda, 3+m)$-Einstein manifolds on the pair vector fields $(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$, i.e.*

$$\frac{m}{f}Hessf(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = Ric(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) - \lambda g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}),$$

and using the equation $Ric(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -\frac{\phi''}{\phi} - \frac{\psi''}{\psi}$ (which is derived by calculation) gives us

$$m\frac{f''}{f} = -\frac{\phi''}{\phi} - \frac{\psi''}{\psi} - \lambda$$

similarly applying Equation (11) *on the pair vectors* $(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_1})$ *gives*

$$\frac{f'\phi'}{f\phi} = -\frac{\phi''}{\phi} - \frac{\phi'\psi'}{\phi\psi} - \lambda$$

and further, applying (11) *on the pair vetors* $(\frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial\theta_2})$ *yields*

$$\frac{f'\psi'}{f\psi} = -\frac{\psi''}{\psi} - \frac{\phi'\psi'}{\phi\psi} - \lambda.$$

Considering suitable initial conditions we can solve the system of above differential equations near $t = 0$. *The initial conditions* $\phi(0) = \psi(0) > 0, \phi'(0) \neq \psi'(0)$, $f(0) > 0$, *and* $f'(0) = 0$ *lead us to a local solution for Equation* (11) *whose Cotton tensor does not vanish, and hence, it is not locally conformally flat. We recall that in dimension 3 locally conformally flatness is equivalent to vanishing the Cotton tensor.*

In the following we observe the classification of Riemannian $(\lambda, n+m)$-Einstein metrics which are also Einstein. Afterwards we recall the corresponding classification for the pseudo-Riemannian setting.

Example 4.45. (*[HPW12], Proposition* 3.1). *Let* $n \geq 2$ *and let* (M^n, g, f) *be a non-trivial* $(\lambda, n+m)$*-Einstein manifold which in addition is* ρ*-Einstein. Then it is isometric to one of the examples in Table 2 for* $\bar{k} = \frac{\lambda-\rho}{m}$.

Proof. By assumption we have $Ric = \rho g$. Thus from Equation (11) we obtain

$$Hessf = -\bar{k}fg \ , \ \bar{k} = \frac{\lambda - \rho}{m}.$$

Let $\bar{k} = 0$ then $\text{Hess}f = 0$. Since the $(\lambda, n+m)$-Einstein structure is non-trivial f is non-constant, hence it is a multiple of a distance function and consequently the metric splits along f. This gives us the $\lambda = 0, \mu > 0$ entry in the table.

Suppose $\bar{k} \neq 0$ then from $\mathcal{L}_{\nabla f}g = 2\text{Hess } f = -2\bar{k}fg$ it follows that $f''(t) = -\bar{k}f$. By a similar discussion as in [Küh88, Lemma 12] we obtain

$$g = dt^2 + (f'(t))^2 g_S \tag{105}$$
$$f = f(t) \tag{106}$$

around regular points of f where g_S is the metric of a regular level set of f, e.g. see $\lambda < 0$, $\mu = 0$ entry in TABLE 2. Around critical points of f using [Küh88, Lemma 18] we get

$$g = dt^2 + \frac{(f'(t))^2}{(f''(0))^2} g_S \tag{107}$$
$$f = f(t) \tag{108}$$

where g_S is the metric on the standard unit sphere, e.g. see $\lambda, \mu < 0$ entry in TABLE 2. Via these lemmas we obtain the solutions as in the following table, cf. [Küh88], [Bes08] and [Bri25]. □

	$\lambda > 0$	$\lambda = 0$	$\lambda < 0$
$\mu > 0$	D^n g=$dt^2 + \sqrt{k} sin^2(\sqrt{k}t) g_{S^{n-1}}$ f(t)=C $cos(\sqrt{k}t)$	[0,∞)×F g=$dt^2 + g_F$ $f(t) = Ct$	[0,∞) ×N g=$dt^2 + \sqrt{-\bar{k}} cosh^2(\sqrt{-\bar{k}}t) g_N$ f(t)=C sinh($\sqrt{-\bar{k}}t$)
$\mu = 0$	None	None	(-∞, ∞) ×F g=$dt^2 + e^{2\sqrt{-\bar{k}}t} g_F$ $f(t) = Ce^{\sqrt{-\bar{k}}t}$
$\mu < 0$	None	None	H^n g=$dt^2 + \sqrt{-\bar{k}} \sinh^2(\sqrt{-\bar{k}}t) g_{S^{n-1}}$ f(t)=C cosh($\sqrt{-\bar{k}}t$)

TABLE 2. Non-trivial $(\lambda, n+m)$-Einstein manifolds which are in addition Einstein. In this table S^{n-1} denotes a round sphere, F is Ricci flat and N denotes an Einstein metric with negative Ricci curvature and $C \in \mathbb{R}^+$ is arbitrary.

Reminder: For details in the relation between sectional curvature of N (or F) and the metric g see [HPW12, Remark 3.2].

Remark 4.46. *Since in Example 4.45 the manifold is considered to be Einstein, every solution in TABLE 2 satisfies the necessary conditions in Corollary 4.42. For example consider the entry $\lambda < 0$, $\mu > 0$ with $M^n = [0, \infty) \times N, g = dt^2 + \sqrt{-\bar{k}}\ cosh^2(\sqrt{-\bar{k}}t) g_N$ where N is an Einstein metric with negative Ricci curvature. Here we have $u(t) = \sqrt{-\bar{k}}\ cosh(\sqrt{-\bar{k}}t)$ and $f(t) = \sinh(\sqrt{-\bar{k}}t)$ for which the necessary conditions, i.e. (80), (81) and (82), hold. I.e. $f' = u(t)$, $\frac{u''}{u} = \frac{u'''}{u'}$ and on the boundary $\{0\} \times N$ the condition $f''(0) = u'(0) = 0$ is also satisfied. This shows that (M, g, f) is a $(\lambda, n+m)$-Einstein manifold.*

Remark 4.47. *Note that in TABLE 2 above the entries $\lambda = 0$, $\mu > 0$ resp. $\lambda < 0$, $\mu > 0$ have the slices $\{0\} \times F$ resp. $\{0\} \times N$ as boundary at which f vanishes, and hence the intervals are closed on the left, i.e. the local warped structure is defined on $[0, \infty) \times F$ resp. $[0, \infty) \times N$.*

Example 4.48. *Under the same assumptions as in Example 4.45 above in the pseudo-Riemannian setting, [KR09, Step 4 and Step 5] give the solutions for f and the corresponding warping functions.*

Example 4.49. *Here we present two examples showing that the zeros of ∇f in a non-trivial pseudo-Riemannian $(\lambda, n+m)$-Einstein manifold (M, g, f) are not in general isolated. These two examples are some extended forms of the one I received from Prof. Kühnel to the case $m \geq 1$:*

1) Consider a Riemannian product $M = M_\star \times H^2$, $g = \begin{pmatrix} g_\star & 0 \\ 0 & g_{-1} \end{pmatrix}$, where H^2 is the hyperbolic plane in polar coordinates carrying $g_{-1} = dt^2 + \sinh^2 t dv^2$ with $\mathrm{Ric}_{-1} = -g_{-1}$. Also we consider $f_{H^2}(t, v) =$ with $t = 0$ as the critical point and giving $\nabla^2 f_{H^2} = f_{H^2} g_{-1}$. The factor $M_\star$ of the product is chosen to be Einstein with $\mathrm{Ric}_\star = -(m+1) g_\star$.

Hence Ric $= \begin{pmatrix} Ric_\star & 0 \\ 0 & \mathrm{Ric}_{-1} \end{pmatrix} = \begin{pmatrix} -(m+1)g_\star & 0 \\ 0 & -g_{-1} \end{pmatrix}$, *consequently we can write*

$$\mathrm{Ric} + (m+1)g = \begin{pmatrix} 0 & 0 \\ 0 & mg_{-1} \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now if we let $f(x,t,v) = f_{H^2}(t) = cosh(t)$ *with* $x \in M_\star$ *then* $\nabla f = \nabla f(t) = \frac{\partial}{\partial t}$ *and it follows that*

$$\nabla^2 f = \begin{pmatrix} 0 & 0 \\ 0 & fg_{-1} \end{pmatrix} = \frac{f}{m}(Ric + (m+1)g), \quad (m+1) = -\lambda.$$

Thus every point with $t = 0$ *and* x *arbitrary is critical which implies then that critical points of* f *are not in general isolated.*

Now we investigate the conditions in Theorem 7.2 on the Weyl tensor, i.e. harmonicity of the weyl tensor W *and* $W(\nabla f, ., .\nabla f) = 0$ *for this example as well. For that purpose we start with Equation*

$$R = W + \frac{2}{n-2} Ric \odot g - \frac{Scal}{(n-1)(n-2)} g \odot g \tag{109}$$

where dim $M = n$. *The divergence of Equation* (109) *will be*

$$\begin{aligned}(divR)(X,Y,Z) =& (divW)(X,Y,Z) + \frac{2}{n-2}(divRic \odot g)(X,Y,Z) \\ & - \frac{1}{(n-1)(n-2)}(divScal \;\; g \odot g)(X,Y,Z). \end{aligned} \tag{110}$$

We have $Scal = Scal_\star + Scal_{-1} = -(m+1)(n-2) - 2 = -mn + 2m - n$. *Via calculation we have*

$$(divRic \odot g)(X,Y,Z) = \frac{1}{2}\{(divRic)(Z)g(X,Y) - (divRic)(Y)g(X,Z) \tag{111}$$

$$+ (\nabla_Z Ric)(X,Y) - (\nabla_Y Ric)(X,Z)\}. \tag{112}$$

Equation (112) *can be simplified using the following corollary of the second Bianchi identity*

$$(divRic)(Z) = \frac{1}{2} dScal(Z) = \frac{1}{2} g(Z, \nabla Scal), \;\; \textit{for every } Z \tag{113}$$

Moreover by similar computation as in [HPW12, page 296] one obtains

$$(\nabla_Z Ric)(X,Y) - (\nabla_Y Ric)(X,Z) = (divR)(Z,Y,X), \tag{114}$$

and via [HPW12, page 297, Remark 7.2]

$$(divR)(Z,Y,X) = \frac{1}{2(n-2)}(g \odot g)(Z,Y,X) \tag{115}$$

if and only if M has harmonic Weyl tensor. Thus Equation (110) *changes into*

$$(divR)(X,Y,Z) = (divW)(X,Y,Z) + \frac{2}{(n-2)}.\frac{1}{2}\{(divRic)(Z).g(X,Y) - (divRic)(Y)g(X,Y) + (\nabla_Z Ric)(X,Y) - (\nabla_Y Ric)(X,Z)\} - \frac{1}{(n-1)(n-2)}(div(-mn+2m-n)g \odot g)(X,Y,Z) \quad (116)$$

and using equations (113), (114) *and* (115) *it further simplifies to*

$$(divR)(X,Y,Z) = (divW)(X,Y,Z) + \frac{1}{(n-2)}\{\frac{1}{2}g(Z,\nabla Scal)g(X,Y) - \frac{1}{2}g(Y,\nabla Scal)g(X,Z) + \frac{1}{2(n-1)}(g(Z,\nabla Scal)g(Y,X) - g(Z,X)g(Y,\nabla Scal))\} \quad (117)$$

and finally to

$$(divR)(X,Y,Z) = (divW)(X,Y,Z) + \frac{n}{2(n-1)(n-2)}\{g(Z,\nabla Scal)g(Y,X) - g(Z,X)g(Y,\nabla Scal)\}. \quad (118)$$

Since $Scal = -mn+2m-2$ is constant we have that $\nabla Scal = 0$ and consequently from Equation (118) *it follows that*

$$(divR)(X,Y,Z) = (divW)(X,Y,Z), \ \textit{for every } X,Y,Z \quad (119)$$

expressing that div $W = 0$ if and only if div $R = 0$. Therefore these two divergences are equivalent in this example.

Now it is turn to investigate the assumption $W(\nabla f,.,.,\nabla f) = 0$ in terms of the curvature tensor R for the example. As $\nabla f \in \mathfrak{X}(H^2)$, if either X or Y, say X, belongs to $\mathfrak{X}(M_\star)$ then by computation we obtain

$$R(\nabla f,X,Y,\nabla f) = W(\nabla f,X,Y,\nabla f) + \frac{2}{(n-2)}\frac{1}{2}\{Ric(\nabla f,\nabla f)g(X,Y) + Ric(X,Y)g(\nabla f,\nabla f) - Ric(\nabla f,Y)g(X,\nabla f) - Ric(X,\nabla f)g(\nabla f,Y)\} + \frac{mn-2m+n}{(n-1)(n-2)}\{g(\nabla f,\nabla f)g(X,Y) - g(\nabla f,Y)g(X,\nabla f)\} \quad (120)$$

where all terms in { }-signs vanish as X is parallel with respect to vector fields Y, ∇f in $\mathfrak{X}(H^2)$. Since $R(\nabla f,X,Y,\nabla f)$ is a mixed curvature tensor for the Riemannian product $M = M_\star \times H^2$, it vanishes. Therefore $W(\nabla f,X,Y,\nabla f)$ vanishes as well, hence this case of Weyl tenor is satisfied. For a mixed curvature tensor of a product manifold see [O'N10, page 89].

If both $X, Y \in \mathfrak{X}(H^2)$ then via (120) *as well as the relations $Ric(\nabla f, \nabla f) = Ric_{-1}(\nabla f, \nabla f) = -g_{-1}(\nabla f, \nabla f)$ and $Ric(X, Y) = Ric_{-1}(X, Y) = -g_{-1}(X, Y)$ we obtain*

$$\begin{aligned} R(\nabla f, X, Y, \nabla f) &= W(\nabla f, X, Y, \nabla f) + \frac{1}{(n-2)}\{-g_{-1}(\nabla f, \nabla f)g_{-1}(X, Y) \\ &- g_{-1}(X, Y)g_{-1}(\nabla f, \nabla f) + g_{-1}(\nabla f, Y)g_{-1}(X, \nabla f) + g_{-1}(X, \nabla f)g_{-1}(\nabla f, Y)\} \\ &+ \frac{mn - 2m + n}{(n-1)(n-2)}\{g_{-1}(\nabla f, \nabla f)g_{-1}(X, Y) - g_{-1}(\nabla f, Y)g_{-1}(X, \nabla f)\} \end{aligned} \tag{121}$$

which after simplification becomes

$$\begin{aligned} R(\nabla f, X, Y, \nabla f) &= W(\nabla f, X, Y, \nabla f) \\ &+ \frac{m-1}{n-1}\{-g_{-1}(\nabla f, \nabla f)g_{-1}(X, Y) + g_{-1}(\nabla f, Y)g_{-1}(\nabla f, X)\} \end{aligned} \tag{122}$$

implying, when $m = 1$ it follows that $W(\nabla f, X, Y, \nabla f) = 0$ if and only if $R(\nabla f, X, Y, \nabla f) = 0$. Therefore in this case if

$$\begin{aligned} R(\nabla f, X, Y, \nabla f) &= -\frac{m-1}{n-1}\{g_{-1}(\nabla f, \nabla f)g_{-1}(X, Y) - g_{-1}(\nabla f, Y)g_{-1}(\nabla f, X)\} \\ &= -\frac{m-1}{n-1}g_{-1} \odot g_{-1}(\nabla f, X, Y, \nabla f), \end{aligned} \tag{123}$$

then the condition $W(\nabla f, X, Y, \nabla f) = 0$ is satisfied.

We finally reach the case $X, Y \in \mathfrak{X}(M_\star)$. Thus by Equation (109) *and using the relations $Ric(\nabla f, \nabla f) = Ric_{-1}(\nabla f, \nabla f) = -g_{-1}(\nabla f, \nabla f)$, $Ric(X, Y) = Ric_\star(X, Y) = -(m+1)g_\star(X, Y)$, $Ric(X, \nabla f) = 0$ and so on, one obtains*

$$R(\nabla f, X, Y, \nabla f) = W(\nabla f, X, Y, \nabla f) + \frac{2 - m - n}{(n-1)(n-2)}\{g_\star(X, Y)g_{-1}(\nabla f, \nabla f)\} \tag{124}$$

which does not satisfy the condition $W(\nabla f, ., ., \nabla f) = 0$. This is because the left side identically vanishes as $R(\nabla f, X, Y, \nabla f)$ is a mixed curvature for the product manifold $M = M_\star \times H^2$, while on the right hand side if $W(\nabla f, ., ., \nabla f) = 0$, then the rest must also vanishes. But it does not vanish as $2 - m - n = 0$ is not possible, due to $m \geq 1$, $n - 2 \geq 3$.

2) As another similar example which also does not satisfy the conditions on the Weyl tensor we can consider the product $M = M_1 \times M_2$, $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$, where in polar coordinates $g_2 = -dt^2 + sinh^2 dv^2$ with $Ric_2 = g_2$. Also we consider $f_2(t, v) =$ with $t = 0$ as critical point and giving $\nabla^2 f_2 = -f_2 g_2$. The factor M_1 of the product is chosen to be Einstein with $Ric_1 = (m+1)g_1$.

So $Ric = \begin{pmatrix} Ric_1 & 0 \\ 0 & Ric_2 \end{pmatrix} = \begin{pmatrix} (m+1)g_1 & 0 \\ 0 & g_2 \end{pmatrix}$, consequently we can write

$$Ric - (m+1)g = \begin{pmatrix} 0 & 0 \\ 0 & -mg_2 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now if we let $f(x,t,v) = f_2(t) = cosh(t)$ with $x \in M_1$ then $\nabla f = \nabla f(t) = -sinht\frac{\partial}{\partial t}$ and it follows that

$$\nabla^2 f = \begin{pmatrix} 0 & 0 \\ 0 & -fg_2 \end{pmatrix} = \frac{f}{m}(Ric - (m+1)g), \ \ (m+1) = \lambda.$$

Thus every point with $t = 0$ and x arbitrary is critical which implies then that critical points of f are not in general isolated.

