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Essays in Information Economics

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Abstract

This thesis consists of three chapters and studies how information, or the lack thereof, affects the decision making of economic agents. By studying the theoretical implications of different information structures, it seeks to contribute to the understanding of how economic agents interact and what optimal decision making implies in situations of uncertainty.

Chapter 1 studies the strategic interaction between two agents/countries deciding whether to take climate action. A climate action is successful in restoring the environment if a critical mass of agents participate, providing a public good. The critical mass needed for a success is interpreted as the current state of the environment and modelled as a continuous variable. Depending on the state, agents may face either free ridding or coordination incentives. If one agent's action is sufficient to restore the environment, actions exhibit strategic substitutes and free riding incentives prevail. If the state is above a critical value, actions exhibit strategic complements; both agents need to coordinate for a success allowing for the possibility of a coordination failure. In a complete information environment, there always exists an equilibrium that exhibits a coordination failure. On the contrary, if agents face some uncertainty about the needed participation, under conditions on their utilities, a coordination failure will be avoided whenever the participation of both agents is needed. We show that risk-dominant actions can be strictly dominant at signals around the parameter value where actions change discontinuously from strategic substitutes to complements, even if they are nowhere strictly dominant in the underlying complete information game, and iteratively strictly dominant in the whole range of signals at which they are risk-dominant. We provide conditions on agents' utilities that warrant this outcome.

Chapter 2 generalizes the insight of chapter 1 in general two-player, two-action environments where agents' payoffs may change discontinuously. In particular, we extend global games *à la* Carlsson and van Damme (1993) to environments where the risk-dominant equilibrium is selected even if there is no dominance solvable game in the underlying class of complete information games. Strict dominance can emerge in the incomplete information game from strategic uncertainty due to discrete payoff changes in underlying games, and we provide sufficient conditions on payoff changes that warrant iterated dominance of the risk-dominant equilibrium. Thus, strategic uncertainty creates strictly dominant actions as well as fostering iterated dominance, in contrast to global games hitherto where strategic uncertainty does only the latter. Discrete payoff changes tend to arise, in particular, in situations where a public good can be provided with varying degrees of coordination

depending on the state, so that coordinating actions can be strategic substitutes and free-riding incentives present. We illustrate our findings in a stylized regime change model.

Chapter 3 studies whether information provided from a better than the agents informed central bank allows the latter to control inflation. We study this question in a monetary economy with asymmetric information and rational expectations. The central bank follows an expected inflation targeting rule and has private, noisy information about the future state of the economy, which communicates to market participants through its forecast about expected inflation (Delphic Guidance). Agents update their beliefs in a Bayesian way and infer the noisy signal for which the central bank has been informed about. Through this mechanism the central bank can shape agents' beliefs about the future state of the economy which affect current realised inflation, and control the stochastic path of inflation. Crucially, conventional inflation targeting policies, without explicit guidance, do not suffice to control the stochastic path of inflation. We characterise situations where a more comprehensive communication policy is called for, where the central bank needs to communicate its forecast about expected inflation as well as its forecasts about expected output.

Dedication and acknowledgements

I dedicate this thesis to my parents, *Michael* and *Popi*. I will always be grateful for your unwavering love and sacrifice. I also thank my brother *Chris* for his encouragement and our many late night conversations. Being part of such a loving family has been the greatest support throughout my life.

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Author's declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

SIGNED:

DATE: 02/03/2023

Contents

Chapter 1

Coordinating Climate Action Under Uncertainty

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1.1 Introduction

Despite continuous discussions, increased public awareness and an increase in the measures goverments take, climate change remains one of the most urgent issues of modern societies. Until 2100, we expect an increase of 4°C in global average temperature compared to pre-industrial levels, a scenario of catastrophic estimated consequences. In the absence of a global institution able to monitor and enforce commitments, countries seem reluctant or unable to implement the proposed policies and the switch to renewable energy sources is slow. With the adoption of carbon markets nowhere near the desired level[1] and countries being unlikely to reach the Paris Agreement goal of limiting global warming to well below 2°C, preferably to 1.5°C increase, compared to pre-industrial levels,[2] scientists urge for higher coordinated effort.

Economic literature has largely studied the free riding incentives that arise in such public good provision problems. Less attention has been given to the coordination mo-

[1]It is estimated that to reach the temperature-limiting level of 2^oC the carbon price should be \$100–\$250 per ton. In 2019 we were less than 10% of that price (Nordhaus (2019)).

[2]The latest UNEP (UN Environment Programme) "Emissions Gap Report" found out that all efforts would prevent only 7.5% of greenhouse gas emissions by 2030. To reach the 1.5°C target, however, it would have to be 55%. The models included the updated nationally determined contributions (NDCs) to climate protection in the context of the 2021's UN Climate Change Conference in Glasgow. Even with these new targets, we would have a warming of 2.7°C within this century.

tives that may occur when free riding may lead to substantial consequences, like in the case of climate change. If agents' benefit from free riding is less than their benefit from coordinating their action towards environmentally friendly production, then strong coordination incentives arise. Yet even in this case, coordination may not occur since there is the possibility of a coordination failure. This paper offers a framework to study those issues in an environment where agents may face either free riding or coordination incentives depending on noisy observations they make about the environment. We examine under which conditions they can avoid a coordination failure.

To make ideas more concrete, consider a game between the US and China deciding whether they should switch from a fossil fuel based production to a renewable energy production. Since climate action remains costly, even if agents enjoy great benefits from mitigation, they would rather free ride others' action if their participation is not pivotal. If for example the prediction for the increase in average global temperature was 2.5^oC (lower than the actual one) then China or the US[3] unilaterally could achieve the goal of 2^oC by switching technologies. In this case free riding incentives prevail. On the other hand, in the worse current 4^oC increase prediction, no agent can unilaterally achieve the goal and avoid the consequences of living in an above 2^oC world. To achieve the goals of the Paris agreement both countries need to coordinate their effort. This opens the door to the possibility of self-fulfilling coordination failures. Crucially, the state of the environment determines whether free riding or coordination incentives prevail. There exists a natural level of pollution, beyond which higher coordination is required to achieve the target goal. On the other hand, the actual state of the environment that determines agents' incentives is not precisely observed, since exact knowledge about future climate states is not possible. In turn, agents may face uncertainty on the exact strategic interaction they face.

We study an incomplete information environment where incentives may change discontinuously depending on the underlying state of nature. We explore equilibrium in these situations and describe under which conditions agents will avoid a coordination failure with certainty. Importantly, we find that in this environment information frictions matter. In a complete information environment, there exist no payoff parameters such that a coordination failure can be avoided with certainty; an equilibrium in which agents coordinate will always exist alongside one which exhibits coordination failure. In the incomplete information version of the game though, when agents only make noisy observations about the needed participation, under conditions on their payoffs, both agents will have a strictly dominant strategy to participate in a climate action and avoid a coordination failure. This provides a positive description of which types of problems can be resolved via coordination in the absence of an institution able to enforce commitments.

The idea that multiple equilibria might not be robust in the introduction of perturbations about the payoffs or information of the game is not new i.e. Harsanyi (1973a), Selten

[3]Accounting for 33% and 12% of total carbon emissions (World bank (2019)).

and Bielefeld (1988) and others. Even amongst strict equilibria that coordination games imply, Carlsson and van Damme (1993) showed that the multiplicity is not robust to the introduction of incomplete information about some payoff relevant parameter. Instead the risk dominant action is the one uniquely selected after iterated deletion of strictly dominated strategies. The result depends on agents' actions being strategic complements, payoffs being continuous to the fundamental and on the existence of extreme regions of the fundamental for which agents have a strictly dominant action. With the predictive power of the theory thus reinstated, global games have been fruitfully applied to various contexts, such as financial markets and social situations.

There are many situations though in which actions are not always strategic complements, potentially changing discontinuously between complements and substitutes. Coordination to mitigate the effects of climate change exhibits such characteristics. Similar strategic interactions emerge in other collective action situations such as contribution to a public good or protest participations. A defining feature of these situations is that the successful outcome of the coordinated actions is a public good that benefits everyone regardless of whether or not they contributed. This precludes the coordinating actions from being invariably strategic complements due to free-riding incentives, which is a crucial departure from the global games literature hitherto.

From a theoretical standpoint, we depart from the Carlsson and van Damme (1993) framework by assuming that there exists a critical fundamental value where actions discontinuously change from being substitutes to complements. The key observation is that, in the incomplete information version of the game, the risk dominant action can be strictly dominant around the discontinuity, even though it is nowhere strictly dominant in the complete information game. This allows for an iterative process similar to the one in global games to select that action for the entire range of the fundamentals in which it is risk dominant. We examine the conditions on the utilities around the critical value that warrant reverberation of the iterative process throughout the risk-dominant region.

We consider a stylised regime change model, vastly studied in the global games literature, modified to include free riding incentives. Two agents/countries simultaneously decide whether to take the costly action of adopting carbon emission reducing policies or not in order to achieve some goal set exogenously by some international authority. Both agents have a benefit if the goal is achieved but if an insufficient number of agents chooses to take action, the policy fails, with no benefit to either.

Given a target goal, the actual state of the environment dictates how many agents are needed to adopt the policy for a success which is modelled as the unobservable (underlying) fundamental; a continuous random variable. Since the actual state of the environment is unobserved, agents face uncertainty on the participation needed to achieve the goal. However, each agent observes a private signal on the state of the environment with a small random noise, from which they make inferences, before deciding whether to adopt a policy. The signal also allows the agent to make inference about the other agent's signal and their inference on the fundamental.

We start by characterizing a class of equilibria in which action to protect the environment will be taken whenever a success is possible (asymptotically as the noise vanishes) provided that acting is the risk-dominant action whenever actions are strategic complements. In this equilibrium, referred to as an interval-threshold equilibrium, one agent participates only when actions feature strategic complements and both agents' participation is needed for a success. The other agent adopts the policy both when actions feature strategic complements, avoiding a coordination failure, and when actions feature strategic substitutes. For large noise these are not the only equiliria that exist.

The main finding is that the coordination failure is always prevented in any equilibrium, because the risk-dominant action uniquely survives iterated eliminations of strictly dominated strategies so long as the cost of adopting the policy is below a bound which we identify, for small enough noise. At the borderline signal, where agents' participations are equally likely to be strategic complements and substitutes, an agent is pivotal in succeeding with one half probability whether the other agent participates or not. Consequently, his minimal expected benefit from participating is bounded away from zero however the other agent mixes between participating and not across her possible signals. If participation cost is below this minimal benefit, therefore, participation is strictly dominant for him upon observing a signal in a small neighborhood of that borderline signal, even though no action is strictly dominant in the complete information version of the game. This allows for an iterative procedure to select the risk dominant action for signals in the interval converging to the values of the fundamentals for which actions feature strategic complements as the noise vanishes. This result applied in the climate action context provides a positive description to the problems that can be resolved with coordination between the agents and whether intervention from an authority is necessary to guarantee coordination.

The rest of the paper is organised as follows. Section 2 discusses the relevant literature. Section 3 describes the model and the equilibrium concept. Section 4 characterizes interval-threshold equilibria and discusses equilibria when noise is large. Section 5 establishes the iterative dominance of risk-dominant actions in the strategic complements region. Section 6 discusses and section 7 concludes.

1.2 Related Literature

The theoretical underpinnings of the model are closely related with the global games framework, firstly studied by Carlsson and van Damme (1993). They showed that equilibrium selection in coordination games is possible if we embed complete information games in incomplete information environments. The risk dominant action (Harsanyi et al. (1988))

is the uniquely selected equilibrium. Their framework was later expanded by Morris and Shin (2001) and Morris and Shin (2002a) who highlighted the effects of public information in such environments. The framework has been utilized to study coordination issues in many theoretical and applied papers studying a variety of phenomena. Examples include Angeletos et al. (2006) who demonstrated how policy interventions can act as an endogenous signal and reintroduce multiplicity, Angeletos et al. (2007) who studied coordination games in a dynamic environment and many others. This line of literature assumes that actions feature strategic complements and that utilities are continuous to the fundamentals. In our environment actions change discontinuously from strategic complements to substitutes. Moreover, main result does not require the existence of two dominance regions, an assumption commonly made in this framework.

This paper is related to the line of literature that attempts to relax the complements assumption in global games. Karp et al. (2007) were the first to consider a global game with the addition of congestion effects. Their result was later challenged by Hoffmann and Sabarwal (2015) who argued that their existence result was incomplete. Bunsupha and Ahuja (2018) completed their result fully characterizing an equilibrium for this game with infinitely switching strategies. They showed that this equilibrium is unique under any strategy in which the aggregate action is monotone to the state of the fundamentals. Harrison and Jara-Moroni (2021) expand the global games framework to games that feature only strategic substitutes with overlapping dominance regions. Unlike us their payoffs are continuous to the fundamental and they do not consider pure free riding alongside with coordination incentives.

Equilibrium existence issues in games that feature both strategic substitutes and complements are discussed in Hoffmann and Sabarwal (2019a). While uniqueness in such environments is considered in Hoffmann and Sabarwal (2019b). Their result is different from ours since they assume that agents' utility is continuous to the state. Moreover, their uniqueness result relies on a contagion argument starting from a dominance region. If an action is strictly dominant for some realizations of the fundamentals, and if that dominance region is strong enough (they use a p-dominance condition to measure the influence of the dominant region to nearby values of the state), then for realizations of the fundamental close to this dominant region agents will take the same action, allowing an iterative argument to select an equilibrium. Our result is different to this line of literature since it relies on the discontinuity between strategic substitutes and complements in order to establish an iterative process.

Moreover, this paper is related to the literature that employs the global games framework to study collective action problems (Tullock (1971), Olson (1965)). Shadmehr (2018) study a collective action game where the strength of the regime is commonly known while there exists uncertainty on the participation cost of the agents. The decision to act depends on that cost and they characterise a symmetric equilibrium with a cutoff strategy.

Their equilibrium is unique if the uncertainty is not too small. Actions can feature either strategic substitutes or complements depending on the commonly known strength of the regime thus both cannot exist in the same model as in our environment. Morris and Shadmehr (2020) study a problem where the uncertainty is about the strength of the regime like us. The benefit that agents receive from a successful collective action however depends on the individual's effort, a continuous variable. Thus actions do not necessarily feature free riding. Their focus is the incentives that a leader needs to provide to heterogeneous agents to induce coordination. Other examples that study different aspects of collective action within this framework include Edmond (2013) who studies information manipulation in regime change movements; Shadmehr and Bernhardt (2011) who study the effects that uncertainty about the alternative regimes can have in the participation decision and others.

Lastly, this paper is subject to Weinstein and Yildiz (2007) critique who demonstrated that the particular departure from the complete information that is assumed in the global games framework is with loss of generality. In their paper, they show that the modelling choice of information can be modified in such a way that any action is uniquely rationalizable. By considering more general perturbations, they were able to recreate the global games result for any action. In a later paper Morris et al. (2016) showed that the particular departure of global games coincides with the epistemic foundation that has players being agnostic about their rank beliefs. That is players do not know whether their type is higher compared to their opponents'. Although we restrict ourselves to a less general class of games by considering the perturbation developed by Carlsson and van Damme, this form of incomplete information is believed to be suitable for the phenomena that this paper considers.

1.3 Model

Two risk neutral agents denoted by $i \in I = \{1, 2\}$ simultaneously make a binary choice $a_i \in \{0, 1\}$. We refer to $a_i = 1$ as the agent i's choice to "adopt the carbon reducing policy", or simply "act" for short, and $a_i = 0$ as his choice to "not adopt/not act," respectively. The two agents' choices succeed in restoring the environment if the number of agents who act exceeds θ. The random variable $\theta \in (0, 3)$ is realized at the beginning of the game. We interpret θ as the state of the environment which dictates how many agents need to act to restore it. Each agent receives a benefit of $b > 0$ if the environment is restored and the climate goal is achieved. Each agent i incurs a cost c_i if they act. We interpret c_i as the needed emission limit that some international institution proposes in order to achieve an exogenously set goal. A strict limit would imply a large c_i for agents.

Thus, agent i's utility is

$$u_i(a_i, a_{-i}, \theta) \;=\; \begin{cases} b - a_i c_i & \text{if } \; a_1 + a_2 \geq \theta \\ -a_i c_i & \text{otherwise} \end{cases}$$

and assume $b \geq c_i$. It is trivial that agent i would never act if $b < c_i$.

We describe the state of nature as "moderate" if $\theta \leq 1$, "critical" if $1 < \theta \leq 2$, and "irreversible" if $\theta > 2$. If the state is moderate, the two agents' choices to act are strategic substitutes as just one acting is enough to restore the environment, generating free-riding incentives for the agents. If the state is critical, choices to act are strategic complements since both agents need to act to succeed. If the state of nature is irreversible, clearly both agents have a strictly dominant choice to not act because regardless of agents' actions the environment cannot be restored. The description above is common knowledge, as is the information structure on θ explained below.

In the complete information benchmark where the value of θ is common knowledge, multiple equilibria arise due to standard coordination issues. When the state is moderate and agents' actions are strategic substitutes, there are two pure-strategy equilibria depending on who acts and a mixed-strategy equilibrium in which both agents randomize between acting and not. When the state is critical and agents' actions are strategic complements, there exist an equilibrium in which neither agent acts (coordination failure) as well as one in which both act. When $\theta > 2$, there is a unique dominant-strategy equilibrium where neither agent acts. Importantly regardless how low the cost of adopting the policy is, all of these equilibria exist since $c_i > 0$. Moreover, notice that 'acting' is not strictly dominant for any θ. This is not the case in the incomplete information environment.

We study an incomplete information environment where each agent privately observes a noisy signal of the underlying fundamental θ drawn from a uniform distribution over $[0, 3]$[4] Specifically, each agent i observes a signal $x_i = \theta + \epsilon_i$ where ϵ_i is an unbiased noise independently and identically distributed according to a cdf F supported on $[-\sigma, \sigma]$, with an associated density function f which is symmetric around and single-peaked at 0. Being interested in the impact of departure from complete information, we assume that the noise is relatively small, in particular, $\sigma \in (0, 1/6)$. With a slight abuse of notation, we denote the cdf of the random variable $\theta + \epsilon_i$ by $F(\cdot|\theta)$ and the density function by $f(\cdot|\theta)$, both with $[\theta - \sigma, \theta + \sigma]$ as their support.

Then, the posterior distribution (cdf) of θ conditional on any signal $x_i \in \mathbb{R}$ is $F(\cdot|x_i)$ is because

$$\frac{\int_{x_i-\sigma}^{\theta} f(x_i|\theta')d\theta'}{\int_{x_i-\sigma}^{x_i+\sigma} f(x_i|\theta')d\theta'} = \frac{\int_{x_i-\sigma}^{\theta} f(x_i-\theta')d\theta'}{\int_{x_i-\sigma}^{x_i+\sigma} f(x_i-\theta')d\theta'} = 1 - F(x_i - \theta) = F(\theta - x_i) = F(\theta|x_i),$$

[4]The distribution of θ is inessential for qualitative results so long as it has continuous and strictly positive density on an interval containing $[0, 2]$, but uniform distribution facilitates exposition greatly. Moreover, for the limit results, as noise vanishes any prior would approximate a uniform distribution.

where the third equality is due to symmetry distribution of noise around 0. That is, upon observing a signal x_i, agent i's posterior belief on θ is also F, centered at $\theta = x_i$ with a support $[x_i - \sigma, x_i + \sigma]$; thus, the posterior distribution $F(\cdot|x_i)$ shifts to the right as x_i increases by the same amount: $F(\theta|x_i) = F(\theta'|x_i')$ if $\theta' - \theta = x_i' - x_i$.

Finally, we assume that the cost of adopting the policy/acting satisfies

$$c_1 + c_2 < 1 \quad \text{and} \quad c_1 \leq c_2.$$

The first inequality ensures that acting is risk dominant for the range of the fundamentals for which agents' choices to act are strategic complements.[5] The second inequality is without loss. When it strictly holds, it implies that agent 1 has a risk dominant action to act whenever $\theta \leq 1$ and it is not risk dominant for agent 2 to act is this range of signals.

A strategy of agent i is a measurable function $s_i : \mathbb{R} \to [0,1]$ that specifies, contingently on every possible signal $x_i \in \mathbb{R}$, a probability with which agent i chooses to act. Agent i's expected utility from taking $a_i \in \{0,1\}$ upon observing a signal x_i, conditional on the other agent's strategy s_{-i}, is

$$U_i(a_i, s_{-i}, x_i) := \int\int \left[s_{-i}(x_{-i})u_i(a_i,1,\theta) + (1 - s_{-i}(x_{-i}))u_i(a_i,0,\theta)\right] dF(x_{-i}|\theta)\, dF(\theta|x_i).$$

Let $U_i(\alpha, s_{-i}, x_i) = \alpha\, U_i(1, s_{-i}, x_i) + (1-\alpha)U_i(0, s_{-i}, x_i)$ for $\alpha \in (0,1)$.

Definition 1 A strategy profile (s_1^*, s_2^*) is a *Bayesian Nash equilibrium* (BNE) if

$$U_i(s_i^*(x_i), s_{-i}^*, x_i) \;\geq\; U_i(a_i, s_{-i}^*, x_i) \qquad \forall a_i \in \{0,1\}, \quad \forall x_i \in \mathbb{R}, \quad i = 1,2.$$

1.4 Interval-threshold equilibrium

We start the analysis with characterising existence of equilibria in the incomplete information environment. Ideally, the two agents would like to coordinate on both acting when $\theta \in (1,2)$ and only one of them acting when $\theta < 1$, but this is infeasible because they observe only noisy signals of θ. Since the noise is small, however, such coordination may be approximated if one agent acts on all signals roughly below 2, and the other agent acts on all signals roughly in the interval $[1,2]$.

We characterize the conditions under which such a strategy profile indeed constitutes a BNE, specifically where one agent i acts below a threshold x_i^* and the other agent $-i$ acts in an interval $[\underline{x}_{-i}, x_{-i}^*]$ where $\max\{x_1^*, x_2^*\} \in (2-\sigma, 2+\sigma)$. We refer to such equilibrium as an *interval-threshold equilibrium.*

Intuitively, upon observing their respective upper threshold signals, the agent with the higher threshold, say i with $x_i^* > x_{-i}^*$, infers that the state is more likely to be irreversible

[5]The case where $c_1 + c_2 > 1$ would imply that "not act" would be the risk dominant action. This would trivialise the problem and no agent would act in the strategic complements region in the incomplete information version of the game.

(i.e., $\theta > 2$ is more likely) and also that the other agent is less likely to act, than the other agent $-i$ does upon observing x^*_{-i}. Hence, the agent with the higher upper threshold takes more risk by acting on his upper threshold signal and therefore, his cost of acting should be lower. We start with this observation stated below (and proved in Appendix).

Lemma 1 *In every interval-threshold equilibrium, $x^*_2 \le x^*_1$ where the inequality is strict if and only if $c_1 < c_2$.*

An agent brings a benefit of $b = 1$ to himself by acting when his acting is pivotal in restoring the environment, namely, when either (i) the state is critical (i.e., $1 < \theta < 2$) and the other agent acts or (ii) the state is moderate (i.e., $\theta < 1$) and the other agent does not. The probability of an agent's action being pivotal is:

$$Pv(x_i) = \text{Prob}(\text{agent } -i \text{ acts}, \, \theta \in (1,2) \,|\, x_i) + \text{Prob}(\text{agent } -i \text{ not act}, \, \theta < 1 \,|\, x_i) \, . \tag{1.1}$$

Hence, conditional on his signal x_i, it is optimal for an agent i to act if the probability that his action is pivotal exceeds his cost of acting $Pv(x_i) > c_i$, not act if $Pv(x_i) < c_i$ and he is indifferent between acting and not if they coincide:

Since the LHS (left-hand side) of (1.1) is continuous in x_i, (1.1) holds at each boundary signals x^*_i, x^*_{-i} and $\underline{x}_{-i}$. We first determine the boundary signal levels from this indifference condition, then verify optimality at other signals.

1.4.1 Optimality at the boundary signals

We start with the configuration that agent 1 acts below a threshold x^*_1, called a "threshold-player," and player 2 acts on signals in an interval $[\underline{x}_2, x^*_2]$, called an "interval-player". Subsequently, we examine the other configuration which is analyzed analogously subject to suitable modifications due to $c_1 \le c_2$.

Agent 1 acts on all signals below $x^*_1 \in (2-\sigma, 2+\sigma)$ in the considered configuration. Observing a signal $x_2 < x^*_1 - 2\sigma$, therefore, agent 2 infers that agent 1 will act for sure and thus, that he is pivotal if and only if the state is critical. Since the state must be critical if $x_2 > 1+\sigma$, he should act at signals $x_2 \in (1+\sigma, x^*_1 - 2\sigma)$, implying that $\underline{x}_2 < 1+\sigma < x^*_1 - 2\sigma < x^*_2$.

Moreover, upon observing $\underline{x}_2$, agent 2 is pivotal with the posterior probability that the state is critical, $1 - F(1|\underline{x}_2)$. Hence, the indifference condition for agent 2 at the lower boundary signal $\underline{x}_2$ simplifies to the first term of (1.1) being equal to c_2:

$$1 - F(1|\underline{x}_2) \;=\; c_2 \quad \Longrightarrow \quad \underline{x}_2 \in \begin{cases} (1-\sigma, 1] & if \;\; c_2 \le 0.5 \\ (1, 1+\sigma) & if \;\; c_2 > 0.5. \end{cases} \tag{1.2}$$

This equation determines the value of $\underline{x}_2$ uniquely and independently of x^*_1 and x^*_2.

To determine the upper threshold levels, note that upon observing their respective upper boundary signal x_i^*, both agents deduce that the state is never moderate (i.e., $\theta > 1$) because $1 + \sigma < x_2^*$ as verified above. Hence, either agent is pivotal if and only if the state is critical ($\theta < 2$) and the other agent acts, simplifying the indifference condition at x_i^* to the first term of (1.1) being equal to c_i:

$$\int_{x_1^*-\sigma}^{2} F(x_2^*|\theta)\, dF(\theta|x_1^*) \;=\; c_1 \quad \text{and} \quad \int_{x_2^*-\sigma}^{2} F(x_1^*|\theta)\, dF(\theta|x_2^*) \;=\; c_2. \tag{1.3}$$

Here, the integrand $F(x_i^*|\theta)$ is the probability that agent i would act conditional on θ, from the perspective of agent $-i$ upon observing x_{-i}^*. This is clear for agent $i = 1$, the threshold-player, because he is supposed to act at all signals below x_1^*; and so is $F(x_2^*|\theta)$ because, upon observing x_1^*, agent 1 infers that x_2 is at most 2σ away from $x_1^* > 2 - \sigma$, hence $x_2 > 2 - 3\sigma = 1 + 3\sigma > \underline{x}_2$. Thus, the upper boundary signals x_1^* and x_2^* are determined as the solution to the two equations in (1.3), independently of $\underline{x}_2$.

As we show in Appendix, there is a unique solution to (1.3) and $1 + 3\sigma < x_2^* < x_1^* \in (2 - \sigma, 2 + \sigma)$. It is clear that $x_1^*, x_2^* < 2 + \sigma$ because if $x_i^* \geq 2 + \sigma$ then the state must be irreversible (i.e., $\theta > 2$) and there is no chance to restore the environment. If $x_1^* \leq 2 - \sigma$ so that $x_2^* < 2 - \sigma$ as well, on the other hand, upon observing their respective upper boundary signal x_i^*, either agent i would infer that the state must be critical and thus that he is pivotal when the other agent observes a signal below x_{-i}^*. The probabilities for the two agents to be pivotal upon observing x_i^* as such are complementary, implying that the LHS of the two equations in (1.3) add up to 1, but this would contradict the assumption that $c_1 + c_2 < 1$.

1.4.2 Optimality at non-boundary signals

We have so far determined the boundary signal levels by (1.2) and (1.3) in an equilibrium where agents 1 and 2 adopt a threshold strategy and an interval strategy, respectively. We now verify optimality of these strategies at other signals.

Conditional on agent 1's strategy of acting on all signals below x_1^*, it is straightforward to see that it is optimal for agent 2 to act precisely at signals $x_2 \in [\underline{x}_2, x_2^*]$ because the expected gain from acting is lower at $x_2 < \underline{x}_2$ than at $\underline{x}_2$ since the state is less likely to be critical (while agent 1 will act for sure because $x_1 \leq \underline{x}_2 + 2\sigma < x_1^*$); and it increases as x_2 increases from $\underline{x}_2$ because the state is more likely to be critical, until x_2 gets high enough so that the state starts to become more likely to be irreversible and/or the other agent starts to be less likely to act; at that point the expected gain starts to decline, down to c_2 at $x_2 = x_2^*$ by (1.3) and lower afterward.

Next, we check optimality of agent 1 acting at every $x_1 < x_1^*$. Conditional on agent 2 acting if and only if $x_2 \in [\underline{x}_2, x_2^*]$, agent 1's expected gain from acting on observing a signal x_1, i.e., the LHS of (1.1), is

$$\int_{-\infty}^{1} F(\underline{x}_2|\theta)dF(\theta|x_1) \ + \ \int_{1}^{2} [F(x_2^*|\theta) - F(\underline{x}_2|\theta)]dF(\theta|x_1). \tag{1.4}$$

It is verified (in Appendix) that (1.4) decreases in $x_1 \leq 1-\sigma$ (when the second integral vanishes), but for $x_1 \geq 1+\sigma$ (when the first integral vanishes) it initially increases then declines (when the posterior probability of agent 2 acting declines), down to c_2 at $x_1 = x_1^*$ and further afterwards. Therefore, it suffices to show that (1.4) exceeds c_1 at every $x_1 \in [1-\sigma, 1+\sigma]$. Note that $F(x_2^*|\theta) = 1$ in (1.4) for $x_1 \leq 1+\sigma$ because $x_2^* > 1+3\sigma$ as noted above.

First, consider the case that $\underline{x}_2 \in (1-\sigma, 1]$, that is, $c_2 \leq 0.5$ by (1.2). Recall that agent 2's expected gain from acting on observing $x_2 = \underline{x}_2$, which equals c_2 by definition of $\underline{x}_2$, is the probability that $\theta \in (1,2)$, i.e., $1 - F(1|\underline{x}_2)$. Thus, upon observing the same signal $x_1 = \underline{x}_2$, if agent 1 is pivotal with a probability at least 0.5 conditional on θ being in a subset with a posterior probability at least $2(1 - F(1|\underline{x}_2))$, then agent 1's expected gain from acting is at least $1 - F(1|\underline{x}_2) = c_2 \geq c_1$. We identify, in Appendix, a subset of θ that works as such (the top end of feasible θ's upon observing $x_1 = \underline{x}_2$), and also show that the argument extends to other signals $x_1 \in (1-\sigma, 1+\sigma)$. In addition, a symmetric logic applies to the case that $\underline{x}_2 \in (1, 1+\sigma)$.

We now consider the alternative configuration in which agent 1 acts in an interval $[\underline{x}_1, x_1^*]$ and agent 2 below a threshold x_2^*. Analogously to the previous configuration, the upper boundary levels x_1^* and x_2^* are determined by (1.3) and $\underline{x}_1$ is determined by the condition $c_1 = 1-F(1|\underline{x}_1)$. In the current configuration, $\underline{x}_1 \in (1-\sigma, 1)$ because $c_1 < 0.5$ by an analogous reasoning behind (1.2), and the previous analysis for the case $\underline{x}_2 \in (1-\sigma, 1)$ applies with the roles of agents 1 and 2 swapped. Specifically, conditional on agent 1 acting if and only if $x_1 \in [\underline{x}_1, x_1^*]$, agent 2's expected gain from acting at signal x_2 is

$$\int_{-\infty}^{2} F(\underline{x}_1|\theta)dF(\theta|x_2) \ + \ \int_{1}^{2} [F(x_1^*|\theta) - F(\underline{x}_1|\theta)]dF(\theta|x_2) \tag{1.5}$$

and the minimum value of (1.5) across all $x_2 < x_2^*$ exceeds c_1.

Note that (1.5) is a function of c_1 because $F(x_1^*|\theta) = 1$ for $x_2 \in (1-\sigma, 1+\sigma)$ and $\underline{x}_1$ is determined by $c_1 = 1 - F(1|\underline{x}_1)$; hence the minimum value of (1.5) across all $x_2 < x_2^*$ is also a function of c_1, which we denote by $\bar{c}_2(c_1)$. Therefore, the current configuration constitutes a BNE if and only if $c_2 \leq \bar{c}_2(c_1)$. Note that in the limit case as $c_1 \to 0$ so that $\underline{x}_1 \to 1-\sigma$, the value of (1.5) at $x_2 = 1-\sigma$ converges to 0.5. This implies that if $c_2 > 0.5$ then the current configuration fails to be a BNE for sufficiently small c_1.

Summarizing the discussion so far, we characterize interval-threshold equilibria as below.

Proposition 1 (a) *There exists an interval-threshold equilibrium in which agent 1 adopts the threshold strategy and agent 2 the interval strategy. This equilibrium is unique and achieves the efficiency of complete information asymptotically as* $\sigma \to 0$.
(b) *It is an equilibrium for agent 2 to adopt the threshold strategy and agent 1 the interval strategy if and only if* $c_2 \in [c_1, \bar{c}_2(c_1)] \neq \emptyset$ *where* $\bar{c}_2(c_1)$ *is the minimum value of (1.5) across all* $x_2 < x_2^*$ *and converges to* 0.5 *from above as* $c_1 \to 0$.

Recall that the upper boundary levels x_1^* and x_2^*, determined by the equation system (1.3), are the same regardless of which agent adopts the interval strategy. Therefore, both agents $i = 1, 2$ act at all signals in their respective range $[\underline{x}_i, x_i^*]$ in any interval-threshold equilibrium, thus largely coordinate when both need to act to restore the environment (since $[\underline{x}_i, x_i^*] \approx [1, 2]$). In the next section, an iterated dominance argument shows that such coordination in the complementary region must prevail in every equilibrium if c_2 is not too large, as noise vanishes.

1.5 Iterative Dominance

Carlsson and van Damme (1993) establish the seminal result in 2-player, 2-action global games where the players' utilities change continuously in an underlying parameter θ and each player observes a noisy signal of θ: if an action, which is risk-dominant in some open range I of underlying parameter values, is strictly dominant at some $\theta \in I$ for at least one player, then it is iteratively dominant at all signals in I for both players in the global game as the noise vanishes.

Their result does not apply to the model analyzed in the previous section (in particular, to the complementary region) because no action is strictly dominant at any parameter values $\theta < 2$. Nevertheless, we show that acting ($a_i = 1$) is strictly dominant at signals near $x_i = 1$ in the global game, and through an iterative process its dominance extends to all signals in the complementary region as σ tends to 0. The key property behind this result is that acting, which is risk-dominant in the complete information game when θ is above the critical value of 1 (where the utilities are discontinuous), is also sufficiently attractive even if θ is slightly below 1 and the other agent switches to not acting ($a_{-i} = 0$). This may hedge the risk-dominant action sufficiently for it to be the dominant action at signals near the critical value, initiating the iterative expansion process.

Continuing with the model analyzed in the previous section, recall that an agent i is pivotal when either $\theta \in (1, 2)$ and the other agent $-i$ acts or $\theta < 1$ and agent $-i$ does not. Given a strategy $s_{-i} : \mathbb{R} \to [0, 1]$ of agent $-i$, therefore, the probability that agent i

is pivotal upon observing a signal $x_i \in (1-\sigma, 1+\sigma)$ is

$$\begin{aligned} P(x_i|s_{-i}) &:= \int_{x_i-\sigma}^{1}\int_{\theta-\sigma}^{\theta+\sigma}[1-s_{-i}(x_{-i})]dF(x_{-i}|\theta)dF(\theta|x_i) + \int_{1}^{x_i+\sigma}\int_{\theta-\sigma}^{\theta+\sigma}s_{-i}(x_{-i})dF(x_{-i}|\theta)dF(\theta|x_i) \\ &= F(1|x_i) + \int_{-\infty}^{\infty} s_{-i}(x_{-i})\Lambda(x_{-i}|x_i)\, dx_{-i} \\ \text{where} \quad \Lambda(x_{-i}|x_i) &:= \int_{1}^{x_i+\sigma} f(x_{-i}|\theta)f(\theta|x_i)d\theta - \int_{x_i-\sigma}^{1} f(x_{-i}|\theta)f(\theta|x_i)d\theta. \end{aligned} \tag{1.6}$$

If $P(x_i|s_{-i}) > c_i$ for every s_{-i}, then it is the dominant strategy for agent i to act at the signal x_i. To examine when this is the case, we observe that $P(x_i|s_{-i})$ is minimized when $s_{-i}(x_{-i}) = 0$ if $\Lambda(x_{-i}|x_i) \geq 0$ and when $s_{-i}(x_{-i}) = 1$ if $\Lambda(x_{-i}|x_i) < 0$.

Since $f(x_{-i}|\theta) = f(\theta|x_{-i})$ due to symmetry, $\Lambda(x_{-i}|x_i)$ is positive (negative, resp) if $\theta > 1$ is more (less, resp) likely than $\theta < 1$ conditional on observing two signals x_i and x_{-i}. Hence, $\Lambda(x_{-i}|x_i) = 0$ when x_{-i} and x_i are equidistant from 1 in opposite directions, i.e., $x_{-i} = 2 - x_i$, because then θ is equally likely to be above or below 1. Consequently,

$$\Lambda(x_{-i}|x_i) \begin{cases} < 0 & \text{if } x_{-i} < 2 - x_i \\ > 0 & \text{if } x_{-i} > 2 - x_i. \end{cases} \tag{1.7}$$

Thus, $P(x_i|s_{-i})$ is minimized when $s_{-i}(x_{-i}) = 0$ for $x_{-i} \geq 2 - x_i$ and $s_{-i}(x_{-i}) = 1$ for $x_{-i} < 2 - x_i$, which we denote by $\check{s}_{-i}$. Let $\underline{P}(x_i) := P(x_i|\check{s}_{-i})$ denote the minimum value of $P(x_i|s_{-i})$ across all s_{-i}.

If $x_i = 1$, in particular, $\check{s}_{-i}$ assigns 0 for $x_{-i} \geq 1$ and 1 for $x_{-i} < 1$. Therefore, $\underline{P}(1)$ is the probability, conditional on $x_i = 1$, that θ is below 1 but x_{-i} is above 1, or the other way around. The two events are equally likely and the probability of the latter is $\int_1^{1+\sigma} F(1|\theta)f(\theta|1)d\theta$. Hence,

$$\underline{P}(1) = 2\int_1^{1+\sigma} F(1|\theta)f(\theta|1)d\theta = 2\int_1^{1+\sigma} F(1-\theta)f(1-\theta)d\theta \;=\; \frac{1}{4}$$

where the last equality follows because $\int_{-\infty}^{a} F(x)f(x)dx = F(a)^2/2$ for any cdf F.[6]

If $x_i = 1-\sigma$ so that $\check{s}_{-i}$ assigns 1 for all $x_{-i} < 1+\sigma$, agent i is never pivotal because $\theta < 1$ for sure and the other agent were to always act, i.e., $\underline{P}(1-\sigma) = 0$. Analogously, $\underline{P}(1+\sigma) = 0$ because if $x_i = 1+\sigma$ then $\theta > 1$ and the other agent never acts according to $\check{s}_{-i}(x_{-i})$.

As such, the function $\underline{P}(x_i)$ is defined continuously on the interval $[1-\sigma, 1+\sigma]$ and assumes strictly positive values in the interior and 0 at the boundaries. For each $c \in (0, \underline{P}(1))$, therefore, a largest interval $(\underline{x}^{(1)}(c), \widehat{x}^{(1)}(c))$ exists on which $\underline{P}(x_i) > c$.[7] Consequently,

[6] Letting $F(x) = t$ so that $f(x)dx = dt$, $\int_{-\infty}^{a} F(x)f(x)dx = \int_{-\infty}^{F(a)} tdt = F(a)^2/2$.

[7] Note that $1 - \underline{x}^{(1)}(c) = \widehat{x}^{(1)}(c) - 1$ by symmetry.

[A] it is strictly dominant for an agent i to act at every signal $x_i \in (\underline{x}^{(1)}(c_i), \widehat{x}^{(1)}(c_i))$ if $c_i < \underline{P}(1)$.

Clearly, $1-\sigma < \underline{x}^{(1)}(c_1) < \underline{x}^{(1)}(c_2) < 1 < \widehat{x}^{(1)}(c_2) < \widehat{x}^{(1)}(c_1) < 1+\sigma$ if $c_1 < c_2 < \underline{P}(1)$.

From this initial range of signals on which acting is dominant, we expand the dominant range of signals iteratively in the usual manner. Given [A], an agent i with a signal $x_i \in [1-\sigma, 1+\sigma]$ is pivotal with a probability at least

$$\underline{P}_i^{(1)}(x_i) := \min_{s_{-i}} P(x_i|s_{-i}) \text{ subject to } s_{-i}(x_{-i}) = 1 \quad \forall x_{-i} \in (\underline{x}^{(1)}(c_{-i}), \widehat{x}^{(1)}(c_{-i})). \tag{1.8}$$

If $2 - x_i < \widehat{x}^{(1)}(c_{-i})$, the constraint in (1.8) requires s_{-i} to assign 1 to an interval of signals x_{-i} to which $\check{s}_{-i}$ assigns 0, increasing the value of $\min_{s_{-i}} P(x_i|s_{-i})$. Therefore, $\underline{P}_i^{(1)}(x_i) > \underline{P}(x_i)$ for all $x_i \in [1, 1+\sigma]$, in particular, and consequently, the range of signals on which acting is (iteratively) strictly dominant for agent i expands to an interval $(\underline{x}^{(2)}(c_i), \widehat{x}^{(2)}(c_i))$ that contains $(\underline{x}^{(1)}(c_i), \widehat{x}^{(1)}(c_i))$ and $\widehat{x}^{(1)}(c_i) < \widehat{x}^{(2)}(c_i)$.

Repeating the process iteratively, one generates an increasing sequence of upper boundaries of dominant ranges $\{\widehat{x}^{(n)}(c_i)\}_n$ for each agent i. Suppose $\widehat{x}^{(n)}(c_i) \geq 1+\sigma$ for both $i = 1, 2$ for some n, so that both agents are certain that $\theta > 1$ upon observing the boundary signal $\widehat{x}^{(n)}(c_i)$. Then, the probability of agent i being pivotal on observing $x_i \geq \widehat{x}^{(n)}(c_i)$ is minimized when agent $-i$ acts only in the then-dominant range of signals (which expands every round). Therefore, from then on, each agent's upper boundary of dominant range increases by at least the same amount as the other agent's boundary increased in the previous round (i.e., $\widehat{x}^{(n+1)}(c_i) - \widehat{x}^{(n)}(c_i) \geq \widehat{x}^{(n)}(c_{-i}) - \widehat{x}^{(n-1)}(c_{-i})$) until it reaches $2-\sigma$, when the expansion slows down and settles at x_i^* for both players, i.e., the upper boundary signals of the interval-threshold equilibrium in the previous section. We show in Appendix that this is indeed the case if $c_1, c_2 < \underline{P}(1) = 1/4$.

Next, we determine $\underline{x}^{(\infty)}(c_i)$, the lower end of the signal range for which acting is iteratively dominant for agent i. Given that it is iteratively dominant for both agents to act at every $x_i \in (\underline{x}^{(1)}(c_i), x_i^*)$ as shown above, upon observing a signal $x_i \in (1-\sigma, \underline{x}^{(1)}(c_i))$, the probability that agent i is pivotal is minimized when $s_{-i}(x_{-i}) \equiv 1$ by (1.6). Thus, the minimized value is $1 - F(1|x_i)$ which increases in x_i from 0 at $x_i = 1-\sigma$ and exceeds c_2 at all $x_i \in (\underline{x}^{(1)}(c_i), 1)$ as shown in [A] above. Consequently, $\underline{x}^{(\infty)}(c_i)$ is the signal $x_i \in (1-\sigma, \underline{x}^{(1)}(c_i))$ that solves $1 - F(1|x_i) = c_i$ for $i = 1, 2$. Note that this is $\underline{x}_i$ defined in the previous section, namely, the lowest signal at which the interval-player acts in the interval-threshold equilibrium, which we now denote as $\underline{x}(c_i)$ to be explicit about its dependence on c_i (but not on i).

Proposition 2 *It is iteratively strictly dominant for agent i to act at every signal in the interval $[\underline{x}(c_i), x_i^*) \supset [1, 2-\sigma]$ if $c_1 \leq c_2 < \underline{P}(1) = 1/4$.*

We stated the result for $c_1, c_2 < 1/4$, but this is not necessary. Note that the lower c_1 is, the larger is the initial signal range where acting is dominant for agent 1, $(\underline{x}^{(1)}(c_1), \widehat{x}^{(1)}(c_1))$.

This in turn means that a larger dominant signal range for agent 2 in the next stage, $(\underline{x}^{(2)}(c_2), \widehat{x}^{(2)}(c_2))$, and so on. As a result, the conclusion of Proposition 2 holds for higher c_2 (that goes above 1/4) if c_1 is lower.

Finally, it is straightforward to show that agent i never acts at any signal $x_i > x_i^*$ in every equilibrium, leading to the following characterization of equilibrium in conjunction with Proposition 2.

Corollary 1 *If $c_2 < 1/4$, in every equilibrium both agents act for sure at all $x_i \in (\underline{x}_i, x_i^*) \supset [1, 2-\sigma]$ and never acts at any signal $x_i > x_i^*$.*

We already established that it constitutes an equilibrium that agent i acts if and only if $x_i \in (\underline{x}_i, x_i^*)$ and agent $-i$ acts if and only if $x_{-i} < x_{-i}^*$. This implies that the range of signals where the risk dominant actions are iteratively dominant cannot be expanded beyond $(\underline{x}_i, x_i^*)$.

Section 5 demonstrated how low agents' costs facilitate uniqueness in the complements region. An important observation is that there is a friction between uniqueness in the strategic complements and substitutes region. By allowing equilibrium selection in the strategic complements region, we make it harder for equilibrium selection in the strategic substitutes region to be achieved. This is a known friction in the literature as demonstrated at Guesnerie (2004). They argue that conditions that facilitate equilibrium selection in a game of strategic complements will have the opposite effect in a game with strategic substitutes. In our game this is incorporated in the players' costs. For high enough costs we can achieve uniqueness in the strategic substitutes region but we will have multiplicity in the complements region. On the contrary, low costs result in a unique equilibrium in the complements region but multiplicity in the substitutes region. This is highlighted in the next proposition.

Proposition 3 *It is iteratively strictly dominant for agent 1 to act and agent 2 not to act at every signal $x_i < 1$ if $c_2 > 3/4$.*

Notice that equilibrium selection in the strategic substitutes region implies multiplicity whenever the fundamentals exhibit strategic complements and vice versa. This in turn in the context of the model discussed above implies a potential source of inefficiency. Notice that in the case where $c_1 < c_2$ the socially optimal outcome would require agent 1 acting whenever $\theta \in (0, 1]$ while both agents act whenever $\theta \in (1, 2)$. This way, the environment is restored with the least possible cost.

1.6 Discussion

The analysis so far provides a description of the situations where agents will coordinate in taking the climate action, avoiding a coordination failure, whenever both agents' participation is needed. The observation that if agents' participation cost is low enough, coordination is guaranteed can be crucial for an institution that attempts to coordinate agents in adopting climate friendly policies. Notice that such an institution would attempt to coordinate agents by setting a goal which would imply the costs that agents will have to occur in order to succeed in lowering global temperature. If the goal is too ambitious, for example, and requires a large change in production, then it is associated with larger costs. The model implies that picking the correct goal may have large consequences for coordination and provides a framework to study these issues.

Moreover, in our attempt to focus on the coordination incentives of the issue, we have abstracted from other strategic interactions that the problem presents. For example, one could expand the analysis by studying a two stage game to include communication or negotiations that may lead to commitment. Even though commitment would likely facilitate coordination, when considering issues like climate change it has been proved hard to implement. Whether meaningful communication in such an environment where agents may face different incentives is possible is a question that we leave for future research.

Lastly, the work so far has focused on a two player binary action model. Naturally, when one thinks of issues like climate change should include more agents that are heterogeneous not only with respect to their cost from acting but also to the respective benefit they get from a resolution of the issue as well as the impact their action has. Inclusion of such elements would significantly complicate the analysis since agents in this case would not only face either strategic substitutes or complements like in the model above but in cases actions can feature both strategic substitutes and complements with those incentives changing at different points for different agents. Nevertheless, this paper indicates that studying those issues in an incomplete information environment, and carefully considering the different strategic interactions between agents can have strong implications about the situations where a coordination failure will be avoided.

1.7 Conclusion

Climate change is one of the most urgent issues of our time. Despite specialists' warnings, countries seem reluctant to implement the measures suggested by international institutions and under the current predictions they will fail to achieve the proposed goal. At the heart of the problem lies the public good characteristic of the issue. All interested parties would like to free ride others' effort and not contribute by participating and taking costly green measures. Even in the presence of large consequences due to the effects of climate change, when agents have a strong incentive to coordinate their effort, it is not clear that they can avoid a coordination failure. This paper studies these issues in an incomplete information environment and describes under which conditions coordination is the unique game theoretic prediction of the strategic interaction.

Two large players decide whether to take climate action or not, after observing noisy signals about the state of the environment. If the state is moderate only one agent needs to adopt climate policies in order to restore the environment, while both agents' participation is needed if the state is critical. Actions can thus exhibit either strategic substitutes or complements with the possibility of a self-fulfilling coordination failure. The key implication of the model is that there exist utility levels such that a coordination failure will always be prevented, in the incomplete information environment of the game. The same utility levels would not guarantee coordination in the complete information game. This provides a description of the types of problems that can be solved with coordination.

From a theoretical standpoint, we study a 2-player, 2-action coordination game in which agents' actions can feature either strategic complements or substitutes. The departure from the previous literature stems from actions changing between substitutes and complements discontinuously to the underlying fundamental. We observe that around the critical level of the fundamental value, where such discontinuity occurs, agents can have a strictly dominant action in the incomplete information game even though no action is strictly dominant in the complete information version. That is because the risk dominant action from one side of the discontinuity, depending on agents' utilities, can be sufficiently attractive to the agents, in the contingency that their opponent takes the opposite action, on the other side of the discontinuity. This allows for an iterative process similar to the one developed in Carlsson, van Damme (1993) to select that action as the unique prediction for all fundamental values for which it remains risk dominant. We derive conditions on the utilities of the agents that allow for such iterative process to take hold.

1.8 Appendix

Proof of Lemma 1. Consider the agent i whose upper threshold is higher, i.e., $x_i^* \geq x_{-i}^*$. Upon observing $x_i^* > 2 - \sigma$, this agent infers $\theta > 1$ and thus that he is pivotal if $\theta < 2$ and the other agent observes a signal to act, the likelihood of which is equal to c_i by (1.1). The probability that the other agent $-i$ observes a signal to act, however, is less than 0.5 because x_{-i} is equally likely to be above and below x_i^* and $x_{-i}^* < x_i^*$, from which we deduce that $c_i < 0.5$. Since this probability is positive, we also deduce that $x_{-i}^* > x_i^* - 2\sigma > 2 - 3\sigma > 1 + \sigma$. Then, upon observing x_{-i}^*, agent $-i$ also infers $\theta > 1$ and thus that he is pivotal if $\theta < 2$ and the other agent observes a signal to act, the likelihood of which is equal to c_{-i}.

Since $x_i^* > x_{-i}^* > 1 + \sigma$, the posterior probability that $\theta \in (1, 2)$ is higher at the signal x_{-i}^* than at x_i^*, and the probability of the other agent observing a signal to act conditional on agent i observing x_i^* is no higher than 0.5. Thus, if the agent i observes a signal to act with a probability exceeding 0.5 conditional agent $-i$ observing x_{-i}^*, then $c_i < c_{-i}$ ensues, i.e., $i = 1$. This is clearly the case if agent i is the threshold-player. If agent i is the interval-player, then $\underline{x}_i < 1 + \sigma$ because he should act upon observing a signal $x_i = 1 + \sigma$ given that $\theta > 1$ for sure and agent $-i$ acts with prob at least 0.5, as well as $c_i < 0.5$. Thus, $[\underline{x}_i, x_i^*]$ is an interval of length exceeding 2σ and contains x_{-i}^*, hence the agent i observes a signal to act with a probability exceeding 0.5 conditional agent $-i$ observing x_{-i}^*. ∎

Proof of Proposition 1. We provide the deferred proofs.

(1) To show there is a unique solution to (1.3) and $2 - 3\sigma < x_2^* < x_1^* \in (2 - \sigma, 2 + \sigma)$.

We have shown in the main text that $x_2^* \leq x_1^* \in (2 - \sigma, 2 + \sigma)$. For agent 1 to be indifferent between acting and not at x_1^*, he should be pivotal with a positive probability, which implies that $x_2^* > x_1^* - 2\sigma > 2 - 3\sigma$.

Next, suppose there are two solutions to (1.3, denoted by (x_1^*, x_2^*) and (x_1', x_2') where $x_1' = x_1^* - r < x_1^*$ wlog. Then, (1.3) dictates that

$$\int_{x_1^*-\sigma}^{2} F(x_2^* - \theta + \sigma) f(\theta - x_1^* + \sigma) d\theta \; = \; c_1 \; = \; \int_{x_1^*-r-\sigma}^{2} F(x_2' - \theta + \sigma) f(\theta - x_1^* + r + \sigma) d\theta.$$

Note that the RHS evaluated at $x_2' = x_2^* - r$, is $\int_{x_1^*-\sigma}^{2+r} F(x_2^* - \tilde{\theta} + \sigma) f(\tilde{\theta} - x_1^* + \sigma) d\tilde{\theta} > c_1$ by change of variable $\tilde{\theta} = \theta + r$. This implies that $x_2' < x_2^* - r$. On the other hand,

$$\int_{x_2^*-\sigma}^{2} F(x_1^* - \theta + \sigma) f(\theta - x_2^* + \sigma) d\theta \; = \; c_2 \; = \; \int_{x_2'-\sigma}^{2} F(x_1^* - r - \theta + \sigma) f(\theta - x_2' + \sigma) d\theta$$

by (1.3), but the RHS evaluated at $x_2' = x_2^* - r$, is $\int_{x_2^*-\sigma}^{2+r} F(x_1^* - \tilde{\theta} + \sigma) f(\tilde{\theta} - x_2^* + \sigma) d\tilde{\theta} > c_2$. This implies that $x_2' > x_2^* - r$ (because the RHS of the previous displayed equation decreases in x_2' due to symmetry and single-peakedness of f), contradicting the earlier assertion $x_2' < x_2^* - r$. Note that this argument presumes $x_2^* + \sigma > 2$. If $x_2^* + \sigma < 2$ then since $\theta < 2$ is evident to agent 2 upon observing x_2^* or x_2', $x_1^* - x_2^* = x_1' - x_2'$ must hold, again contradicting $x_2' < x_2^* - r$.

(2) To show (1.4) decreases in $x_1 \leq 1 + \sigma$; for $x_1 \geq 1 + \sigma$, it initially increases then declines.

The derivative of (1.4) wrt x_1 is

$$-\int_{-\infty}^{1} F(\underline{x}_2|\theta) f'(\theta|x_1) d\theta \;-\; \int_{1}^{2} [F(x_2^*|\theta) - F(\underline{x}_2|\theta)] f'(\theta|x_1) d\theta. \tag{1.9}$$

Note that f is symmetric around and single-peaked at $\theta = x_1$, that is, $f'(\theta|x_1) = -f'(2x_1 - \theta|x_1) > 0$ for $\theta \in (x_1 - \sigma, x_1]$, which is used repeatedly in the reasoning below. For $x_1 \leq 1 - \sigma$, only the first term is relevant (the second term vanishes) which is negative because $F(\underline{x}_2|\theta)$ decreases in $\theta \in [x_1 - \sigma, x_1 + \sigma]$. For $x_1 \geq 1 + \sigma$, only the second term is relevant (the first term vanishes). $F(x_2^*|\theta) = 1$ for $\theta \leq x_2^* - \sigma$, decreases for $\theta \in (x_2^* - \sigma, x_2^* + \sigma)$ and is 0 for $\theta \geq x_2^* + \sigma$. $F(\underline{x}_2|\theta) = 1$ for $\theta \leq \underline{x}_2 - \sigma$, decreases for $\theta \in (\underline{x}_2 - \sigma, \underline{x}_2 + \sigma)$ and is 0 for $\theta \geq \underline{x}_2 + \sigma$. Since $x_2^* - \underline{x}_2 > 2\sigma$, $F(x_2^*|\theta) - F(\underline{x}_2|\theta)$ increases for $\theta \in (1, \underline{x}_2)$ if nonempty, then stay constant at 1 until $\theta = x_2^* - \sigma$ (hence, for an interval of θ of length at least 2σ), from which point it declines down to 0 at $\theta = x_2^* + \sigma$. Due to symmetric and single-peaked f, therefore, as x_1 increases from $1 + \sigma$ the second term of (1.9) is positive, then 0 for a while before turning to negative. This means that for $x_1 \geq 1 + \sigma$, (1.4) initially increases then declines down to c_2 at $x_1 = x_1^*$ and further afterwards.