Now we investigate the condition $W(\nabla f, ., ., \nabla f) = 0$. Specifically we discuss the case where both $X, Y \in \mathfrak{X}(M_1)$. Using again Equation (109) and noting that dim $M = n$ and $Scal = Scal_1 + Scal_2 = (m+1)(n-2) + 2 = mn - 2m + n$ we obtain:

$$R(\nabla f, X, Y, \nabla f) = W(\nabla f, X, Y, \nabla f) + \frac{m+n-2}{(n-1)(n-2)}\{g_2(\nabla f, \nabla f)g_1(X, Y)\}. \tag{125}$$

Similar to the last example, as $R(\nabla f, ., ., \nabla f)$ is a mixed curvature for the product $M = M_1 \times M_2$ we have $R(\nabla f, X, Y, \nabla f) = 0$. It follows then that $W(\nabla f, X, Y, \nabla f) = 0$ if and only if $\frac{(m+n-2)}{(n-1)(n-2)} = 0$ which is not possible, as $m + n - 2 = 0$ can not happen due to $n > 2$, $m > 1$.

5 New tensors for $(\lambda, n+m)$-Einstein manifolds with $m > 1$ and their applications

In the following we recall some results which are originally structured in the Riemannian setting where either the same formulations work for the pseudo-Riemannian setting or via subtle changes they are adjusted to pseudo-Riemannian manifolds. In each statement we will clarify the setting.

We already know that the scalar curvature of a $(\lambda, n+m)$-Einstein manifold is constant when $m = 1$ (in this case it is $Scal = (n-1)\lambda$). When $m > 1$ we consider the following tensors in the pseudo-Riemannian setting which are originally defined in the Riemannian case by [HPW12, section 5]

$$\begin{aligned} \rho(x) =& \frac{1}{m-1}((n-1)\lambda - Scal) \\ P =& Ric - \rho g. \end{aligned} \tag{126}$$

Using the above equation for ρ one may write Equation (3.12) in [CSW11] in terms of ρ and P, which is the key to prove Proposition 4.41 in the pseudo-Riemannian case:

Proposition 5.1. (*The pseudo-Riemannian version of Equation* (3.12) *in [CSW11]*). *Suppose (M, g, f) is a pseudo-Riemannian $(\lambda, n+m)$-Einstein manifold then we have*

$$\frac{f}{2}\nabla\rho = P(\nabla f). \tag{127}$$

Next we note the following formula for the tensor P.

Proposition 5.2. (*[HPW12], The pseudo-Riemannian version of Proposition 5.4). For a pseudo-Riemannian $(\lambda, n+m)$-Einstein manifold (M, g, f) we have*

$$div(f^{m+1}P) = 0.$$

Proof. Consider an orthonormal basis $\{E_i\}$ with $\epsilon_i = g(E_i, E_i) \in \{\pm 1\}$. Hence

$$\begin{aligned}
&div(f^{m+1}P)(X) = \text{trace } (V \longrightarrow \#(\nabla f^{m+1}P)(V, ., X)) \\
&\sum_i g(E_i, \#(\nabla(f^{m+1}P))(E_i, ., X)) = \sum_i \epsilon_i \nabla_{E_i}(f^{m+1}P)(E_i, X) \\
&= \sum_i \epsilon_i (\nabla_{E_i} f^{m+1})P(E_i, X) + f^{m+1} \sum_i \epsilon_i \nabla_{E_i} P(E_i, X) \\
&= P(\nabla f^{m+1}, X) + f^{m+1} divP(X)
\end{aligned}$$

which in combination with Equation (127) and [O'N10, Corollary 3.54] gives us

$$\begin{aligned}
div(f^{m+1}P) &= f^{m+1} divP + P(\nabla f^{m+1}) \\
&= f^{m+1} div(Ric) - f^{m+1}\nabla\rho + (m+1)f^m P(\nabla f) \\
&= \frac{1}{2} f^{m+1}\nabla Scal - f^{m+1}\nabla\rho + \frac{(m+1)}{2} f^{m+1}\nabla\rho \\
&= -\frac{m-1}{2} f^{m+1}\nabla\rho - f^{m+1}\nabla\rho + \frac{(m+1)}{2} f^{m+1}\nabla\rho \\
&= 0
\end{aligned}$$

□

At this moment we consider a new algebraic curvature tensor Q in the pseudo-Riemannian setting satisfying

$$div(f^{m+1}Q) = 0$$

with the additional property that its trace is a multiple of P. As in [HPW12, section 6] we define it to be

$$\begin{aligned}
Q &= R + \frac{2}{m} Ric \odot g - \frac{(\lambda+\rho)}{m} g \odot g \\
&= R + \frac{2}{m} P \odot g + \frac{(\rho-\lambda)}{m} g \odot g
\end{aligned} \tag{128}$$

where R is the curvature tensor.

The following proposition approves the latter property of Q for pseudo-Riemannian manifolds.

Proposition 5.3. (*[HPW12], The pseudo-Riemannian version of Proposition* 6.1). *Suppose E_i is an orthonormal basis with $\epsilon_i = g(E_i, E_i)$, then we have the following properties in the pseudo-Riemannian case*

$$\sum_{i=1}^{n} \epsilon_i Q(X, E_i, E_i, Y) = \frac{m+n-2}{m} P(X, Y)$$

$$\sum_{i,j=1}^{n} \epsilon_i \epsilon_j Q(E_j, E_i, E_i, E_j) = \frac{m+n-2}{m(m-1)}((m+n-1)Scal - (n(n-1))\lambda)$$

Proof. We consider an orthonormal basis E_i with $\epsilon_i = g(E_i, E_i) \in \{\pm 1\}$ for our argument. By Equation (128) we obtain the following

$$\begin{aligned}
\sum_{i=1}^{n} \epsilon_i Q(X, E_i, E_i, Y) &= Ric(X,Y) + \frac{1}{m}\sum_{i=1}^{n}(Ric(X,Y)\epsilon_i g(E_i,E_i) + \epsilon_i Ric(E_i,E_i)g(X,Y) \\
&- \epsilon_i Ric(X,E_i)g(Y,E_i) - \epsilon_i Ric(Y,E_i)g(X,E_i)) \\
&- \frac{\lambda+\rho}{m}\sum_{i=1}^{n}(g(X,Y)\epsilon_i g(E_i,E_i) - \epsilon_i g(X,E_i)g(Y,E_i)) \\
&= Ric(X,Y) + \frac{1}{m}((n-2)Ric(X,Y) + Scal\ \ g(X,Y)) \\
&- \frac{\lambda+\rho}{m}(n-1)g(X,Y) \\
&= \frac{m+n-2}{m}Ric(X,Y) \\
&+ \frac{1}{m}(Scal - (n-1)\lambda - (n-1)\rho)g(X,Y) \\
&= \frac{m+n-2}{m}(Ric(X,Y) - \rho g(X,Y)). \qquad (129)
\end{aligned}$$

On the other hand the trace of P satisfies the following

$$tr(P)(= Scal - n\rho) = (n-1)\lambda - (m+n-1)\rho$$

which after substitution in (129) and doing some calculations we get the second identity. □

For a pseudo-Riemannian $(\lambda, n+m)$-Einstein manifold the following formula involves Q and P.

Proposition 5.4. (*[HPW12], The pseudo-Riemannian version of Proposition* 6.2). *Suppose (M, g, f) is a pseudo-Riemannian $(\lambda, n+m)$-Einstein manifold. Then we have the following property:*

$$\begin{aligned}
&\frac{f}{m}((\nabla_X P)(Y,Z) - (\nabla_Y P)(X,Z)) \\
&\quad = -Q(X,Y,Z,\nabla f) - \frac{1}{m}(g \odot g)(X,Y,Z,P(\nabla f)).
\end{aligned}$$

Proof. We consider Equation (11) of a pseudo-Riemannian $(\lambda, n+m)$-Einstein metric and do calculation as following:

$$
\begin{aligned}
&R(X,Y,\nabla f,Z)\\
=&(\nabla_X(\frac{f}{m}(Ric-\lambda g)))(Y,Z)-(\nabla_Y(\frac{f}{m}(Ric-\lambda g)))(X,Z)\\
=&\frac{f}{m}((\nabla_X P)(Y,Z)-(\nabla_Y P)(X,Z))\\
&+\frac{1}{m}g(X,\nabla f)P(Y,Z)-\frac{1}{m}g(Y,\nabla f)P(Y,Z)\\
&-\frac{1}{m}g(X,\nabla(f(\lambda-\rho)))g(Y,Z)+\frac{1}{m}g(Y,\nabla(f(\lambda-\rho)))g(X,Z)\\
=&\frac{f}{m}((\nabla_X P)(Y,Z)-(\nabla_Y P)(X,Z))-\frac{\lambda-\rho}{m}(g\odot g)(X,Y,Z,\nabla f)\\
&+\frac{1}{m}g(X,\nabla f)P(Y,Z)-\frac{1}{m}g(Y,\nabla f)P(X,Z)\\
&+\frac{1}{m}g(X,f\nabla\rho)g(Y,Z)-\frac{1}{m}g(Y,f\nabla\rho)g(X,Z)\\
=&\frac{f}{m}((\nabla_X P)(Y,Z)-(\nabla_Y P)(X,Z))-\frac{\lambda-\rho}{m}(g\odot g)(X,Y,Z,\nabla f)\\
&+\frac{2}{m}(P\odot g)(X,Y,Z,\nabla f)+\frac{1}{m}(g\odot g)(X,Y,Z,P(\nabla f)).
\end{aligned}
$$

By transferring the suitable terms of the right side of the last equality to the very left side, i.e. $R(X,Y,\nabla f,Z)$, and by definition of Q in (128) the identity follows. □

Now we turn our attention to the proof of the former property of Q involving the divergence of Q.

Proposition 5.5. (*[HPW12], The pseudo-Riemannian version of Proposition* 6.3). *Suppose we have a pseudo-Riemannian $(\lambda, n+m)$-Einstein manifold (M,g,f). Then*

$$div(f^{m+1}Q)=0.$$

Proof. For an orthonormal basis $\{E_i\}$ with $\epsilon_i=g(E_i,E_i)\in\{\pm1\}$ in the pseudo-Riemannian setting, by definition, one obtains

$$
\begin{aligned}
&div(f^{m+1}Q)(X,Y,Z)=\text{trace }(V\longrightarrow \#(\nabla f^{m+1}Q)(V,.,X,Y,Z))\\
&\sum_i g(E_i,\#(\nabla(f^{m+1}Q))(E_i,.,X,Y,Z))=\sum_i \epsilon_i\nabla_{E_i}(f^{m+1}Q)(E_i,X,Y,Z)\\
&=\sum_i \epsilon_i(\nabla_{E_i}f^{m+1})Q(E_i,X,Y,Z)+f^{m+1}\sum_i\epsilon_i\nabla_{E_i}Q(E_i,X,Y,Z)\\
&=f^m((m+1)Q(X,Y,Z,\nabla f)+f^{m+1}divQ(X,Y,Z)).
\end{aligned}
$$

Hence the assertion is equivalent to the equation

$$f divQ(X,Y,Z)=-(m+1)Q(X,Y,Z,\nabla f). \tag{130}$$

To prove (130) we first calculate the divergence of the second term in (128), i.e. $div(\frac{2}{m}P \odot g)$:

$$\begin{aligned}
f\frac{2}{m}&(div(P \odot g))(X,Y,Z)\\
=&\frac{f}{m}(divP)(X)g(Y,Z) - \frac{f}{m}(divP)(Y)g(X,Z)\\
&+ \frac{f}{m}(\nabla_X P)(Y,Z) - \frac{f}{m}(\nabla_Y P)(X,Z)\\
=&-\frac{m+1}{m}P(X,\nabla f)g(Y,Z) + \frac{m+1}{m}P(Y,\nabla f)g(X,Z)\\
&-(Q(X,Y,Z,\nabla f) + \frac{1}{m}(g \odot g)(X,Y,Z,P(\nabla F)))\\
=&-Q(X,Y,Z,\nabla f) - \frac{m+2}{m}(g \odot g)(X,Y,Z,P(\nabla f))
\end{aligned}$$

and similarly the divergence of the last term, i.e. $div(\frac{(\rho-\lambda)}{m}g \odot g)$, becomes

$$\begin{aligned}
\frac{f}{m}(div((\rho-\lambda)g \odot g))(X,Y,Z) &= \frac{f}{m}(g \odot g)(X,Y,Z,\nabla\rho)\\
&= \frac{2}{m}(g \odot g)(X,Y,Z,P(\nabla f)).
\end{aligned}$$

Now through taking divergence of Equation (128) and using the last equations for the divergence of the last two terms we obtain

$$\begin{aligned}
f(divQ)(X,Y,Z) =&f(divR)(X,Y,Z) - Q(X,Y,Z,\nabla f)\\
&-(g \odot g)(X,Y,Z,P(\nabla f)). \qquad (131)
\end{aligned}$$

Moreover from proposition 5.4 we

$$\begin{aligned}
f(&divR)(X,Y,Z)\\
=&f(\nabla_X Ric)(Y,Z) - f(\nabla_Y Ric)(X,Z)\\
=&f(\nabla_X P)(Y,Z) - f(\nabla_Y P)(X,Z)\\
&+ fg(X,\nabla\rho)g(Y,Z) - fg(Y,\nabla\rho)g(X,Z)\\
=&f(\nabla_X P)(Y,Z) - f(\nabla_Y P)(X,Z) + f(g \odot g)(X,Y,Z,\nabla\rho)\\
=&f(\nabla_X P)(Y,Z) - f(\nabla_Y P)(X,Z) + 2(g \odot g)(X,Y,Z,P(\nabla f))\\
=&-mQ(X,Y,Z,\nabla f) - (g \odot g)(X,Y,Z,P(\nabla f))\\
&+ 2(g \odot g)(X,Y,Z,P(\nabla f))\\
=&-mQ(X,Y,Z,\nabla f) + (g \odot g)(X,Y,Z,P(\nabla f))
\end{aligned}$$

which in combination with Equation (131) above gives us the result. □

In the following we observe applications of the previous calculations and formulations for new assertions. Here we recall a definition.

Definition 5.6. *For a pseudo-Riemannian manifold* (M^n, g) *of dimension* $n \geq 3$ *the Schouten tensor* S *is a* $(0,2)$*-tensor defined by*

$$S = Ric - \frac{Scal}{2(n-1)} g.$$

(M^n, g) *is said to have harmonic Weyl tensor if the Schouten tensor* S *is a Codazzi tensor, i.e*

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) \ \forall X, Y, Z. \tag{132}$$

Remark 5.7. (*[HPW12], The pseudo-Riemannian version of Remark* 7.1). *A three dimensional pseudo-Riemannian manifold* (M^3, g) *has harmonic Weyl tensor if and only if it is locally conformally flat. For dimension* $n = 3$ *we always have* $W = 0$. *Additionally,* $div(W) = 0$ *if and only if* (M^n, g) *has harmonic Weyl tensor.*

Remark 5.8. (*[HPW12], The pseudo-Riemannian version of Remark* 7.2). *A pseudo-Riemannian* (M, g) *has harmonic Weyl tensor if and only if the following holds*

$$divR(X, Y, Z) = \frac{1}{2(n-1)} (g \odot g)(X, Y, Z, \nabla Scal).$$

Next we investigate the Weyl tensor of a one dimensional basis warped product metric with Einstein fibre.

Lemma 5.9. (*[Bes08],* 16.26(i)). *A warped product metric of the form* $g = \epsilon dt^2 + u^2(t) g_N$, *where* g_N *is Einstein, has harmonic Weyl tensor and satisfies* $W(\nabla f, ., ., \nabla f) = 0$.

Proof. As g_N is Einstein, inserting the relation $\partial_t = \frac{\nabla f}{|\nabla f|}$ in Lemma 4.27 immediately implies that $W(\nabla f, ., ., \nabla f) = 0$. In order to show that (M, g) has harmonic Weyl tensor we alternatively prove div$W$=0. For vector fields $X, Y, Z \perp \partial_t$ and an orthonormal frame field $(E_i)_1^n$

$$divW(X, Y, Z) = \sum_i^n \epsilon_i (\nabla_{E_i} W)(E_i, X, Y, Z)$$
$$= \sum_i^n \epsilon_i \Big\{ \nabla_{E_i}(W(E_i, X, Y, Z)) - W(\nabla_{E_i} E_i, X, Y, Z)$$
$$- W(E_i, \nabla_{E_i} X, Y, Z) - W(E_i, X, Y, \nabla_{E_i} Z) \Big\} = 0 - 0 - 0 - 0 = 0.$$

Where in the calculation above we again used Lemma 4.27 and the fact that g_N is Einstein and hence has harmonic Weyl tensor, i.e. $div(W_N) \equiv 0$. □

Harmonic Weyl tensor has an essential role in the main characterizations in this thesis, e.g. in Proposition 7.2 and Theorem 7.9. We first investigate the role of harmonic Weyl tensors in connecting the tensors P and Q.