(3) To show that (1.4) exceeds c_1 at every $x_1 \in [1 - \sigma, 1 + \sigma]$.

Focus on the highest possible θ's with a posterior probability $2(1 - F(1|\underline{x}_2))$ upon observing $x_1 = \underline{x}_2$, that is, the interval $[\widehat{\theta}, \underline{x}_2 + \sigma]$ where $1 - F(\widehat{\theta}|\underline{x}_2) = 2(1 - F(1|\underline{x}_2))$. Agent 1's action is pivotal with a probability greater than 0.5 conditional on $\theta \in [\widehat{\theta}, \underline{x}_2 + \sigma]$, because then θ is equally likely to be above and below 1 (by construction) and

$$F(\underline{x}_2|\theta) > F(\underline{x}_2|\theta') \iff F(\underline{x}_2|\theta) + 1 - F(\underline{x}_2|\theta') > 1 \quad \text{if} \quad \theta < 1 < \theta', \tag{1.10}$$

that is, the average probability that agent 1's action is pivotal between any two $\theta, \theta' \in [\widehat{\theta}, \underline{x}_2 + \sigma]$, one below 1 and the other above 1, exceeds 0.5. This implies that (1.4) exceeds $1 - F(1|\underline{x}_2) = c_2$ at $x_1 = \underline{x}_2$.

The same conclusion obtains when agent 1 observes $x_1 > \underline{x}_2$ as well, because then θ is more likely to be above than below 1 subject to θ being in the top interval of possible θ's of measure $2(1 - F(1|\underline{x}_2))$ and, in addition to (1.10), we have $1 - F(\underline{x}_2|\theta) > 0.5$ for all $\theta > 1$. The same also holds at $x_1 < \underline{x}_2$, because then it is straightforward to verify that agent 1's action is pivotal with a probability exceeding 0.5 both conditional on $\theta < 1$ and conditional on $\theta \in (1, 2)$.

Therefore, if $\underline{x}_2 \in (1-\sigma, 1)$, i.e., $c_2 \leq 0.5$, then the minimum value of (1.4) across all $x_1 < x_1^*$ exceeds c_2, hence exceeds c_1 as well, establishing it to be an equilibrium for agent 1 to adopt the threshold strategy below x_1^* and agent 2 the interval strategy on $[\underline{x}_2, x_2^*]$.

Next, consider the case that $\underline{x}_2 \in (1, 1+\sigma)$ so that $c_2 > 0.5$ by (1.2). Note the symmetry between this and the previous case: agent 1's action is pivotal if both θ and x_{-i} are one the same side (below or above) of 1 and $\underline{x}_2$, respectively, except that $\underline{x}_2$ is on the opposites of 1 in the two cases. From this symmetry it follows that the value of (1.4) at $x_1 \in (1-\sigma, 1+\sigma)$ in one case coincides with the value of (1.4) in the other case when x_1 is equidistant from 1 in the other direction and consequently, that the minimum value of (1.4) among all $x_1 < x_1^*$ is also the same in the two cases. Since this minimum value has been shown to exceed c_1 when $c_2 < 0.5$ above, so it must when $c_2 > 0.5$ as well, establishing it to be an equilibrium for agent 1 to adopt the threshold strategy below x_1^* and agent 2 the interval strategy on $[\underline{x}_2, x_2^*]$. ∎

Proof of Proposition 2. Recall the iterative process that generates an increasing sequence of upper boundaries of dominant ranges $\{\widehat{x}^{(n)}(c_i)\}_n$ for each agent i. It remains to verify that $\lim_{n\to\infty} \widehat{x}^{(n)}(c_i) = x_i^*$ for $i = 1, 2$ if $c_1, c_2 < \underline{P}(1) = 1/4$.

Note that this will indeed be the case if the upper boundary $\widehat{x}^{(1)}(c_i)$ is already above $1+\sigma$ after the first round, i.e., $c_i \leq \min_{x_i \in [1-\sigma, 1+\sigma]} \underline{P}_i^{(1)}(x_i)$ for $i = 1, 2$. For $c_i > 0$ small enough, this is the case because $\widehat{x}^{(1)}(c_i) \to 1+\sigma$ as $c_i \to 0$ and thus, $\underline{P}_i^{(1)}(x_i)$ is bounded away from 0 on $[1-\sigma, 1+\sigma]$.

From construction of the sequence of dominant intervals $\{(\underline{x}^{(n)}(c_i), \widehat{x}^{(n)}(c_i))\}_n$, it is clear that $(\underline{x}^{(n)}(c_i), \widehat{x}^{(n)}(c_i)) \subset (\underline{x}^{(n)}(c_i'), \widehat{x}^{(n)}(c_i'))$ for each n and $i = 1, 2$, if $c_i \geq c_i'$ for $i = 1, 2$. Therefore, if $\widehat{x}^{(\infty)}(c_i') < 1+\sigma$ for some i and some (c_1', c_2'), then $\widehat{x}^{(\infty)}(c_1) = \widehat{x}^{(\infty)}(c_2) < 1+\sigma$ for $c_1 = c_2 = \min\{c_1', c_2'\}$. Moreover, since $\widehat{x}^{(n)}(c_i)$ is continuous in c_i when $c_1 = c_2$, there is some $c > 0$ such that $\widehat{x}^{(\infty)}(c_1) = \widehat{x}^{(\infty)}(c_2) = 1+\sigma$ for $(c_1, c_2) = (c, c)$. This means that for $(c_1, c_2) = (c, c)$, we have c being equal to

$$
\begin{aligned}
\underline{P}_i^{(\infty)}(1+\sigma) &= \min_{s_{-i}} P(1+\sigma|s_{-i}) \ \text{ subject to } \ s_{-i}(x_{-i}) = 1 \quad \forall x_{-i} \in (\underline{x}^{(\infty)}(c_{-i}), \widehat{x}^{(\infty)}(c_{-i})) \\
&\geq P(1+\sigma|s_{-i}) \ \text{ where } \ s_{-i}(x_{-i}) = 1 \quad \Leftrightarrow \quad x_{-i} \in (-\infty, 1-\sigma] \cup [1, 1+\sigma] \\
&> 1/4
\end{aligned}
$$

where the weak equality is due to (1.6) and the strict inequality ensues because the regime is strong for sure on $x_i = 1+\sigma$, given which $Prob(x_{-i} \in [1, 1+\sigma]) > 1/4$, contradicting $c < \underline{P}(1) = 1/4$. ∎

Proof of Proposition 3.

See Corollary 2.2 and its discussion in the next Chapter. ∎

Chapter 2

Global Games without Dominance Solvable Games

Statement of co-authorship: *This chapter is co-authored with In-Uck Park. Both authors contributed in all parts of the chapter.*

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2.1 Introduction

Game theory has long been a primary theoretic framework in economic analysis, yet its predictive power and usefulness are marred by multiple equilibria that often arise even in basic economic models such as coordination games. The problem is attributed at least partly to excessive rationality and perfect knowledge of the game assumed on players, and equilibrium selection has been fruitful through requiring robustness to slight perturbations in such aspects, e.g., Harsanyi (1973a) and Selten (1975), among others.

Even between two strict Nash equilibria, Carlsson and van Damme (1993) show that the risk-dominant equilibrium (Harsanyi and Selten, 1988; defined later) can be selected if the payoffs of the game to be played are determined continuously by an underlying state/fundamental and the players observe only private signals of the underlying state with small noises. This framework, known as "global games," has been widely adopted to study coordination issues in various economic and social phenomena (see below) owing to its appealing structure of informational incompleteness. The authors note, however, that *the result depends critically on the existence of a subclass of dominance solvable games* (for some underlying states) *that serve as take-offs for the iterated dominance argument* (p. 992). Then, the equilibrium action of the dominance solvable games will become iteratively dominant when nearby states are observed if it is risk-dominant in those states, because the players will then assign high enough probability to their opponent playing that equilibrium given the strategic uncertainty caused by small noise in observation.

We show that the same result can be obtained even if no dominance solvable game exists because the initial dominance could be forged from strategic uncertainty in the global game. To see the basic idea, consider two friends who will commonly benefit when a task is completed, which may require just one or both of them to work depending on the underlying state. Individually, work will be beneficial only if the other also works in case both are required, but only if the other doesn't otherwise (free riding). Hence, work is dominant in neither state (whether one or both are needed). If both states are equally likely based on the observed signals, however, work would bring benefit with 50% chance no matter what, thus would be the dominant action so long as the cost of work is small relative to the benefit. We study when the iterated dominance may start off from dominance in the global (rather than the underlying) game due to strategic uncertainty and progress to cover the entire risk-dominant region.

If payoffs of the game are continuous everywhere in the underlying state, dominant actions cannot emerge in the global game without dominance solvable games (hence the quote above) because players know the payoffs arbitrarily precisely for small noise. Therefore, our results rely on discontinuous changes in payoffs for the iterated dominance to take off. In financial markets, for instance, returns from investments may change discontinuously if different monetary policies will be adopted depending on whether the value of the fundamental turns out to be above or below a threshold. Discrete changes in payoffs also tend to arise in situations where a public good can be provided with varying degrees of coordination depending on the underlying state. For example, in a democracy movement aimed at toppling a repressive regime, a full uprising may be needed to succeed if the regime is strong (e.g., if the fundamental, capturing the regime's strength, is above a threshold level) but a modest turnout may suffice otherwise. Then, participations in the protest are strategic complements in the former case, but strategic substitutes in the latter where free-riding incentives are inherent. Our analysis pertains to such situations which have not yet been addressed in the global game literature.

Formally, we extend two-person, two-action global games studied in Carlsson and van Damme (1993) to environments where strategic uncertainty in the global game kicks off (as well as advances) iterated elimination of strictly dominated strategies. Suppose an action pair, say (a_1, a_2), is the risk-dominant equilibrium for underlying states above a certain critical level. A risk-dominant equilibrium is a strict equilibrium with a greater product of deviation losses than the other strict equilibrium. Hence, the actions a_1 and a_2 are optimal relative to each other but suboptimal relative to the other action of the opponent, say b_i, for states above the critical level.

Let us ask when a player, say 1, may find the action a_1 to be his dominant choice in the global game upon observing the critical level as his signal (so that the underlying state may be on either side). Dominance requires a_1 to be uniquely optimal whatever agent 2 does, accounting for uncertain underlying state. Since a_1 is optimal relative to a_2 but not to b_2 in states above the critical level as mentioned, at minimum a_1 has to be optimal relative to b_2 in some states below the critical level. This observation leads to a necessary condition for an iterated elimination process to take off, namely, either the risk-dominant action α_i is strictly dominant or the two actions are strategic substitutes immediately below the critical level for at least one player.

Once the iteration process progressed enough so that, conditional on the boundary signal of the dominance range, the dominant choices must be risk-dominant for any possible underlying state (given small noise), then a standard logic in global games ensures that the iteration continues to cover the entire risk-dominant range of signals. In Carlsson and van Damme (1993), this is warranted because the initial dominance range starts from dominance solvable states which are even more conducive (than risk-dominant states) to subsequent dispersion of the dominant choices. This logic carries over to our environments provided that the risk-dominant action (above the critical level) suddenly becomes strictly dominant below the critical level.

Our main results concern the alternative case that the initial dominance stems from strategic uncertainty in the global game, without strict dominance anywhere. In such circumstances, extra conditions are needed for the iteration to progress enough because initial dominance and early expansion can be more limited. Essentially, the actions need be sufficiently strong strategic substitutes below the critical level, so that the risk-dominant action is more beneficial in case the other player plays the other action. As a result, it is dominant more widely, facilitating the expansion.

Combining the two conditions discussed above (i.e., for take-off and enough progress), we provide sufficient conditions on payoffs at the critical level, that warrant iterated selection of the risk-dominant equilibrium. Unsurprisingly, the details of the conditions depend on certain key aspects of players' payoffs and on noise structure. We also provide, where possible, more restrictive sufficient conditions that can be easily checked with payoffs only (independently of noise structure) and apply them to a stylized regime change model to illustrate their potential usefulness.

Related Literature. Global games were first introduced by Carlsson and van Damme (1993) for two-person, two-action games as an equilibrium selection mechanism, and extended to arbitrary number of players and actions by Frankel et al. (2003). The framework has subsequently been fruitfully utilized to study coordination issues in various economic and social situations. Morris and Shin (1998) apply it to study potential causes of currency attacks and policy implications for curtailing them; Morris and Shin (2002) and Hellwig (2002) scrutinize the impact of public signals that players may receive in addition to private signals; Angeletos, et al. (2006) examine signalling effects of policy interventions in financial contexts; Angeletos, et al. (2007) extend the analysis to dynamic settings where agents take actions in multiple periods and learn about the fundamental over time;Jullien and Pavan (2019) study the effects of information management in platform markets. This is only a partial list.

Several authors adopt the framework to specifically address collective action/regime change issues. Shadmehr and Bernhardt (2011) investigate novel implications of the strategic interactions emerging from uncertain payoffs. Edmond (2013) studies how information technology interacts with regime's survival. Morris and Shadmehr (2020) characterize how to optimally inspire heterogeneous revolutionaries.

Almost all papers above focus on environments where actions feature strategic complements, but studies also exist that focus on strategic substitutes. Hoffman and Sabarwal (2019) extend the original global game arguments to multi-person, multi-action games allowing strategic substitutes by imposing a suitable p-dominance condition. Harrison and Jara-Moroni (2021) obtain unique equilibrium in multi-person, binary action games of strategic substitutes under a certain payoff asymmetry.

Our model accommodates both strategic complements and substitutes, in particular, allowing for free-riding incentives which have not been addressed in global games hitherto. Crucially, we show how and when the global game approach works even without dominance solvable games, opening up new scope for application.

The paper is organized as follows. Section 2 illustrates the core insights behind our main result in a stylized regime change model. Section 3 describes the model. Section 4 presents the main findings and analysis. Section 5 concludes and Appendix contains deferred proofs.

2.2 Illustration – a stylized regime change model

Two agents indexed by $i \in \{1, 2\}$ simultaneously make a binary choice $a_i \in \{0, 1\}$. We refer to $a_i = 1$ as the agent i's choice to "attack the regime," or simply "act" for short, and $a_i = 0$ as his choice to "not attack/not act." The two agents succeed in overthrowing the regime if the number of agents who attack/act exceeds the strength of the regime, denoted by θ, which is a random variable uniformly distributed over $(0, 2)$. Each agent i receives a benefit of $b = 1$ if the regime is overthrown and incurs a cost $c_i \in (0, 1)$ if he acts. Thus, agent i's utility is

$$u_i(a_i, a_{-i}, \theta) \;=\; \begin{cases} b - a_i \cdot c_i & \text{if } \; a_1 + a_2 > \theta \\ -\, a_i \cdot c_i & \text{otherwise.} \end{cases}$$

We say the regime is "weak" if $\theta \in (0, 1)$ and "strong" if $\theta \in [1, 2)$. If the regime is weak, the two agents' choices to act are strategic substitutes as just one attack is enough to topple the regime. If the regime is strong, choices to act are strategic complements since both agents need to act to succeed. The description above is common knowledge, as is the information structure on θ explained below.

In the complete information benchmark where the value of θ is common knowledge, multiple equilibria arise due to standard coordination issues. If the regime is weak and agents' actions are strategic substitutes, there are two pure-strategy equilibria depending on who acts. When the regime is strong and agents' actions are strategic complements, there exist an equilibrium in which neither agent acts (coordination failure) as well as one in which both act.

We study an incomplete information environment, or a global game, where each agent privately observes a noisy signal of the regime's strength θ before action choice, but not θ. Specifically, each agent i observes a signal $x_i = \theta + \epsilon_i$ where ϵ_i is an unbiased noise identically and independently distributed according to an atomless cdf F supported on $[-\sigma, \sigma]$, with an associated density function f assumed to be symmetric around 0 in this illustration. Hence, the two agents observe signals within 2σ of each other, i.e., $|x_1 - x_2| < 2\sigma$. We consider small noise, i.e., σ is small.

The agents' costs of acting are assumed to satisfy

$$c_1 + c_2 \ < \ b = 1 \quad \text{and} \quad c_1 \ < \ c_2. \tag{2.1}$$

By the first inequality, it is the risk-dominant equilibrium for both agents to act if $\theta \in [1, 2)$, i.e., if agents' choices to act are strategic complements; by the second, it is risk-dominant for agent 1 to act and agent 2 to not act if $\theta < 1$, i.e., if choices to act are strategic substitutes. (We define risk-dominant equilibrium formally in (2.4) below.)

A strategy s_i of agent i specifies a probability $s_i(x_i)$ with which agent i chooses to act, contingently on every possible signal $x_i \in X := (-\sigma, 2+\sigma)$. Agent i's expected utility from taking $a_i \in \{0, 1\}$ upon observing a signal x_i, conditional on the other agent's strategy s_{-i}, is

$$U_i(a_i, s_{-i}, x_i) := \iint \big[s_{-i}(x_{-i})u_i(a_i, 1, \theta) + (1 - s_{-i}(x_{-i}))u_i(a_i, 0, \theta)\big]\, dF(x_{-i}|\theta)\, d\tilde{F}(\theta|x_i).$$

where $F(\cdot|\theta)$ is the distribution of each agent's signal conditional on θ and $\tilde{F}(\cdot|x_i)$ is the posterior distribution of θ conditional on observing the signal x_i. Upon observing a signal x_i, it is strictly dominant for agent i to act ($a_i = 1$) if

$$U_i(1, s_{-i}, x_i) \ > \ U_i(0, s_{-i}, x_i) \ \text{ for every strategy } s_{-i} \text{ of agent } -i.$$

Clearly, no action choice is strictly dominant in the complete information game (i.e., when θ is known) for any $\theta \in (0, 2)$. However, we show that it is strictly dominant for both agents to act upon observing the signal $x_i = 1$ in the global game if $c_i < 1/4$; and also that it is iteratively dominant for both to act at every signal $x_i \in (1, 2)$ as σ tends to 0.

Each agent i brings a benefit of $b = 1$ to himself by acting when his acting is pivotal in overthrowing the regime, namely, when either

(i) the regime is strong (i.e., $1 \leq \theta$) and the other agent acts, or

(ii) the regime is weak (i.e., $\theta < 1$) and the other agent does not act.

Thus, the probability that his action is pivotal conditional on his signal x_i is

$$Pv(x_i) := \text{Prob}(\text{agent } -i \text{ acts}, 1 \leq \theta \,|\, x_i) + \text{Prob}(\text{agent } -i \text{ not act}, \theta < 1 \,|\, x_i),$$

and upon observing his signal x_i, it is uniquely optimal for agent i to act if $Pv(x_i) > c_i$, and uniquely optimal for agent i to not act if $Pv(x_i) < c_i$.

Note from (i) and (ii) above that each agent i is least likely to be pivotal if the other agent were to act (not act, resp.) when the regime is more likely to be weak (strong, resp.). Suppose that agent i observes the borderline signal $x_i = 1$, so that the regime is equally likely to be weak or strong based on his signal alone. Then, the regime is more likely to be weak (strong, resp.) if the other agent observes a signal $x_{-i} < 1$ ($x_{-i} > 1$, resp.). Hence, agent i is least likely to be pivotal upon observing $x_i = 1$ if the other agent were to act when $x_{-i} < 1$ and not otherwise: in this case agent i is pivotal when θ is above 1 but x_{-i} is below 1, or the other way around. Since the two events are equally likely and the probability of the former is $\int_1^{1+\sigma} F(1|\theta)d\tilde{F}(\theta|1)$, agent i is pivotal at $x_i = 1$ with a probability at least

$$2\int_1^{1+\sigma} F(1|\theta)d\tilde{F}(\theta|1) \;=\; 2\int_1^{1+\sigma} F(1-\theta)f(1-\theta)d\theta \;=\; 2\int_{-\sigma}^{0} F(\theta)f(\theta)d\theta \;=\; \frac{1}{4}.$$

Here, the first equality follows because $\tilde{F}(\theta|1) = 1 - F(1-\theta)$ so that $d\tilde{F}(\theta|1) = f(1-\theta)d\theta$, and the last equality obtains because for any cdf F on $\mathbb{R}$ we have

$$\int_{-\infty}^{z} F(\theta)f(\theta)d\theta \;=\; \int_0^{F(z)} \vartheta d\vartheta \;=\; \frac{F(z)^2}{2} \tag{2.2}$$

by a change of variable as $\vartheta = F(\theta)$ so that $d\vartheta = f(\theta)d\theta$. Thus, $Pv(1) \geq 1/4$.

If $c_i < 1/4$, therefore, it is strictly dominant for agent i to act on the borderline signal $x_i = 1$, and so it is for nearby signals as well by continuity. Hence, there is a largest interval containing 1, denoted by $(\underline{x}_i^1(c_i), \widehat{x}_i^1(c_i))$, such that

[A] it is strictly dominant for agent i to act at every signal $x_i \in (\underline{x}_i^1(c_i), \widehat{x}_i^1(c_i))$ if $c_i < 1/4$.

From this initial range of signals on which acting is dominant, we expand the dominance range of signals iteratively. Recall that each agent is least likely to be pivotal when the other agent acts when θ is more likely to be below 1 and not act otherwise. Thus, both agent acting as per [A], in particular on signals in $[1, \widehat{x}_i^1(c_i))$, increases each other's likelihood to be pivotal at signals above 1. Hence, the dominance range of signals expands, in particular, to a higher upper boundary denoted by $\widehat{x}_i^2(c_i) > \widehat{x}_i^1(c_i)$.

Repeating the process iteratively, one generates an increasing sequence of upper boundaries of dominance ranges, $\{\widehat{x}_i^n(c_i)\}_n$. If $\widehat{x}_i^n(c_i) \geq 1 + \sigma$ for both $i \in \{1, 2\}$ in some round n, observing a signal above his boundary $\widehat{x}_i^n(c_i)$, agent i is certain that the regime is strong ($1 \leq \theta$) and thus he is pivotal whenever the other agent $-i$ acts. Since agent

$-i$ acts on her dominance range at minimum, agent i's upper boundary moves up to a level where he is indifferent between acting and not when agent $-i$ acts precisely on her dominance range. From then on, each agent's upper boundary of the dominance range increases at least as much as the other agent's boundary increased in the pervious round (i.e., $\widehat{x}_i^{n+1}(c_i) - \widehat{x}_i^n(c_i) \geq \widehat{x}_{-i}^n(c_{-i}) - \widehat{x}_{-i}^{n-1}(c_{-i})$) until at least it reaches $2-\sigma$, because the probability that the other agent's signal falls in the range with an increased boundary is no lower if one's own signal increased as much or less.

It remains to verify that the upper boundaries indeed surpass $1+\sigma$, so that acting is risk-dominant for any possible underlying state when the boundary signal is observed. This would be nonissue if the underlying game was continuous in θ and acting was dominance solvable at $\theta = 1$ (as in the framework of Carlsson and van Damme) because then, conditional on the boundary signal of the initial dominance range, acting would be either risk-dominant or dominance solvable in underlying states for small noise, and the latter states are even more conducive to acting.

The situation differs in our setting because initial dominance stems from strategic uncertainty without dominance solvability, thus is governed by the size and structure of noise. But if, in each round n with $\widehat{x}_i^n(c_i) < 1+\sigma$, the other agent $-i$ finds it dominant to act upon observing a signal $x_{-i} = \widehat{x}_i^n(c_i)$, two agents' upper boundaries top each other every round until they reach $1+\sigma$. This is verified to be the case if $c_i < 1/4$ according to a sufficient condition obtained in the next section (Corollary 2).

To recap, acting is iteratively dominant in the risk-dominant region if $c_i < 1/4$, because then (i) acting is strictly dominant at the borderline signal $x_i = 1$ in the global game, starting off an iteration process, and (ii) the upper boundaries of the dominance range of signals surpass $1+\sigma$, allowing the iteration cruise all the way. In the next section, by elaborating these two conditions we derive sufficient conditions for risk-dominant choices to be iteratively dominant in general 2-player, 2-action global games. Applied to the model in this illustration, the findings also imply that if $3/4 < c_2$, it is iteratively dominant for agent 1 to act and agent 2 to not on all signals $x_i < 1$.

Remark. The regime change game may be modified as follows: the underlying state $\theta \in (0,1)$ is the probability that it takes both agents to act to topple the regime (a contingency described as the regime being strong), while either agent acting alone is enough with probability $1-\theta$ (a contingency described as the regime being weak); and the payoffs of the underlying game at θ are the expected benefit from toppling the regime given θ, net of any acting cost. Then, the global game conforms to the framework of Carlsson and van Damme (1993), i.e., the underlying game payoffs are continuous in θ. However, this global game depicts a different situation: the regime's strength is inherently uncertain, captured by a binary *random variable* parameterized by the probability θ of being strong; and observing any signal $x_i \in (0,1)$ agents know almost precisely what kind of random variable it is as $\sigma \to 0$ and also that the other player knows about it with the same precision, too. In our model, by contrast, the regime's strength is determined without residual uncertainty by the underlying state (i.e., strong if $\theta \geq 1$ and weak if $\theta < 1$), and signals give clues about the regime's strength with widely varying degrees of precision. In particular, observing a signal $x_i = 1$ the agents get no clue themselves (other than equally likely to be strong and weak) even as $\sigma \to 0$, but are aware that the opponent could have a pretty good idea as x_{-i} can be near $1 \pm \sigma$. Thus, strategic uncertainty is unabated around the critical signal even as the noise vanishes, and we show that iterated dominance may spring from such unabated strategic uncertainty, too.

2.3 Model and preliminaries

Carlsson and van Damme (1993) establish a seminal result in 2-player, 2-action global games: if an action pair is either risk-dominant or the unique strict equilibrium on an open set of underlying states and is strictly dominant for one player at some state therein, it is iteratively dominant on that set (of signals) in the global game. However, this result is not useful in analyzing the regime change model of the previous section because no action is strictly dominant at any underlying state, nonetheless iterative dominance of risk-dominant actions prevails.

This finding stems from a key departure from Carlsson and van Damme (1993): payoffs of the game change *discontinuously* at some critical state and the iterative process takes off from the strategic uncertainty around this critical level. Such discontinuities often arise in environments where a public good can be provided with varying degrees of coordination depending on the underlying state. Generalizing the insight from the regime change model, we aim to provide conditions on discontinuous payoffs that warrant iterative dominance of risk-dominant actions. We view our result as complementing that of Carlsson and van Damme (1993, CvD for short).

As in CvD, we consider 2-player, 2-action games (2×2 games) where each player $i \in \{1, 2\}$ chooses one of two available actions, α_i and β_i. For easy comparison, we follow their notation closely below. A specification of utility levels, $g \in \mathbb{R}^8$, defines a "game," where each coordinate of g corresponds to one of eight utility levels $u_i(\gamma_1, \gamma_2)$ for $\gamma_i \in \{\alpha_i, \beta_i\}$ and $i \in \{1, 2\}$. For compact exposition, define "deviation loss" from the strategy profiles $\alpha = (\alpha_i, \alpha_{-i})$ and $\beta = (\beta_i, \beta_{-i})$, respectively, as

$$g_i^\alpha = u_i(\alpha_i, \alpha_{-i}) - u_i(\beta_i, \alpha_{-i}) \;\text{ and }\; g_i^\beta = u_i(\beta_i, \beta_{-i}) - u_i(\alpha_i, \beta_{-i}) \;\text{ for }\; i = 1, 2. \tag{2.3}$$

Then, $\gamma \in \{\alpha, \beta\}$ is a Nash equilibrium of g if $g_i^\gamma \geq 0$ for $i = 1, 2$. It is well-known that a generic 2×2 game $g \in \mathbb{R}^8$ has either a unique equilibrium or three equilibria (two strict and one mixed), e.g.,Harsanyi (1973b). When there are two strict Nash equilibria, we denote them by α and β (relabelling strategies as needed).

For $\gamma \in \{\alpha, \beta\}$, let $G^\gamma \subset \mathbb{R}^8$ be the set of games g for which γ is a strict Nash equilibrium, that is, $g_1^\gamma, g_2^\gamma > 0$. For $g \in G^\alpha \cap G^\beta$ so that both α and β are strict equilibria,

α is said to *risk-dominate* β if the product of deviation losses is larger from α than from β, i.e.,

$$g_1^\alpha g_2^\alpha \;>\; g_1^\beta g_2^\beta. \tag{2.4}$$

Let $R^{\alpha\succ\beta}$ denote the set of games in which α risk-dominates β and let R^α denote the set of games in which α is the only strict equilibrium or risk-dominates β, i.e.,

$$R^{\alpha\succ\beta} := \{g \in G^\alpha \cap G^\beta : a \text{ risk-dominates } b\} \quad \text{and} \quad R^\alpha := (G^\alpha \setminus G^\beta) \cup R^{\alpha\succ\beta}.$$

The underlying game is determined by a state $\theta \in \Theta$ through a "game function" $g : \Theta \to \mathbb{R}^8$, where θ is a random variable uniformly distributed over an open interval $\Theta \subset \mathbb{R}$.[1] We assume that $g : \Theta \to \mathbb{R}^8$ is piece-wise C^1 with bounded derivative: Θ is partitioned into intervals $\Theta_1, \Theta_2, \cdots$, where $\sup \Theta_\ell = \inf \Theta_{\ell+1}$, and g is C^1 and the derivative $dg/d\theta$ is bounded in the interior of each Θ_ℓ. Thus, g may be discontinuous at each borderline state $\theta = \sup \Theta_\ell = \inf \Theta_{\ell+1}$ for some ℓ.

Each player does not observe θ but observes a noisy private signal $x_i = \theta + \sigma\epsilon_i$ where ϵ_i is distributed by an atomless cdf F and corresponding density f with support $[-1, 1]$ that satisfies the standard monotone likelihood ratio property (MLRP), i.e.,

$$\frac{f(x-\theta)}{f(x-\theta')} \;\text{ increases in } x \;\text{ if }\; \theta > \theta' \tag{2.5}$$

(which warrants that a higher signal is more likely from a higher state); and $\sigma > 0$ is a scale factor. Upon observing a signal x_i, therefore, player i infers that the underlying state θ is in the interval $(x_i - \sigma,\, x_i + \sigma)$, the posterior distribution of which derived by Bayes rule from F (to be specified later).

A tuple (g, Θ, F, σ) defines a global game in which each player selects a strategy (measurable function) $s_i : X \to [0, 1]$ that specifies a probability of choosing α_i contingently on the observed signal $x_i \in X := \cup_{\theta\in\Theta}(\theta - \sigma, \theta + \sigma)$.

Let $U_i(\gamma_i, s_{-i}|x_i)$ denote player i's expected utility from $\gamma_i \in \{\alpha_i, \beta_i\}$ conditional on observing a signal x_i and the other player's strategy s_{-i}. We define the following process of prescribing *iteratively dominant action* for some signals in the global game: for $n \in \mathbb{N}$,

$$\begin{aligned}
&S_i^{\sigma,0} := \{s \,|\, s : X \to [0,1]\},\\
&A_i^{\sigma,n} = \{x_i \in X \mid U_i(\alpha_i, s_{-i}|x_i) > U_i(\beta_i, s_{-i}|x_i) \;\; \forall s_{-i} \in S_{-i}^{\sigma,n-1}\},\\
&B_i^{\sigma,n} = \{x_i \in X \mid U_i(\alpha_i, s_{-i}|x_i) < U_i(\beta_i, s_{-i}|x_i) \;\; \forall s_{-i} \in S_{-i}^{\sigma,n-1}\},\\
&S_i^{\sigma,n} = \{s_i \in S_i^{\sigma,0} \mid s_i(x_i) = 1 \text{ if } x_i \in A_i^{\sigma,n} \text{ and } s_i(x_i) = 0 \text{ if } x_i \in B_i^{\sigma,n}\ \},\\
&A_i^\sigma = \cup_{n=1}^\infty A_i^{\sigma,n}, \qquad B_i^\sigma = \cup_{n=1}^\infty B_i^{\sigma,n}.
\end{aligned}$$

We say that a strategy s_i is "admissible" in round n if $s_i \in S_i^{\sigma,n-1}$.

[1] Our results extend to the case that θ admits a bounded, strictly positive and C^1 density h on Θ, because key arguments are local where h is approximately uniform as explained in CvD.

Carlsson and van Damme (1993) characterize when risk-dominant actions are iteratively dominant in the case that g is continuous everywhere, as presented below for the environment described above.

Theorem CvD *Suppose $I \subset \Theta_\ell$ is an open interval such that $g(I) \subset R^\alpha$. If α_i is strictly dominant for at least one player in the underlying game $g(\theta)$ for some $\theta \in I$, then every signal $x_i \in I$ is in $A_1^\sigma \cap A_2^\sigma$ for sufficiently small σ.*

The theorem above requires α_i to be strictly dominant for at least one player in some underlying game $g(\theta)$ where $\theta \in \Theta_\ell$, which serves as the take-off for iterative elimination of dominated actions. As illustrated, even if α_i is not strictly dominant in any underlying game, the process may take off in the global game from a critical signal (which will be a boundary of some interval Θ_ℓ) and continue all the way. We try to understand when this is indeed the case by delineating the conditions on the underlying game payoffs at the limit as θ tends to the relevant boundary.

We present our findings for the case that α risk-dominates β at the *lower* end of some partition element Θ_ℓ, so that the pertinent conditions are on the limit deviation losses at the lower boundary, denoted by $x = \inf \Theta_\ell$, from both directions, namely, $g_i^\gamma(x^+) = \lim_{\theta \downarrow x} g_i^\gamma(\theta)$ and $g_i^\gamma(x^-) = \lim_{\theta \uparrow x} g_i^\gamma(\theta)$. The result can be restated symmetrically to work for the other case in which α risk-dominates β at the upper end of Θ_ℓ.

Suppose that α risk-dominates β at x^+, that is,

$$g(x^+) \in G^\alpha \cap G^\beta \quad \text{and} \quad g_1^\alpha(x^+) g_2^\alpha(x^+) > g_1^\beta(x^+) g_2^\beta(x^+) \tag{2.6}$$

where $x = \inf \Theta_\ell = \sup \Theta_{\ell-1}$ for some $\ell > 1$. Then, since g is continuous in the interior of Θ_ℓ, α risk-dominates β on an open interval of θ in Θ_ℓ, denoted by $I \subset \Theta_\ell$, with x as the lower boundary, i.e., $x = \inf I = \inf \Theta_\ell$.

We say that "α is iteratively dominant (in I)" if every signal $x_i \in I$ is in $A_1^\sigma \cap A_2^\sigma$ for sufficiently small σ. We characterize when this is the case for games that satisfy (2.6), in terms of conditions on the values $g_i^\gamma(x^+)$ and $g_i^\gamma(x^-)$ for $\gamma \in \{\alpha, \beta\}$ and $i \in \{1, 2\}$.

Strategic interactions in global games rely on each player's inference on the distribution of the other player's signal conditional on his own signal. We close this section with a useful lemma that it is distributed symmetrically around a single peak at one's own signal x_i (which is proved in Appendix).

Lemma 2 *The density of x_{-i} conditional on player i's signal x_i is symmetric around and single-peaked at $x_{-i} = x_i$ if $(x_i - \sigma, x_i + \sigma) \subset \Theta$.*

2.4 When are risk-dominant actions iteratively dominant?

We study when α is iteratively dominant in the full range of signals for which it is risk-dominant. This obviously requires emergence of initial dominance of α_i to kick off the iteration. In the model of CvD, the initial dominance emerges from α_i being strictly dominant in some underlying games adjacent to those in which α is risk-dominant or the unique strict equilibrium. In our context where g is discontinuous at $\theta = x$ and α is risk-dominant at $\theta > x$, this corresponds to α_i becoming strictly dominant in $g(\theta)$ as θ dips below the critical level x. We start with this case and discuss the extent to which the logic of CvD extends to situations where g is discontinuous (Section 4.1), prior to analyzing other environments where the logic of CvD is insufficient (Section 4.2). To facilitate the exposition we explain the core arguments for our results heuristically in the main text, deferring a formal proof to Appendix.

2.4.1 The case that α_i is strictly dominant in $g(x^-)$.

Let us consider a global game where α_i is strictly dominant in underlying games $g(\theta)$ for θ immediately below the critical level x, while α is risk-dominant in $g(\theta)$ for θ above x as per (2.6). In the case that g is continuous and α remains a strict equilibrium at $\theta = x$ as considered in Theorem CvD, this means that $g_i^\beta(x) = 0 < g_i^\alpha(x)$ and $g_i^\beta(\theta)$ increases strictly at $\theta = x$, so that $g_i^\beta(\theta) < 0 < g_i^\alpha(\theta)$ for θ in an interval immediately below x. For small enough σ, therefore, when player i observes most of the signals in that interval up to $x - \sigma$, he infers that α_i is strictly dominant in the underlying game, thus in the global game as well. In fact, α_i is strictly dominant for slightly higher signals as well due to continuity, say up to $\widehat{x}_i \in (x - \sigma, x)$. Starting from this strict dominance, an iterative process sequentially renders both α_i and α_{-i} strict dominant throughout the entire range of signals for which α is risk-dominant. We first elaborate on this core logic of Theorem CvD and extend it to our environments.

To elaborate on the logic, we observe that $g_i^\alpha(\theta) + g_i^\beta(\theta) > 0$ for $\theta \geq x$ in I for both players due to risk dominance, (2.6), and also for θ slightly below x due to continuity of g. The inequality $g_i^\alpha(\theta) + g_i^\beta(\theta) > 0$ means that the two actions are *strategic complements* in $g(\theta)$: the inequality is equivalent to

$$u_i(\alpha_i, \alpha_{-i}) - u_i(\beta_i, \alpha_{-i}) = g_i^\alpha(\theta) \ > \ -g_i^\beta(\theta) = u_i(\alpha_i, \beta_{-i}) - u_i(\beta_i, \beta_{-i}) \qquad (2.7)$$

which means that player i' net payoff gain from playing α_i rather than β_i is larger when player $-i$ also plays α_{-i} as opposed to when she plays β_{-i}. Hence, α_i is more attractive the more likely player $-i$ is to play α_{-i}, and is uniquely optimal for player i if player $-i$ plays α_{-i} with a probability exceeding a threshold level that makes two actions equivalent for player i, namely $\frac{g_i^\beta(\theta)}{g_i^\alpha(\theta)+g_i^\beta(\theta)}$.[2] For later use, we also note that the two actions are *strategic substitutes* if the reverse inequality of (2.7) holds, i.e., $g_i^\alpha(\theta) + g_i^\beta(\theta) < 0$, in which case α_i is more attractive the less likely player $-i$ is to play α_{-i}, and thus is uniquely optimal if α_{-i} is played with a probability lower than $\frac{g_i^\beta(\theta)}{g_i^\alpha(\theta)+g_i^\beta(\theta)}$.

Therefore, upon observing signals x_i above the critical level x or slightly below x in the global game, either player i infers that actions are strategic complements and plays α_i as the unique optimal action if the other player is expected to play α_{-i} with a probability higher than the relevant threshold. Recall that one player, i, plays α_i as the strictly dominant action for signals up to $\widehat{x}_i \in (x-\sigma, x)$. Thus, there is a signal, say $\widehat{x}_{-i}$ (near $\widehat{x}_i$), such that the probability with which player i observes a signal $x_i < \widehat{x}_i$ (hence, takes α_i as the dominant action) is the threshold level for player $-i$ when she observes $\widehat{x}_{-i}$. Consequently, player $-i$ plays α_{-i} as the (iteratively) dominant action upon observing signals up to $\widehat{x}_{-i}$ (because the lower x_{-i} is, the higher is the probability that player i observes $x_i < \widehat{x}_i$). This renders α_i more attractive for player i due to strategic complementarity, making it the dominant action for signals up to a higher boundary than $\widehat{x}_i$ in the next round. By the same token, the ranges of signals at which α_i is dominant expand iteratively for both players in every round due to strategic complementarity, throughout the risk dominant range of signals if σ is small enough.

A key logic in this process is that the probabilities with which each player observes a signal below their upper boundary (hence plays α_i as the dominant action) conditional on the other player's boundary signal sum up to one (cf. Lemma 2) in each round, whereas their threshold probabilities $\frac{g_i^\beta(\theta)}{g_i^\alpha(\theta)+g_i^\beta(\theta)}$ sum less than 1 due to risk dominance. This warrants that the risk-dominant action is uniquely optimal for at least one player i upon observing his boundary signal, expanding the boundary in the next round. So long as this logic is preserved in our context when g changes discontinuously at $\theta = x$, the iterative dominance result of CvD should prevail. This logic is indeed preserved if the threshold probability does not jump too much as θ dips below x, as stated in the next result (and proved in Appendix).

Proposition 4 *In a global game (g, Θ, F, σ) with an open interval $I \subset \Theta_\ell$ such that $x = \inf I = \inf \Theta_\ell$ and $g(I) \cup g(x^+) \subset R^{\alpha \succ \beta}$, α is iteratively dominant in I if*

$$\begin{cases} g_i^\alpha(x^-) > 0 \;\; \text{for} \;\; i \in \{1,2\}, \;\; \text{and} \\ g_i^\beta(x^-) < 0 \text{ for one } i, \text{ and } g_{-i}^\beta(x^-) < 0 \text{ or } \frac{g_{-i}^\beta(x^-)}{g_{-i}^\alpha(x^-)+g_{-i}^\beta(x^-)} < 1 - \frac{g_i^\beta(x^+)}{g_i^\alpha(x^+)+g_i^\beta(x^+)}. \end{cases} \tag{2.8}$$

The condition (2.8) ensures that α_i is strictly dominant in $g(x^-)$ for one agent i, i.e., $g_i^\beta(x^-) < 0 < g_i^\alpha(x^-)$, while for the other player, either α_{-i} is also strictly dominant in

[2] Note that α_i is strictly dominant if this threshold is negative (i.e., $g_i^\beta(\theta) < 0 < g_i^\alpha(\theta)$), while β_i is strictly dominant if the threshold exceeds 1 (i.e., $g_i^\alpha(\theta) < 0 < g_i^\beta(\theta)$). An analogous comment applies for the case of strategic substitutes.

$g(x^-)$ or the threshold probabilities sum less than 1 for underlying states θ in a neighborhood of the critical level x. Note that (2.8) is satisfied by a much larger set of game functions g's than those that satisfy the conditions of Theorem CvD.

2.4.2 The case that α_i is nowhere strictly dominant

Proposition 4 extends Theorem CvD to environments where α_i turns strictly dominant from being risk-dominant in the underlying game $g(\theta)$ discontinuously for at least one player i. We now study the alternative case that this does not happen, that is, in our context α_i is not strictly dominant in $g(\theta)$ for θ's immediately below x for either player in the sense that

$$\text{if } \; g_i^\alpha(\theta) > 0 \; \text{ then } \; g_i^\beta(\theta) \geq 0 \; \text{ for } \; \theta \; \text{ in an interval } \; (\tilde{\theta}, x) \neq \emptyset, \tag{2.9}$$

while α is risk-dominant in $g(\theta)$ for $\theta > x$ as per (2.6).

We aim to understand when and how iterative dominance of risk-dominant actions may result from strategic uncertainty due to discontinuous payoffs, in the absence of nearby underlying states where the action is strictly dominant and the iteration takes off from. It is possible, however, that an iteration process may take off elsewhere due to a cause unrelated to the strategic uncertainty, yet travel through x and into the range of risk-dominant signals, I. For example, the iteration process may start from some states in $\Theta_{\ell-1}$ far below x and expand upward all the way to x (*à la* Theorem CvD), then jump over x and continue throughout I.

In order to present a cleaner picture of how payoff discontinuity may lead to iterative dominance of α without such confounding factors, we proceed our discussion presuming that α_i is never iteratively dominant at signals outside a small neighborhood of I, namely, $(x-\sigma, \sup I)$: that is, $A_i^{\sigma,n} \cap (x-\sigma, \sup I) = A_i^{\sigma,n}$ for all n and $i = 1, 2$. However, this is for expositional convenience and our main result, Proposition 5, is independent of this presumption (as will be clear in the discussion below).

The discussion revolves around how $A_i^{\sigma,n}$, the set of signals where α_i is (iteratively) dominant in each round n, evolves. This necessitates checking when α_i is dominant in the global game. For this, we define $V_i(s_{-i}|x_i) := U_i(\alpha_i, s_{-i}|x_i) - U_i(\beta_i, s_{-i}|x_i)$ as the expected net gain of player i from playing α_i rather than β_i upon observing a signal x_i, conditional on the other player's strategy s_{-i}. Since the said net gain conditional on θ is $g_i^\alpha(\theta)$ or $-g_i^\beta(\theta)$ depending on whether player $-i$ chooses α_{-i} or β_{-i} (cf. (2.3)),

$$V_i(s_{-i}|x_i) \;=\; \int_\theta \int_{x_{-i}} [s_{-i}(x_{-i}) g_i^\alpha(\theta) - (1 - s_{-i}(x_{-i})) g_i^\beta(\theta)] dF_\sigma(x_{-i}|\theta) d\tilde{F}_\sigma(\theta|x_i) \tag{2.10}$$

where $F_\sigma(x_{-i}|\theta) = F(\frac{x_{-i}-\theta}{\sigma})$ is the distribution of signal x_{-i} conditional on θ and $\tilde{F}_\sigma(\theta|x_i) = 1 - F(\frac{x_i-\theta}{\sigma})$ is the posterior distribution of the state θ conditional on player i observing the signal x_i.[3]

By definition, α_i is initially strictly dominant for player i upon observing a signal x_i, i.e., $x_i \in A_i^{\sigma,1}$, if $V_i(s_{-i}|\, x_i) > 0$ for all $s_{-i} : X \to [0,1]$. We start by identifying a specific s_{-i} that minimizes $V_i(s_{-i}|\, x_i)$, denoted by $\check{s}_{-i}(\cdot|x_i)$, which is instrumental in presenting our arguments.[4]

The strategy $\check{s}_{-i}$ that minimizes $V_i(s_{-i}|\, x_i)$

For our analysis, it suffices to focus on signals x_i in the interval $(x-\sigma, x+\sigma)$, so that θ may be on either side of the critical level x. For small enough σ, the value of $g_i^\gamma(\theta)$ is approximated by $g_i^\gamma(x^+)$ if θ is just above x, i.e., if $\theta \in (x, x_i+\sigma)$, and by $g_i^\gamma(x^-)$ if $\theta \in (x_i-\sigma, x)$ for $\gamma \in \{\alpha,\beta\}$. Consequently, $V_i(s_{-i}|x_i)$ is first-order approximated as $\sigma \to 0$, by the integral in (2.10) when $g_i^\gamma(\theta)$ is replaced as such, which we denote by $\widetilde{V}_i(s_{-i}|x_i)$. With this replacement, $\widetilde{V}_i(s_{-i}|x_i)$ is a weighted sum of the four limit values of conditional net gain from playing α_i, namely, $g_i^\alpha(x^+)$, $g_i^\alpha(x^-)$, $-g_i^\beta(x^+)$ and $-g_i^\beta(x^-)$, where their respective weights are the probabilities, conditional on x_i, that θ is on the relevant side of x and player $-i$ plays γ_{-i} according to s_{-i} for $\gamma \in \{\alpha,\beta\}$.

To specify $\check{s}_i$ that minimizes $\widetilde{V}_i(s_{-i}|x_i)$ invariantly to σ, we introduce an alternative scale of state θ and signals x_i by their distance from x in multiples of σ, which we refer to as the "relative scale": state $\theta = x+\vartheta\sigma$ is represented as state $\vartheta \in \mathbb{R}$ and signal $x_i = x+\lambda_i\sigma$ is represented as signal $\lambda_i \in \mathbb{R}$ for $i = 1,2$. With this change of scale/variables, we have

$$\widetilde{V}_i(s_{-i}|x+\lambda_i\sigma) := -F(\lambda_i)g_i^\beta(x^+) - \big[1-F(\lambda_i)\big]g_i^\beta(x^-) + \int_{\lambda_{-i}} s_{-i}(x+\lambda_{-i}\sigma)\Psi_i(\lambda_{-i}|\lambda_i)d\lambda_{-i} \quad (2.11)$$

$$\text{where} \qquad \Psi_i(\lambda_{-i}|\lambda_i) := [g_i^\alpha(x^+)+g_i^\beta(x^+)]\int_0^\infty f(\lambda_{-i}-\vartheta)f(\lambda_i-\vartheta)d\vartheta \quad (2.12)$$
$$+ [g_i^\alpha(x^-)+g_i^\beta(x^-)]\int_{-\infty}^0 f(\lambda_{-i}-\vartheta)f(\lambda_i-\vartheta)d\vartheta.$$

Note that $\widetilde{V}_i(s_{-i}|x+\lambda_i\sigma)$ is invariant to σ as a function of λ_i, so long as s_{-i} is treated as a function of λ_{-i}. Regardless of σ, therefore, $\widetilde{V}_i(s_{-i}|x+\lambda_i\sigma)$ is minimal when $s_{-i}(x+\lambda_{-i}\sigma) = 0$ if $\Psi_i(\lambda_{-i}|\lambda_i) \geq 0$ and $s_{-i}(x+\lambda_{-i}\sigma) = 1$ if $\Psi_i(\lambda_{-i}|\lambda_i) < 0$, which is the strategy we denote by $\check{s}_{-i}(\cdot|x_i)$ for $x_i = x+\lambda_i\sigma \in (x-\sigma, x+\sigma)$.

The first and second integrals of $\Psi_i(\lambda_{-i}|\lambda_i)$ in (2.12) are the probabilities that a given signal pair $(x_i, x_{-i}) = (x+\lambda_i\sigma, x+\lambda_{-i}\sigma)$ is observed when $\theta > x$ (i.e., $\vartheta > 0$) and when $\theta < x$ (i.e., $\vartheta < 0$), respectively. Since $g_i^\alpha(x^+)+g_i^\beta(x^+) > 0$ from (2.6), if $g_i^\alpha(x^-)+g_i^\beta(x^-) \geq 0$ then $\Psi_i(\lambda_{-i}|\lambda_i) \geq 0$ and thus, $\check{s}_{-i}(\cdot|x_i) \equiv 0$ for every $x_i = x+\lambda_i\sigma \in (x-\sigma, x+\sigma)$. This is intuitively clear: if the actions are strategic complements on both sides of x, $V_i(s_{-i}|x_i)$ is minimized when α_{-i} is never played according to s_{-i}.

[3]Formally, $\tilde{F}_\sigma(\theta|x_i) = \frac{\int_{x_i-\sigma}^{\theta} f(\frac{x_i-\vartheta}{\sigma})d\vartheta}{\int_{x_i-\sigma}^{x_i+\sigma} f(\frac{x_i-\vartheta}{\sigma})d\vartheta} = 1 - F(\frac{x_i-\theta}{\sigma})$ and thus, $\tilde{f}_\sigma(\theta|x_i) = \sigma^{-1}f(\frac{x_i-\theta}{\sigma})$.

[4]Since $|x_i - x_{-i}| \leq 2\sigma$, we define $\check{s}_{-i}(\cdot|x_i)$ on $x_{-i} \in [x_i-2\sigma, x_i+2\sigma]$.

If $g_i^\alpha(x^-) + g_i^\beta(x^-) < 0$, on the other hand, the actions are certain to be substitutes if $x_i = x - \sigma$ (so that the underlying state θ must be below x) but complements if $x_i = x + \sigma$ (so that $\theta > x$); moreover, the actions are more likely to be complements as the observed signals $x_i = x + \lambda_i \sigma$ and/or $x_{-i} = x + \lambda_{-i}\sigma$ increase from $x - \sigma$ to $x + \sigma$, i.e., as λ_i and/or λ_{-i} increase from -1 to 1. Given a signal $x_i = x + \lambda_i \sigma \in (x - \sigma, x + \sigma)$, therefore, there is a unique value of λ_{-i}, denoted by $T_{-i}(\lambda_i) \in (-1, 1)$, such that $\Psi_i(T_{-i}(\lambda_i)|\lambda_i) = 0$. Hence, as λ_{-i} increases above $T_{-i}(\lambda_i)$, or equivalently, as x_{-i} increases above the threshold $x + T_{-i}(\lambda_i)\sigma$, the actions become sufficiently likely to be strategic complements so that $\breve{s}_{-i}(x_{-i}|x + \lambda_i\sigma)$ switches from 1 to 0. Since a higher signal $x_i = x + \lambda_i\sigma \in (x - \sigma, x + \sigma)$ also renders actions more likely to be complements as noted above, $T_{-i}(\lambda_i)$ decreases in $\lambda_i \in (-1, 1)$. The next lemma summarizes the findings on $\breve{s}_{-i}(\cdot|x_i)$.

Lemma 3 *For $x_i = x + \lambda_i\sigma \in (x - \sigma, x + \sigma)$ where $\lambda_i \in (-1, 1)$, $\widetilde{V}_i(s_{-i}|x + \lambda_i\sigma)$ is minimized by*

$$\breve{s}_{-i}(\cdot|x + \lambda_i\sigma) \equiv 0 \quad \text{if} \quad g_i^\alpha(x^-) + g_i^\beta(x^-) \geq 0;$$
$$\breve{s}_{-i}(\cdot|x + \lambda_i\sigma) = \left\{ \begin{array}{ll} 1 & \text{for } x_{-i} < x + T_{-i}(\lambda_i)\sigma \\ 0 & \text{for } x_{-i} > x + T_{-i}(\lambda_i)\sigma \end{array} \right\} \quad \text{if} \quad g_i^\alpha(x^-) + g_i^\beta(x^-) < 0$$

where $T_{-i}(\lambda_i) \in (-1, 1)$ satisfies $\Psi_i(T_{-i}(\lambda_i)|\lambda_i) = 0$ and decreases in λ_i.

Having pinned down $\breve{s}_{-i}$, we now examine when the risk-dominant actions are iteratively dominant. We organize the analysis depending on whether the actions are strategic complements or substitutes in $g(x^-)$ for each player.

α is not iteratively dominant if $g_i^\alpha(x^-) + g_i^\beta(x^-) > 0$ for $i = 1, 2$.

Suppose the actions are complements for player i in $g(x^-)$, i.e., $g_i^\alpha(x^-) + g_i^\beta(x^-) > 0$. Since α_i is not strictly dominant immediately below x, i.e., (2.9), it follows that $g_i^\beta(\theta) \geq 0$ for θ in a small interval immediately below x.[5] Since $g_i^\beta(\theta) > 0$ for $\theta \in I$ as well, observing a signal x_i above x (within I) or slightly below x in the global game, player i infers that $g_i^\beta(\theta) \geq 0$ in the underlying game, that is, β_i is optimal if the other player always plays β_{-i} regardless of her signal x_{-i}. If σ is small enough, therefore, α_i is not strictly dominant at any signal $x_i \in (x - \sigma, \sup I)$ in the global game.

Consequently, if the actions are complements for both players in $g(x^-)$, then $A_i^{\sigma,1} = \emptyset$ for both $i = 1, 2$ (under the presumption that α_i is never iteratively dominant outside $(x - \sigma, \sup I)$). Then, it is admissible in round 2 for the other player $-i$ to play β_{-i} regardless of her signal and thus, α_i is not dominant at any signal $x_i \in (x - \sigma, \sup I)$ by the same reasoning as before, i.e., $A_i^{\sigma,2} = \emptyset$ for both $i = 1, 2$. Recursively, $A_i^{\sigma,n} = \emptyset$ in

[5]To elaborate, if $g_i^\beta(\theta) < 0$ for $\theta < x$ arbitrarily close to x, we would have $g_i^\beta(\theta) < 0 < g_i^\alpha(\theta)$ for θ immediately below x because $g_i^\alpha(x^-) + g_i^\beta(x^-) > 0$, contradicting (2.9).

every round n for both players and an iteration process never takes off from the strategic uncertainty in this case. To recap, if the actions are strategic complements for both players in $g(x^-)$, iterative dominance of risk-dominant actions in I cannot result from the strategic uncertainty due to discontinuous payoffs at $\theta = x$.