Proposition 5.10. (*[HPW12], The pseudo-Riemannian version of Proposition* 7.1). *Suppose (M, g, f) is a pseudo-Riemannian $(\lambda, n+m)$-Einstein manifold with harmonic Weyl tensor. Then we have*

$$\begin{aligned} Q(X, Y, Z, \nabla f) &= \frac{m+n-2}{m(n-1)}(P(\nabla f, X)g(Y, Z) - P(\nabla f, Y)g(X, Z)) \\ &= \frac{m+n-2}{m(n-1)}(g \odot g)(X, Y, Z, P(\nabla f)) \end{aligned} \tag{133}$$

Proof. See the last calculation in the proof of Proposition 5.5 which gives us

$$f(divR)(X, Y, Z) = -mQ(X, Y, Z, \nabla f) + (g \odot g)(X, Y, Z, P(\nabla f)) \tag{134}$$

also by Remark 5.8 we obtain

$$\begin{aligned} f(divR)(X, Y, Z) &= f\frac{1}{2(n-1)}(g \odot g)(X, Y, Z, \nabla Scal) \\ &= -f\frac{m-1}{2(n-1)}(g \odot g)(X, Y, Z, \nabla \rho) \\ &= -\frac{m-1}{n-1}(g \odot g)(X, Y, Z, P(\nabla f)). \end{aligned} \tag{135}$$

As the left hand sides of equations (134) and (135) are equal, the right sides will also be equal from which the identity in the pseudo-Riemannian setting follows. □

By inspiration of Proposition 5.10 we have the first important property of a pseudo-Riemannian $(\lambda, n+m)$-Einstein manifold under the assumption of harmonicity of the Weyl tensor which will be used in our characterizations.

Lemma 5.11. (*[HPW12], The pseudo-Riemannian version of Corollary* 7.1). *Suppose (M, g, f) is a pseudo-Riemannian $(\lambda$,n+$m)$-Einstein manifold with harmonic Weyl tensor. Then at a point where $\nabla f \neq 0$, ∇f is an eigenvector for the tensor P. Additionally for vector fields $X, Y, Z \perp \nabla f$ we have*

$$Q(X, Y, Z, \nabla f) = 0 \tag{136}$$

$$Q(\nabla f, Y, Z, \nabla f) = \frac{m+n-2}{m(n-1)}P(\nabla f, \nabla f)g(Y, Z) \tag{137}$$

Proof. Let $Z = \nabla f$ in (133) for a pseudo-Riemannian manifold. This gives us

$$P(\nabla f, X)g(\nabla f, Y) - P(\nabla f, Y)g(X, \nabla f) = 0 \quad \text{for any } X, Y$$

which expresses that ∇f is an eigenvector for P. Hence $P(X, \nabla f) = 0$ for $X \perp \nabla f$. Respectively, using this in (133) for $X, Y, Z \perp \nabla f$ gives us Equation (136). Also, if we let $X = \nabla f$ and $Y, Z \perp \nabla f$ in (133) then we obtain (137). □

Remark 5.12. (*[HPW12], The pseudo-Riemannian version of Remark* 7.3). *∇f is an eigenfield for the tensor P (or for the Schouten tensor) if and only if ∇f is an eigenfield for the tensor Hess f. If this holds then $|\nabla f|^2$ is constant on the connected components of the level sets of f. Because for any $X \perp \nabla f$*

$$\nabla_X |\nabla f|^2 = 2Hessf(\nabla f, X) = 2\mu_1 g(\nabla f, X) = 0$$

where, using Lemma 5.11, μ_1 denotes the eigenvalue of Hessf with respect to ∇f (Moreover if the normal vector field ∇f is null on some level set, say $f^{-1}(c)$, then $|\nabla f|^2$ is the constant zero on it).

In [HPW12] the reason for taking repeatedly the Weyl tensor into consideration as an assumption is to control the other eigenvalues of the tensor P (other than ∇f), hence those of Ricci and Schouten tensors by Equation (11). Similarly here in the pseudo-Riemannian case we use the same assumptions on the Weyl tensor for the same purpose, i.e. to get control on the number of eigenvalues of the tensor P and consequently of Ricci tensor. Next result presents the decomposition of Q in terms of the Weyl tensor W and P.

Proposition 5.13. (*[HPW12], The pseudo-Riemannian version of Proposition* 7.2). *Let $m > 1$ and let (M, g, f) be a pseudo-Riemannian $(\lambda, n+m)$-Einstein manifold with harmonic Weyl tensor, then we have*

$$Q = W + \frac{2(n+m-2)}{m(n-2)}(P \odot g) - \frac{n+m-2}{m(n-1)(n-2)} tr(P)(g \odot g). \qquad (138)$$

Proof. In the pseudo-Riemannian setting, by definition we have

$$Q = R + \frac{2}{m} P \odot g + \frac{\rho - \lambda}{m} g \odot g,$$
$$R = W + \frac{2}{n-2} P \odot g + (\frac{2\rho}{n-2} - \frac{Scal}{(n-1)(n-2)}) g \odot g.$$

Combining these two equations together and using the relations

$$Scal = (n-1)\lambda - (m-1)\rho,$$
$$tr(P) = -(m+n-1)\rho + (n-1)\lambda$$

we obtain

$$\begin{aligned} Q =& W + \frac{2(m+n-2)}{m(n-2)}(P \odot g) \\ &+ (\frac{((n+2m-2)(n-1) + m(m-1))\rho}{m(n-1)(n-2)} - \frac{(n+m-2)\lambda}{m(n-2)}) g \odot g \\ =& W + \frac{2(m+n-2)}{m(n-2)}(P \odot g) \\ &+ \frac{m+n-2}{m(n-1)(n-2)}((m+n-1)\rho - (n-1)\lambda) g \odot g \\ =& W + \frac{2(m+n-2)}{m(n-2)}(P \odot g) - \frac{m+n-2}{m(n-1)(n-2)} tr(P) g \odot g. \end{aligned}$$

which completes the proof. □

At this step, we see that via the Weyl tensor there exists a subset of a pseudo-Riemannian manifold (M, g) consisting of points at which the number of mutually distinct eigenvalues of the tensor P (and consequently the Hessian tensor by Equation (11)) is at most two.

Lemma 5.14. (*The pseudo-Riemannian version of [HPW12, Lemma 7.1]*). *Let (M, g) be a pseudo-Riemannian manifold. Also let (M, g, f) be a $(\lambda, n+m)$-Einstein manifold with harmonic Weyl tensor and $W(\nabla f, Y, Z, \nabla f) = 0$. Then at a point p where $g(\nabla f, \nabla f) \neq 0$, P (or Ricci tensor or Schouten tensor) has at most two eigenvalues. If it has two eigenvalues then one has multiplicity 1 with eigenvector ∇f, and the other one has multiplicity $n-1$ with vectors orthogonal to ∇f. If it has only one eigenvalue then it is with multiplicity n and* Ric *is proportional to g.*

Proof. By Lemma 5.11 we already know that ∇f is an eigenvector for P at p. By assumption we have $W(\nabla f, Y, Z, \nabla f) = 0$, therefore using (138) for vector fields $Y, Z \perp \nabla f$ we obtain

$$\begin{aligned} Q(\nabla f, Y, Z, \nabla f) =& \frac{2(n+m-2)}{m(n-2)}(P \odot g)(\nabla f, Y, Z, \nabla f) \\ &- \frac{n+m-2}{m(n-1)(n-2)} tr(P)(g \odot g)(\nabla f, Y, Z, \nabla f) \\ =& \frac{(n+m-2)}{m(n-2)}(P(\nabla f, \nabla f)g(Y, Z) + P(Y, Z)|\nabla f|^2) \\ &- \frac{n+m-2}{m(n-1)(n-2)} tr(P)|\nabla f|^2 g(Y, Z). \end{aligned}$$

Also remember equation (137), i.e.

$$Q(\nabla f, Y, Z, \nabla f) = \frac{m+n-2}{m(n-1)} P(\nabla f, \nabla f)g(Y, Z).$$

Through equating the right sides of the last two equations, as their left hand sides are equal, it follows that

$$(n-1)P(Y, Z)|\nabla f|^2 = (tr(P)|\nabla f|^2 - P(\nabla f, \nabla f))g(Y, Z)$$

expressing that Y and Z are eigenvectors for P and have the same eigenvalue. If P has only on eigenvalue, say μ, at p then $P = Ric - \rho g = \mu g$ saying μ has multiplicity n, and also, $Ric = (\rho + \mu)g$. Moreover we get the same results for Hess$f$ using the identity

$$Hessf = \frac{f}{m}(P + (\rho - \lambda)g)$$

and respectively the same results for the Schouten tensor S by

$$Hessf = \frac{f}{m}(S + (\frac{Scal}{2(n-1)} - \lambda)g)$$

where by definition $P(X, Y) = Ric(X, Y) - \rho g(X, Y)$, and $S(X, Y) = Ric(X, Y) - \frac{Scal}{2(n-1)} g(X, Y)$ for all vector fields X, $Y \in TM$. □

Under the assumptions in Lemma 5.14 we saw that the tensor P, and hence the Schouten tensor S, has at most two distinct eigenvalues. In case there are two different eigenvalues, denote by σ_1 and σ_2 the eigenvalue functions of S and let

$$O = \{x \in M : g(\nabla f, \nabla f)_x \neq 0, \sigma_1(x) \neq \sigma_2(x)\}.$$

Note that for a Riemannian manifold (M, g) the condition $g(\nabla f, \nabla f) \neq 0$ reduces naturally to $\nabla f \neq 0$. If (M, g) is Einstein, then there is only one eigenvalue, i.e. $\sigma_1 = \sigma_2$. In this case Hess$f$ is proportional to the metric, cf. Proposition 4.45 in the Riemannian case or [KR09, Section 4 and Section 5] in the pseudo-Riemannian setting for the relevant results.

Remark 5.15. *Under the assumptions of Lemma 5.14, at a point p where $g(\nabla f, \nabla f) \neq 0$ the Hessian tensor Hess$f$ of f has at most two eigenvalues, say μ_1 and μ_2, such that ∇f is an eigenvector with eigenvalue μ_1 and vectors orthogonal to this direction, i.e. $X \perp \nabla f$, correspond to eigenvalue μ_2. In more details, for $X, Y \perp \nabla f$*

$$\begin{aligned} Hessf(\nabla f, X) &= g(\nabla_{\nabla f} \nabla f, X) = \mu_1 g(\nabla f, X) \\ \Rightarrow \nabla_{\nabla f} \nabla f &= \mu_1 \nabla f \qquad (139) \\ Hessf(X, Y) &= g(\nabla_X \nabla f, Y) = \mu_2 g(X, Y) \\ \Rightarrow \nabla_X \nabla f &= \mu_2 X. \qquad (140) \end{aligned}$$

6 Local and global characterizations of Riemannian $(\lambda, n+m)$-Einstein metrics and some investigation

In this section we first shortly investigate geodesic polar coordinates system which is an essential key in upcoming discussions. Afterwards we inquire local and global characterizations in the Riemannian case. In addition, we give a brief investigation on $(\lambda, n+m)$-Einstein metrics of constant scalar curvature.

6.1 Geodesic polar coordinates on pseudo-Riemannian manifolds and some applications

Let $\mathbb{R}^n_k$ be the pseudo-Euclidean space of signature k with the standard metric, i.e. $g(x, x) = -(x_1^2 + ... + x_k^2) + x_{k+1}^2 + ... + x_n^2$ for $x \in \mathbb{R}^n$. For $n \geq 2, \epsilon \in \{\pm 1\}$ let $S(\epsilon) := \{x \in \mathbb{R}^n_k | g(x, x) = \epsilon\}$ and $|x| := \sqrt{|g(x, x)|} \geq 0$ denotes the pseudo-norm.

Similar to [KR97a, Section 3] we denote by $S^0(1)$ the connected component of $S(1)$ which contains the point $(0, ..., 0, 1)$ and by $S^0(-1)$ the connected component of $S(-1)$ containing $(1, 0, ..., 0)$. Then we let $\Sigma := S^0(1) \cup S^0(-1)$ and $C := \{x \in \mathbb{R}^n_k | g(x, x) = 0\}$ be the light cone. We have the polar coordinates on the complement of the light cone $\mathbb{R}^n_k - C$ in the pseudo-Euclidean space $\mathbb{R}^n_k$ as follows

$$y \in \mathbb{R}^n_k - C \to \Phi(y) = (r(y), \phi(y)) \in \mathbb{R} \times \Sigma$$

where $r(y)$ denotes the radial part, i.e. the absolute value of $r(y)$ is the pseudo-norm $|y|$. For a detailed investigation on the image $G := \Phi(\mathbb{R}^n_k - C) \subset \mathbb{R} \times \Sigma$ of the polar coordinates see [KR97a, page 8].

Similar to [KR97a] we consider geodesic polar coordinates around any point $p \in M$ of a pseudo-Riemannian manifold using the exponential map. $C_p := \{X \in T_pM | g(X,X) = 0\}$ denoting the light cone at p, there exists an open neighborhood $\tilde{U}$ of the zero vector in $T_pM \cong \mathbb{R}^n_k$ so that

$$\phi : \Phi(\tilde{U} - C_p) \subset G \longrightarrow U \subset M$$

$\phi(t,x) = exp_p(\Phi^{-1}(t,x))$ defines geodesic polar coordinates.

Remark 6.1. *In these coordinates we consider local warped structure of the form* $\epsilon dt^2 + u_\epsilon(t)^2.g_1(x)$, $\epsilon \in \{\pm 1\}$, $(t,x) \in G \subset \mathbb{R} \times \Sigma$, *where* g_1 *is the standard metric on* Σ *and where* $u_\epsilon(t) = 0$ *on the light cone on which* $t = 0$.

First we study a pair of functions $u_\pm$ which define a smooth metric in a neighborhood of the origin, i.e. sufficient and necessary conditions for the functions $u_\pm$ such that the metric extends onto the light cone.

NOTE: In the following unlike the argument in [KR97a, Section 3] the function $u(t)$ is odd, i.e. its even derivatives vanish $u(0) = 0, u''(0) = 0...$, just for adjusting to this text as the warping functions is $u(t)$ rather than $u'(t)$ (in [KR97a] the authors use the warping function ψ' wherever there is a critical point for ψ).

Definition 6.2. (*[KR97a], Definition* 3.1). *1.) We define the following set* $\mathscr{F}$ *of two* C^∞*-functions* $u = (u_+, u_-) : \mathbb{R} \longrightarrow \mathbb{R}$ *which satisfies the following conditions:* $u_\pm^{(2m)}(0) = 0$, $u_+^{(2m+1)}(0) = (-1)^{m+1} u_-^{(2m+1)}(0)$ *for all* $m \geq 0$ *and* $u'_+(0) = -u'_-(0) \neq 0$.

2.) We define the set $A_u \subset \mathbb{R}^n_k - C$ *in geodesic polar coordinates* $(t,x) \in G \subset \mathbb{R} \times \Sigma$ *such that:* $(t,x) \in A_u$ *if and only if* $u_\epsilon, \epsilon = g(x,x)$ *does not vanish between 0 and* t.

Lemma 6.3. (*[KR97a], Lemma* 3.4). *Let a smooth pseudo-Riemannian metric* g *be given in geodesic polar coordinates* $(t,x) \in G \subset \mathbb{R} \times \Sigma$ *by*

$$g(t,x) = \epsilon dt^2 + \frac{u_\epsilon(t)^2}{u'_\epsilon(0)^2} g_* \tag{141}$$

with a C^∞*-metric* g_* *on* Σ. *Then* g_* *coincides with the standard metric* g_1 *on* Σ *of constant sectional curvature* ϵ.

Proof. Suppose σ is a plane spanned by the orthogonal vectors X, Y with $\epsilon = g(X,X)g(Y,Y) = \pm 1$ which are both orthogonal at $t = t_0$ to the radial geodesic $t \mapsto (t, x_0)$ for a fixed $x_0 \in \Sigma$. Also let $K(\sigma)$ resp. $K_*(\sigma)$ denotes the sectional curvature of σ in (M, g) resp. in (Σ, g_*). From formula (36) in Lemma 4.23 we have the following for the curvature of such a warped product:

$$K(\sigma) = g(R(X,Y)Y,X) = g(R_*(X,Y)Y,X) - \epsilon(\frac{u'_\epsilon(t_0)}{u_\epsilon(t_0)})^2 \tag{142}$$

$$= \frac{\epsilon}{u_\epsilon(t_0)^2}\Big(K_*(\sigma)u'_\epsilon(0)^2 - \epsilon u'_\epsilon(t_0)^2\Big). \tag{143}$$

Since $K_*(\sigma)$ is independent of t, when $t_0 \to 0$ by non-degeneracy of the metric it follows that $K_*(\sigma) = \epsilon$. Hence (Σ, g_*) has constant sectional curvature ϵ, i.e. g_* is isometric to g_1. $\square$

Now we recall an assertion which is essential for extending a metric in geodesic polar coordinates onto the light cone.

Proposition 6.4. (*[KR97a], some part of Proposition* 3.5). *Let $u_\pm$ be two smooth real functions with $u'_+(0) = -u'_-(0)$. Then we define the function $u(t,x) = u_\epsilon(t)$, $\epsilon \in \{\pm 1\}$, on the complement $\mathbb{R}^n_k - C$ of the light cone C in the pseudo-Euclidean space. Here $(t,x) \in G \subset \mathbb{R} \times \Sigma$ are geodesic polar coordinates of the pseudo-Euclidean space. We also define the metric*

$$g(t,x) := g_u(t,x) = \epsilon dt^2 + \frac{u_\epsilon(t)^2}{u'_\epsilon(0)^2} g_1$$

where $\epsilon = g(x,x) \in \{\pm 1\}$ on the subset A_u, on which u_ϵ does not vanish, see Definition 6.2 2.)*. Then the following holds:*

1. The function u extends smoothly onto $B_u := A_u \cup C \subset \mathbb{R}^n_k$, i.e. onto the light cone, if and only if $u_\pm \in \mathscr{F}$, i.e.

$$u_\pm^{(2m)}(0) = 0, u_+^{(2m+1)}(0) = (-1)^{m+1} u_-^{(2m+1)}(0) \tag{144}$$

for all $m \geq 0$.