In light of the above, we now consider cases in which the actions are strategic substitutes in $g(x^-)$ for at least one player. Then, for α to be iteratively dominant as a result of an iteration process stemming from the strategic uncertainty, three key steps need to take place as follows for all small enough σ:

(S1) α_i is initially dominant at some signals in $(x-\sigma, x+\sigma)$, i.e., $A_i^{\sigma,1} \cap (x-\sigma, x+\sigma) \neq \emptyset$, for at least one player i.
(S2) The ranges of iteratively dominant signals, $A_i^{\sigma,n}$, expand above $x+\sigma$ for both players in some round, say N, i.e., $A_i^{\sigma,N} \cap [x+\sigma, \sup I) \neq \emptyset$ for both $i = 1, 2$.
(S3) The iterative process continues throughout the risk-dominant signals.

Recall that $A_i^{\sigma,n} = \emptyset$ if the actions are complements for player i in $g(x^-)$ so long as $A_{-i}^{\sigma,n-1} = \emptyset$ as discussed above. Therefore, if the two actions are strategic substitutes in $g(x^-)$ for only one player, say i, then $A_i^{\sigma,1} \neq \emptyset$ and $A_{-i}^{\sigma,1} = \emptyset$, whereas $A_i^{\sigma,1}$ may be nonempty for both players if the actions are strategic substitutes for both players in $g(x^-)$. In either case, the sets $A_i^{\sigma,n}$ grow in every round n in the sense that $A_i^{\sigma,n-1} \subset A_i^{\sigma,n}$. This process generates a sequence of upper boundary signals $\{\bar{x}_i^n\}$ for each player i, where $\bar{x}_i^n = \sup A_i^{\sigma,n}$ is the least upper bound of player i's signals for which α_i is iteratively dominant in round n.

The step (S2) requires that $\bar{x}_i^N \geq x + \sigma$ for both i in some round N. A precise and simple characterization of whether and when this happens seems out of reach because it would require tracking how the sets $A_i^{\sigma,n}$ change each round, which depends on the sets $A_{-i}^{\sigma,n-1}$ and $B_{-i}^{\sigma,n-1}$ of the previous round and recursively, of all previous rounds. The task is further complicated by the fact that the sets $A_i^{\sigma,n}$ may consist of multiple disjoint intervals.

Instead, we look for sufficient conditions for (S2) by delineating how a single interval of initially dominant signals for each player may expand to surpass $x + \sigma$ in some round. The details of analysis differ depending on whether the actions are strategic substitutes for both players or for just one player in $g(x^-)$. We start with the former case. It proves useful to express signals in the relative scale in this discussion, in particular, the boundary signals as $\bar{x}_i^n = x + \bar{\lambda}_i^n \sigma$.

When $g_i^\alpha(x^-) + g_i^\beta(x^-) < 0$ for $i = 1, 2$

Suppose that the actions are strategic substitutes for both players in $g(x^-)$, i.e., $g_i^\alpha(x^-) + g_i^\beta(x^-) < 0$ for $i = 1, 2$. Let us consider a simple iteration process (as in the illustration of Section 2) that takes off from a single interval of initially dominant signals, $A_i^{\sigma,1}$, that contains the critical signal x for both players; then the ranges of iteratively dominant signals, $A_i^{\sigma,n}$, evolve as single intervals with the following feature: the upper boundary $\bar{x}_i^n$ of each player's dominant signals exceeds the upper boundary $\bar{x}_{-i}^{n-1}$ of the other player's dominant signals of the previous round.

Such "overtaking" of each other's upper boundary will expand $A_i^{\sigma,n}$ over $x+\sigma$ as in the step (S2), unless both players' upper boundaries stall prematurely at the same limit, say $\bar{x} = \lim_{n\to\infty} \bar{x}_1^n = \lim_{n\to\infty} \bar{x}_2^n < x+\sigma$. We provide a sufficient condition that precludes such premature stalling, which will ensure that the iteration process expands over $x+\sigma$ and continues throughout the range of risk-dominant signals. The condition is then generalized to preclude premature stalling of the players' upper boundaries even at different limits.

If the upper boundaries stall at the same limit $\bar{x} < x+\sigma$, the minimal value of the net gain $V_i(s_{-i}|x_i)$ among all admissible s_{-i}'s is nil at $x_i = \bar{x}$ for both players. Denoting the limit signal as $\bar{x} = x + \bar{\lambda}\sigma$ in the relative scale, this minimal value is obtained when s_{-i} replicates $\breve{s}(\cdot|x+\bar{\lambda}\sigma)$ specified in Lemma 3, subject to prescribing α_{-i} at (i.e. assigning 1 to) all signals $x_{-i} \in A_{-i}^{\sigma,\infty}$. If this minimal value is warranted to be positive, the premature stalling at $\bar{x} = x+\bar{\lambda}\sigma$ would be precluded.

However, the set $A_{-i}^{\sigma,\infty}$ is not known precisely, except that it contains the interval $[x, x+\bar{\lambda}\sigma)$. Hence, if $V_i(s_{-i}|x+\bar{\lambda}\sigma)$ is positive when s_{-i} replicates $\breve{s}(\cdot|x+\bar{\lambda}\sigma)$ subject to prescribing α_{-i} at signals x_{-i} in $[x, x+\bar{\lambda}\sigma)$, the value $V_i(s_{-i}|x+\bar{\lambda}\sigma)$ is higher when s_{-i} prescribes α_{-i} to all signals x_{-i} in $A_{-i}^{\sigma,\infty}$, thus precluding premature stalling at the level $\bar{x} = x+\bar{\lambda}\sigma$.

Likewise, premature stalling can be precluded at any level $\bar{x} \in [x, x+\sigma]$ if the relevant condition holds for every $\bar{\lambda} \in [0,1]$, that is, if

$$\widetilde{V}_i(\breve{s}_{-i}^{\lambda}|x+\lambda\sigma) > 0 \quad \text{for all} \quad \lambda \in [0,1] \quad \text{for} \quad i \in \{1,2\} \tag{2.13}$$

where $\breve{s}_{-i}^{\lambda}(\cdot)$ denotes the strategy s_{-i} that assigns 1 to x_{-i} if $x_{-i} \in [x, x+\lambda\sigma)$ or $x_{-i} \leq x + T_{-i}(\lambda)\sigma$, and 0 otherwise.

It is straightforward to verify that the steps (S1) and (S2) are satisfied if (2.13) holds in a global game, even when the ranges of iteratively dominant signals, $A_i^{\sigma,n}$, consist of multiple intervals. First, since $\breve{s}_{-i}^{\lambda} = \breve{s}_{-i}$ when $\lambda = 0$, the inequality at $\lambda = 0$ is the condition that α_i is initially dominant at $x_i = x$, hence (S1). To check (S2), we let $\widehat{x}_i^n$ denote the upper boundary of the largest interval in $A_i^{\sigma,n}$ that contains x and define $\widehat{x}_i^\infty = \lim_{n\to\infty} \widehat{x}_i^n$. If $\widehat{x}_i^\infty < x+\sigma$ for one player i and $\widehat{x}_i^\infty \leq \widehat{x}_{-i}^\infty$, then (2.13) would imply that α_i is iteratively dominant at $x_i = \widehat{x}_i^\infty$, thus at slightly higher signal as well by continuity, contradicting the definition of $\widehat{x}_i^\infty$. Hence, we deduce that $\widehat{x}_i^\infty \geq x+\sigma$ for both i, establishing (S2).

Moreover, it then follows that the iterative expansion continues throughout the risk-dominant range of signals by an argument standard in global games as briefly explained earlier (Section 4.1) and detailed in Appendix. Therefore, (2.13) serves as a sufficient condition for α to be iteratively dominant in I.

The condition (2.13), however, is specific to the cases where the iteration process takes off from signals around the critical signal x and the two players' upper boundaries of dominant signals "overtake" each other's in each round n. We now reformulate (2.13) to accommodate more general iterative processes.

Consider an iteration process represented by a sequence of $A_i^{\sigma,n}$ for each player i, that satisfies (S2) in round N. Fix an initially dominant signal $x_i^0 = x + \lambda_i^0 \sigma \in A_i^{\sigma,1}$ and let $\widehat{x}_i^n = x + \widehat{\lambda}_i^n \sigma$ denote the upper boundary of the largest interval in $A_i^{\sigma,n}$ that contains x_i^0. We now convert labelling of each player's signals to a common benchmark labelling in a way that $\widehat{x}_i^n$ exceeds $\widehat{x}_{-i}^{n-1}$ after respective conversion as follows.

For each player i, define a continuous and strictly increasing bijective function ρ_i : $[-1,1] \to [-1, \max\{\widehat{\lambda}_{-i}^N, \widehat{\lambda}_i^N\}]$ with the following properties: assuming $\widehat{\lambda}_{-i}^N \le \widehat{\lambda}_i^N$,

(i) $\rho_i(0) = \lambda_i^0$, and

(ii) $\rho_i^{-1}(\widehat{\lambda}_i^{n-1}) \le \rho_{-i}^{-1}(\widehat{\lambda}_{-i}^n) \le \rho_i^{-1}(\widehat{\lambda}_i^n)$ for each $n \le N$.

That is, each ρ_i maps "benchmark" labelling of signals on $[-1,1]$ to actual signals of player i in such a way that (i) the critical signal in the benchmark labelling ($\lambda = 0$) is initially dominant, and (ii) the upper boundary $\widehat{x}_i^n = x + \widehat{\lambda}_i^n \sigma$ of each player i exceeds the previous boundary $\widehat{x}_{-i}^{n-1}$ of the other player, when converted to their respective benchmark labels, namely, $x + \rho_i^{-1}(\widehat{\lambda}_i^n)\sigma$ exceeds $x + \rho_{-i}^{-1}(\widehat{\lambda}_{-i}^{n-1})\sigma$. Then, the condition (2.13) can be generalized as below to preclude premature stalling of iterative expansion by ensuring that the two players' upper boundary signals $\widehat{x}_i^n$ of iterative dominance "overtake" each other's via suitable conversions to a common benchmark labeling:

$$\widetilde{V}_i(\breve{s}_{-i}^{\rho_{-i}(\lambda)} | x + \rho_i(\lambda)\sigma) > 0 \quad \text{for all} \quad \lambda \in [0,1] \quad \text{for} \quad i \in \{1,2\}, \tag{2.14}$$

where $\breve{s}_{-i}^{\rho_{-i}(\lambda)}$ is a strategy that assigns 1 to x_{-i} if $x_{-i} \in [x + \rho_{-i}(0)\sigma, x + \rho_{-i}(\lambda)\sigma)$ or $x_{-i} \le x + T_{-i}(\rho_i(\lambda))\sigma = 1$ in case $\rho_i(\lambda) < 1$, and 0 otherwise.

Proposition 5 *Consider a global game (g, Θ, F, σ) with an open interval $I \subset \Theta_\ell$ such that $x = \inf I = \inf \Theta_\ell$ and $g(I) \cup g(x^+) \subset R^{\alpha \succ \beta}$. If $g_i^\alpha(x^-) + g_i^\beta(x^-) < 0$ for both i, α is iteratively dominant in I if (2.14) holds for some pair of functions $\rho_i : [-1,1] \to [-1,\Lambda]$, $i \in \{1,2\}$, where each ρ_i is a continuous and strictly increasing bijection for $\Lambda > 1$ and $\rho_i(0) < 1$. If (2.14) holds for ρ_1 and ρ_2, so does it when $g_i^\beta(x^-)$ is reduced for either i.*

The inequality in (2.14) ensures that player i's net gain from playing α_i (rather than β_i) is positive observing his signal $x_i = x + \rho_i(\lambda)\sigma$ subject to player $-i$ playing α_{-i} in a specific interval, $[x + \rho_{-i}(0)\sigma, x + \rho_{-i}(\lambda)\sigma)$. Recall that $\widetilde{V}_i(s_{-i}|x_i)$ is a weighted sum of the four conditional net gain values from playing α_i, namely, $g_i^\alpha(x^+)$, $g_i^\alpha(x^-)$, $-g_i^\beta(x^+)$ and $-g_i^\beta(x^-)$, where the weights are the probabilities with which relevant contingencies arise for each value conditional on x_i and s_{-i}. Given the values $g_i^\alpha(x^+)$ and $g_i^\beta(x^+)$ as in (2.6), therefore, $\widetilde{V}_i(\breve{s}^{\lambda}_{-i}|x+\rho_i(\lambda)\sigma)$ is higher as $g^{\alpha}_i(x^-)$ and/or $-g^{\beta}_i(x^-)$ are higher. Insofar as $g^{\alpha}_i(x^-)$ cannot be increased above 0 (for α_i not to be strictly dominant in $g(x^-)$), (2.14) is more likely to hold as $g^{\beta}_i(x^-)$ is lower, i.e., as strategic substitutability intensifies in the sense that $g^{\alpha}_i(x^-)+g^{\beta}_i(x^-)<0$ is reduced. The last claim of Proposition 5 captures this observation.

In principle, the condition (2.14) can be checked from the values $g^{\alpha}_i(x^+)$, $g^{\beta}_i(x^+)$, $g^{\alpha}_i(x^-)$ and $g^{\beta}_i(x^-)$ in conjunction with the noise distribution F, once ρ_1 and ρ_2 are specified. However, it is hard to know a priori which ρ_1 and ρ_2 would work, and also cumbersome to check whether (2.14) holds for each and every $\lambda\in[0,1)$. In fact, since F affects how $T_{-i}(\lambda)$ changes in λ, it is impossible to derive a condition equivalent to (2.14) that can be applied independently of F.

Instead, distribution-free conditions may be sought for a subclass of environments of economic interest. In particular, within the class of noise distributions that are symmetric around 0, which is a sensible property, we provide a condition independent of F that warrants (2.14) for the case that both ρ_i's are identity functions, i.e., (2.13), in Corollary 2. We apply this condition to the regime change model of Section 2 and derive further implications.

Corollary 2 *In the situation considered in Proposition 5 where ϵ_i is distributed symmetrically around 0, α is iteratively dominant in I if either of the following conditions hold:*

$$(a)\quad\begin{cases} g^{\alpha}_i(x^+)+g^{\beta}_i(x^+)\ge -g^{\alpha}_i(x^-)-g^{\beta}_i(x^-)>0, \ \textit{and} \\ -g^{\alpha}_i(x^+)\le g^{\beta}_i(x^-)<\frac{-3g^{\beta}_i(x^+)-g^{\alpha}_i(x^+)}{2} \quad \textit{or} \quad g^{\beta}_i(x^-)\le -g^{\alpha}_i(x^+)<-3g^{\beta}_i(x^+). \end{cases}$$

$$(b)\quad\begin{cases} g^{\alpha}_i(x^+)+g^{\beta}_i(x^+)\le -g^{\alpha}_i(x^-)-g^{\beta}_i(x^-), \ \textit{and} \\ g^{\beta}_i(x^-)<\frac{-3g^{\beta}_i(x^+)-g^{\alpha}_i(x^+)}{2}<-3g^{\beta}_i(x^+) \quad \textit{or} \quad g^{\beta}_i(x^-)\le -g^{\alpha}_i(x^+)<-3g^{\beta}_i(x^+). \end{cases}$$

If $g^{\alpha}_i(x^+)+g^{\beta}_i(x^+)\ge -g^{\alpha}_i(x^-)-g^{\beta}_i(x^-)>0$ as considered in part (a), $\widetilde{V}_i(s^{\lambda}_{-i}|x+\lambda\sigma)$ in (2.11) is higher than the value of the same formula when $g^{\alpha}_i(x^-)+g^{\beta}_i(x^-)$ is replaced by $-[g^{\alpha}_i(x^+)+g^{\beta}_i(x^+)]$. We show (in Appendix) that this latter value can be bounded below uniformly across λ when the noise is symmetric. If this uniform lower bound is positive, (2.13) holds and α is iteratively dominant in I by Proposition 5. The condition that this uniform bound is positive is stated in part (a), which differs depending on whether $g^{\alpha}_i(x^+)\ge -g^{\beta}_i(x^-)$ or $g^{\alpha}_i(x^+)\le -g^{\beta}_i(x^-)$.

As an intermediate step to part (b), observe that Corollary 2 (a) applies when $g^{\alpha}_i(x^+)+g^{\beta}_i(x^+)=-g^{\alpha}_i(x^-)-g^{\beta}_i(x^-)$ and thus, (2.13) holds if the condition in part (a) holds. If $g^{\beta}_i(x^-)$ is reduced, so that $g^{\alpha}_i(x^+)+g^{\beta}_i(x^+)\le -g^{\alpha}_i(x^-)-g^{\beta}_i(x^-)$ as part (b) postulates, (2.13) continues to hold by the last claim of Proposition 5. Therefore, the condition in part (a), when $g^{\beta}_i(x^-)$ is reduced to any lower level, is sufficient for (2.13) to hold. This is the condition stated in part (b).

The environment of Corollary 2 covers the regime change model of Section 2: the critical state is $x = 1$ and $g_i^\alpha(x^+) = 1 - c_i,\ g_i^\beta(x^+) = c_i,\ g_i^\alpha(x^-) = -c_i$ and $g_i^\beta(x^-) = c_i - 1$. Note that $g_i^\alpha(x^+) + g_i^\beta(x^+) = -g_i^\alpha(x^-) - g_i^\beta(x^-)$ so that both parts (a) and (b) of Corollary 2 apply. Since $g_i^\alpha(x^+) = -g_i^\beta(x^-)$, every condition therein prescribes the same condition for "acting" to be iteratively dominant in the complementary region, namely, $c_i < 1/4$ as stated in Section 2.

Corollary 2 also reveals that it is iteratively dominant for agent 1 to act and for agent 2 to not act on signals in the *substitutive* region, i.e., $x_i \in (0, 1)$, if $c_2 > 3/4$. To see this, we need to relabel the choices for agent 2: since it is risk-dominant for agent 1 to act and agent 2 not act on signals below 1 (as noted in Section 2), we relabel the choice to act as β_2 and to not act as α_2 for agent 2. In addition, we need to reorient θ and x_i in the reverse direction, say as $\vartheta = -\theta$ and $y_i = -x_i$, so that α risk-dominates β for $\vartheta \in (-1, 0)$, and focus on $g_i^\gamma(y^+)$ and $g_i^\gamma(y^-)$ at the discontinuity point $y = -1$. Then, $g_1^\alpha(y^+) = 1 - c_1,\ g_1^\beta(y^+) = c_1,\ g_1^\alpha(y^-) = -c_1$ and $g_1^\beta(y^-) = c_1 - 1$; $g_2^\alpha(y^+) = c_2,\ g_2^\beta(y^+) = 1 - c_2,\ g_2^\alpha(y^-) = c_2 - 1$ and $g_2^\beta(y^-) = -c_2$. Once again, $g_i^\alpha(y^+) + g_i^\beta(y^+) = -g_i^\alpha(y^-) - g_i^\beta(y^-)$ and $g_i^\alpha(y^+) = -g_i^\beta(y^-)$, hence every condition prescribes that if $c_1 < 1/4$ and $c_2 > 3/4$ then the risk-dominant equilibrium in the substitutive region is iteratively dominant.

When $g_i^\alpha(x^-) + g_i^\beta(x^-) < 0 < g_{-i}^\alpha(x^-) + g_{-i}^\beta(x^-)$

Lastly, we consider the case where the two actions are strategic substitutes for one player, say 1, but complements for player 2. In this case, the initial dominance should arise for player 1 because, as explained earlier, it cannot arise for a player for whom the actions are complements. As in the previous cases, for iterative dominance of α, it is crucial that the range of iteratively dominant signals, $A_i^{\sigma,n}$, expand over $x + \sigma$ for both players in some round, and a sufficient condition can be formulated analogously to (2.14). However, extra conditions are needed to ensure that the range of player 1's signals at which α_1 is initially dominant, $A_1^{\sigma,1}$, is large enough to trigger iterative dominance of α_2 for the other player in the subsequent round.

Specifically, the range of initial dominance, $A_1^{\sigma,1}$, contains an interval which is large enough so that, conditional on player 1 playing α_1 at signals in this interval, player 2 finds α_2 dominant at some of her signals in the subsequent round. We formalize this as follows: there is an interval of signals at which α_1 is initially dominant, that is converted via ρ_1 to an interval immediately below x in benchmark labelling without loss, denoted by $[x + \underline{\lambda}\sigma, x] \subset (x - \sigma, x]$, or equivalently,

$$\widetilde{V}_i(\check{s}_{-i}|x + \rho_i(\lambda)\sigma) > 0 \quad \text{for all} \quad \lambda \in [\underline{\lambda}, 0] \neq \emptyset \quad \text{for} \quad i = 1. \tag{2.15}$$

Then, this triggers dominance of α_2 if α_2 is dominant for player 2 at some signal, denoted x_2^0, conditional on player 1 playing α_1 at the signals in the aforesaid interval of initial dominance, namely, $[x + \rho_1(\underline{\lambda})\sigma, x + \rho_1(0)\sigma]$ converted back from the benchmark labelling.

This amounts to $\widetilde{V}_2(s_1^{\rho_1(\underline{\lambda},0)}|x_2^0) > 0$ where $s_1^{\rho_1(\underline{\lambda},0)}$ denotes a strategy s_1 that assigns 1 precisely to signals $x_1 \in [x + \rho_1(\underline{\lambda})\sigma, x + \rho_1(0)\sigma]$ and 0 to all other signals, because $\check{s}_1(\cdot) \equiv 0$ by Lemma 3 given that the actions are strategic complements on both sides of x for player 2. Converting the signal x_2^0 to x in the benchmark labelling without loss, we formalize this condition as $\widetilde{V}_2(s_1^{\rho_1(\underline{\lambda},0)}|x + \rho_2(0)\sigma) > 0$.

Finally, the condition (2.14) needs to be modified for player $i = 2$ as follows because, unlike in the previous case, $\check{s}_1(\cdot) \equiv 0$ and α_1 is initially dominant in the interval $[x + \rho_1(\underline{\lambda})\sigma, x + \rho_1(0)\sigma]$:

$$\widetilde{V}_i(s_{-i}^{\rho_{-i}(\underline{\lambda},\lambda)}|x + \rho_i(\lambda)\sigma) > 0 \quad \text{for all} \quad \lambda \in [0,1] \quad \text{for} \quad i = 2 \tag{2.16}$$

where $s_{-i}^{\rho_{-i}(\underline{\lambda},\lambda)}$ is a strategy that assigns 1 to x_{-i} if $x_{-i} \in [x + \rho_{-i}(\underline{\lambda})\sigma, x + \rho_{-i}(\lambda)\sigma)$ and 0 otherwise. Note that (2.16) subsumes $\widetilde{V}_2(s_1^{\rho_1(\underline{\lambda},0)}|x + \rho_2(0)\sigma) > 0$. We now state a sufficient condition for the iterative dominance of α when the actions are strategic complements for one player and strategic substitutes for the other in $g(x^-)$.

Proposition 6 *Consider a global game (g, Θ, F, σ) with an open interval $I \subset \Theta_\ell$ such that $x = \inf I = \inf \Theta_\ell$ and $g(I) \cup g(x^+) \subset R^{\alpha \succ \beta}$. If $g_1^\alpha(x^-) + g_1^\beta(x^-) < 0 < g_2^\alpha(x^-) + g_2^\beta(x^-)$, α is iteratively dominant in I if (2.14) holds for $i = 1$ as well as (2.15) and (2.16), for some pair of functions $\rho_i : [-1,1] \to [-1,\Lambda]$, $i \in \{1,2\}$, where each ρ_i is a continuous and strictly increasing bijection for $\Lambda > 1$ and $\rho_i(0) < 1$. If all three conditions hold for ρ_1 and ρ_2, so do they when $g_i^\beta(x^-)$ is reduced for either i.*

2.5 Conclusion

We demonstrated in a stylized regime change model that the risk-dominant equilibrium can be uniquely selected in the global game even if underlying game is not dominance solvable for any fundamental value/underlying state. This requires a departure from the standard global game framework: specifically, discrete changes in payoffs of the game at a critical fundamental value (in appropriate directions and magnitudes) may hedge the risk-dominant action sufficiently for it to be the dominant action at signals near the critical level, initiating an iterated dominance process from strategic uncertainty in the global game.

Essentially, the hedging emanates from the actions becoming strategic substitutes as the fundamental crosses the critical level, so that the risk-dominant action can be optimal whichever action the other player takes if the fundamental is on the appropriate side of the critical level. Based on this insight, we extend two-person, two-action global games studied in Carlsson and van Damme (1993) by providing sufficient conditions for the iterated dominance argument to take off without dominance solvable games and cover the risk-dominant region.

As discussed in Introduction, the global game framework has been fruitful in studying coordination issues in various economic and social situations where actions feature strategic complements, although some recent studies consider strategic substitutes. Our model accommodates both strategic complements and substitutes, in particular, allowing for free-riding incentives which have not been addressed in global games hitherto. Crucially, we examine how and when the iterative coordination may arise from the strategic uncertainty as to whether the actions are complements or substitutes.

Insofar as such strategic uncertainties tend to arise in certain situations of economic interest, for instance, where a public good can be provided with varying degrees of coordination depending on the underlying state, our findings open new scope for fruitful applications of global games. The scope of application will be further enlarged by extending the analysis to models of more players and/or actions, which we leave for future research.

2.6 Appendix

Proof of Lemma 2:

We assume $x_i = 0$ and $\sigma = 1$ without loss. The cdf of x_{-i} conditional on $x_i = 0$ is

$$\int_{-\sigma}^{\sigma} F(x_{-i}|\theta) d\widetilde{F}(\theta|0) = \int_{-\sigma}^{\sigma} F(x_{-i} - \theta) f(-\theta) d\theta.$$

Differentiating this wrt x_{-i}, we get conditional density of $x_{-i} \in (-2\sigma, 0)$ as

$$\chi(x_{-i}|0) \;=\; \int_{-\sigma}^{x_{-i}+\sigma} f(x_{-i} - \theta) f(-\theta) d\theta \quad \text{for} \quad x_{-i} \in (-2\sigma, 0).$$

We show below that this increases in $x_{-i} < 0$.

Fix $x_{-i} \in (-2\sigma, 0)$. Since MLRP, (2.5), implies $\frac{f'(z)f(y)-f(z)f'(y)}{f(y)^2} \geq 0$ when $y > z$,

$$\frac{\partial}{\partial \epsilon}[f(x_{-i} - \theta + \frac{\epsilon}{2}) f(-\theta - \frac{\epsilon}{2})]\Big|_{\epsilon=0} \;=\; \frac{f'(x_{-i} - \theta) f(-\theta) - f(x_{-i} - \theta) f'(-\theta)}{2} \;\geq\; 0. \quad (2.17)$$

Hence,

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}\chi(x_{-i} + \epsilon|0)\Big|_{\epsilon=0} \;&=\; \frac{\partial}{\partial \epsilon}\int_{-\sigma}^{x_{-i}+\epsilon+\sigma} f(x_{-i} + \epsilon - \theta) f(-\theta) d\theta\Big|_{\epsilon=0} \\
&=\; \frac{\partial}{\partial \epsilon}\int_{-\sigma-\epsilon/2}^{x_{-i}+\epsilon/2+\sigma} f(x_{-i} - \tilde{\theta} + \frac{\epsilon}{2}) f(-\tilde{\theta} - \frac{\epsilon}{2}) d\tilde{\theta}\Big|_{\epsilon=0} \\
&=\; [f(-\sigma) f(-x_{-i} - \sigma) + f(x_{-i} + \sigma) f(\sigma)]/2 \\
&\quad + \int_{-\sigma}^{x_{-i}+\sigma} \frac{f'(x_{-i} - \theta) f(-\theta) - f(x_{-i} - \theta) f'(-\theta)}{2} d\theta \qquad [*] \\
&\geq\; 0
\end{aligned}$$

where the second equality is due to a change of variables ($\tilde{\theta} = \theta - \epsilon/2$) and the inequality due to (2.17). Since $\frac{\partial}{\partial x_{-i}}\chi(x_{-i}|0) = \frac{\partial}{\partial \epsilon}\chi(x_{-i} + \epsilon|0)\big|_{\epsilon=0}$, this proves that $\chi(x_{-i}|0)$ increases in $x_{-i} < 0$.