2. The metric g_u extends smoothly onto the light cone if and only if $u_\pm \in \mathscr{F}$ and g_u is conformally flat.

6.2 Local characterization of Riemannian $(\lambda, n{+}m)$-Einstein metrics (M, g, f) around regular points of f based on [HPW12, Theorem 7.1]

[HPW12, Theorem 7.1] gives local forms of the metric g and f for a Riemannian $(\lambda, n+m)$-Einstein metric (M, g, f) where it has harmonic Weyl tensor and satisfies $W(\nabla f, ., ., \nabla f) = 0$ around points in the set O. We recall that $O = \{x \in$

$M : g(\nabla f, \nabla f)_x \neq 0, \sigma_1(x) \neq \sigma_2(x)\}$ where σ_1, σ_2 represent the first and the second eigenvalues of the Schouten tensor respectively. We generalize it to a local characterization with the weaker condition $\nabla f|_p \neq 0$, instead of the stronger one $p \in O$ which is used in [HPW12, Theorem 7.1]. In this generalization, in addition we formulate relation between the warping function of a local warped product form of g and the function f.

Theorem 6.5. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g) be a Riemannian manifold with a smooth function f defined on M. Then the following conditions are equivalent:*

1) (M, g, f) satisfies Equation (11) *of a non-trivial $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ in a neighborhood of $p \in M$ with $\nabla f|_p \neq 0$.*

2) (a) Case $p \in int(M)$: There exist local coordinates $(t, t_1, ..., t_{n-1})$ with $t \in (-\varepsilon, \varepsilon)$ in a neighborhood of $p \in M$ and an Einstein Riemannian hypersurface $(N, g_N = g_N(t_1, ..., t_{n-1}))$ of (M, g) with normalized scalar curvature $\varrho_N = k$ and a function $u = u(t) > 0$, in addition $f = f(t) > 0$ satisfying (55) *and* (56) *in Proposition 4.35 such that*

I) $g(\partial_t, \partial_t) = 1$
II) $g(\partial_t, \partial_{t_i}) = 0, \quad$ *for* $i = 1, ..., n-1$
III) $g(\partial_{t_i}, \partial_{t_j}) = u^2(t) g_N(\partial_{t_i}, \partial_{t_j})(t_1, ..., t_{n-1}) \quad i, j = 1, ..., n-1.$

(b) Case $p \in \partial M$: There exist local coordinates $(t, t_1, ..., t_{n-1})$ with $t \in [0, \varepsilon)$ in a neighborhood of p and an Einstein Riemannian hypersurface $(N, g_N = g_N(t_1, ..., t_{n-1}))$ of (M, g) with normalized scalar curvature $\varrho_N = k$ and a function $u = u(t) > 0$, in addition $f(t) > 0$ for all $t \in (0, \varepsilon)$ satisfying (55) & (56) *as well as $f(0) = 0$ satisfying* (57) *at $t = 0$ such that the conditions I, II, III in* (a) *holds.*

Any case of 2) implies that $g = dt^2 + u^2(t) g_N$ around p. If the metric is locally conformally flat in a neighborhood of p then N must be a space of constant curvature.

Proof. 1) $\Rightarrow$ 2): By Lemma 5.14 the Schouten tensor has at most two eigenvalues σ_1 and σ_2 at each point in a neighborhood $\widetilde{\mathscr{U}}$ of p consisting of regular points of f. If $\sigma_1 = \sigma_2$ in an open subset U of $\widetilde{\mathscr{U}}$ then via Schur's lemma g is Einstein on U. In addition, the derivative of $\sigma = \sigma_1 - \sigma_2 = 0$ vanishes on U. By analyticity of g, see [HPW12, Proposition 2.4], $d\sigma$ and hence σ vanish on $\widetilde{\mathscr{U}}$. Therefore it would be Einstein, i.e. $\sigma_1 = \sigma_2$, on the whole $\widetilde{\mathscr{U}}$. In this case, using Corollary 4.42 and Example 4.45 we see that the conclusion of this theorem is satisfied. Therefore we can assume that the open set $O \cap \widetilde{\mathscr{U}}$ is dense in $\widetilde{\mathscr{U}}$.

For more convenience we first suppose that $p \in int(M)$. Without loss of generality assume $p \in O \cap \widetilde{\mathscr{U}}$ (otherwise we may start by a point $p_1 \in O \cap \widetilde{\mathscr{U}}$ and through the same procedure as follows we get the same result on $\widetilde{\mathscr{U}} \ni p$

including the point p) , thus the Schouten tensor S has two different eigenvalues σ_1 and σ_2 in a neighborhood $\mathscr{U}$ of p in $\widetilde{\mathscr{U}}$. By Lemma 5.14 we know also that the dimension of the eigenspace corresponding to σ_2 is bigger than one when $\dim M > 1$, hence [Bes08, 16.11(iii)] proves that σ_2 is locally constant on the level sets of f in $\mathscr{U}$.

As the Schouten tensor S has two distinct eigenvalues in $\mathscr{U}$, via the relation $Hessf = \frac{f}{m}(S + (\frac{Scal}{2(n-1)} - \lambda)g)$ it follows that Hess$f$ has also two distinct eigenvalues in $\mathscr{U}$, call them μ_1 and μ_2, where the eigenspaces for μ_i correspond to eigenspaces for σ_i by

$$\mu_i = \frac{f}{m}(\sigma_i + \frac{Scal}{2(n-1)} - \lambda) \quad i = 1, 2. \tag{145}$$

We already know by Remark 5.12 that $|\nabla f|$ is locally constant on the level sets of f in $\mathscr{U}$ which in turn concludes that μ_1 is also locally constant on the level sets of f in $\mathscr{U}$. In more details, from

$$\frac{1}{2}D_{\nabla f}|\nabla f|^2 = Hessf(\nabla f, \nabla f) = \mu_1|\nabla f|^2$$

we get

$$\mu_1 = \frac{1}{2}\frac{1}{|\nabla f|^2}D_{\nabla f}|\nabla f|^2$$

and hence

$$D_X\mu_1 = \frac{1}{2}\frac{1}{|\nabla f|^2}D_X D_{\nabla f}|\nabla f|^2 = \frac{1}{2}\frac{1}{|\nabla f|^2}D_{\nabla f}D_X|\nabla f|^2 = 0, \;\; X \perp \nabla f. \tag{146}$$

Moreover if $X \perp \nabla f$ then

$$D_X\rho = \frac{2}{f}P(\nabla f, X) = 0$$

expressing that ρ and hence the scalar curvature $Scal$ are locally constant on the level sets of f, hence by Equation (145) σ_1 and μ_2 are locally constant on the level sets of f. So $|\nabla f|^2$, μ_1 and μ_2 are all locally constant on the level sets of f in $\mathscr{U}$.

Let $c := f(p)$ and $N \subset \mathscr{U}$ be the connected component of $f^{-1}(c)$ containing p in $\mathscr{U}$. Since $|\nabla f| \neq 0$ on N it follows that (N, g_N) is a Riemannian hypersurface of (M, g). One can choose a coordinate chart $(t_1, ..., t_{n-1})$ on the level hypersurface N. We are interested to extend this chart to a neighborhood of p in M using f. For that purpose, we note that as the norm $|\nabla f|$ is locally constant on the level sets of f in $\mathscr{U}$ it may be considered as a function of f, i.e. $|\nabla f| = |\nabla f|(f)$. It follows that

$$d(\frac{1}{|\nabla f|}df) = d(\frac{1}{|\nabla f|}) \wedge df = -\frac{d|\nabla f|}{|\nabla f|^2}df \wedge df = 0 \tag{147}$$

meaning that $\frac{1}{|\nabla f|}df$ is locally closed and hence exact in $\mathscr{U}$. Therefore, there exists a smooth function t on $\mathscr{U}$ such that

$$dt = \frac{1}{|\nabla f|}df \text{ with } t = t(f) = \int_N^f \frac{df}{|\nabla f|} \tag{148}$$

respectively $|\nabla t| = 1$. Additionally, the symmetry of Hess t together with the equation $|\nabla t| = 1$ implies

$$Hesst(\nabla t, X) = g(\nabla_{\nabla t}\nabla t, X) = g(\nabla_X \nabla t, \nabla t) = \frac{1}{2}\nabla_X |\nabla t|^2 = 0; \;\; X \in TM$$

from which by the non-degeneracy property of g it follows that $\nabla_{\nabla t}\nabla t = 0$. Accordingly the trajectories of $\frac{\nabla f}{|\nabla f|}$, i.e. integral curves of ∇t, are geodesics which are normal to level sets of f in $\mathscr{U}$.

Consequently we may extend the coordinate chart on N to geodesic parallel coordinates $(t, t_1, ..., t_{n-1})$ in a neighborhood of p satisfying:
- the t-lines are geodesics with t as arc length.
-$\frac{\partial}{\partial t}$ is orthogonal to every set $\{(t, t_1, ..., t_{n-1})| \;\; t = constant\}$, i.e. $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t_i}) = 0, \;\; i = 1, ..., n-1$.

This shows that the different t-levels are parallel to each other and the distance between them equals the difference of t-values.

Now consider the f-levels $\{q| \;\; f(q) = constant\}$ where the t-level containing p, for which $t = 0$, coincides with $f^{-1}(c)$. As $|\nabla f|$ is constant along the level sets of f in $\mathscr{U}$, see Remark 5.12, they are also parallel to each other. Therefore, the t-levels coincide with the f-levels and we can consider f as a function of t alone:

$$f(t, x) = f(t) \text{ and } \nabla f(t, x) = f'(t)\frac{\partial}{\partial t} \tag{149}$$

where $f > 0$ because M is boundaryless. Since μ_1 and μ_2 are locally constant on the level sets of f in $\mathscr{U}$ they may also be considered as functions of f by which (149) it follows then that they are functions of t, e.g. $\mu_2(t) = (\mu_2 o f)(t)$, which would be given then by (58) & (59).

From $dt = \frac{1}{|\nabla f|}df$, i.e. t a distance function, it follows that the metric g in $\mathscr{U} \ni p$ can be decomposed into

$$g = \frac{1}{|\nabla f|^2}df \otimes df + g_f \tag{150}$$

where g_f represents a Riemannian metric on a level set of f in $\mathscr{U}$ with tangent space orthogonal complement to the space generated by the unit normal vector field $\frac{\nabla f}{|\nabla f|}$. By (150) and the fact that the eigenvalue μ_1 of Hess$f$ corresponds to ∇f and μ_2 corresponds to vector fields orthogonal to it, we obtain

$$Hessf = \mu_1.\frac{1}{|\nabla f|^2}df \otimes df + \mu_2 g_f. \tag{151}$$

From Equation (151) one obtains $\mathcal{L}_{\nabla f} g_f = 2Hessf|_{g_f} = 2\mu_2 g_f$ by which Equation (150) gives out a local warped structure in $\mathscr{U}$ in terms of t:

We replace the first term on the line element (150) by $\frac{df}{|\nabla f|} = dt$ giving

$$g = dt^2 + g_t \tag{152}$$

where $t = 0$ corresponds to $N \subset f^{-1}(c)$. Now we work on g_t to acquire the desired structure. Let X be a lift of a vector field on N, then $g(\partial t, X) = 0$ by the Gauss-Lemma. Also for vectors X_1, X_2 tangent to N at x_0 let $X_i(t) = d\ exp(t, x_0)(X_i), i = 1, 2$ then

$$\begin{aligned} \frac{d}{dt}|_{t=s} g(X_1, X_2)(t) &= \mathcal{L}_{\partial t} g(X_1, X_2)(s) = \frac{1}{f'(s)} \mathcal{L}_{\nabla f} g(X_1, X_2)(s) \\ &= \frac{2}{f'(s)} Hessf(X_1, X_2)(s) = \frac{2}{f'(s)} \mu_2(s) g(X_1, X_2)(s) \end{aligned} \tag{153}$$

where $\mathcal{L}_Z g(X, Y) = g(\nabla_X Z, Y) + g(X, \nabla_Y Z)$ is the Lie derivative of the metric in direction of the vector field Z. By an integration step from (153) we obtain

$$g_t = \left(e^{\int_0^t \frac{1}{f'(s)} \mu_2(s) ds}\right)^2 g_{N \subset f^{-1}(c)}.$$

Therefore we may write (152) as the warped structure

$$g = dt^2 + u^2(t) g_N, \;\; t \in (-\varepsilon, \varepsilon) \tag{154}$$

where

$$u(t) = e^{\int_0^t \frac{1}{f'(s)} \mu_2(s) ds}. \tag{155}$$

To confirm (154) is a warped product metric it remains to show that g_N is independent of t, and also, is non-degenerate. For X_1 and X_2 as above, the mapping $t \longmapsto (u(t))^{-2} g(X_1, X_2)(t)$ satisfies the differential equation

$$\left(\frac{g(X_1, X_2)}{u^2}\right)'(t) = \frac{\frac{d}{ds}|_{s=t} g(X_1, X_2)(s)}{u^2(t)} - \frac{2u'(t)}{u^3(t)} g(X_1, X_2)(t) = 0 \tag{156}$$

expressing $g_N(X_1, X_2) = \frac{g(X_1, X_2)}{u^2(t)}$ is independent of the coordinate function t and hence one may introduce this expression as $g_N(t_1, ..., t_{n-1})$. To see g_N is non-degenerate, suppose $g_N(X, Y) = 0$ for some X and all Y tangent to N. On the other hand by (154) we have $g(X, \partial t) = 0$. As the metric g is non-degenerate we obtain $X = 0$. By Proposition 4.35 the equations (55) and (56) are satisfied for a boundaryless manifold.

Now suppose that $p \in \partial M$. Then through the same discussion as above and noting that due to $N \subseteq \partial M$ the trajectory geodesics of ∇f starting at N point only to one side, e.g. to its positive side, it follows that there exist geodesic parallel coordinates $(t, t_1, t_2, ..., t_{n-1})$ in a neighborhood of p with $t \in [0, \epsilon)$,

$t(p) = 0$ for which the result holds. Moreover the conditions (55), (56), (57) in Proposition 4.35 hold on $\mathscr{U}$.

Considering the way of defining the function t at above, we see that t is smooth as far as $\nabla f \neq 0$, cf. Equation (148), including the points at which $\sigma_1 = \sigma_2$.

By smoothness of the metric g and t it follows that we have a warped product, where g_N is an Einstein Riemannian hypersurface of (M, g), along all of t as long as $\nabla f \neq 0$, therefore on the whole $\widetilde{\mathscr{U}}$. In parallel, to extend the relations $f = f(t)$, and (55), (56), (57) as far as $\nabla f \neq 0$ (so on the whole $\widetilde{\mathscr{U}}$) we use again a similar discussion as above.

Now we see that any metric of this form whose Schouten and hence the Ricci tensor has two distinct eigenvalues must have g_N Einstein. By Lemma 4.23.(41) the first eigenvalue of the Ricci tensor would be $\gamma_1 = -(n-1)\frac{u''(t)}{u(t)}$. As the second eigenvalue γ_2 corresponds to vectors $X, Y \perp \partial_t$, using Lemma 4.23.(36) we have

$$Ric(X,Y) = \gamma_2 g(X,Y) = Ric_N(X,Y) - \frac{1}{u^2}[(n-2)u'^2 + uu'']g(X,Y)$$

thus

$$Ric_N(X,Y) = (\gamma_2 + \frac{1}{u^2}[(n-2)u'^2 + uu''])g(X,Y)$$

implying that g_N is Einstein. Moreover, using Lemma 4.29 a metric in this form is conformally flat if and only if g_N has constant curvature. Therefore the conclusion of the theorem is satisfied on the whole $\widetilde{\mathscr{U}}$.

2) $\Rightarrow$ 1): Suppose by contradiction that $\nabla f(p) = 0$. If it is Case (b) then by Proposition 4.9 it is a contradiction. If it Case (a), then we may consider geodesic polar coordinates with origin at p, hence it must satisfy $u(a) = 0$, $a = t(p)$, cf. Remark 6.1. This is a contradiction with the assumption.

Let the metric be isometrically $g = dt^2 + u^2(t)g_N$ where g_N is an Einstein hypersurface with $\varrho_N = k$, and equations (55), (56), (57) be satisfied by $u = u(t) > 0$ and $f = f(t) \geq 0$. Then Proposition 4.35 implies that (M, g, f) satisfies Equation (11) of $(\lambda, n+m)$-Einstein manifolds around p.

By Lemma 5.9 the manifold (M, g) has harmonic Weyl tensor and satisfies $W(\nabla f, ., ., \nabla f) = 0$ around p. □

In the Riemannian case, Catino proved existence of a local warped product metric with $(n-1)$ dimensional Einstein fibre around regular points of f in a $(\lambda, n+m)$-Einstein manifold (M, g, f), see [Cat12, Theorem 1.1]. In this Theorem Catino assumes harmonicity of the Weyl tensor W and $W(\nabla f, ., ., .) = 0$. The assumption $W(\nabla f, ., ., .) = 0$ by Catino is stronger than the corresponding one, i.e. $W(\nabla f, ., ., \nabla f) = 0$, in [HPW12, Theorem 7.1] and Theorem 6.5. Also, the condition $p \in O$ in [HPW12, Theorem 7.1] is stronger than the condition $\nabla f|_p \neq 0$ in Theorem 6.5. Therefore, Theorem 6.5 is a stronger and more general result.

Lemma 6.6. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g, f) satisfies Equation (11) of a non-trivial $(\lambda, n+m)$-Einstein manifold with harmonic Weyl tensor and $(\nabla f, ., ., \nabla f) = 0$. In addition, suppose that for a coordinate system $(t, t_1, t_2, ..., t_{n-1})$ the metric is of the form*

$$g = dt^2 + u^2(t)g_N, \ \ t \in [-\epsilon, \epsilon], \ \ \epsilon \in \mathbb{R}^+ \tag{157}$$

where $u(t) > 0$ on $(-\epsilon, \epsilon)$ and the function satisfies

$$f(t, x) = f(t), \ \ x \in N. \tag{158}$$

If $u(a) = 0$, $a \in \{-\epsilon, \epsilon\}$, then $f'(a) = 0$.

Proof. By contradiction suppose that $f'(a) \neq 0$. Then by the proof of Theorem 6.5 we see that the warped product is extendable to a neighborhood of $t = a$, hence the warping function must satisfy $u(a) > 0$, which is a contradiction to the assumption $u(a) = 0$. □

Corollary 6.7. *Under the same assumptions as Theorem 6.5.1) we have $u(t) > 0$ if and only if $f'(t) \neq 0$ (equivalently $u(t_0) = 0$ if and only if $f'(t_0) = 0$ where t_0 is the first zero for them).*

Proof. By combination of Theorem 6.5, Lemma 6.6 and Remark 6.1 the result follows. □

6.3 Local characterization of Riemannian $(\lambda, n+m)$-Einstein metrics (M, g, f) around critical points of f

In the Riemannian setting, as in subsection 6.1, in geodesic polar coordinates $(t, x) \in \mathbb{R} \times S(1)$ we consider local warped product metrics of the form $dt^2 + u^2(t)g_1(x)$, $(t, x) \subset \mathbb{R} \times S^{n-1}(1)$, where g_1 is the induced metric on the standard sphere $S^{n-1}(1)$. Hence at critical points of f being located at the origin of local geodesic polar coordinates the warping function vanishes, i.e. $u(0) = 0$, cf. Remark 6.1,

In this subsection we first show that under the assumptions of harmonicity of the Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$, critical points of f in a Riemannian $(\lambda, n + m)$-Einstein metric, in geodesic polar coordinates with origin located at the critical points, are isolated and the level sets close to critical points are isometric to spheres. Around critical points of f, under some conditions on the warping function of a given warped product the metric is conformally flat. Via these properties we then locally characterize non-trivial $(\lambda, n+m)$-Einstein manifolds (M, g, f) around critical points of f in the Riemannian case.

Lemma 6.8. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g) be a connected Riemannian manifold with a smooth function f defined on M. Assume that (M, g, f) satisfies Equation (11) of a non-trivial $(\lambda, n + m)$-Einstein metric with harmonic Weyl tensor satisfying $W(\nabla f, ., ., \nabla f) = 0$ in a neighborhood of $p \in M$ with $\nabla f|_p = 0$.*

Then there exists a neighborhood $\mathscr{U}$ of p such that

(i) p is the only critical point of f in $\mathscr{U}$,

(ii) The level hypersurfaces of f in $\mathscr{U}$ coincide with the geodesic distance spheres around p.

Proof. At first we consider geodesic polar coordinates with origin at p. We choose $\mathscr{U}$ such that every point in $\mathscr{U}$ has a unique and shortest geodesic joining it with p. Then we consider q to be a regular point of f. By Remark 5.12 we have $f(p) \neq f(q)$. Let $\mathscr{A} := \{\bar{q} \in \mathscr{U} | f(\bar{q}) = f(q)\}$ consisting of only regular points. Consider the trajectory geodesics of ∇f starting at the hypersurface $\mathscr{A}$ and pointing to one of its sides, without loss of generality to the side containing p. $\mathscr{A}$ contains a point q_0 realizing the distance

$$s_0 := d(p, q_0) = d(p, \mathscr{A}) > 0.$$

EITHER there is a minimizing geodesic γ_0 joining p and q_0 consisting only of regular points of f (the case considering the possibility of existing a critical point of f between p and q_0 along γ_0 will be investigated in the rest of this proof). This realizes the distance between $\gamma_0(0) = p$ and $\gamma_0(s_0) = q_0$. By the Gauss lemma γ_0 meets $\mathscr{A}$ perpendicularly. Consequently by a discussion in the proof of Theorem 6.5 γ_0 is the same curve (up to parameterization) as the trajectory of ∇f through q_0. Any other point $q_1 \in \mathscr{A}$ yields similarly a geodesic trajectory γ_1 of ∇f. Let $\gamma_1(s_0) = q_1$. Then the claim is that $\gamma_1(0) = p$. To see this let d_M and $d_{\mathscr{A}(s)}$ denote the distance functions in M and the level set $\mathscr{A}(s)$ corresponding to the parameter s, respectively. Then for any $s > 0$

$$\begin{aligned} d_M(\gamma_0(s), \gamma_1(s)) &\leq d_{\mathscr{A}(s)}(\gamma_0(s), \gamma_1(s)) \\ &= \frac{u(s)}{u(s_0)} d_{\mathscr{A}(s_0)}(\gamma_0(s_0), \gamma_1(s_0)). \end{aligned} \tag{159}$$

For the last equality in (159) we used the warped product metric according to Theorem 6.5. Since the critical point p is located at the origin of geodesic polar coordinates we have $u(0) = 0$, cf. Remark 6.1. It follows then

$$\begin{aligned} d_M(\gamma_0(0), \gamma_1(0)) &= \lim_{s \to 0} d_M(\gamma_0(s), \gamma_1(s)) \\ &\leq \lim_{s \to 0} \frac{u(s)}{u(s_0)} d_{\mathscr{A}}(\gamma_0(s_0), \gamma_1(s_0)) = 0. \end{aligned} \tag{160}$$

Therefore, $\gamma_1(0) = \gamma_0(0) = p$, and $\mathscr{A}$ is contained in the geodesic distance sphere with radius s_0 around p. On the other hand it follows that the arc length parameter on the trajectories is just the geodesic distance to p. Therefore p is the only critical point in $\mathscr{U}$, and the f-levels coincide there with the geodesic distance spheres around p.

OR the same argument as above shows that in a certain minimal distance s_1

($< s_0$) there are critical points, the same distance on each trajectory, and ultimately all trajectories pass through the same critical point p_1. Then there are only regular points between p_1 and q. This implies that in an open neighborhood p_1 is surrounded by non-critical level sets of f (all diffeomorphic with the $(n-1)$-sphere), so this critical point p_1 is also isolated. But by connectedness there can not be two critical points p and q_1 at the same side of $\mathscr{A}$. □

Corollary 6.9. *The same procedure on the other side of $\mathscr{A}$ shows that either there is no critical point or there is precisely one other critical point p' with the same properties. In combination this seems to show that three or more critical points are impossible if $\mathscr{A}$ is connected (compare Theorem 6.10 and Theorem 6.12). If $\mathscr{A}$ is not connected one has the same situation for each component separately.*

Theorem 6.10. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g) be a Riemannian manifold with a smooth function f defined on M. Then the following conditions are equivalent:*

1) (M, g, f) is conformally flat and satisfies Equation (11) of a non-trivial (λ,n+m)-Einstein metric in a neighborhood of $p \in M$ with $\nabla f|_p = 0$.

2) There exist polar coordinates $(t, t_1, ..., t_{n-1}) \in I \times S^{n-1}(1)$, $I \subseteq \mathbb{R}$ being an open interval, in a neighborhood of p and an odd function $u = u(t)$, i.e. $u(0) = u^{(even)}(0) = 0$, with $u(t) > 0$ on $t \in I - \{0\}$ and $0 \neq (u')^2(0) = k$, such that in these coordinates $f = f(t)$ and

$$g = dt^2 + \frac{u^2(t)}{k} g_{S^{n-1}(1)} \tag{161}$$

where $g_{S^{n-1}(1)}$ denotes the line element of the standard unit sphere $S^{n-1}(1)$; In addition, the conditions (55) and (56) in Proposition 4.35 hold.

Proof. 1) ⇒ 2) We consider a neighborhood $\mathscr{U}$ of p such that $g|_{\mathscr{U}}$ be conformally flat. This provides g with harmonic Weyl tensor and the property $W(\nabla f, ., ., \nabla f) = 0$ on $\mathscr{U}$. Hence, by Lemma 6.8 we already know that (after restriction of $\mathscr{U}$ if necessary) p is the only critical point of f in $\mathscr{U}$. In addition, by Theorem 6.5 and Lemma 6.8 we may introduce locally coordinates such that for $t \neq 0$

$$\begin{aligned} g &= dt^2 + u^2(t) g_N \\ f(t, x) &= f(t), \;\; x \in N \end{aligned} \tag{162}$$

where g_N is the induced metric on a regular level set N of f. By smoothness of f and g it follows that the equations in (162) for f and g hold at the time $t = 0$ as well.

Let X, Y be two orthonormal vectors in M which are tangent to a level hypersurface $N = \{q | f(q) = t_0 > 0\}$ for sufficiently small t_0. By Equation (36)

($\epsilon = 1$) the sectional curvatures Sec resp. Sec_N of the (X, Y)-plane in (M, g) resp. (N, g_N) satisfy

$$\begin{aligned} Sec &= g(R(X,Y)Y,X) \\ &= g(R_N(X,Y)Y,X) - \frac{(u'(t_0))^2}{(u(t_0))^2} \\ &= \frac{1}{(u^2(t_0))^2}(Sec_N - (u'(t_0))^2). \end{aligned}$$

On the other hand, g_N is independent of t when it tends to zero, cf. the proof of Theorem 6.5. Since $u(0) = 0$, cf. Remark 6.1, it follows that

$$0 = \lim_{t\to 0}(Sec_N - (u'(t))^2) = Sec_N - (u'(0))^2.$$

It implies that (N, g_N) is a space of constant curvature $Sec_N = (u'(0))^2$, and hence either $Sec_N > 0$ or $Sec_N = 0$. We already know that N is a geodesic distance sphere which is diffeomorphic to S^{n-1}, hence the case $Sec_N = 0$ can not occur. Because in this case N will be flat with Euclidean space as its universal cover, while we know that the universal cover of any sphere is itself. Therefore $Sec_N > 0$.

Consequently $(u'(0))^2 > 0$ and $g_N = \frac{1}{(u'(0))^2} g_{S^{n-1}(1)}$. Moreover using Proposition 4.35 via multiplying both sides of Equation (65) with $u^2(t)$ and then taking the limit while t tends to 0 as well as noting Proposition 4.9, which implies $f(0) \neq 0$, one obtains $u'^2(0) = k$. Hence $g_N = \frac{1}{k} g_{S^{n-1}(1)}$. By our assumption the metric g is everywhere smooth and has no singularity at $t = 0$. Therefore using the same calculation as [Pet16, 1.4.4] we conclude that $u(t)$ is an odd function at $t = 0$, i.e. $u^{(even)}(0) = 0$ and Equation (161) is valid for all $t \geq 0$ as the usual expression of the Euclidean metric in polar coordinates. Since $u(t)$ and $f(t)$ are continuous and since by assumption (M, g, f) satisfies Equation (11), by Proposition 4.35 the conditions (55) and (56) are satisfied.

2) $\Rightarrow$ 1) In order to see that $\nabla f|_p = 0$ we use the assumption $u(0) = 0$ together with Lemma 6.6. In order to see that Equation (161) together with $f = f(t)$, $t \in I$, satisfies Equation (11) of a $(\lambda, n + m)$-Einstein manifold in polar coordinates, one may apply Theorem 6.5 for all points except $t = 0$. The oddness of the function $u(t)$, i.e. $u^{(even)}(0) = 0$, and $0 \neq u'^2(0) = k$ yields that the right hand side of (161) has no proper singularity at $t = 0$. Thus by continuity Equation (11) holds at $t = 0$ as well. Moreover, since by assumption the function $u(t)$ in (161) satisfies $u'(0) \neq 0$ and $u^{(even)}(0) = 0$, via similar calculations to the proof of [KR97a, Proposition 3.5] we see that the local warped metric (161) is conformally flat. □

Reminder: In Theorem 6.10 if in addition (M, g) is Einstein then Hess$f$ would be proportional to the metric g. In this situation there is already a characterization by [Küh88, Lemma 18] in terms of a local warped decomposition of g with $f'(t)$ as the warping function.

6.4 Global characterization of Riemannian $(\lambda, n+m)$-Einstein metrics (M, g, f) based on [HPW12, Theorem 7.1]

Theorem 6.11. (*[HPW12], Theorem* 7.2). *Let $m > 1$ and suppose that (M, g) is complete, simply connected Riemannian manifold and has harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$, then (M, g, f) is a non-trivial $(\lambda, n+m)$-Einstein metric if and only if it is of the form*

$$g = dt^2 + u^2(t) g_N$$

$$f = f(t)$$

where g_N is an Einstein metric. Moreover, if $\lambda \geq 0$ then (N, g_N) has non-negative Ricci curvature, and if it is Ricci flat, then u is a constant, i.e, (M, g) is a Riemannian product.

There are some points to be discussed on Theorem 6.11. The first point: In Theorem 6.11 the authors assume the manifold is simply connected, while through the next result we see that "simply connected" is not needed in the formulation of the theorem.

The second point: Under the assumptions of the theorem the number of critical points of f can be at most two, cf. Corollary 6.9 or Theorem 6.12. In particular the warped product structure is global, i.e. complete, if there are no critical points for f, see Corollary 6.13.

The third point: It is necessary to show that the critical points of f are isolated, cf. Lemma 6.8. Additionally, in order that the local warped product $g = dt^2 + u^2(t) g_N$ on $M - \{\text{critical points of } f\}$ can be extended to a metric on M we need that the warping function $u(t)$ be odd on the critical points of f, i.e. $u^{(even)}(\gamma_0) = 0$ where $\gamma_0 = t(q)$ with $\nabla f(q) = 0$, as well as $u'(\gamma_0) \neq 0$, cf. Theorem 6.10.

The fourth point: Although the relations between $u(t)$ and $f(t)$ are investigated for some specific cases such as $(0, n+m)$-Einstein and $(\lambda, 2+m)$-Einstein metrics in [HPW12], but for a $(\lambda, n+m)$-Einstein metric they are not in general formulated. To generally relate $u(t)$ and $f(t)$ in the formulation of the theorem we may use (55), (56) and (57) of Proposition 4.35.

The fifth point: By Lemma 5.9 we may include the properties harmonicity of the Weyl tensor of (M, g) and $W(\nabla f, ., ., \nabla f) = 0$ in the characterization equivalence. In other words these two properties may be moved from being as assumption to be used in the equivalence relation of the characterization. This is due to Lemma 5.9 which says that any warped product $g = dt^2 + u^2(t) g_N$ with Einstein fibre g_N has harmonic Weyl tensor and satisfies $W(\nabla f, ., ., \nabla f) = 0$.

Now considering all these points together we can restate the global statement [HPW12, Theorem 7.2] as follows $\big($ [(means either [or (, similarly does)]$\big)$.

Theorem 6.12. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g) be a connected Riemannian manifold with a smooth function f on M. Then the following conditions are equivalent:*

1) (M, g, f) is a non-trivial $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$.

2) If C denotes the set of critical points of f then $N' := |C| \leq 2$, and $(M \setminus C, g)$ is isometric with a warped product metric

$$g = dt^2 + u^2(t) g_N \tag{163}$$

$$f = f(t) \tag{164}$$

on $I \times N$ where (N, g_N) is a complete Einstein Riemannian hypersurface of (M, g) with normalized scalar curvature $k = \varrho_N$ and $I = [(\alpha_0, \beta_0)] \subset \mathbb{R}$ which is unlimited in both sides, i.e. $I = (-\infty, \infty)$ if there is niether a critical point for f nor a boundary point of M. Otherwise, it is closed in the left i.e. $I = [\alpha_0, \beta_0)]$ with $\alpha_0 \in \mathbb{R}$ if there exists a point $q_0 \in \partial M$ with $f(q_0) = f(\alpha_0) = 0$ (or similarly $I = [(\alpha_0, \beta_0]$ with $\beta_0 \in \mathbb{R}$ for a boundary point q_0 with $f(q_0) = f(\beta_0) = 0$). Or, $I = (\alpha_0, \beta_0)]$ has finite α_0 with open left side (or $I = [(\alpha_0, \beta_0)$ has finite β_0 with open right side) only if it corresponds to a minimum (or maximum) point q_0 of f with $f(q_0) = f(\alpha_0)$ (or $f(q_0) = f(\beta_0)$). In addition, in the latter case where $\gamma_0 = \alpha_0$ (or $\gamma_0 = \beta_0$) is finite and corresponds to a critical point q_0, $u = u(t)$ is odd at γ_0, i.e. $u^{(even)}(\gamma_0) = 0$, with $u'(\gamma_0) \neq 0$. In all cases $f((\alpha_0, \beta_0)) > 0$, and, $f(\alpha_0) = 0$ if $\{\alpha_0\} \times N \in \partial M$ (or $f(\beta_0) = 0$ if $\{\beta_0\} \times N \in \partial M$).
The product $I \times N$ becomes complete if we add the set C of critical points to it. In addition, $f = f(t)$ and $u = u(t)$ satisfy the equations (55), (56) and (57) in Proposition 4.35.