Next, we prove that the density $\chi(x_{-i}|0)$ is symmetric around 0, i.e., $\chi(x_{-i}|0) = \chi(-x_{-i}|0)$. It suffices to show that $\frac{\partial}{\partial x_{-i}}\chi(x_{-i}|0)\big|_{x_{-i}=y} = -\frac{\partial}{\partial x_{-i}}\chi(x_{-i}|0)\big|_{x_{-i}=-y}$ for $y \in (0, 2\sigma)$. Since

$$\chi(x_{-i}|0) \;=\; \int_{x_{-i}-\sigma}^{\sigma} f(x_{-i} - \theta) f(-\theta) d\theta \quad \text{for} \quad x_{-i} \in (0, 2\sigma),$$

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\chi(x_{-i}+\epsilon|0)\Big|_{\epsilon=0} &= \frac{\partial}{\partial\epsilon}\int_{x_{-i}+\epsilon/2-\sigma}^{\sigma-\epsilon/2} f(x_{-i}-\tilde{\theta}+\frac{\epsilon}{2})f(-\tilde{\theta}-\frac{\epsilon}{2})d\tilde{\theta}\Big|_{\epsilon=0}\\ &= -[f(x_{-i}-\sigma)f(-\sigma)+f(\sigma)f(-x_{-i}+\sigma)]/2\\ &\quad+\int_{x_{-i}-\sigma}^{\sigma}\frac{f'(x_{-i}-\tilde{\theta})f(-\tilde{\theta})-f(x_{-i}-\tilde{\theta})f'(-\tilde{\theta})}{2}d\tilde{\theta}\\ &= -[f(x_{-i}-\sigma)f(-\sigma)+f(\sigma)f(-x_{-i}+\sigma)]/2\\ &\quad+\int_{-\sigma}^{-x_{-i}+\sigma}\frac{f'(-\theta)f(-x_{-i}-\theta)-f(-\theta)f'(-x_{-i}-\theta)}{2}d\theta\end{aligned}$$

which is negative of [*] as desired, where the last equality is due to change of variable $\theta=\tilde{\theta}-x_{-i}$. This completes the proof. ∎

Proof of Proposition 4:

Let $\tau_i^+ = \frac{g_i^\beta(x^+)}{g_i^\alpha(x^+)+g_i^\beta(x^+)} \in (0,1)$ denote the threshold probability for player i in $g(x^+)$. The sum of threshold probabilities is $\tau_1^+ + \tau_2^+ < 1$ because $g_1^\alpha(x^+)g_2^\alpha(x^+) > g_1^\beta(x^+)g_2^\beta(x^+)$ by (2.6).

By assumption, α_i is strictly dominant $g(x^-)$ for at least one player who we denoted by 1. For the other player 2, either (i) $g_2^\beta(x^-)<0$ so that α_2 is dominant in $g(x^-)$, or (ii) $g_2^\beta(x^-)\geq 0$ and $0 \leq \frac{g_2^\beta(x^-)}{g_2^\alpha(x^-)+g_2^\beta(x^-)} < 1 - \frac{g_1^\beta(x^+)}{g_1^\alpha(x^+)+g_1^\beta(x^+)}$, in which case α_2 is uniquely optimal if player 1 plays α_1 with a probability greater than the threshold level $\tau_2^- = \frac{g_2^\beta(x^-)}{g_2^\alpha(x^-)+g_2^\beta(x^-)}$. In case (i), since α_2 is uniquely optimal regardless of the probability with which player 1 plays α_1, we may set the threshold level as $\tau_2^- = 0$. In either case, $\tau_1^+ + \tau_2^- < 1$. By the same token, we set $\tau_1^- = 0$.

The threshold probability $\tau_i(\theta) := \frac{g_i^\beta(\theta)}{g_i^\alpha(\theta)+g_i^\beta(\theta)}$ in $g(\theta)$ is arbitrarily closely approximated by τ_i^+ as θ converges to x from above, and by τ_i^- as θ converges to x from below if $\tau_i^- > 0$. For small enough σ, therefore, upon observing $x_i \in (x-7\sigma, x+3\sigma)$,

[B] player i finds α_i uniquely optimal if α_{-i} is played with a probability exceeding $\max\{\tau_i^-, \tau_i^+\}+\epsilon$ for arbitrarily small $\epsilon>0$ such that $\max\{\tau_i^-,\tau_i^+\}+\epsilon<1$.

Recall that α_1 is initially strictly dominant for player 1 upon observing signals up to a threshold $\widehat{x}_1 \in (x-\sigma, x)$ if σ is small enough, in particular, upon observing signals $x_1 \in (x-7\sigma, \widehat{x}_1)$, i.e., $(x-7\sigma,\widehat{x}_1) \subset A_1^{\sigma,1}$.

For $x_2 \in (x-7\sigma, x+3\sigma)$, let $Prob(x_1 < \widehat{x}_1|x_2)$ denote the probability with which player 1 observes a signal $x_1 < \widehat{x}_1$ (hence, plays α_1) conditional on player 2 observing her signal x_2. By Lemma 2, there is $\widehat{x}_2 \in (\widehat{x}_1 - 2\sigma, \widehat{x}_1+2\sigma)$ such that $Prob(x_1 < \widehat{x}_1|\widehat{x}_2) = \max\{\tau_2^-,\tau_2^+\}+\epsilon$. Since $Prob(x_1<\widehat{x}_1|x_2)$ decreases in $x_2 \in (x-5\sigma,\widehat{x}_2)$ by Lemma 2, α_2 is dominant for player 2 upon observing signals $x_2 \in (x-5\sigma,\widehat{x}_2)$, i.e., $(x-5\sigma,\widehat{x}_2)\subset A_2^{\sigma,2}$. Note that $x-\sigma<\widehat{x}_1$ and $x-3\sigma<\widehat{x}_2$.

Let $\widehat{x}_2^{\sigma,2} \geq \widehat{x}_2$ denote the upper boundary of the largest interval in $A_2^{\sigma,2}$ that contains $\widehat{x}_2$ in its support, and let $\widehat{x}_1^{\sigma,2} \geq \widehat{x}_1$ denote the upper boundary of the largest interval in $A_1^{\sigma,2}$ that contains $\widehat{x}_1$ in its support. Likewise, for both $i \in \{1,2\}$, let $\widehat{x}_i^{\sigma,n} \geq \widehat{x}_i^{\sigma,n-1}$ denote the upper boundary of the largest interval in $A_i^{\sigma,n}$ that contains $\widehat{x}_i$ in its support for $n > 2$ and let $\widehat{x}_i^{\sigma,\infty} = \lim_{n\to\infty} \widehat{x}_i^{\sigma,n}$. Note that $(x - 5\sigma, x - 3\sigma) \subset (x - 5\sigma, \widehat{x}_i^{\sigma,n}) \subset A_i^{\sigma,n}$ for all $n \geq 2$ for both $i = 1, 2$. We show below that (i) $\widehat{x}_i^{\sigma,\infty} > x + \sigma$ and (ii) $\widehat{x}_i^{\sigma,\infty} \geq \widehat{X}$ for both i for any $\widehat{X} \in (x + \sigma, \sup I)$ if σ is small enough, which will establish the Proposition.

(i) With a view to reaching a contradiction, suppose $\widehat{x}_i^{\sigma,\infty} \leq x + \sigma$ for some i for arbitrarily small σ. Assume $\widehat{x}_i^{\sigma,\infty} \leq \widehat{x}_{-i}^{\sigma,\infty}$ without loss. If $2\sigma \leq \widehat{x}_{-i}^{\sigma,\infty} - \widehat{x}_i^{\sigma,\infty}$, in round n large enough so that $\widehat{x}_{-i}^{\sigma,n} - \widehat{x}_i^{\sigma,n}$ is at least arbitrarily close to 2σ, $Prob(x_{-i} < \widehat{x}_{-i}^{\sigma,n} | \widehat{x}_i^{\sigma,n})$ is arbitrarily close to 1, hence exceeds $\max\{\tau_i^-, \tau_i^+\} + \epsilon$. Consequently, α_i is dominant at $x_i = \widehat{x}_i^{\sigma,n}$ and also at slightly higher signals by continuity, contradicting $\widehat{x}_i^{\sigma,n}$ being a boundary signal of $A_i^{\sigma,n}$.

If $\widehat{x}_{-i}^{\sigma,\infty} - \widehat{x}_i^{\sigma,\infty} < 2\sigma$, in round n large enough so that $\widehat{x}_{-i}^{\sigma,n} - \widehat{x}_i^{\sigma,n} < 2\sigma$, we have $Prob(x_i < \widehat{x}_i^{\sigma,n} | \widehat{x}_{-i}^{\sigma,n}) + Prob(x_{-i} < \widehat{x}_{-i}^{\sigma,n} | \widehat{x}_i^{\sigma,n}) = 1$ by Lemma 2, that is, the probability that player i observes $x_i < \widehat{x}_i^{\sigma,n}$ (hence play α_i) conditional on player $-i$ observing $x_{-i} = \widehat{x}_{-i}^{\sigma,n}$, and the converse probability are complimentary. Since $\tau_1^+ + \max\{\tau_2^-, \tau_2^+\} < 1$ as asserted above, so that $\tau_1^+ + \max\{\tau_2^-, \tau_2^+\} + 2\epsilon < 1$ for small enough ϵ, at least one player i infers the other player playing α_{-i} with a probability exceeding $\max\{\tau_i^-, \tau_i^+\} + \epsilon$ upon observing $x_i = \widehat{x}_i^{\sigma,n}$ if σ is small enough, thus finds α_i iteratively dominant by [B]. As this would contradict $\widehat{x}_i^{\sigma,n}$ being a boundary signal of $A_i^{\sigma,n}$, we have established that $\widehat{x}_i^{\sigma,\infty} > x + \sigma$ for both $i = 1, 2$.

(ii) In the underlying game $g(\theta)$ for $\theta \in I$, α_i is uniquely optimal if player $-i$ plays α_{-i} with a threshold probability $\tau_i(\theta) = \frac{g_i^\beta(\theta)}{g_i^\alpha(\theta) + g_i^\beta(\theta)} \in (0,1)$. Upon observing $x_i \in [x + \sigma, \widehat{X}]$, therefore, α_i is uniquely optimal if the other player plays α_{-i} with a probability exceeding $\tau_i(x_i) + \epsilon$ for arbitrarily small $\epsilon > 0$ if σ is sufficiently small. The set $[x + \sigma, \widehat{X}]$ being closed, the value of $\epsilon > 0$ can be chosen uniformly for all $x_i \in [x + \sigma, \widehat{X}]$ given σ, in such a way that $\epsilon \to 0$ as $\sigma \to 0$. Since $\tau_i(\theta) + \tau_{-i}(\theta) < 1$ for every $\theta \in \{x^+\} \cup (x, \widehat{X}]$ due to risk dominance as noted earlier and $\tau_i(\theta)$ and $\tau_{-i}(\theta)$ are continuous in θ, there is $\eta > 0$ such that $\tau_i(\theta) + \tau_{-i}(\theta) + 2\epsilon < 1 - \eta$ for all $\theta \in [x + \sigma, \widehat{X}]$ for all sufficiently small σ.

With a view to reaching a contradiction, suppose $\widehat{x}_i^{\sigma,\infty} \in (x + \sigma, \widehat{X})$ for some i for arbitrarily small σ. Assume $\widehat{x}_i^{\sigma,\infty} \leq \widehat{x}_{-i}^{\sigma,\infty}$ without loss. If $2\sigma \leq \widehat{x}_{-i}^{\sigma,\infty} - \widehat{x}_i^{\sigma,\infty}$, in round n large enough so that $\widehat{x}_{-i}^{\sigma,n} - \widehat{x}_i^{\sigma,n}$ is at least arbitrarily close to 2σ, $Prob(x_{-i} < \widehat{x}_{-i}^{\sigma,n} | \widehat{x}_i^{\sigma,n})$ is arbitrarily close to 1, hence exceeds $\tau_i(\widehat{x}_i^{\sigma,n}) + \epsilon$. Consequently, α_i is dominant at $x_i = \widehat{x}_i^{\sigma,n}$ and also at slightly higher signals by continuity, contradicting $\widehat{x}_i^{\sigma,n}$ being a boundary signal of $A_i^{\sigma,n}$.

If $\widehat{x}_{-i}^{\sigma,\infty} - \widehat{x}_i^{\sigma,\infty} < 2\sigma$, in round n large enough so that $\widehat{x}_{-i}^{\sigma,n} - \widehat{x}_i^{\sigma,n} < 2\sigma$, we have $Prob(x_i < \widehat{x}_i^{\sigma,n} | \widehat{x}_{-i}^{\sigma,n}) + Prob(x_{-i} < \widehat{x}_{-i}^{\sigma,n} | \widehat{x}_i^{\sigma,n}) = 1$ by Lemma 2 as before. In addition, $\tau_i(\widehat{x}_i^{\sigma,n}) + \tau_{-i}(\widehat{x}_{-i}^{\sigma,n}) + 2\epsilon < 1$ for small enough σ because $\tau_i(\theta) + \tau_{-i}(\theta) + 2\epsilon < 1 - \eta$ for all $\theta \in [x+\sigma, \widehat{X}]$ as asseted above and $|\widehat{x}_i^{\sigma,n} - \widehat{x}_{-i}^{\sigma,n}| \to 0$ as $\sigma \to 0$. Hence, in round n for large enough n, either α_i is dominant for player i upon observing $x_i = \widehat{x}_i^{\sigma,n}$ or α_{-i} is dominant for player $-i$ upon observing $x_{-i} = \widehat{x}_{-i}^{\sigma,n}$. As this would contradict $\widehat{x}_i^{\sigma,n}$ being a boundary signal of $A_i^{\sigma,n}$ for at least one player i, we have established that $\widehat{x}_i^{\sigma,\infty} \geq \widehat{X}$ for both $i = 1, 2$. ∎

Proof of Lemma 3:

As asserted in the main text, $\widetilde{V}_i(s_{-i}|x + \lambda_i\sigma)$ is minimized with $\breve{s}_{-i}$ such that $\breve{s}_{-i}(x + \lambda_{-i}\sigma|x + \lambda_i\sigma) = 0$ if $\Psi_i(\lambda_{-i}|\lambda_i) \geq 0$ and $\breve{s}_{-i}(x + \lambda_{-i}\sigma|x + \lambda_i\sigma) = 1$ if $\Psi_i(\lambda_{-i}|\lambda_i) < 0$. Hence, it is straightforward that $\breve{s}_{-i}(\cdot|x + \lambda_i\sigma) \equiv 0$ if $g_i^\alpha(x^-) + g_i^\beta(x^-) \geq 0$.

Next, suppose $g_i^\alpha(x^-)+g_i^\beta(x^-) < 0$. It is clear from (2.12) that $\Psi_i(-1|-1) = g_i^\alpha(x^-)+g_i^\beta(x^-) < 0$ because $\int_0^\infty f(-1-\vartheta)f(-1-\vartheta)d\vartheta = 0$, and $\Psi_i(1|1) = g_i^\alpha(x^+) + g_i^\beta(x^+) > 0$ because $\int_{-\infty}^0 f(1-\vartheta)f(1-\vartheta)d\vartheta = 0$. Also, $\Psi_i(\lambda_{-i}|\lambda_i)$ is continuous in λ_i and λ_{-i} and symmetric between them. Hence, if $\Psi_i(\lambda_{-i}|\lambda_i)$ strictly increases in $\lambda_{-i} \in (-1, 1)$ for any given $\lambda_i \in (-1, 1)$, the specification of $\breve{s}_{-i}(\cdot|x + \lambda_i\sigma)$ in the Lemma 3 verified.

It remains to verify that $\partial\Psi_i(\lambda_{-i}|\lambda_i)/\partial\lambda_{-i} > 0$. Note that $Pr^+ + Pr^- = 1$ where $Pr^+ = \int_0^\infty f(\lambda_{-i}-\vartheta)f(\lambda_i-\vartheta)d\vartheta$ and $Pr^- = \int_{-\infty}^0 f(\lambda_{-i}-\vartheta)f(\lambda_i-\vartheta)d\vartheta$. Due to MLRP, f is single-peaked, say at $\hat{\lambda} \in (-1, 1)$. First, consider the case that $\lambda_i \geq \hat{\lambda}$. If $\lambda_{-i} \geq \hat{\lambda}$ as well, then $\partial Pr^-/\partial\lambda_{-i} = \int_{-\infty}^0 f'(\lambda_{-i}-\vartheta)f(\lambda_i-\vartheta)d\vartheta < 0$ because $f'(\lambda) < 0$ for $\lambda > \hat{\lambda}$, hence $\partial Pr^+/\partial\lambda_{-i} > 0$ because $Pr^+ + Pr^- = 1$. Hence, $\partial\Psi_i(\lambda_{-i}|\lambda_i)/\partial\lambda_{-i} > 0$. If $\lambda_{-i} < \hat{\lambda}$, on the other hand, $\partial Pr^+/\partial\lambda_{-i} = \int_0^\infty f'(\lambda_{-i}-\vartheta)f(\lambda_i-\vartheta)d\vartheta > 0$ because $f'(\lambda) > 0$ for $\lambda < \hat{\lambda}$, hence $\partial Pr^-/\partial\lambda_{-i} < 0$ because $Pr^+ + Pr^- = 1$. Hence, again $\partial\Psi_i(\lambda_{-i}|\lambda_i)/\partial\lambda_{-i} > 0$. Next, $\partial\Psi_i(\lambda_{-i}|\lambda_i)/\partial\lambda_{-i} > 0$ is proved analogously for the alternative case that $\lambda_i < \hat{\lambda}$. This completes the proof. ∎

Proof of Proposition 5:

In a global game considered in the proposition, suppose $g_i^\alpha(x^-) + g_i^\beta(x^-) < 0$ for $i = 1, 2$. The condition (2.14) for $\lambda = 0$, i.e., $\widetilde{V}_i(\breve{s}_{-i}^{\rho_{-i}(0)}|x + \rho_i(0)\sigma) > 0$, implies $x_i^0 = x + \rho_i(0)\sigma \in A_i^{\sigma,1}$. Let $\widehat{x}_i^{\sigma,n} > x_i^0$ be the upper boundary of the largest interval in $A_i^{\sigma,n}$ containing x_i^0. Denote $\widehat{x}_i^{\sigma,n} = x + \widehat{\lambda}_i^n\sigma$. Clearly, $\widehat{\lambda}_i^n \in (0, 1]$ increases in n. We show that (i) $\widehat{\lambda}_i^\infty = \lim_{n\to\infty} \widehat{\lambda}_i^n > 1$ and then (ii) $\widehat{x}_i^{\sigma,\infty} = x+\widehat{\lambda}_i^\infty\sigma \geq \widehat{X}$ for both i for any $\widehat{X} \in (x+\sigma, \sup I)$ if σ is small enough, which will establish iterative dominance of α in I.

(i) With a view to reaching a contradiction, suppose $\widehat{\lambda}_i^\infty \leq 1 < \Lambda$, so that $\rho_i^{-1}(\widehat{\lambda}_i^\infty) < 1$, for some i for arbitrarily small σ, and $\rho_i^{-1}(\widehat{\lambda}_i^\infty) \leq \rho_{-i}^{-1}(\min\{\widehat{\lambda}_{-i}^\infty, \Lambda\})$ without loss. If $\rho_i^{-1}(\widehat{\lambda}_i^\infty) < \rho_{-i}^{-1}(\min\{\widehat{\lambda}_{-i}^\infty, \Lambda\})$, in round $n+1$ large enough so that $\rho_i^{-1}(\widehat{\lambda}_i^n) \leq \rho_i^{-1}(\widehat{\lambda}_i^\infty) < \rho_{-i}^{-1}(\min\{\widehat{\lambda}_{-i}^n, \Lambda\})$, we have $V_i(s_{-i}|x + \rho_i(\widehat{\lambda})\sigma) \geq V_i(\breve{s}_{-i}^{\rho_{-i}(\widehat{\lambda})}|x + \rho_i(\widehat{\lambda})\sigma)$ for every $\widehat{\lambda} \in [\rho_i^{-1}(\widehat{\lambda}_i^n), \rho_i^{-1}(\widehat{\lambda}_i^\infty)]$ and every admissible s_{-i} in round $n+1$ by Lemma 3, because player $-i$ plays α_{-i} on $A_{-i}^{\sigma,n} \supset [x_{-i}^0, \widehat{x}_{-i}^{\sigma,n}) \supset [x+\rho_{-i}(0)\sigma, x+\rho_{-i}(\widehat{\lambda})\sigma)$ given that $x+\rho_{-i}(\rho_i^{-1}(\widehat{\lambda}_i^\infty))\sigma <$

$x+\rho_{-i}(\rho_{-i}^{-1}(\min\{\widehat{\lambda}_{-i}^n,\Lambda\}))\sigma = x+\min\{\widehat{\lambda}_{-i}^n,\Lambda\}\sigma \le x+\widehat{\lambda}_{-i}^n\sigma = \widehat{x}_{-i}^{\sigma,n}$. Since $V_i(\breve{s}_{-i}^{\rho_{-i}(\widehat{\lambda})}|x+\rho_i(\widehat{\lambda})\sigma) \to \widetilde{V}_i(\breve{s}_{-i}^{\rho_{-i}(\widehat{\lambda})}|x+\rho_i(\widehat{\lambda})\sigma)$ as $\sigma \to 0$ and $\widetilde{V}_i(\breve{s}_{-i}^{\rho_{-i}(\widehat{\lambda})}|x+\rho_i(\widehat{\lambda})\sigma) > 0$ by (2.14), we would have $V_i(\breve{s}_{-i}^{\rho_{-i}(\widehat{\lambda})}|x+\rho_i(\widehat{\lambda})\sigma)) > 0$ for $\widehat{\lambda} \in [\rho_i^{-1}(\widehat{\lambda}_i^n), \rho_i^{-1}(\widehat{\lambda}_i^\infty)]$ so that $\widehat{x}_i^{\sigma,\infty} = x+\rho_i(\rho_i^{-1}(\widehat{\lambda}_i^\infty))\sigma \in A_i^{\sigma,n+1}$, a contradiction.

It remains to consider the case that $\widehat{\lambda}_i^\infty \le 1 < \Lambda$ and $\rho_i^{-1}(\widehat{\lambda}_i^\infty) = \rho_{-i}^{-1}(\widehat{\lambda}_{-i}^\infty) < 1$. In this case, the minimal value of $V_i(s_{-i}|x+\widehat{\lambda}_i^\infty\sigma)$ in each round n obtains when s_{-i} assigns 1 only to $x_{-i} \in (-\infty, x+T_{-i}(\widehat{\lambda}_i^\infty)\sigma) \cup A_{-i}^{\sigma,n-1}$ by Lemma 3. By continuity of $V_i(\cdot|x+\widehat{\lambda}_i^\infty\sigma)$, this minimal value converges to a value no lower than $\widetilde{V}_i(\breve{s}_{-i}^{\widehat{\lambda}_i^\infty}|x+\widehat{\lambda}_i^\infty\sigma)$ as $n \to \infty$ because $\widehat{x}_i^{\sigma,n} \to \widehat{x}_i^{\sigma,\infty}$. Since $\widetilde{V}_i(\breve{s}_{-i}^{\widehat{\lambda}_i^\infty}|x+\widehat{\lambda}_i^\infty\sigma) > 0$ by (2.14), we would have $\widehat{x}_i^{\sigma,\infty} = x+\widehat{\lambda}_i^\infty\sigma \in A_i^{\sigma,n}$ for some n if σ is small enough, again a contradiction. This establishes that $\widehat{\lambda}_i^\infty > 1$ for $i = 1,2$.

(ii) With a view to reaching a contradiction, suppose $\widehat{x}_i^{\sigma,\infty} \in (x+\sigma, \widehat{X})$ for some i for arbitrarily small σ. Assume $\widehat{x}_i^{\sigma,\infty} \le \widehat{x}_{-i}^{\sigma,\infty}$ without loss. Upon observing $x_i \in (x+\sigma,\widehat{X})$, each agent is certain that $\theta \in I$, thus α risk-dominates β.

If $\widehat{x}_i^{\sigma,\infty} \ge x+3\sigma$ for both i, upon observing $x_i = \widehat{x}_i^{\sigma,\infty}$ each agent i knows that agent $-i$ observes $x_{-i} > x+\sigma$ and thus chooses α_{-i} with probability at least $Prob(x_{-i} < \widehat{x}_{-i}^{\sigma,\infty} \,|\, \widehat{x}_i^{\sigma,\infty})$. An argument analogous to that in the proof for Proposition 4 leads to a contradictory conclusion that α_i should be dominant at $\widehat{x}_i^{\sigma,\infty}$ for at least one i.

Thus, suppose $\widehat{x}_i^{\sigma,\infty} \in (x+\sigma, x+3\sigma)$ for at least one i. First, if $\widehat{x}_i^{\sigma,\infty} = x+\widehat{\lambda}_i^\infty\sigma \le x+\Lambda\sigma$ for either player i, let i denote the sole such player or the player with a lower $\rho_i^{-1}(\widehat{\lambda}_i^\infty)$. Then, $V_i(s_{-i}|\widehat{x}_i^{\sigma,\infty} = x+\widehat{\lambda}_i^\infty\sigma)$ is minimal in round n large enough when s_{-i} assigns 1 only at $x_{-i} \in A_{-i}^{\sigma,n} \supset (x_{-i}^0, x+\widehat{\lambda}_{-i}^n\sigma) \supset (x_{-i}^0, x+\rho_{-i}(\rho_i^{-1}(\widehat{\lambda}_i^n))\sigma)$, which is no lower than $V_i(\breve{s}_{-i}^{\rho_{-i}(\rho_i^{-1}(\widehat{\lambda}_i^n))}|x+\widehat{\lambda}_i^\infty\sigma)$. Since $\widetilde{V}_i(\breve{s}_{-i}^{\rho_{-i}(\rho_i^{-1}(\widehat{\lambda}_i^\infty))}|x+\widehat{\lambda}_i^\infty\sigma) > 0$ by (2.14), we have $\widetilde{V}_i(\breve{s}_{-i}^{\rho_{-i}(\rho_i^{-1}(\widehat{\lambda}_i^n))}|x+\widehat{\lambda}_i^\infty\sigma) > 0$ for large enough n by continuity, hence $V_i(\breve{s}_{-i}^{\rho_{-i}(\rho_i^{-1}(\widehat{\lambda}_i^n))}|x+\widehat{\lambda}_i^\infty\sigma) > 0$ for large enough n as $\sigma \to 0$. This would mean that α_i is iteratively dominant at $x_i = \widehat{x}_i^{\sigma,\infty}$, a contradiction.

Second, if $\widehat{x}_i^{\sigma,\infty} = x+\widehat{\lambda}_i^\infty\sigma \in (x+\Lambda\sigma, x+3\sigma)$ for both players, assume $\widehat{x}_i^{\sigma,\infty} \le \widehat{x}_{-i}^{\sigma,\infty}$. Then, $V_i(s_{-i}|\widehat{x}_i^{\sigma,\infty})$ is minimal in round n large enough when s_{-i} assigns 1 only at $x_{-i} \in A_{-i}^{\sigma,n} \supset (x_{-i}^0, x+\rho_{-i}(\widehat{\lambda}_{-i}^n)\sigma)$, which is no lower than that when s_{-i} assigns 1 only at $x_{-i} \in (x_{-i}^0, x+\rho_{-i}(\widehat{\lambda}_{-i}^n)\sigma)$. This, in turn, is no lower than $V_i(s_{-i}|x+\Lambda\sigma)$ when s_{-i} assigns 1 only at $x_{-i} \in (x_{-i}^0, x+\Lambda\sigma)$, which is positive by (2.14). This would mean that α_i is iteratively dominant at $x_i = \widehat{x}_i^{\sigma,\infty}$ by the same token as above, a contradiction.