Proof. 1) $\Rightarrow$ 2): By Lemma 6.8, C is a set of isolated points. For every fixed point $q \in M \setminus C$, by Theorem 6.5 there is an open neighborhood $\mathscr{U} \ni q$ in which equations (163) and (164) hold. Where g_N, $N := \{x \in M | f(x) = f(q)\}$, is an Einstein Riemannian hypersurface of (M, g) say with normalized scalar curvature $k = \varrho_N$. The hypersurface (N, g_N) is complete as every Cauchy sequence in N converges in M. Accordingly we have $\mathscr{U} = (\alpha, \beta) \times N$. The trajectory through q is the unique geodesic with tangent $\frac{\partial}{\partial t}$. By completeness this is defined for every parameter t as far as does not hit the boundary ∂M.

We define α_0 and β_0 to be the infimum and supremum of α, β such that (163) holds for $(\alpha, \beta) \times N$. Here the extension to $(\alpha_0, \beta_0) \times N$ is regardless of whether or not the points belong to the set O. In fact, it continues as long as the points lie in regular level sets of f, cf. Theorem 6.5. Moreover, a similar discussion as in Theorem 6.5 implies that $f(t, x) = f(t)$ on $(\alpha_0, \beta_0) \times N$ where $x \in N$.

If α_0 (or β_0) is finite then there is a limit point q_0 on this geodesic with $f(q_0) = f(\alpha_0)$ (or $f(\beta_0)$). If q_0 is boundary point i.e. $f(q_0) = f(\alpha_0) = 0$ (or $f(q_0) = f(\beta_0) = 0$) by Proposition 4.9 it follows that $\nabla f(q_0) \neq 0$. Thus Equation (148) in the proof of Proposition 6.5 implies that t would be also smooth at α_0 (or β_0). Therefore by smoothness of g, f, t the condition (163) and (164) are valid at α_0 (or β_0) as well. In addition, since $\{\alpha_0\} \times N$ (or $\{\beta_0\} \times N$) is a component of ∂M by completeness we have $f^{-1}((-\infty, \beta_0)]) = f^{-1}((\alpha_0, \beta_0)])$ (or $f^{-1}([(\alpha_0, \infty)) = f^{-1}([(\alpha_0, \beta_0)))$.

If q_0 is not a boundary point then it must be a critical point of f because otherwise by the argument in Theorem 6.5 Equation (163) could be extended to a neighborhood of q_0 which is a contradiction. Furthermore q_0 is a minimum (a maximum) of f, because by Theorem 6.10 $u'(q_0) \neq 0$ and hence Hess f is definite at q_0. By connectedness no other critical points can occur, cf. the proof of Lemma 6.8. Hence the number of critical points of f is at most two, i.e. $N' := |C| < 2$. Moreover, by our assumption the metric g and f are everywhere smooth and have no singularity at $\gamma_0 = \alpha_0$. Hence through the same calculation as in [Pet16, 1.4.4] it follows that $u(t)$ is an odd function at $\gamma_0 = \alpha_0$ (or β_0), i.e. $u^{(even)}(\gamma_0) = 0$, where $\gamma_0 \in \mathbb{R}$ and $u'(\gamma_0) \neq 0$ and moreover equations (163) and (164) are valid for all $t \in (\alpha_0, \beta_0)$ and for $\alpha_0 \in \mathbb{R}$ (or $\beta_0 \in \mathbb{R}$).

$I \times N$ becomes complete if it be added with the set C of critical points of f. In more details, when $N' = |C| = 0$ it would be global, i.e. complete. If $N' = 1$ then it would be complete by adding the only critical point which is the minimum of level $f(\alpha_0)$ (or the maximum of level $f(\beta_0)$). If $N' = 2$ then $I \times N$ becomes complete by adding the minimum of level $f(\alpha_0)$ as well as the maximum of level $f(\beta_0)$.

Moreover, due to smoothness of g, f and t as well as oddness of $u(t)$ at finite $\gamma_0 = \alpha_0$ (or $\gamma_0 = \beta_0$), i.e. $u^{(even)}(\gamma_0) = 0$, the functions $u = u(t)$ and $f = f(t)$ satisfy the equations (55), (56) and (57) in Proposition 4.35.

2) $\Rightarrow$ 1): By assumption $(M \setminus C, g)$ is isometric with $dt^2 + u^2(t)g_N$ and $f(t,x) = f(t)$ on $I \times N$; $x \in N$ where $I = [(\alpha_0, \beta_0)]$. Also, by assumption even derivatives of $u(t)$ vanish, i.e. $u(t)$ is odd, at finite end points of the interval I when they correspond to a critical point of f. Therefore, the warped product metric extends smoothly to a metric on M.

Since by assumption $I \times N$ becomes complete after addition with the set C of critical points of f we conclude that the metric g is complete as well.

By assumption, $u(t)$ and $f(t)$ satisfy equations (55), (56) and (57) in Proposition 4.35 on I. On the other hand $u(t)$ is odd on the critical points of f. Therefore, by Proposition 4.35 it follows that (M, g, f) is a $(\lambda, n+m)$-Einstein manifold.

As g_N is Einstein, by Lemma 5.9 and smoothness of the metric (163) due to oddness of $u(t)$ at critical points of f we conclude that (M, g) has harmonic Weyl tensor and satisfies $W(\nabla f, ., ., \nabla f) = 0$. □

Corollary 6.13. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g) be a connected Riemannian manifold (without boundary) on which a smooth function f is defined. Then the following conditions are equivalent:*

1) (M, g, f) is a non-trivial $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ and where f has no critical point.

2) $g = dt^2 + u^2(t)g_N$ on $\mathbb{R} \times N$ where (N, g_N) is a complete Einstein Riemannain hypersurface of (M, g) say with normalized scalar curvature $\varrho_N = k$, *in addition $f = f(t) : \mathbb{R} \to \mathbb{R}^+$ which together with $u = u(t) : \mathbb{R} \to \mathbb{R}^+$ satisfies the two equations in Proposition 4.35.1).*

Remark 6.14. *For a manifold (M, g) with boundary suppose that the metric locally satisfies $g = dt^2 + u^2(t)g_N$. If after some time $f(t_0) = 0$ then by Proposition 4.9 it follows that $f'(t_0) \neq 0$, and hence via Corollary 6.7 $u(t_0) > 0$. This implies that $\{t \in I | u(t) = 0 = f(t)\} = \emptyset$. Moreover, if $(t_0, n) \in \partial M$ then $\{t_0\} \times N$ is a level set of f, also, a component of ∂M. Therefore by completeness of (M, g) every maximal geodesic $\gamma(t)$ hitting the boundary at $\{t_0\} \times N$ must stop when reaches $\gamma(t_0)$ and there is no level set of f past $t = t_0$.*

6.5 Some investigation on Riemannian $(\lambda, n + m)$-Einstein metrics of constant scalar curvature satisfying additional curvature conditions

Proposition 6.15. *Let $m > 1$, $\lambda > 0$ and (M, g, f) be a Riemannian non-trivial $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$. Then (M, g) is diffeomorphic to sphere with the standard metric.*

Proof. By Theorem 4.18 M is compact, thus the function f must have at least two critical points. Through the same argument as [Küh88, Theorem 21.(i)] the result follows. □

Proposition 6.16. *Let $m > 1$, $\lambda > 0$ and let (M, g, f) be a Riemannian non-trivial $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$. In addition, suppose (M, g) is of constant scalar curvature and has no boundary. Then (M, g) is isometric with the standard sphere of certain radius.*

Proof. From Proposition 6.15 we know that (M, g) is diffeomorphic to the sphere. Additionally, by the same argument as Theorem 6.10 we obtain in $M \setminus \{p, q\}$ the expression $g = dt^2 + \frac{u^2(t)}{k} g_{S^{n-1}(1)}$ where the elements p, q show the critical points of f on a compact manifold M. Then using the same calculation as [Küh88, Theorem 24] together with Lemma 4.26.(iv) the result follows. □

Corollary 6.17. (*[CSW11], Proposition* 3.6). *Let $m > 1$ and (M, g, f) be a Riemannian $(\lambda, n + m)$-Einstein metric with constant scalar curvature and $\lambda \neq 0$, then the scalar curvature is bounded by $n\lambda$ and $n\rho$. Furthermore if $Scal = n\lambda$ or $Scal = n\rho$, then the manifold is Einstein.*

If a Riemannian $(\lambda, n + m)$-Einstein metric (M, g, f) is of constant scalar curvature it may be classified by rigidity in the sense of [HPW14], see also [HPW15] for further results.

7 Local characterization of pseudo-Riemannian $(\lambda, n+m)$-Einstein metrics and the Brinkmann case

In this section we generalize the characterization around a regular point of f from Riemannian to pseudo-Riemannian setting. We then discuss a related result in dimension 4. We classify non-trivial pseudo-Riemannian $(\lambda, n+m)$-Einstein manifolds (M, g, f) with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ around critical points of f which in addition are assumed to be isolated. We already know that at least in the case f is a Morse function critical points will automatically be isolated. In addition, we investigate the so-called Brinkmann spaces for $(\lambda, n+m)$-Einstein manifolds in the pseudo-Riemannian setting. In this case in particular the metric can be written in the form of $g = 2dt_1dt_2 + g_*(t_1)$ where the $(n-2)$-dimensional metric $g_*(t_1)$ does not depend on t_2.

7.1 Local characterization of pseudo-Riemannian $(\lambda, n+m)$-Einstein metrics (M, g, f) around points at which $g(\nabla f, \nabla f) \neq 0$

In this subsection we generalize our local result Theorem 6.5 from the Riemannian to the pseudo-Riemannian setting by means of assuming $|\nabla f| \neq 0$ instead of $\nabla f \neq 0$. In fact, the generalization excludes not only the critical points of f but also the points at which ∇f is null. We start with the following statement which plays an essential role.

Proposition 7.1. *Let (M^n, g) be a pseudo-Riemannian manifold of dimension $n > 3$. Also let (M, g, f) be a $(\lambda, n+m)$-Einstein manifold with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$. Also suppose $g(\nabla f, \nabla f) \neq 0$ in the connected component $\mathscr{A}_p$ of the level set of f containing $p \in M$. Then the Schouten tensor S has at most two eigenvalue functions defined in $\mathscr{A}_p$, say σ_1 and σ_2. If it has two eigenvalues, then dim $V_{\sigma_1} = 1$ and dim $V_{\sigma_2} = n-1$ (> 1) where V_{σ_i}, $i = 1, 2$ denotes the eigenspace corresponding to eigenvalue function σ_i and where as in Lemma 5.14 σ_1 signifies the eigenvalue function relevant to ∇f. Moreover σ_2 is constant along the level sets of f in $\mathscr{A}_p$. If it has only one eigenvalue, say σ, then dim $V_\sigma = n$ and σ would be constant along the level sets of f in $\mathscr{A}_p$.*

Proof. By Lemma 5.14 there are at most two eigenvalue functions σ_1 and σ_2 in $\mathscr{A}_p$. Suppose $\sigma_1 \neq \sigma_2$. Let $x \mapsto V_{\sigma_i(x)}(x)$, $i = 1, 2$ denotes the smooth eigenspace distribution V_{σ_i} for the eigenvalue function σ_i. Then, by Lemma 5.14 dim $V_{\sigma_1} = 1$ and dim $V_{\sigma_2} > 1$ when $n > 2$. For eigenvalue functions $\lambda, \mu \in \{\sigma_i\}$, $i = 1, 2$ and for vector fields X, Y, Z where $Y \in C^\infty(V_\mu)$, $X \in C^\infty(V_\lambda)$ using the Leibniz rule one obtains

$$\begin{aligned}
&(\nabla_Z S)(X, Y) = \nabla_Z(S(X, Y)) - S(D_Z X, Y) - S(X, D_Z Y) \\
&= \nabla_Z(\lambda g(X, Y)) - \mu g(\nabla_Z X, Y) - \lambda g(X, \nabla_Z Y) \Rightarrow \\
&(\nabla_Z S)(X, Y) = Z(\lambda) g(X, Y) + (\lambda - \mu) g(\nabla_Z X, Y). \qquad (165)
\end{aligned}$$

Now let $\lambda = \mu = \sigma_2$ and $X \in C^\infty(V_{\lambda=\sigma_2})$ not lightlike. Then the restriction of the metric to the orthogonal of X inside V_{σ_2} is non-degenerate. Therefore, we can find locally a non-zero $Y \in C^\infty(V_{\sigma_2})$ there which is not a null vector and which satisfies $g(X,Y) = 0$. Since dim $(V_\lambda) > 2$ we can always choose such $Y \in C^\infty(V_{\sigma_2})$. Furthermore using Equation (165) for $\lambda = \mu$ $(= \sigma_2)$ and for $Z = X$, and since, the tensor S is Codazzi we obtain $|Y|^2.X(\lambda) = (D_X S)(Y,Y) = (D_Y S)(X,Y) = 0$ which by $|Y|^2 \neq 0$ it follows that $X(\sigma_2) = X(\lambda) = 0$. Hence $X(\sigma_2) = X(\lambda) = 0$ for all X in V_{σ_2} which are not lightlike. However, any lightlike vector can be approximated by a non lightlike vector. In this regard there are two statements:
1) Every non-zero vector X is the limit of a sequence of non lightlike vectors X_n.
2) If $X_n(\sigma_2) = 0$ for all n, then passing to the limit $X(\sigma_2) = 0$ also holds.

Statement 1) follows since in an orthonormal basis the metric has the form $x^2 - y^2$ where X=(x,y) are coordinates of the vector. Therefore, if $x^2 - y^2 = 0$ and, for instance, $x \neq 0$, then $X_n = X + (\frac{x}{n}, 0)$ will be the approximation we are looking for.

Statement 2) follows since $X_n(\sigma_2) = d\sigma_2(X_n)$. Now the differential $d\sigma_2$ of σ_2 is linear and hence it is continuous and therefore

$$0 = \lim_n X_n(\sigma_2) = \lim_n d\sigma_2(X_n) = d\sigma_2(\lim_n X_n)) = d\sigma_2(X) = X(\sigma_2).$$

Therefore, $X(\sigma_2) = 0$ for all vectors $X \in V_{\sigma_2}$, consequently σ_2 is constant along the level sets of f in $\mathscr{A}_p$.

If $\sigma_1 = \sigma_2 = \sigma$ in $\mathscr{A}_p$, then by the proof of Lemma 5.14 we have $\dim V_\sigma = n$. Moreover, via calculation we get $Ric = (\frac{Scal}{2(n-1)} + \sigma)g$ through which taking the trace of both sides gives out $\sigma = \frac{n-2}{2n(n-1)} Scal$. So it becomes $Ric = \frac{Scal}{n} g$ implying that Scal and hence σ are constant along the level sets of f in $\mathscr{A}_p$. In the literature we may find that the relation $Ric = \frac{Scal}{n} g$ on a neighborhood implies that $Scal$ is constant on the neighborhood. □

From the literature we already know that around a boundary point $p \in \partial M$ there exists a local coordinate system $(t, t_1, ..., t_{n-1})$ with $t \geq 0$ where ∂M corresponds to $t = 0$. In the generalization of Theorem 6.5, since we use very similar steps we just provide a brief proof.

Theorem 7.2. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g) be a pseudo-Riemannian manifold and f a smooth function on M. Then the following conditions are equivalent:*

1) (M, g, f) satisfies Equation (11) of a non-trivial $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ in a neighborhood of p with $g(\nabla f, \nabla f)|_p \neq 0$.

2) (a) Case $p \in int(M)$: There exist local coordinates $(t, t_1, ..., t_{n-1})$ with $t \in (-\varepsilon, \varepsilon)$ in a neighborhood of $p \in M$ and an Einstein Riemannian hypersurface $(N, g_N = g_N(t_1, ..., t_{n-1}))$ of (M, g) with normalized scalar curvature

$\varrho_N = k$ and a function $u = u(t) > 0$, in addition $f = f(t) > 0$ satisfying (55) *and* (56) *in Proposition 4.35 such that*

I) $g(\partial_t, \partial_t) = \epsilon, \quad \epsilon := sign\ \ g(\nabla f(p), \nabla f(p)) \in \{\pm 1\}$
II) $g(\partial_t, \partial_{t_i}) = 0, \quad$ *for* $i = 1, ..., n-1$
III) $g(\partial_{t_i}, \partial_{t_j}) = u^2(t) g_N(\partial_{t_i}, \partial_{t_j})(t_1, ..., t_{n-1}) \quad i, j = 1, ..., n-1.$

(b) Case $p \in \partial M$: There exist local coordinates $(t, t_1, ..., t_{n-1})$ with $t \in [0, \varepsilon)$ in a neighborhood of p and an Einstein Riemannian hypersurface $(N, g_N = g_N(t_1, ..., t_{n-1}))$ of (M, g) with normalized scalar curvature $\varrho_N = k$ and a function $u = u(t) > 0$, in addition $f(t) > 0$ for all $t \in (0, \varepsilon)$ satisfying (55) & (56) *as well as $f(0) = 0$ satisfying* (57) *at $t = 0$ such that the conditions I, II, III in* (a) *holds.*

Any case of 2) implies that $g = \epsilon dt^2 + u^2(t) g_N$ around p. If the metric is locally conformally flat in a neighborhood of p then N must be a space of constant curvature.

Proof. 1) $\Rightarrow$ 2): By Lemma 5.14 the Schouten tensor has at most two eigenvalues σ_1 and σ_2 in a neighborhood $\widetilde{\mathscr{U}}$ of p with $g(\nabla f, \nabla f)|_{\widetilde{\mathscr{U}}} \neq 0$. Without loss of generality suppose $p \in O \cap \widetilde{\mathscr{U}}$, i.e. $\sigma_1(p) \neq \sigma_2(p)$ (otherwise we may start with a point $p_0 \in O \cap \widetilde{\mathscr{U}}$ and still get the same result).

For more convenience we first assume $p \in int(M)$. Let $\mathscr{U} \subset O \cap \widetilde{\mathscr{U}}$ be a neighborhood of p with compact closure. Thus the Schouten tensor S has two distinct eigenvalues σ_1 and σ_2 in $\mathscr{U}$ where σ_1 denotes the eigenvalue of S with eigenfield ∇f and σ_2 the eigenvalue for eigenfields in the orthogonal complement of ∇f. By Lemma 5.14 the tangent space to every point $q \in \mathscr{U}$ is the direct sum of the eigenspaces corresponding to σ_1 and σ_2, denoted by V_{σ_1} and V_{σ_2} respectively, where dim V_{σ_2} is bigger than one in $\mathscr{U}$. Hence Proposition 7.1 says that σ_2 is locally constant on the level sets of f in $\mathscr{U}$.

Since S has two distinct eigenvalues in $\mathscr{U}$, via the relation $Hess f = \frac{f}{m}(S + (\frac{Scal}{2(n-1)} - \lambda) g)$ it follows that Hess$f$ has also two distinct eigenvalues in $\mathscr{U} \cap int(M)$, call them μ_1 and μ_2, where the eigenspaces for μ_i correspond to those for σ_i. Through the same discussion as in Theorem 6.5 we see that $|\nabla f|^2$, μ_1 and μ_2 are all locally constant on the level sets of f in $\mathscr{U}$.

Let $c := f(p)$ and $N \subset \mathscr{U}$ be the connected component of $f^{-1}(c)$ containing p. Since $|\nabla f| \neq 0$ on N one can choose a coordinate chart $(t_1, ..., t_{n-1})$ on the level hypersurface N. As the pseudo-norm $|\nabla f| := \sqrt{|g(\nabla f, \nabla f)|}$ is locally constant on the level sets of f it may be considered as a function of f, i.e. $|\nabla f| = |\nabla f|(f)$. Similar to the proof of Theorem 6.5, it follows that there exists a smooth function t on $\mathscr{U}$ such that

$$dt = \frac{1}{|\nabla f|} df \ \text{ with } t = t(f) = \int_N^f \frac{df}{|\nabla f|} \tag{166}$$

respectively $|\nabla t| = 1$, in addition $\nabla_{\nabla t}\nabla t = 0$. Accordingly the integral curves of ∇t which are normal to level sets of f in $\mathscr{U}$ are geodesics. Then we may extend the coordinate chart on N to geodesic parallel coordinates $(t, t_1, ..., t_{n-1})$ in a neighborhood of p in which $f(t,x) = f(t)$ and $\nabla f(t,x) = \epsilon f'(t)\partial_t$.

Since the trajectories of unit normal vector fields $\frac{\nabla f}{|\nabla f|} = \nabla t$ to level sets of f in $\mathscr{U}$ are geodesics, we may decompose the n dimensional tangent spaces to points of $\mathscr{U}$ into the 1 dimensional space generated by $\frac{\nabla f}{|\nabla f|}$ and its orthogonal complement.

Furthermore, since $t \mapsto exp(t\frac{\nabla f}{|\nabla f|}(x))$ is a geodesic and hence preserves the causal character of the velocity vector fields, it follows that $g(\partial_t, \partial_t) = \epsilon = sign\ g(\nabla f(p), \nabla f(p))$. Moreover, similar to the proof of Theorem 6.5 we see that around p the metric is of the form

$$g = \epsilon dt^2 + u^2(t)g_N \tag{167}$$

where g_N is Einstein. Now suppose that $p \in \partial M$. Then through the same discussion as above and noting that due to $N \subseteq \partial M$ the trajectory geodesics of ∇f starting at N point only to one side, e.g. to its positive side, it follows that there exist geodesic parallel coordinates $(t, t_1, t_2, ..., t_{n-1})$ in a neighborhood of p with $t \in [0, \epsilon)$, $t(p) = 0$ for which the result holds. Moreover the conditions (55), (56), (57) in Proposition 4.35 hold on $\mathscr{U}$.

Now consider the closed set $O_1 = \{x \in \widetilde{\mathscr{U}} : g(\nabla f, \nabla f)_x \neq 0, \sigma_1(x) = \sigma_2(x)\}$. If the interior of $O_1 = \partial O_1 \cup int(O_1)$ is not empty, then Hess$f$ would be proportional to the metric g on the interior. [KR97a, Lemma 2.7] together with the same discussion on the boundary points in the last paragraph implies a warped product $g = \epsilon dt^2 + u^2(t)g_N(t_1, ..., t_{n-1})$ where (N, g_N) is a pseudo-Riemannian hypersurface of (M, g) say with $\varrho_N = k$ as well as $u(t) = f'(t) \geq 0$. Again, the same argument as in the proof of Theorem 6.5 implies that g_N is Einstein. Moreover, $u(t)$ and $f(t)$ satisfy (55), (56), (57) in Proposition 4.35 on the interior of O_1. Since, in this case Hess$f$ is a multiple of $g|_{\mathscr{U}}$ the equations in Proposition 4.35 reduce to equations (80), (81) and (82) of Corollary 4.42. Therefore we can assume that both the open set $O \cap \widetilde{\mathscr{U}}$ and the interior of the closed set $O_1 \cap \widetilde{\mathscr{U}}$ are non-empty.

Considering the way of defining the function t at above, we see that t is smooth as far as $g(\nabla f, \nabla f) \neq 0$, cf. Equation (166), including the points at which $\sigma_1 = \sigma_2$. Therefore t will be smooth on the whole $\widetilde{\mathscr{U}}$.

By smoothness of the metric g and t it follows that we have a warped product of the form (167), where g_N is Einstein say with $\varrho_N = k$, along all of t as long as $g(\nabla f, \nabla f) \neq 0$, therefore on the whole $\widetilde{\mathscr{U}}$. In parallel, to extend the relations $f = f(t)$ and (55), (56), (57) in Proposition 4.35 as far as $g(\nabla f, \nabla f) \neq 0$ (so on the whole $\widetilde{\mathscr{U}}$) we use again a similar discussion as above. Moreover, using Lemma 4.29 a metric in this local form is conformally flat if and only if g_N has constant curvature.

2) $\Rightarrow$ 1): Suppose by contradiction that $g(\nabla f(p), \nabla f(p)) = 0$. Therefore we

may consider geodesic polar coordinates with origin at p, hence it must satisfy $u(a) = 0$, $a = t(p)$, cf. Remark 6.1. This is a contradiction with the assumption.

Let the metric be isometrically $g = \epsilon dt^2 + u^2(t) g_N(t_1, ..., t_{n-1})$ where g_N is an Einstein hypersurface with $\varrho_N = k$ and the conditions (55), (56), (57) in Proposition 4.35 be satisfied by $u(t) > 0$ and $f = f(t) \geq 0$. Then Proposition 4.35 implies that (M, g, f) satisfies Equation (11) of $(\lambda, n+m)$-Einstein manifolds around p.

By Lemma 5.9 the manifold (M, g) has harmonic Weyl tensor and satisfies $W(\nabla f, ., ., \nabla f) = 0$ around p. □

One could regard Proposition 7.2 specifically for the definite case of Riemannian manifolds which then (in the case of $\partial M = \emptyset$) gives out Theorem 6.5.

Lemma 7.3. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g, f) satisfies Equation* (11) *of a non-trivial $(\lambda, n+m)$-Einstein manifold with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$. In addition, suppose that for a coordinate system $(t, t_1, t_2, ..., t_{n-1})$ the metric is of the form*

$$g = \epsilon dt^2 + u^2(t) g_N, \;\; \epsilon \in \{\pm 1\}, \;\; t \in [-\varepsilon, \varepsilon], \;\; \varepsilon \in \mathbb{R}^+ \tag{168}$$

where $u(t) > 0$ on $(-\varepsilon, \varepsilon)$ and the function satisfies

$$f(t, x) = f(t), \;\; x \in N. \tag{169}$$

If $u(a) = 0$, $a \in \{-\varepsilon, \varepsilon\}$, then $g(\nabla f(a), \nabla f(a)) = 0$.

Proof. By contradiction suppose $g(\nabla f(a), \nabla f(a)) \neq 0$. Then by the proof of Theorem 7.2 we see that the warped product is extendable to a neighborhood of $t = a$, hence the warping function must satisfy $u(a) > 0$, which is a contradiction to the assumption $u(a) = 0$. □

Corollary 7.4. *Under the same assumptions as Theorem 7.2.1) we have $u(t) > 0$ if and only if $g(\nabla f(t), \nabla f(t)) \neq 0$ (equivalently $u(t_0) = 0$ if and only if $g(\nabla f(t_0), \nabla f(t_0)) = 0$ where t_0 is the first zero for them).*

Proof. By combination of Theorem 7.2, Lemma 7.3 and Remark 6.1 the result follows. □

Corollary 7.5. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M, g) be a pseudo-Riemannian manifold and f a smooth function on M. Then the following conditions are equivalent:*

1) (M, g, f) is a non-trivial $(\lambda, n+m)$-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$, and $|\nabla f| \neq 0$ on the whole M.

2) $g = \epsilon dt^2 + u^2(t) g_N$, $\epsilon \in \{\pm 1\}$, on $I \times N$ where $I = (-\infty, \infty)$ provided $f^{-1}(0) = \emptyset$ or $I = [\alpha, \infty)$ with $f(\alpha) = 0$ or $I = (-\infty, \beta]$ with $f(\beta) = 0$ or $I = [\alpha, \beta]$ with $f(\alpha) = f(\beta) = 0$ and (N, g_N) is a complete Einstein pseudo-Riemannian hypersurface of (M, g) say with normalized scalar curvature $\varrho_N = k$, and $u : I \to \mathbb{R}^+$, $f : I \to \mathbb{R}^+ \cup \{0\}$ satisfying the equations (55), (56), (57) *in Proposition 4.35.*

Proof. 1) $\Rightarrow$ 2): By Theorem 7.2 we have a global warped product $g = \epsilon dt^2 + u^2(t)g_N$ on $I \times N$ where I is the maximal interval on which the trajectories of $\frac{\nabla f}{|\nabla f|}$ are defined and (N, g_N) is an Einstein pseudo-Riemannian hypersurface of (M, g) say with normalized scalar curvature $\varrho_N = k$ and where $u : I \to \mathbb{R}^+$. In addition, $f : I \to \mathbb{R}^+ \cup \{0\}$ which together with $u(t)$ satisfies (55), (56), (57) by Proposition 4.35. Since by assumption g is complete it follows that g_N is also complete.

If $f(t_0) = 0$, i.e. $\{t_0\} \times N$ is a component of the boundary ∂M, then the trajectories of $\frac{\nabla f}{|\nabla f|}$ stop at t_0. In more details, If $(t_0, n) \in \partial M$ then by completeness of (M, g) every maximal geodesic $\gamma(t)$ hitting the boundary at $\{t_0\} \times N$ stops when $\gamma(t_0)$ and hence there is no level set of f past $t = t_0$. Otherwise, if $f(t)$ is non-vanishing everywhere then $I = \mathbb{R}$. Therefore, the possible forms of I would be $I = [\alpha, \infty)$ with $f(\alpha) = 0$ or $I = (-\infty, \beta]$ with $f(\beta) = 0$ or $I = [\alpha, \beta]$ with $f(\alpha) = f(\beta) = 0$, and in the latter case $I = (-\infty, \infty)$.

2) $\Rightarrow$ 1) Suppose by contradiction that $g(\nabla f(p), \nabla f(p)) = 0$ for some $p \in M$. Therefore we may consider geodesic polar coordinates with origin at p, hence it must satisfy $u(a) = 0$; $a = t(p)$, cf. Remark 6.1 . This is a contradiction with the assumption that $g = \epsilon dt^2 + u^2(t)g_N$ is a global metric.

Suppose the metric is isometrically $g = \epsilon dt^2 + u^2(t)g_N$ on $I \times N$ where I is as in the assumption and where g_N is a complete Einstein pseudo-Riemannian hypersurface of (M, g) with $\varrho_N = k$. Hence g is complete. Additionally, since the functions $u = u(t) : I \to \mathbb{R}^+$ and $f = f(t) : I \to \mathbb{R}^+ \cup \{0\}$ satisfy the equations in Proposition 4.35 it follows that (M, g, f) satisfies Equation (11) of $(\lambda, n + m)$-Einstein manifolds.

By Lemma 5.9 the manifold (M, g) has harmonic Weyl tensor and satisfies $W(\nabla f, ., ., \nabla f) = 0$. $\square$

7.2 A specific result in dimension 4

Lemma 7.6. *([Bes08], Proposition 1.120). A 3-dimensional (connected) pseudo-Riemannian manifold (M, g) is Einstein if and only if it has constant sectional curvature.*

Proof. As in the literature, we know that any Einstein manifold $Ric = \gamma g$ with dim $M > 2$ has constant Einstein function γ. Now in order to complete the discussion one may instead use the known statement in the literature "(M, g) has constant sectional curvature k if and only if it satisfies"

$$R(X, Y, Z, V) = k(g \odot g)(X, Y, Z, V) \text{ for all } X, Y, Z, V \in TM. \tag{170}$$

As the dimension is $n = 3$ we obtain

$$g^{kl} g_{kl} R_{kijl} = \gamma g_{kl} g_{ij} \Rightarrow 3R_{kijl} = \gamma g_{kl} g_{ij}.$$

Similarly one has

$$3R_{kilj} = \gamma g_{kj} g_{il} \tag{171}$$

giving $-3R_{kilj} = -\gamma g_{kj} g_{il}$. By the properties of the curvature tensor we know that $-R_{kilj} = R_{kijl}$, so we obtain

$$R_{kijl} = \frac{\gamma}{6}(g_{kl}g_{ij} - g_{kj}g_{il}). \tag{172}$$

Therefore (M, g) is of constant curvature $k = \frac{\gamma}{6}$.

For the converse, suppose (M, g) has constant curvature k. Hence we can use the identity (170) in local coordinates giving

$$R_{kijl} = k(g \odot g)(\partial_k, \partial_i, \partial_j, \partial_l) \Rightarrow R_{kijl} = k(g_{kl}g_{ij} - g_{kj}g_{il})$$

which after tracing becomes $R_{ij} = 2kg_{ij}$. □

Corollary 7.7. *Let $m > 1$ and (M, g) be a pseudo-Riemannian manifold with dim $M = 4$. Also let (M, g, f) be a non-trivial $(\lambda$,n+m)-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ in neighborhood of p with $g(\nabla f, \nabla f)(p) \neq 0$, then (M, g) is conformally flat around p.*

Proof. By Theorem 7.2 the metric is locally of the form

$$g = \epsilon dt^2 + u^2(t)g_N$$

around p where g_N is Einstein of dimension 3, hence by Lemma 7.6 is of constant sectional curvature. Now Lemma 4.29 implies that g is conformally flat around p. □

Hint: In Corollary 7.7 above if in addition (M, g) is Einstein then it would be of constant curvature around p. Because we know from the literature that a manifold is both locally conformally flat and Einstein if and only if it has constant sectional curvature.

7.3 Classification of $(\lambda, n + m)$-Einstein metrics (M, g, f) around critical points of f in the pseudo-Riemannian setting

By Lemma 6.8 we already know that under the conditions harmonicity of the Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$, critical points of f in a Riemannian non-trivial $(\lambda, n + m)$-Einstein manifold are isolated. But, still we do not know whether under the same assumptions on the Weyl tensor as in Lemma 6.8, critical points of f are isolated in the pseudo-Riemannian setting. Example 4.49 shows that critical points of f in a non-trivial $(\lambda, n + m)$-Einstein manifold are not in general isolated. Any way in case a pseudo-Riemannian $(\lambda, m + n)$-Einstein manifold (M, g, f) is also Einstein, the Hessian tensor will be proportional to the metric and hence by [KR97a, Proposition 2.3] critical points of f would be isolated.

Remark 7.8. *Suppose (M,g) is a pseudo-Riemannian manifold in which geodesic polar coordinates (t,x) is of the form*

$$g = \epsilon dt^2 + u^2(t) g_N. \tag{173}$$

We know from the literature that the function $\psi(t) = \int_0^t u(s)ds$ satisfies $\nabla^2\psi = \epsilon u'(t)g$. Thus one can use [KR97a, Proposition 2.1 and Remark 2.2] to obtain $\nabla\psi(\gamma(t)) = (\kappa + \theta_\gamma(t))\gamma'(t)$ where $\theta_\gamma(t) := \int_0^t \epsilon u'(\gamma(s))ds$ and where κ, $\gamma(t)$ are defined as in [KR97a, Proposition 2.1]. Moreover if $\gamma(0)$ is a critical point of ψ i.e. $\nabla\psi(\gamma(0)) = 0$ then $\kappa = 0$, $\nabla\psi(\gamma(t)) = \theta_\gamma(t)\gamma'(t)$ and if $\gamma(t)$ is a null geodesic then $\psi(t)\Big(:= \psi(\gamma(t))\Big) = \psi(0)$ for all t.