Finally, to prove the last part of the proposition, suppose (2.14) holds for ρ_1 and ρ_2 in a global game. By Lemma 3, $\widetilde{V}_i(\breve{s}_{-i}^{\rho_{-i}(\lambda)}|x+\rho_i(\lambda)\sigma)$ is the minimal value of $\widetilde{V}_i(s_{-i}|x+\rho_i(\lambda)\sigma)$ among all s_{-i}'s subject to assigning 1 to all $x_{-i} \in (x+\rho_{-i}(0)\sigma, x+\rho_{-i}(\lambda)\sigma)$. Given any such s_{-i}, note from (2.11) that $\widetilde{V}_i(s_{-i}|x+\rho_i(\lambda)\sigma)$ is higher when $g_i^\beta(x^-)$ is reduced because the coefficient of $g_i^\beta(x^-)$ in (2.11) is the negative of the probability that $\theta < x$ and player $-i$ plays β_{-i} according to s_{-i} conditional on player i's signal $x_i = x+\rho_i(\lambda)\sigma$. Therefore, the minimal value of $\widetilde{V}_i(s_{-i}|x + \rho_i(\lambda)\sigma)$ among all such s_{-i}'s is no lower when $g_i^\beta(x^-)$ is reduced for either i, that is, (2.14) continues to hold when $g_i^\beta(x^-)$ is reduced for either i. This completes the proof. ∎

Proof of Corollary 2:

(a) Suppose that density f of ϵ_i symmetric around 0. Then, if the two players observe signals x_i and x_{-i} that are equidistant from x in the opposite direction, θ is equally likely to be above or below x and thus, $\Psi_i(x_{-i}|x_i) \geq 0$ if $g_i^\alpha(x^+) + g_i^\beta(x^+) \geq -g_i^\alpha(x^-) - g_i^\beta(x^-) > 0$. This implies that x_{-i} needs to be below x by more than x_i exceeds x for $\Psi_i(x_{-i}|x_i) = 0$ to hold, that is, $T_i(x + \lambda\sigma) \leq x - \lambda\sigma$. Hence, $F_\sigma\big((-\infty, T_i(x+\lambda\sigma)) \cup (x, x+\lambda\sigma)|\theta\big) = F_\sigma(x + \lambda\sigma|\theta) - F_\sigma(x|\theta) + F_\sigma(T_i(x+\lambda\sigma)|\theta)$ and replacing $g_i^\alpha(x^-) + g_i^\beta(x^-)$ with $-[g_i^\alpha(x^+) + g_i^\beta(x^+)]$ in $\widetilde{V}_i(\check{s}_{-i}^\lambda|x + \lambda\sigma)$ gives

$$\widetilde{V}_i(\check{s}_{-i}^\lambda|x + \lambda\sigma) \;\geq\; [g_i^\alpha(x^+) + g_i^\beta(x^+)]\Delta_i(\lambda) - g_i^\beta(x^-) + [g_i^\beta(x^-) - g_i^\beta(x^+)]F(\lambda) \quad (2.18)$$

$$\text{where}\;\; \Delta_i(\lambda) \;:=\; \int_x^{x+\lambda\sigma+\sigma} [F_\sigma(x + \lambda\sigma|\theta) - F_\sigma(x|\theta) + F_\sigma(T_i(x + \lambda\sigma)|\theta)]d\tilde{F}_\sigma(\theta|x + \lambda\sigma)$$
$$- \int_{x+\lambda\sigma-\sigma}^{x} [F_\sigma(x + \lambda\sigma|\theta) - F_\sigma(x|\theta) + F_\sigma(T_i(x + \lambda\sigma)|\theta)]d\tilde{F}_\sigma(\theta|x + \lambda\sigma).$$

We show below that (i) $\Delta_i(\lambda) \geq \Delta^*(\lambda)$ where $\Delta^*(\lambda)$ is $\Delta_i(\lambda)$ evaluated when $T_i(x + \lambda\sigma) = x - \lambda\sigma$, and (ii) $\Delta^*(\lambda)$ increases in $\lambda \in [0, 1)$ at a rate no lower than $f(\lambda)$, i.e., $\frac{d}{d\lambda}\Delta^*(\lambda) \geq f(\lambda)$. By property (ii), $\Delta^*(\lambda)$ is bounded below by

$$\Delta^*(0) + \int_0^\lambda f(z)dz \;=\; \Delta^*(0) + F(\lambda) - F(0) \;=\; F(\lambda) - \frac{3}{4}$$

because, given symmetric noise which implies $\tilde{F}_\sigma(\theta|x) = F_\sigma(\theta|x) = 1 - F_\sigma(x|\theta)$, we have $\Delta^*(0) = \int_x^{x+\sigma}[1 - F_\sigma(x|\theta)]dF_\sigma(\theta|x) - \int_{x-\sigma}^{x}[1 - F_\sigma(x|\theta)]dF_\sigma(\theta|x) = -1/4$ by (2.2) and $F(0) = 1/2$. Hence, from (i) we deduce that the RHS of (2.18) is bounded below by

$$[g_i^\alpha(x^+) + g_i^\beta(x^+)]\Big(F(\lambda) - \frac{3}{4}\Big) - g_i^\beta(x^-) + [g_i^\beta(x^-) - g_i^\beta(x^+)]F(\lambda). \quad (2.19)$$

Consequently, (2.13) holds if (2.19) is positive for all $\lambda \in [0, 1]$. (2.19) increases in λ if $g_i^\alpha(x^+) + g_i^\beta(x^+) \geq -[g_i^\beta(x^-) - g_i^\beta(x^+)] > 0$, i.e., $g_i^\alpha(x^+) \geq -g_i^\beta(x^-) > 0$, but decreases in λ if $g_i^\alpha(x^+) + g_i^\beta(x^+) \leq -[g_i^\beta(x^-) - g_i^\beta(x^+)]$, i.e., $g_i^\alpha(x^+) \leq -g_i^\beta(x^-)$. Hence, (2.19) is minimal at $\lambda = 0$ in the former case and at $\lambda = 1$ in the latter; and the minimized values are $[g_i^\alpha(x^+) + g_i^\beta(x^+)]\big(F(0) - \frac{3}{4}\big) - g_i^\beta(x^-) + [g_i^\beta(x^-) - g_i^\beta(x^+)]F(0) = \frac{-g_i^\alpha(x^+) - 3g_i^\beta(x^+) - 2g_i^\beta(x^-)}{4}$ and $[g_i^\alpha(x^+) + g_i^\beta(x^+)]\big(1 - \frac{3}{4}\big) - g_i^\beta(x^+) = \frac{g_i^\alpha(x^+) - 3g_i^\beta(x^+)}{4}$, respectively. Therefore, (2.13) holds if $-3g_i^\beta(x^+) - 2g_i^\beta(x^-) > g_i^\alpha(x^+)$ or $g_i^\alpha(x^+) > 3g_i^\beta(x^+)$ depending on whether $g_i^\alpha(x^+) \geq -g_i^\beta(x^-)$ or not, as summarized in part (a) of Corollary 2.

We now prove that (i) $\Delta_i(\lambda) \geq \Delta^*(\lambda)$ and (ii) $\frac{d\Delta^*(\lambda)}{d\lambda} > f(\lambda)$.

(i) Replace $T_i(x+\lambda\sigma)$ with τ in $\Delta_i(\lambda)$. Treating τ as an independent variable, it suffices to show that $\frac{\partial \Delta_i(\lambda)}{\partial \tau} \leq 0$ for $\tau \in (x-\sigma, x+\sigma)$ because $T_i(x+\lambda\sigma) \leq x - \lambda\sigma$. Denoting $y = x + \lambda\sigma$ for notational ease, we have

$$\frac{\partial \Delta_i(\lambda)}{\partial \tau} = \int_x^{y+\sigma} \sigma^{-1} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y) - \int_{y-\sigma}^{x} \sigma^{-1} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y).$$

Note that $f(\frac{\tau-\theta}{\sigma}) = 0$ for $\theta > \tau + \sigma$ because then $\frac{\tau-\theta}{\sigma} < -1$. If $y > x + \sigma/2$, we have

$$\begin{aligned}
\frac{\partial \Delta_i(\lambda)}{\partial \tau} &= \sigma^{-1}\left[\int_x^{\tau+\sigma} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y) - \int_{\tau-x+y}^{x} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y) - \int_{y-\sigma}^{\tau-x+y} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y)\right] \\
&= -\int_{\tau-x+y}^{x} \sigma^{-1} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y) \ \leq \ 0
\end{aligned}$$

where the second equality obtains because $\int_x^{\tau+\sigma} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y) = \int_{y-\sigma}^{\tau-x+y} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y)$. If $y \leq x + \sigma/2$, on the other hand,

$$\begin{aligned}
\frac{\partial \Delta_i(\lambda)}{\partial \tau} &= \sigma^{-1}\left[\int_x^{x+2(y-x)} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y) + \int_{x+2(y-x)}^{\tau+\sigma} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y)\right. \\
&\quad \left. - \int_{\tau-x+y}^{x} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y) - \int_{\tau+x-y}^{\tau-x+y} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y) - \int_{y-\sigma}^{\tau+x-y} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y)\right] \\
&= -\int_{\tau-x+y}^{x} \sigma^{-1} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y) \ \leq \ 0
\end{aligned}$$

where the second equality obtains because $\int_x^{x+2(y-x)} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y) = \int_{\tau+x-y}^{\tau-x+y} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y)$ and $\int_{x+2(y-x)}^{\tau+\sigma} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y) = \int_{y-\sigma}^{\tau+x-y} f(\frac{\tau-\theta}{\sigma}) d\tilde{F}_\sigma(\theta|y)$.

(ii) Since $\Delta^*(\lambda)$ is invariant to x and σ, we rewrite it for $x = 0$ and $\sigma = 1$ as

$$\Delta^*(\lambda) = \int_0^{\lambda+1} \Xi(\lambda,\theta) d\tilde{F}(\theta|\lambda) - \int_{\lambda-1}^{0} \Xi(\lambda,\theta) d\tilde{F}(\theta|\lambda) \ \text{ where } \ \Xi(\lambda,\theta) := F(\lambda-\theta) - F(-\theta) + F(-\lambda-\theta).$$

Differentiate the first integral by taking the limit of the following quotient as $\eta \to 0$:

$$\begin{aligned}
&\frac{\int_0^{\lambda+\eta+1} \Xi(\lambda+\eta,\theta) d\tilde{F}(\theta|\lambda+\eta) - \int_0^{\lambda+1} \Xi(\lambda,\theta) d\tilde{F}(\theta|\lambda)}{\eta} \\
&= \frac{\int_{-\eta}^{\lambda+1} [F(\lambda-\theta') - F(-\theta'-\eta) + F(-\lambda-\theta'-2\eta)] d\tilde{F}(\theta'|\lambda) - \int_0^{\lambda+1} [F(\lambda-\theta) - F(-\theta) + F(-\lambda-\theta)] d\tilde{F}(\theta|\lambda)}{\eta} \\
&= \frac{\int_0^{\lambda+1} [F(-\theta) - F(-\theta-\eta) + F(-\lambda-\theta-2\eta) - F(-\lambda-\theta)] d\tilde{F}(\theta|\lambda)}{\eta} + \frac{\int_{-\eta}^{0} [F(\lambda-\theta) - F(-\theta-\eta) + F(-\lambda-\theta-2\eta)] d\tilde{F}(\theta|\lambda)}{\eta} \\
&\to \int_0^{\lambda+1} [f(-\theta) - 2f(-\lambda-\theta)] d\tilde{F}(\theta|\lambda) + [F(\lambda) - F(0) + F(-\lambda)] f(\lambda) \quad \text{as} \quad \eta \to 0,
\end{aligned}$$

where the first equality is due to a change of variable $\theta' = \theta - \eta$. Differentiating the second integral analogously, we get

$$\frac{\int_{\lambda+\eta-1}^{0}\Xi(\lambda+\eta,\theta)d\tilde{F}(\theta|\lambda+\eta)-\int_{\lambda-1}^{0}\Xi(\lambda,\theta)d\tilde{F}(\theta|\lambda)}{\eta}$$

$$=\frac{\int_{\lambda-1}^{0}[F(-\theta)-F(-\theta-\eta)+F(-\lambda-\theta-2\eta)-F(-\lambda-\theta)]d\tilde{F}(\theta|\lambda)}{\eta}-\frac{\int_{-\eta}^{0}[F(\lambda-\theta)-F(-\theta-\eta)+F(-\lambda-\theta-2\eta)]d\tilde{F}(\theta|\lambda)}{\eta}$$

$$\to\int_{\lambda-1}^{0}[f(-\theta)-2f(-\lambda-\theta)]d\tilde{F}(\theta|\lambda)\;-\;[F(\lambda)-F(0)+F(-\lambda)]f(\lambda)\quad\text{as}\quad\eta\to 0.$$

Subtracting the second from the first, given symmetry of f around 0,

$$\begin{aligned}\frac{d\Delta^*(\lambda)}{d\lambda}&=\int_0^{\lambda+1}[f(-\theta)-2f(-\lambda-\theta)]d\tilde{F}(\theta|\lambda)-\int_{\lambda-1}^{0}[f(-\theta)-2f(-\lambda-\theta)]d\tilde{F}(\theta|\lambda)+2[1-F(0)]f(\lambda)\\&=\Big[\int_0^{\lambda+1}f(\theta)d\tilde{F}(\theta|\lambda)-\int_{\lambda-1}^{0}f(\theta)d\tilde{F}(\theta|\lambda)\Big]+2\Big[\int_{\lambda-1}^{0}f(\lambda+\theta)d\tilde{F}(\theta|\lambda)-\int_0^{\lambda+1}f(\lambda+\theta)d\tilde{F}(\theta|\lambda)\Big]+f(\lambda)\\&>f(\lambda)\end{aligned}$$

where the inequality ensues because the first bracketed differential of two integrals of the preceding expression is positive and the second is zero for $\lambda\in[0,1)$.

(b) Suppose $g_i^{\alpha}(x^+)+g_i^{\beta}(x^+)\leq -g_i^{\alpha}(x^-)-g_i^{\beta}(x^-)$. Increase $g_i^{\beta}(x^-)$ to $\widetilde{g}_i^{\beta}(x^-)$ so that $g_i^{\alpha}(x^+)+g_i^{\beta}(x^+)=-g_i^{\alpha}(x^-)-\widetilde{g}_i^{\beta}(x^-)$. By part (a) of Corollary 2, if

$$-g_i^{\alpha}(x^+)\leq\widetilde{g}_i^{\beta}(x^-)<\frac{-3g_i^{\beta}(x^+)-g_i^{\alpha}(x^+)}{2}\quad\text{or}\quad\widetilde{g}_i^{\beta}(x^-)\leq -g_i^{\alpha}(x^+)<-3g_i^{\beta}(x^+),\qquad(2.20)$$

then (2.13) holds and thus, α is iteratively dominant in I.

By Proposition 5, (2.13) continues to hold and α is iteratively dominant in I when a lower $g_i^{\beta}(x^-)$ replaces $\widetilde{g}_i^{\beta}(x^-)$. The condition in (b) is (2.20) when a lower $g_i^{\beta}(x^-)$ replaces $\widetilde{g}_i^{\beta}(x^-)$ because $-g_i^{\alpha}(x^+)<\frac{-3g_i^{\beta}(x^+)-g_i^{\alpha}(x^+)}{2}\Leftrightarrow\frac{-3g_i^{\beta}(x^+)-g_i^{\alpha}(x^+)}{2}<-3g_i^{\beta}(x^+)$. This establishes part (b). ∎

Proof of Proposition 6:

In a global game considered in the proposition, assume $g_1^{\alpha}(x^-)+g_1^{\beta}(x^-)<0<g_2^{\alpha}(x^-)+g_2^{\beta}(x^-)$. The condition (2.15) implies $[x_1^0=x+\rho_1(\underline{\lambda})\sigma,x+\rho_1(0)\sigma]\subset A_i^{\sigma,1}$, and (2.16) for $\lambda=0$ implies $x_2^0=x+\rho_2(0)\sigma\in A_2^{\sigma,2}$. Let $\widehat{x}_i^{\sigma,n}>x_i^0$ be the upper boundary of the largest interval in $A_i^{\sigma,n}$ containing x_i^0. Denote $\widehat{x}_i^{\sigma,n}=x+\lambda_i^n\sigma$. Clearly, $\lambda_i^n\in(0,1]$ increases in n. We show that (i) $\lambda_i^{\infty}=\lim_{n\to\infty}\lambda_i^n>1$ and then (ii) $\widehat{x}_i^{\sigma,\infty}=x+\lambda_i^{\infty}\sigma\geq\widehat{X}$ for both i for any $\widehat{X}\in(x+\sigma,\sup I)$ if σ is small enough, which will establish iterative dominance of α in I.

(i) With a view to reaching a contradiction, suppose $\lambda_i^{\infty}\leq 1<\Lambda$, so that $\rho_i^{-1}(\lambda_i^{\infty})<1$, for some i for arbitrarily small σ, and $\rho_i^{-1}(\lambda_i^{\infty})\leq\rho_{-i}^{-1}(\min\{\lambda_{-i}^{\infty},\Lambda\})$. If $\rho_i^{-1}(\lambda_i^{\infty})<\rho_{-i}^{-1}(\min\{\lambda_{-i}^{\infty},\Lambda\})$, in round $n+1$ large enough so that $\rho_i^{-1}(\lambda_i^{n})\leq\rho_i^{-1}(\lambda_i^{\infty})<\rho_{-i}^{-1}(\min\{\lambda_{-i}^{n},\Lambda\})$, we have $V_i(s_{-i}|x+\rho_i(\lambda)\sigma)\geq V_i(\breve{s}_{-i}^{\rho_{-i}(\lambda)}|x+\rho_i(\lambda)\sigma)$ if $i=1$ and $V_i(s_{-i}|x+\rho_i(\lambda)\sigma)\geq V_i(s_{-i}^{(\underline{\lambda},\lambda)}|x+\rho_i(\lambda)\sigma)$ if $i=2$ for every $\lambda\in[\rho_i^{-1}(\lambda_i^n),\rho_i^{-1}(\lambda_i^{\infty})]$ and every admissible

s_{-i} in round $n+1$ by Lemma 3, because player $-i$ plays α_{-i} on $A^{\sigma,n}_{-i} \supset [x^0_{-i}, \widehat{x}^{\sigma,n}_{-i}) \supset [x^0_{-i}, x + \rho_{-i}(\lambda)\sigma)$, given that $x + \rho_{-i}(\rho_i^{-1}(\lambda_i^\infty))\sigma < x + \rho_{-i}(\rho_{-i}^{-1}(\min\{\lambda^n_{-i}, \Lambda\}))\sigma = x + \min\{\lambda^n_{-i}, \Lambda\}\sigma \le x + \lambda^n_{-i}\sigma = \widehat{x}^{\sigma,n}_{-i}$. Therefore, we would have $V_i(s_{-i}|x + \rho_i^{-1}(\lambda_i^\infty)\sigma) > 0$ for every admissible s_{-i} for small enough σ by (2.14) or (2.16), so that $\widehat{x}^{\sigma,\infty}_i = x + \rho_i(\rho_i^{-1}(\lambda_i^\infty))\sigma \in A^{\sigma,n+1}_i$, a contradiction.

It remains to consider the case that $\lambda_i^\infty \le 1 < \Lambda$ and $\rho_i^{-1}(\lambda_i^\infty) = \rho_{-i}^{-1}(\lambda_{-i}^\infty) < 1$. In this case, the minimal value of $V_i(s_{-i}|x + \lambda_i^\infty\sigma)$ in each round n obtainswhen s_{-i} assigns 1 only to $x_{-i} \in (-\infty, x + T_{-i}(\lambda_i^\infty)\sigma) \cup A^{\sigma,n-1}_{-i}$ if $i = 1$ or only to $x_{-i} \in A^{\sigma,n-1}_{-i}$ if $i = 2$ by Lemma 3. By continuity of $V_i(\cdot|x + \lambda_i^\infty\sigma)$, this minimal value converges to a value no lower than $\widetilde{V}_i(\breve{s}^{\lambda_i^\infty}_{-i}|x + \lambda_i^\infty\sigma)$ or $\widetilde{V}_i(\breve{s}^{(\underline{\lambda},\lambda_i^\infty)}_{-i}|x + \lambda_i^\infty\sigma)$ as $n \to \infty$ because $\widehat{x}^{\sigma,n}_i \to \widehat{x}^{\sigma,\infty}_i$. Since $\widetilde{V}_i(\breve{s}^{\lambda_i^\infty}_{-i}|x + \lambda_i^\infty\sigma) > 0$ by (2.14) and $\widetilde{V}_i(\breve{s}^{(\underline{\lambda},\lambda_i^\infty)}_{-i}|x + \lambda_i^\infty\sigma) > 0$ by (2.16), we would have $\widehat{x}^{\sigma,\infty}_i = x + \lambda_i^\infty\sigma \in A^{\sigma,n}_i$ for some n if σ is small enough, a contradiction. This establishes that $\lambda_i^\infty > 1$ for $i = 1, 2$.

(ii) The proof of part (ii) of Proposition 5 also applies here, establishing iterative dominance of α in I.

The last part of the proposition can be proved in the same way as the last part of the Proposition 5 is proved, hence the details are omitted. This completes the proof. ∎

Chapter 3

Controlling Inflation with Central Bank Communication

Statement of co-authorship: *This chapter is co-authored with Nikolaos Kokonas and Michael Rousakis. All co-authors contributed equally to the chapter.*

3.1 Introduction

The recent global rise in inflation has sparked an interesting debate among academics and policymakers regarding its causes and the extent of monetary policy tightening needed to control inflation. One possible explanation of why central banks delayed in their reaction to prevent the burst in inflation was the belief that inflation expectations were firmly anchored and rises of inflation would be temporary. However, recent data suggests that expectations were not so well-anchored, with a rising share of households expecting that inflation to be higher in the future (Reis, 2022).

The contribution of the paper is to show that central bank announcements regarding its forecasts about likely movements of future macroeconomic variables, in our set-up, inflation and output, what has been dubbed Delphic Guidance in the literature (Campbell et al., 2012), convey information to market participants about the state of the economy that is essential to control the stochastic path of inflation. In that respect, we treat the problem of the determination of the stochastic path of inflation as an information, signal extraction problem. Crucially, we show that reliance solely on conventional inflation targeting policies to control inflation, without explicit guidance about the likely future path of the economy, allows the heterogeneous expectations of households and firms about the future state of the economy to affect arbitrarily the realised path of inflation and, as a result, the central bank loses control of its main target.

We consider an infinite horizon, cashless economy populated by a representative household which consists of a consumer/worker and a producer, and the central bank. The state of the economy is described by productivity, which consists of a permanent, AR(1) component and is subject to zero-mean iid shocks. The model's only source of inefficiency is the asymmetry of information about productivity between agents. At the beginning of each period only the consumer learns productivity. The producer has to form expectations about productivity and maximise expected profits, and they only learn productivity after production decisions have been taken. The central bank steps in after production has taken place and sets its policy according to an expected inflation targeting rule. Furthermore, the central bank might receive noisy signals about future productivity which it communicates to the agents before production takes place, without observing current productivity at the time of the announcement, thus, respecting the informational restrictions of the producer. All agents are Bayesian, and the equilibrium concept is rational expectations.

The log-linear equilibrium of the economy reduces to the standard IS block and a Lucas-type Philips curve (PC), where output increases (decreases) if the producer's estimate about the current price level exceeds (falls below) the realised level of prices. We focus on linear rational expectations equilibria, where inflation and output conjectures are functions of realised productivity and agents' expectations about the permanent component of productivity, which, in turn, is the best estimate about future productivity.

In our environment, conventional monetary policy can only pin down, if at all, expected inflation from the IS block (stochastic Fisher equation), allowing a continuum of inflation paths consistent with equilibrium. In turn, output, being a function of inflation from the Philips curve, is left indeterminate as well. Equivalently, as realised inflation depends on current productivity and estimates about future productivity, conventional policy can only determine their joint impact (sum of coefficients) on realised inflation, leaving the coefficients on agents' estimates about future productivity free and, as a result, the stochastic path of inflation arbitrary. The inability of the central bank to control inflation leads to suboptimal, belief-driven fluctuations.

Central bank communication, in the form of forecasts about expected inflation or output, is powerful because it reveals the central banks' noisy information about the future state of the economy, shaping agents' expectations about future productivity and, through updating of beliefs, renders these expectations irrelevant, thus determining the stochastic path of inflation.

The argument reduces to counting equations and unknowns. Conventional policy, without explicit guidance – either because the central bank does not possess noisy information about the future state of the economy or because it withholds its private information intentionally –, cannot pin down the coefficients on the agents' estimates about future productivity that affect realised inflation and output. Explicit guidance through announcements of forecasts about expected inflation or output allow agents to back out the noisy signals, and through Bayesian updating, adds non-trivial restrictions equal to the number of free coefficients. In turn, consistency between equilibrium conjectures for inflation and output and the IS and PC block, require that the coefficients on the agents' expectations about future productivity that, in turn, affect realised inflation and output become zero, thus, rendering these expectations irrelevant.

We extend the previous argument to an economy with productivity as well as demand shocks, and show that a more comprehensive communication is called for, namely, the central bank must announce forecasts about expected inflation as well as expected output to control the stochastic path of inflation.

The empirical literature has demonstrated that central banks have additional information about inflation beyond what is known to market participants, and that policy actions can modify market participants' forecasts (Romer and Romer, 2000).[1] Our theoretical results uncover the fundamental role that central bank forecasts play in modifying agents' forecasts about the likely path of the economy, and thus, adding additional restrictions to the equilibrium set that pins down a unique path for inflation and output.

It should be stressed that the mechanism to control inflation in our paper is different from the corresponding mechanism in the canonical New Keynesian framework. Specifically, our economy features a continuum of bounded inflation paths in the absence of explicit guidance, thus, the need of a Taylor-type principle argument to select the unique bounded solution out of a continuum of unbounded solutions, as in the canonical New Keynesian model, is not relevant here. In fact, Castillo-Martinez and Reis (2019), in a review of the relevant literature, make clear the distinction between models where arbitrage and interest rate setting by the central bank does not suffice to pin down the stochastic path of inflation, namely, models that rely on the stochastic Fisher equation to pin down inflation (which is our case), and models where the Taylor-type principle, coupled with a terminal condition on inflation selects the unique bounded solution out of a continuum of explosive solutions (which is the mechanism to control inflation in the canonical New Keynesian framework).

The rest of the paper is organized as follows. Section 2 discusses the related literature, Section 3 presents the model, Section 4 presents equilibrium under various announcements from the central bank. Section 5 discusses extensions and section 6 concludes.

3.1.1 Related Literature

Our paper is related to the literature that studies the ability of a central bank to control inflation, under different specifications of monetary policy. A non-extensive list of contributions includes Sargent and Wallace (1975), McCallum (1981), Woodford (1994), Clarida et al. (2000), Cochrane (2011), Cantoni et al. (2019), Castillo-Martinez and Reis (2019), Angeletos and Lian (2021), amongst others. It is worth emphasising that our

[1] Recent empirical work, see, for example, Coenen et al. (2017) and Jain and Sutherland (2020), argues that state-contingent forward guidance and central bank projections provide additional information to market participants and manage to modify their forecasts.

indeterminacy results do not derive from the stability of the steady state or even the infinity of the horizon.[2] In contrasts, our work is closest to Nakajima and Polemarchakis (2005), who showed that in finite or infinite horizon stochastic monetary economies, and under "Ricardian" fiscal policy, interest rate or money supply rules can only pin down an average value of inflation, leaving its distribution across states of the world indeterminate. In that context, Adão et al. (2014) and Magill and Quinzii (2014) showed that fixing the term structure of interest rates determines the path of inflation. We expand this line of literature by treating the determination of the stochastic path of inflation as an information, signal-extraction problem, and show that conventional inflation targeting policies, supplemented with central bank forecasts about the likely future movements of macroeconomic variables, suffice to control the path of inflation.

Our paper is also related to the important literature that studies the social value of public information. The seminal work of Morris and Shin (2002b, 2005) argued that the welfare effects of increased public information is ambiguous, since the release of precise public signals may be welfare impairing in an environment that features strategic complementarities between agents' actions. Woodford (2005) and Morris et al. (2006) questioned the previous anti-transparency result, while Hellwig (2005) argued that the welfare effects may be improving due to reduced price dispersion. Furthermore, Angeletos and Pavan (2007) argued that the welfare consequences of public information depend, crucially, on the nature of strategic interactions between agents. Our approach abstracts from strategic interactions and the resulting externality that arises from individuals trying to second-guess the actions of others. Public information in our set up convey information to agents about the future state of the world, that, in turn, perturbs agents' expectations away from prior beliefs, and adds equilibrium restrictions needed to pin down the path of inflation and output. Our framework offers an example where public information has social value since it eliminates the possible emergence of suboptimal, belief-driven fluctuations, but remains silent about how precise public signals should be. As argued by the literature, the latter point would require careful consideration of agents' interactions.

Bassetto (2019) studies a cheap talk game between a central bank and agents when their incentives are not aligned and finds that indeed information transmission is possible. We take this as given in assuming that information transmission is possible and instead focus on the effect that such communication has in controlling inflation.

Lastly, the theoretical model in the next section is related with the literature on dispersed information and shares elements with Angeletos and La'o (2010), Lorenzoni (2009) and Lorenzoni (2010). Even though we use a similar to those papers friction to allow for belief driven fluctuations, in contrast with this work we focus on classes of equilibria that exhibit indeterminacy without any communication from the better informed central bank.