On the other hand we have $\nabla\psi(t,x) = \nabla\psi(t) = \epsilon u(t)\partial_t$ and $\nabla u(t,x) = \nabla u(t) = \epsilon u'(t)\partial_t$. Therefore along geodesic γ

$$\begin{aligned}\nabla u(t,x)_{|\gamma(t)} = \epsilon u'(t)\partial_{t|\gamma(t)} = \frac{u'(t)}{u(t)}\epsilon u(t)\partial_{t|\gamma(t)} = \frac{u'(t)}{u(t)}\nabla\psi(\gamma(t))\\ = \frac{u'(t)}{u(t)}(\kappa + \theta_\gamma(t))\gamma'(t)\end{aligned} \tag{174}$$

hence in case $\gamma(0)$ is a critical point of ψ i.e. $\nabla\psi(\gamma(0)) = 0$, $\kappa = 0$ then

$$\nabla u(t,x)_{|\gamma(t)} = \frac{u'(t)}{u(t)}\theta_\gamma(t)\gamma'(t). \tag{175}$$

In (175) the coefficient of $\gamma'(t)$ is non-singular at $t = 0$, as we can see it by the L'Hospital's Rule.

Considering isolatedness of critical points of f as an additional assumption, we classify the triples (M,g,f) satisfying Equation (11) of a non-trivial $(\lambda, n+m)$-Einstein manifold with harmonic Weyl tensor and $(\nabla f, ., ., \nabla f) = 0$ around critical points of f in the pseudo-Riemannian setting. It shows that in geodesic polar coordinates whose origin is located at a critical point of f the level sets of f with the induced metric have constant sectional curvature, and in particular the metric is conformally flat.

Proposition 7.9. *Let $m > 1$, $\lambda \in \mathbb{R}$ and (M,g) be a pseudo-Riemannian manifold with a smooth non-constant f on M whose critical points are isolated. In addition suppose that (M,g,f) satisfies Equation (11) of a (λ,n+m)-Einstein metric with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ in a neighborhood of $p \in M$ with $\nabla f|_p = 0$. Then there are functions $u_\pm \in \mathscr{F}$ such that the metric in geodesic polar coordinates $(t,x) \in A_u \subset \mathbb{R}\times\sum$ in a neighborhood $\mathscr{U}$ of p has the form*

$$g(t,x) = g_u(t,x) = \epsilon dt^2 + \frac{u_\epsilon(t)^2}{u'_\epsilon(0)^2} g_1(x); \epsilon = g(x,x) \in \{\pm 1\} \tag{176}$$

where $u(t,x) = u_\epsilon(t)$, $\epsilon = g(x,x)$. If all geodesics through p are defined on the whole real line $\mathbb{R}$, then the metric g is of the form (176) for all $(t,x) \in A_u$, i.e.

as long as u_ϵ does not vanish. Also, all the conditions in Proposition 4.35 are satisfied.

Proof. We consider a neighborhood $\mathscr{U}$ of p in which $g|_{\mathscr{U}}$ has harmonic Weyl tensor and satisfies $W(\nabla f,.,.,\nabla f) = 0$. By assumption (after possibly restriction of $\mathscr{U}$) p is the only critical point of f in $\mathscr{U}$. Hence, by Theorem 7.2 we may introduce locally coordinates such that for $t \neq 0$

$$g = \epsilon dt^2 + u^2(t) g_N$$
$$f(t,x) = f(t), \;\; x \in N \tag{177}$$

where g_N is the induced metric on a regular level set N of f. By smoothness of f and g it follows that the equations in (177) hold at the time $t = 0$ as well. Via Equation (175) in the notation of Remark 7.8 one has

$$\nabla u(t,x)_{|\gamma(t)} = \Big(\frac{u'(t)}{u(t)}\theta_\gamma(t)\Big)\gamma'(t) \tag{178}$$

expressing that the normal vectors of the level hypersurfaces $u^{-1}(u(t_0))$ and of the distance spheres (the sets $\{t = t_0\}$) are proportional. Thus the connected components of $\{t = t_0\} \cap \mathscr{U}$ and of $u^{-1}(u(\gamma(t_0)))$ which contain $\gamma(t_0)$ coincide. Consequently there should be two smooth real functions $u_{\pm 1}$ satisfying $u(t,x) = u_{g(x,x)}(t)$. Furthermore as the metric g is smooth everywhere, from Proposition 6.4 it follows that $u_{\pm}^{(2m)}(0) = 0$ and $u_{+}^{(2m+1)}(0) = (-1)^{m+1}u_{-}^{(2m+1)}(0)$ for all $m \geq 0$.

Therefore, g in geodesic polar coordinates in $\mathscr{U}$ is of the form

$$g(t,x) = g_u(t,x) = \epsilon dt^2 + \frac{u_\epsilon(t)^2}{u_\epsilon'^2(0)} g_N(x); \epsilon = g(x,x) \tag{179}$$

for a C^∞-metric g_N on $\sum = S^0(1) \cup S^0(-1)$. Here the warping function has the denominator $u_\epsilon'(0)$ because the two components of the hypersurface $N = \sum = S^0(1) \cup S^0(-1)$ are of constant sectional curvature ϵ, cf. Lemma 6.3. Now by Lemma 6.3 the metric g_N on the hypersurface $\sum$ coincides with the standard metric g_1 of constant sectional curvature ϵ, hence we obtain the warped product (176). Since the metric g is smooth and hence extends to a neighborhood $\mathscr{U}$ of p, Proposition 6.4 implies that $u_\pm \in \mathscr{F}$.

Now assume $(t_0, x_0) \in A_u$, i.e. $u_\epsilon(t) \neq 0$ for all $t \in (0, t_0)$; $\epsilon = g(x,x)$. Then there is $t_1 \in (0, t_0)$ such that $(t_1, x_0) \in \mathscr{U}$. Assume t_* is the supremum of the numbers $t > 0$ such that for a neighborhood of the radial geodesic segment $r \in [0,t) \mapsto \phi(r, x_0) = (r(\nabla u)^1(x_0), ..., r(\nabla u)^n(x_0)) \in M$ ($(\nabla u)^i$ denotes the i-th component of ∇u in an orthonormal basis) the metric has the warped product representation (176). Here note the geodesically completeness property. Then we have $u_\epsilon(t_*) = 0$, because otherwise (under assumptions of harmonicity of the Weyl tensor and $W(\nabla f,.,.,\nabla f) = 0$) via Theorem 7.2 there would be $t_{*_1} > t_*$ such that $t_{*_1} \in A_u$ respectively the warped product is valid on $(0, t_{*_1})$.

To see that $\phi : A_u \longrightarrow M$ is injective, suppose $\phi(t_1, x_1) = \phi(t_2, x_2)$, $(t_j, x_j) \in A_u$. Two cases are possible:

1). If x_1, x_2 belong to the same component of $\sum$, then as $u(t)$ is strictly monoton from $u(t_1) = u(t_2)$ it follows that $t_1 = t_2$.
2). If x_1, x_2 belong to different components of $\sum$, assume $\gamma_1(t) = \phi(t, x_1)$, $\gamma_2(t) = \phi(t, x_2)$ be the two geodesics starting from p with $\gamma_1(t_1) = \gamma_2(t_2) = \phi(t_1, x_1) = \phi(t_2, x_2) = q$, $t_1 < 0 < t_2$. Then via Equation (178) it follows that

$$\nabla u(q) = -u'_{\epsilon_1}(t_1)\gamma'_1(t_1) = u'_{\epsilon_2}(t_2)\gamma'_2(t_2), \;\; \epsilon_j = g(\gamma'_j, \gamma'_j). \tag{180}$$

As x_1 and x_2 belong to different components we get $\gamma'_1(t_1) \neq \gamma'_2(t_2)$. On the other hand Equation (180) implies that $\gamma'_1(t_1)$ and $\gamma'_2(t_2)$ are parallel, and moreover, says that $\epsilon_j = g(\gamma'_j, \gamma'_j)$ meaning that $\gamma'_1(t_1)$ and $\gamma'_2(t_2)$ have pseudo-norm one. Consequently $\gamma'_1(t_1) = -\gamma'_2(t_2)$. Therefore from Equation (180) above it follows that $u'_{\epsilon_1}(t_1) = u'_{\epsilon_2}(t_2)$. This is not possible as by Equation (144) in Proposition 6.4 we see that u'_ϵ changes sign at 0. Since $u(t)$ and $f(t)$ are continuous and that by assumption (M, g, f) satisfies Equation (11), the conditions in Proposition 4.35.1) should be satisfied. □

Hint: In Proposition 7.9 if in addition (M, g) is Einstein, then Hess$f$ is proportional to the metric g, see [KR09, Step 4 and Step 5] for the corresponding results.

7.4 A note in the specific case of dim$M = 2$

When a manifold (M^2, g) is of dimension 2, by multi-linearity the Ricci tensor satisfies $Ric = \frac{Scal}{2}g$. Thus for every (M^2, g, f) satisfying Equation (11) of a $(\lambda, 2 + m)$-Einstein manifold the tensor Hess$f$ is proportional to the metric g, hence by [KR97a, Lemma 2.7] resp. [KR97a, Proposition 6.1] around points at which $|\nabla f| \neq 0$ resp. at which $|\nabla f| = 0$ the metric is locally a warped product. This in turn implies that, unlike Theorem 7.2 resp. Proposition 7.9 around points at which $|\nabla f| \neq 0$ resp. at which $|\nabla f| = 0$ we do not need any more the additional assumptions on the Weyl tensor in order that the metric splits into a local warped product. Similarly, in our results in the Riemannian case like theorems 6.5, 6.10 and 6.12 we do not need any more the assumptions of harmonicity of the Weyl tensor and $W(\nabla f, \nabla f) = 0$ in dimension 2. For a classification of $(\lambda, 2 + m)$-Einstein metrics see [Bes08, 9.118].

7.5 A comparison between $(\lambda, n+m)$-Einstein structure in the Riemannian and pseudo-Riemannian settings

We already know that in a connected Riemannian $(\lambda, n + m)$-Einstein manifold (M, g, f) with harmonic Weyl tensor and $W(\nabla f, ., ., \nabla f) = 0$ the function f has at most two critical points, cf. Theorem 6.12. But this simple behavior of f in the Riemannian case does not generalize to the pseudo-Riemannian setting. In fact, [KR97a, Theorem 4.3] implies that there exists a smooth pseudo-Riemannian manifold carrying smooth non-constant functions f and γ satisfying $\nabla^2 f = \gamma g$ such that the set of critical points of f is in natural bijection with either the set $J = \{1, ..., m\}$ or $J = \mathbb{N}$ or $J = \mathbb{Z}$. Since the proof of the theorem is independent from the Einstein property, one may choose a $(\lambda, n+m)$-Einstein manifold which is also Einstein. Then the corresponding Equation (11) becomes $\nabla^2 f = \gamma g$ where $\gamma = \frac{f}{m}\{\frac{Scal}{n} - \lambda\}$ and allows us to apply Theorem 4.3 in [KR97a]. In addition to these categories J of critical points for f in the pseudo-Riemannian setting, it may also happen that ∇f is a null vector at some points of M.

Therefore, considering the behavior of f, it seems we may not have a nice characterization like Theorem 6.12 for the pseudo-Riemannian setting. Let $\mathscr{N}$ shows the set of critical points of f and the points at which ∇f is null. Even if we can show that the critical points of f in the pseudo-Riemannian setting under the assumptions of harmonicity of the Weyl tensor and $(\nabla f, \nabla f) = 0$ are isolated, we do not know whether and how the points at which ∇f is null are isolated. Comparing to Theorem 6.12, if we assume the very strong condition that the points in $\mathscr{N}$ be isolated, then we may only characterize with the properties that the warping function $u(t)$ is odd at the points in $\mathscr{N}$, cf. Proposition 6.4.2). In addition, the warped product is complete if we add the set $\mathscr{N}$ to it, and, the equations in Proposition 4.35 are satisfied. But this situation for a characterization is not interesting.

7.6 The Brinkmann case in the pseudo-Riemannian setting

Lemma 7.10. *[Bri25]. If (M, g) admits a lightlike parallel vector field V, then there are local coordinates $t_1, t_2, ..., t_n$ $(n := dimM > 2)$ such that $V = \frac{\partial}{\partial t_1}$ and*

$$(g_{ij}) = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & (g_{\alpha\beta}) & \\ 0 & 0 & & & \end{array}\right)$$

where $\alpha, \beta \in \{3, ..., n\}$ and $\frac{\partial g_{\alpha\beta}}{\partial t_1} = 0$

Proposition 7.11. *Let $m > 1$, $n \geq 3$ and let (M, g, f) be a pseudo-Riemannian non-trivial $(\lambda, n+m)$-Einstein manifold which is also Einstein. Also let ∇f be a non-vanishing and isotropic (i.e. null) vector field on an open subset of M. Then ∇f is parallel, and the metric tensor can be converted in to the form $g = 2dt_1 dt_2 + g_*(t_1)$ where $\nabla f = \partial_{t_2} = \nabla t_1$ and where the $(n-2)$-dimensional metric $g_*(t_1)$ does not depend on t_2. Moreover, (M, g) is Ricci flat.*

Proof. As the manifold (M, g) is Einstein and hence of constant scalar curvature, via Proposition 4.41 (after adaptation to the pseudo-Riemannian case) the Einstein constant is ρ, i.e. $Ric = \rho g$. Therefore

$$\nabla^2 f = \frac{f}{m}(Ric - \lambda g) \Rightarrow$$

$$\nabla^2 f = \gamma g \quad \text{where} \quad \gamma = \frac{f}{m}(\rho - \lambda). \tag{181}$$

By assumption $g(\nabla f, \nabla f) = 0$ on an open set $\mathscr{U}$, hence $0 = \nabla_X(g(\nabla f, \nabla f)) = 2g(\nabla_X \nabla f, \nabla f) = 2\gamma g(X, \nabla f)$ for any vector field X. By non-degeneracy of g it follows that $\gamma \nabla f = 0$, hence $\gamma = 0$ as ∇f is non-vanishing on $\mathscr{U}$ by assumption. On the other hand, following equation (181) we have $\nabla^2 f(X, Y) = \gamma g(X, Y)$ and respectively $\nabla_X \nabla f = \gamma X$ for any X by the non-degeneracy of g. Thus $\nabla_X \nabla f = 0$ for any X which means ∇f is parallel.

As ∇f is non-vanishing on $\mathscr{U}$, we may use the function f as a coordinate t_1, then using Lemma 7.10 the metric can be converted into the form $g = 2dt_1 dt_2 + g_*(t_1)$. Since $t_1 = f$, by definition of f we see that the coordinate function t_1 vanishes on the boundary points and is positive elsewhere. Moreover via the calculation

$$\nabla t_1 = g^{ij} \frac{\partial t_1}{\partial t_i} \frac{\partial}{\partial t_j} = g^{12} \frac{\partial t_1}{\partial t_1} \frac{\partial}{\partial t_2} = 1.1.\partial_{t_2}; \;\; i, j \in \{1, ..., n\}$$

we see that $\nabla t_1 = \partial_{t_2}$ ($\partial_{t_2} := \frac{\partial}{\partial t_2}$). Hence using the equation $\nabla_{\partial_{t_i}} \partial_{t_j} = \nabla_{\partial_{t_j}} \partial_{t_i}$ as well as the fact that $\partial_{t_2} = \nabla t_1$ $(= \nabla f)$ is parallel one obtains

$$\partial_{t_2} g(\partial_{t_i}, \partial_{t_j}) = g(\nabla_{\partial_{t_i}} \partial_{t_2}, \partial_{t_j}) + g(\partial_{t_i}, \nabla_{\partial_{t_j}} \partial_{t_2}) = 0$$

expressing the metric does not depend on t_2.

Using again the fact that ∂_{t_2} is parallel, i.e. $\nabla_X \partial_{t_2} = 0$, one obtains $R(X, Y)\partial_{t_2} = 0$ for all X, Y. Since the manifold is Einstein with $Ric = \rho g$ it follows that $\rho = Ric(\partial_{t_1}, \partial_{t_2}) = 0$. □

Such spaces which admit a parallel isotropic vector field are called Brinkmann spaces. Here we did not consider the usual assumptions on the Weyl tensor because ∇f is isotropic and hence does not meet the conditions of Lemma 5.14. So we can not see the Hessian tensor has at most two eigenvalue functions to use it then to show ∇f is parallel on the open set. Instead we let (M, g) be Einstein by which ∇f becomes parallel, as explained in the proof of Proposition 7.11.

Reminder: For the case where (M, g) is not necessarily Einstein and carries a function $f \in C^\infty(M)$ satisfying $\nabla^2 f = \gamma g$ and in addition results in a Brinkmann space see [KR09, Theorem 3.12].

Remark 7.12. *For more details on transition from a non-isotropic gradient to an isotropic gradient see [Cat06, Theorem 3.1]. There, Catalano proves that for a function $f \in C^\infty(M)$ satisfying $\nabla^2 f = \gamma g$ around a point $p \in M$ with $\nabla f|_p \neq 0$ there are local coordinates $t_1, t_2, ..., t_n$ such that $\nabla f = \frac{\partial}{\partial t_1} + a \frac{\partial}{\partial t_2}$, for a function $a = a(t_2)$, and in addition $g = -a(t_2)dt_2^2 + 2dt_1 dt_2 + g_*(t_2)$. Then the transition corresponds to passing to the limit $a(t_2) \to 0$.*

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