[2]All our results follow intact in a finite horizon version of the model.

3.2 The model

We consider a cashless, competitive, monetary economy. The representative household consists of one producer and one consumer/worker. The consumer/worker, supplies labor to a representative firm they own but is managed by the producer. The firm produces a single non-storable commodity. There exists a short term nominal bond market with the bond price set by the monetary authority according to an expected inflation targeting rule. Time is discrete and infinite with each period denoted by $t = 0, 1, \dots$. Consumer's preferences are given by:

$$U(C_t, N_t) = E_t\big[\sum_{t=0}^{\infty} \beta^t (logC_t - \frac{N_t^{1+\zeta}}{1+\zeta})\big], \tag{3.1}$$

where C_t denotes consumption and N_t denotes employment at period t. The constant $\zeta > 0$ is the inverse of Frisch elasticity of labor supply and $\beta \in (0, 1)$ is a discount factor. The consumer faces a sequence of budget constraints:

$$P_tC_t + Q_tB_{t+1} = B_t + W_tN_t + \Psi_t, \tag{3.2}$$

where P_t denotes commodity prices, B_{t+1} denotes holdings of nominal bonds purchased at period t and maturing at $t + 1$, Q_t denotes the nominal bond price, W_t denotes the nominal wage and Ψ_t denotes the firm's profits. The firm's technology and profits are given respectively by:

$$Y_t = A_tN_t \tag{3.3}$$

$$\Psi_t = P_tY_t - W_tN_t, \tag{3.4}$$

where A_t denotes productivity. The Central bank targets future inflation and sets the nominal bond price according to an interest-rate rule:

$$Q_t = \beta E_t^{cb}\big[\Pi_{t+1}\big]^{-\phi_\pi}\big(\frac{Y_t}{Y_t^*}\big)^{-\phi_y}, \tag{3.5}$$

where Π_{t+1} denotes inflation in period $t + 1$, Y_t^* denotes the natural level of output and the ratio between the natural and current level of output is the output gap. The monetary policy parameter ϕ_π is and ϕ_y are assumed to be non-negative. Without loss for the rest of the text we will assume that $\phi_y = 0$ since whether the central bank targets output gap or not is inessential to our analysis.

Shocks and Signals: The economy is characterised by asymmetric information. Specifically, only the consumer can observe their productivity, A_t. Let $a_t = logA_t$ and define similarly any lowercase variable henceforth. Productivity consists of a permanent component x_t and a temporary component ϵ_t:

$$a_t = logA_t = x_t + \epsilon_t \quad \text{with} \quad \epsilon_t \sim N(0, \sigma_\epsilon^2)$$

The consumer/worker observes a_t but not its decomposition. Aggregate productivity follows a random walk:

$$x_t = \rho x_{t-1} + e_t \quad \text{with} \quad e_t \sim N(0, \sigma_e^2), \quad \rho \in (0, 1]$$

The Central Bank at the beginning of each period receives information about productivity up to τ periods ahead, in the form of noisy observations about future productivity shocks[3] at each period τ in the form of a signal:

$$s_{t+\tau} = \epsilon_{t+\tau} + u_{t+\tau} \quad \text{with} \quad u_{t+\tau} \sim N(0, \sigma_u^2).$$

The terms u_t, e_t, ϵ_t are mutually independent and serially uncorrelated noise.

Timing: Each period is divided in two stages. In stage 1, the central bank makes its announcements which we discuss in the next section. The consumer, but not the producer, observes productivity a_t. Production takes place, producers maximise expected profits, forming expectations about current, unobserved productivity a_t, and households choose labour supply optimally. Crucially, the nominal wage is independent from the consumer's labour supply due to the linearity of technology, thus the producer cannot infer current productivity by observing the current wage. Implicitly it is assumed that the producer and the consumer are physically separated at the beginning of each period. This allows us to abstract from the "Lucas-Phelps" islands framework and focus on a representative agent model instead.[4]

In stage 2, the Central Bank steps in and sets interest rates according to the targeting rule (5) and commodity and bonds markets open.[5] If the central bank acted at the beginning of each period, its actions would be rendered neutral. Consumers choose consumption and bond holdings optimally, taking prices as given, and prices adjust to clear commodity and bond markets. At the beginning of stage 2, the Central Bank and the producer can infer current productivity a_t since production has already taken place. However, no agent in the economy observes permanent productivity x_t. The timing is summarized at the graph bellow.

(t,I):
Central bank announces $s_{t+\tau}$
Consumers observe a_t
Production takes place.

(t,II):
Central bank sets Q_t.
Commodity and bond markets clear.

t

[3]One could allow the central bank to announce observations about the actual value of future productivity ($a_{t+\tau}$) but this would complicate agents' learning problem without adding any insight.

[4]Alternatively, one could introduce idiosyncratic shocks in labour supply such as preference shocks in order for labour supply not to reveal productivity.

[5]This timing is in line with the literature as seen in Lorenzoni (2010)

3.3 Equilibrium

3.3.1 Information and Beliefs

In order to define equilibrium, we first define agents' information sets. Let the state of the economy at time t be $\Omega_t = \left\{(a_t)_{t=0}^t, (x_t)_{t=0}^t, (s_{t+\tau})_{\tau=1}^{\tau}\right\}$. Let the information of agent i, with $i \in \{cb, p, c\}$ denoting respectively the central bank, producer and consumer, be denoted by $I_{t,s}^i$ with $s \in \{1, 2\}$ denoting the stage within a given period t.

Let $\{\mathcal{A}_t\}$ denote the Central Bank's announcement, allowing this set to be empty when we study the case where the central bank transmits no information. We assume that "no announcements" do not have any informational value and agents do not update their beliefs upon hearing such message. This in turn implies that we can focus only in the case where the central bank has some information and transmits it and the case where the central bank does not possess any information and its announcement is empty. In our setup any private information that the central bank has will not affect equilibrium thus the second case where the central bank has no information and makes no announcements is equivalent with the case where the central bank withholds its information[6] thus we use this as the benchmark case of where the central bank offers no announcements. Lastly, in our setup the central bank can credibly convey its information to the agents. We take our cue from Bassetto (2019) who shows that in a cheap talk game where the incentives of a central bank are misaligned with those of the agents, information transmission from a superior informed central bank is possible. We discuss how informativeness of the announcement affects our setup in section 4.

The information sets of the agents are given by:

$$
\begin{aligned}
I_{t,1}^{cb} &= \left\{(a_t)_{t=0}^{t-1}, \mathcal{A}_t)_{t=0}^t\right\} \\
I_{t,2}^{cb} &= \left\{(a_t)_{t=0}^{t}, \mathcal{A}_t)_{t=0}^t\right\} \\
I_{t,1}^{c} &= \left\{(a_t)_{t=0}^{t} \cup (\mathcal{A}_t)_{t=0}^t\right\} \\
I_{t,1}^{p} &= \left\{(a_t)_{t=0}^{t-1} \cup (\mathcal{A}_t)_{t=0}^t\right\} \\
I_{t,2}^{p} &= I_{t,1}^{c}
\end{aligned}
$$

With slight abuse of notation we denote the expectation of agent q at period t and stage s, conditional on their information $I_{t,s}^i$ with $E_{t,s}^i[.] = E_{t,s}^i[.|I_{t,s}^i]$. Agents' expectations are formed as follow:

Each day the consumer observe their productivity and they update their belief about the permanent component of productivity:

$$E_t^c[x_t] = (1-\mu)E_{t-1}^c[x_t] + \mu a_t.$$

[6]For an example where the central bank has private information see Appendix 6.4

Where μ is the Kalman gain which measures the relative weight that agents place between past and new information in order to estimate the permanent component of productivity which we derive in the Appendix. The producer observes productivity only after production has taken place:

$$E^p_{t,1}[a_t] = E^p_{t,1}[x_t] = \rho E^p_{t,1}[x_{t-1}]$$
$$E^p_{t,2}[x_t] = E^c_t[x_t] = (1-\mu)E^c_{t-1}[x_t] + \mu a_t$$

Lastly notice that the producer can make inferences about the consumer's beliefs:

$$E^p_{t,1}\left[E^c_{t,1}[x_t]\right] = (1-\mu)E^c_{t-1}[x_t] + \mu E^p_{t,1}[a_t] = E^p_{t,1}[x_t].$$

Importantly without any communication, no agent can distinguish between the permanent and temporary component one period ahead, regardless of the stage within date t.

$$E^c_t[x_{t+1}] = E^c_t[a_{t+1}] = \rho E^c_t[x_t],$$
$$E^p_t[x_{t+1}] = E^p_t[a_{t+1}] = \rho E^p_t[x_t],$$

The last equations in both agents' learning problem do not hold in the case where the central bank makes announcements about future productivity shocks of the economy. Since agents are Bayesian, they will update their beliefs about future realized productivity ($a_{t+\tau}$), while their beliefs about the permanent component of productivity will remain unchanged. Their expectation about realized productivity in period τ in the case that $((\mathcal{A}_t)^t_{t=0} = \{s_{t+1}, s_{t+2}, ...\})$ is given by:

$$E^i_t[a_{t+\tau}] = \frac{(\sigma^2_x + \sigma^2_\epsilon)^{-1}E^i_t[x_{t+\tau}] + (\sigma^2_u)^{-1}s_{t+\tau}}{(\sigma^2_x + \sigma^2_\epsilon)^{-1} + (\sigma^2_u)^{-1}}$$

Lastly, note that in the case of transparency, the producer's estimate about the beliefs of the consumer is different than before since $E^p_t[a_t] \neq E^p_t[x_t]$:

$$E^p_{t,1}[E^c_t[x_t]] = E^p_{t,1}[(1-\mu)E^c_{t-1}[x_t] + \mu a_t] = (1-\mu)E^c_{t-1}[x_t] + \mu E^p_{t,1}[a_t] \neq E^p_{t,1}[x_t].$$

3.3.2 Equilibrium and Optimality Conditions

Given agents' learning problems and different information sets we proceed to describe equilibrium.

Definition 2 *A rational expectations equilibrium under an interest rate rule $Q(I^{cb}_{t,2})$, consists of prices $\left\{P(I^{cb}_{t,2}), W_t(I^p_{t,1}), Q(I^{cb}_{t,2})\right\}^\infty_{t=0}$, an allocation for the producer $\left\{N^d_t(I^p_{t,1}), Y_t(\Omega_t)\right\}^\infty_{t=0}$ and an allocation for the consumer $\left\{C_t(I^c_{t,2}), N^s_t(I^c_{t,1}), B_{t+1}(I^c_{t,2})\right\}^\infty_{t=0}$ such that:*

1. *Allocations solve the agents' problems at the stated prices*

2. *Markets clear:* $Y_t = C_t$, $N_t^d = N_t^s$, $B_{t+1} = 0$ *for all* t *and* $B_0 = 0$.

In order to characterise equilibria, we start by deriving agents' optimality conditions. The consumer maximises their dicounted expected utility (1) subject to a series of budget constraints (2) and the no-Ponzi scheme constraint $B_{t+1} > -\Gamma$ for any $\Gamma \neq 0$, since $B_t = 0$ at equilibrium. From consumer's optimisation problem one acquires:

$$N_t^{\zeta} = \frac{W_t}{P_t C_t} \tag{3.6}$$

$$Q_t = \beta E_{t,1}^c \Big[\frac{1}{\Pi_{t+1}} \frac{C_t}{C_{t+1}}\Big] \tag{3.7}$$

Equation (3.6) is the intratemporal optimality condition that equates the real wage, in terms of consumption, to the disutility of supplying one additional unit of labour. Equation (3.7) is the intertemporal Euler equation.

The producer chooses labour input to maximise expected profits, $E_t^p[\lambda_t \Psi_t]$, using their expectation of household's marginal utility of wealth, $\lambda_t = (P_t C_t)^{-1}$, as the appropriate discount rate. More formally, we obtain:

$$W_t = \frac{E_{t,1}^p[\lambda_t P_t A_t]}{E_{t,1}^p[\lambda_t]}. \tag{3.8}$$

Due to linearity, the firm accommodates any labour supplied at the given wage as long as expected profits are zero (realised profits are typically not zero since the real wage is not equal to productivity). Notice that (3.8) in combination with (3.6) implies that there exists a wedge between the marginal product of labour and the marginal rate of substitution between consumption and leisure. This in turn means that belief based fluctuations imply inefficient output fluctuations.

We will focus our analysis on linear rational expectations equilibria. Doing so considerably simplifies the analysis and enables the use of Kalman filters for the agents learning problem. To this end we write the agents problems (3.6) – (3.8) alongside with (3.5) in log-linear form[7] as follows:

$$\zeta n_t = w_t - p_t - c_t, \tag{3.9}$$

$$q_t = log\beta + c_t - E_{t,1}^c[\pi_{t+1} + c_{t+1}] + const., \tag{3.10}$$

$$w_t = E_{t,1}^p[a_t] + E_{t,1}^p[p_t] + const.', \tag{3.11}$$

[7]Given the conjectures about inflation and output we consider, we confirm in the Appendix that Π_t and C_t are indeed log-normally distributed.

$$q_t = log\beta - \phi_\pi E^{cb}_{t,2}[\pi_{t+1}] - \phi_y(y_t - y^*_t) + const.'', \tag{3.12}$$

where $const.$ denotes constants. Technology and market clearing conditions can be written:

$$y_t = n_t + a_t \tag{3.13}$$

$$y_t = c_t \tag{3.14}$$

Notice that the producer and the consumer both take decisions, given their information set, at stage 1, thus we are only interested in their beliefs at that stage. We denote with $E^p_t[.] = E^p_{t,1}[.]$ and similarly for the consumer, unless stated otherwise. Substitute (3.9) into (3.11) and subtract and add p_{t-1} to acquire:

$$(1+\zeta)y_t = E^p_t[a_t] + E^p_t[\pi_t] - \pi_t + \zeta a_t. \tag{3.15}$$

Lastly, the information set of the central bank coincides with the information set of the consumer at the time the central bank steps in $I^{cb}_{t,2} = I^c_{t,1}$, thus $E^{cb}_t[.] = E^c_t[.]$. Combining (3.10) with (3.12) one acquires:

$$E^c_t[y_{t+1}] - y_t = (\phi_\pi - 1)E^c_t[\pi_{t+1}] + \phi_y(y_t - y^*_t). \tag{3.16}$$

Equation (3.15) is a Lucas-type Philips curve, where output increases above trend whenever producer's beliefs about inflation exceeds realised inflation at the time production decisions take place; and equation (3.16) is the standard IS equation combined with the central bank targeting rule. These two equations characterize the log-linear equilibrium in our environment. To proceed, we will conjecture linear solutions for inflation and output as functions of shocks and use the method of undetermined coefficient to show that our conjectures are consistent with (3.15) and (3.16) and, as a result, represent a competitive equilibrium for this economy.

Firstly, we discuss all our key results in the benchmark environment of symmetric information, where classical dichotomy applies and monetary policy has no real effects; and subsequently, we proceed to characterise the case of asymmetric information, where communication allows the central bank to stabilise the economy from inefficient, self-fulfilling fluctuations.

3.3.3 Symmetric Information

In order to highlight the type of indeterminacy we have in mind, we begin our analysis by considering an economy with symmetric information and assume that the central bank transmits no information about future shocks. In this case the consumer, the producer and the central bank observe a_t thus we drop the superscripts from agents' expectations. Note that even though information is symmetric, no agent observes the permanent component of productivity x_t. In the absence of asymmetric information $y_t = y_t^* = a_t$, and nominal variables do not affect the real side of the economy. Moreover, since there is no difference in information, $E_t^p[\pi_t] = \pi_t$, thus the equilibrium conditions are:

$$\begin{aligned} y_t &= a_t \\ E_t[y_{t+1}] - y_t &= (\phi_\pi - 1)E_t[\pi_{t+1}]. \end{aligned} \tag{3.17}$$

Importantly, notice that from the Fisher equation $r_t = q_t - E_t[\pi_{t+1}]$ the central bank by choosing q_t can determine only expected inflation. The actual realizations of inflation can take arbitrary values depending on the realizations of uncertainty. To enable this indeterminacy, we allow inflation to depend arbitrarily in both shocks in permanent and temporary productivity. We consider linear equilibria of the following form:

$$\pi_{t+1} = \xi_1 a_t + \xi_2 a_{t+1} + \Xi_2 E_{t+1}[x_{t+1}]. \tag{3.18}$$

That is, inflation depends on the past productivity, a_t, current realisations a_{t+1} and on agents' expectation about permanent productivity $E_{t+1}[x_{t+1}]$. This summarizes all the information agents have for the economy. Taking expectations of (3.18) as of time t one acquires:

$$E_t[\pi_{t+1}] = \xi_1 a_t + (\xi_2 + \Xi_2)E_t[x_{t+1}]. \tag{3.19}$$

The last expression uses the fact that since $E_t[\epsilon_{t+1}] = 0$, agents' estimate about productivity next period coincides with their estimate of its permanent component, $E_t[a_{t+1}] = E_t[x_{t+1}]$. The key observation here is that from the perspective of period t agents cannot distinguish between the permanent and the temporary shocks that will occur one period ahead and agents' beliefs about these shocks will be different once they have realized.

Matching coefficients between (3.19) and (3.17) one acquires $\xi_1 = \dfrac{-1}{\phi_\pi - 1}$, $(\xi_2 + \Xi_2) = \dfrac{1}{\phi_\pi - 1}$. Importantly, $(\xi_2 + \Xi_2)$ can be determined only as a sum. Any combinations of $\xi_2 + \Xi_2$ that satisfies the above equality can be an equilibrium. The particular combination of ξ_2, Ξ_2 is subject to agents' coordination. The equilibrium inflation is given by:

$$\pi_{t+1} = \frac{-1}{\phi_\pi - 1}a_t + \xi_2 a_{t+1} + (\frac{1}{\phi_\pi - 1} - \xi_2)E_{t+1}[x_{t+1}].$$

Where ξ_2 takes arbitrary values. The distribution of inflation across date events is undeterminate since the central bank with its rule cannot uniquely pin down ξ_2. Agents might coordinate their actions on the realisation of extrinsic signals, which might induce excessive inflation volatility that is not attributed to the volatility of fundamentals. Notice that in this simple economy, indeterminacy has no effect on real output. We show that in the case where inflation affects real allocations, this indeterminacy implies that output will be indeterminate as well.

Regardless, this example is enough to hint our key result. Information that shifts agents' expectations away from the priors can break the coincidence of the permanent and temporary component of the economy and restore determinacy. Communication and transparency in our environment act as an extra restriction on the Fisher equation pinning down inflation. The same result we show can be acquired with indirect communication via announcements about the central bank's expectation about future inflation. In stochastic economies that the path of inflation is indeterminate, transparency can be an important tool for a policy maker in order to pin down inflation.

Next we proceed in applying that insight in the case with asymmetric information where indeterminacy affects real allocations, and examine the effect that announcements have in determining the path of inflation.

3.3.4 Asymmetric Information

In the asymmetric information the producer does not observe current productivity and they make decisions based on their expectation about current productivity, $E_t^p[a_t]$. The central bank acts at the end of each period when production has already taken place thus it can infer productivity and shares the same information set with the consumer. Moreover, realized output depends on the difference between realized inflation and the expectations of the producer about realized inflation $E_t^p[\pi_t] - \pi_t$ as can be seen from the Philips curve (3.16). If the central bank with its policy is unable to pin down inflation, output will be indeterminate as well, while fluctuations in inflation will imply suboptimal output fluctuations in the economy.

The equilibrium conditions that describe the environment with asymmetric information are (3.15) alongside with (3.16) in their log-linear form, which we rewrite bellow for convenience:

$$(1+\zeta)y_t = E_t^p[a_t] + E_t^p[\pi_t] - \pi_t + \zeta a_t \tag{15}$$

$$E_t^c[y_{t+1}] - y_t = (\phi_\pi - 1)E_t^c[\pi_{t+1}] + \phi_y(y_t - y_t^*). \tag{16}$$

Without loss of generality, we will assume for the rest of the analysis we set $\phi_y = 0$.

Equilibria without communication

We first characterize the equilibrium without communication. We consider linear equilibria of the following form:

$$y_t = \theta_0 + \theta_1 a_t + \Theta_1 E_t^c[x_t] + \kappa_1 E_t^p[a_t] + K_1 E_t^p[x_t] \tag{3.20}$$

$$\begin{aligned}\pi_{t+1} =& \xi_0 + \xi_1 a_t + \Xi_1 E_t^c[x_t] + \omega_1 E_t^p[a_t] + \Omega_1 E_t^p[x_t] + \\ &+ \xi_2 a_{t+1} + \Xi_2 E_{t+1}^c[x_{t+1}] + \omega_2 E_{t+1}^p[a_{t+1}] + \Omega_2 E_{t+1}^p[x_{t+1}].\end{aligned} \tag{3.21}$$

Here, similarly to the symmetric information, we allow inflation to depend arbitrarily on agents' beliefs about both the permanent and temporary shocks that the economy faces. Thus we index π_{t+1} with the expectation of the consumer at $t+1$ about x_{t+1}, a_{t+1}. Next we turn our attention to (3.20). Notice that from the Philips curve (3.15), current output y_t depends on the wedge between the producer's expectation about inflation and the realized inflation, $E_t^p[\pi_t] - \pi_t$. Moreover, notice that $E_t^p\big[E_t^c[x_t]\big] = E_t^p[x_t]$ and $E_t^p\big[E_t^c[a_t]\big] = E_t^p[a_t]$. Thus we index y_t with both the beliefs of the consumer and the producer about the permanent component of productivity x_t and their beliefs about realized productivity a_t.

Lastly, from the IS curve (3.16), expected inflation in period $t+1$ depends both on realized output y_t and the expectations of the consumer about future output $E_t^c[y_{t+1}]$. For this reason, we add the expectations of the consumer and the producer about the permanent component of productivity alongside realized productivity in period t as well with the expectations of both agents about those variables.

Crucially, $E_{t+1}^c[x_{t+1}] \neq a_{t+1}$ since the term a_{t+1} is observed at period $t+1$ while from the perspective of period t, $E_t^c[x_{t+1}] = E_t^c[a_{t+1}]$. Lastly, notice that on period $t+1$, when they act, the producer shares the information set that the consumer has in period t since they receive information about realized productivity on period t at the end of that period. Thus the consumer in period t is able to calculate the expectations of the producer one period ahead. Having this in mind and taking expectations as of date t of equation (3.21) one acquires:

$$\begin{aligned} E_t^c[\pi_{t+1}] =& \xi_0 + \xi_1 a_t + \Xi_1 E_t^c[x_t] + (\xi_2 + \Xi_2) E_t^c[x_{t+1}] + \\ & \omega_1 E_t^p[a_t] + \Omega_1 E_t^p[x_t] + (\omega_2 + \Omega_2) E_t^p[x_{t+1}]. \end{aligned} \tag{3.22}$$

Substituting (3.20) and (3.22) in (3.15) – (3.16) and matching coefficients, yields the following class of equilibria:

$$y_t = \frac{\zeta - \xi_2}{1+\zeta} a_t + \frac{\xi_2 - \tilde{\xi}}{1+\zeta} E_t^c[x_t] + \frac{1+\xi_2}{1+\zeta} E_t^p[a_t] + \frac{\tilde{\xi} - \xi_2}{1+\zeta} E_t^p[x_t] \tag{3.23}$$

$$\begin{aligned} \pi_{t+1} =& \frac{-\zeta + \xi_2}{(1+\zeta)(\phi_\pi - 1)} a_t + \xi_2 a_{t+1} + \frac{\tilde{\xi} - \xi_2}{(1+\zeta)(\phi_\pi - 1)} E_t^c[x_t] + (\tilde{\xi} - \xi_2) E_{t+1}^c[x_{t+1}] + \\ & \frac{-(1+\tilde{\xi})}{(1+\zeta)(\phi_\pi - 1)} E_t^p[x_t] + \frac{1-\tilde{\xi}}{(1+\zeta)(\phi_\pi - 1)} E_{t+1}^p[x_{t+1}]. \end{aligned} \tag{3.24}$$

Where $\tilde{\xi} = (\xi_2 + \Xi_2) = \dfrac{\zeta}{(1+\zeta)(\phi_\pi - 1) + 1}$ and ξ_2 remains undetermined and can take arbitrary values while the constants ξ_0, θ_0 are suppressed for brevity and are described in the Appendix.

Proposition 7 *In the absence of communication, $\mathcal{A} = \emptyset$, the equilibrium stochastic path of inflation and output is indeterminate and indexed by ξ_2.*

Similarly to the symmetric information, in the absence of further information agents at period t cannot distinguish between changes in the temporary and permanent component of productivity that is $E_t^i[a_{t+1}] = E_t^i[x_{t+1}]$, an equality that does not hold at period $t+1$. The related coefficients $\kappa_1, K_1, \kappa_2, K_2, \theta_2, \Theta_2, \omega_1, \omega_2, \Omega_1, \Omega_2$ in equilibrium depend on the pair ξ_2, Ξ_2 which we can only pin down as a sum $\tilde{\xi} = (\xi_2 + \Xi_2)$. Thus all these coefficients remain undetermined. Lastly, the indeterminacy of the nominal side of the economy makes output indeterminate as well since y_t depends on realised π_t from the Philips curve (3.16).

Indeterminacy here implies that the economy is subject to sunspot equilibria. One can construct an equilibrium in which agents coordinate arbitrarily to some ξ_2 which may be different from the optimal equilibrium from a welfare point of view. The indeterminacy of inflation may lead to suboptimal volatility in the economy since agents can coordinate in equilibria with suboptimal output.

Equilibrium with communication

Next we consider the case in which the central bank communicates $s_{t+1} = \epsilon_{t+1} + u_{t+1}$; that is each period t it makes an announcement about the shock in the temporary component of productivity one period ahead, ϵ_{t+1}. This communication shifts expectations away from the priors allowing for the agents to uniquely pin down the different effect of the permanent and temporary component of the economy with $\xi_2 = \dfrac{\zeta}{(\phi_\pi - 1)(1+\zeta) + 1}$ while $\Xi_2 = 0$, in equilibrium.

Taking expectations as of date t of equation (21), substituting (20) – (21) into (15) – (16) and matching coefficients yields the following class of equilibria:

$$y_t = \frac{(\phi_\pi - 1)\zeta}{(1+\zeta)(\phi_\pi - 1) + 1} a_t + \frac{(1+\zeta)(\phi_\pi - 1) + 1 + \zeta}{(1+\zeta)^2(\phi_\pi - 1) + 1} E_t^p[a_t] \tag{3.25}$$

$$\pi_{t+1} = \frac{\zeta}{(1+\zeta)(\phi_\pi - 1) + 1}(a_{t+1} - a_t) + \frac{(1+\zeta)(\phi_\pi - 1) + \zeta}{(1+\zeta)^2(\phi_\pi - 1)^2 + 1} E_{t+1}^p[(a_{t+1} - a_t)]. \tag{3.26}$$

Proposition 8 *The central bank can uniquely determine the stochastic path of inflation and output by communicating noisy signals about future productivity shocks at each date event, $\mathcal{A}_t = \{s_{t+1}\}$.*

Under communication, the stochastic path of inflation and output is determined. Notice that the equilibrium inflation and allocations do not depend on the permanent component of the economy outside the way it helps estimate productivity. In this equilibrium, agents will disregard shocks in their beliefs about the permanent component of the economy and output will only depend on current productivity and the expectations of the producer about it.

In this simple economy, if the central bank transmits information about the idiosyncratic component of the economy, it is able to determine the stochastic path of inflation. Notice that the result does not depend on the noise of the signal u_{t+1} as long as $\sigma_u^2 < \infty$. As long as this signal is informative, equilibrium is determined. At first glance this may seem to contrast with other literature like Angeletos et al. (2007), Morris and Shin (2002) and others that have argued that transparency in the form of precise public signals, may not be beneficial for the economy since in our environment any information is useful. This is due to our modelling choice to abstract from strategic interactions between the agents which ultimately determine how much transparency (in terms of variance) is beneficial from a welfare point of view. By abstracting from agents' interactions in this manner, we contrast the usefulness of transparency against the case of no transparency at all. This way we are able to highlight the importance of Delphic forward guidance in determining the stochastic path of inflation as we discuss in the next section. One could include such strategic interactions in the model by allowing for non-linear technology such that agents react to other's choice of labour and by introducing some idiosyncratic noise on agents' observation so that incomplete information matters.

Remark : One could consider a different model in which the central bank has information and makes announcements about the permanent component of productivity. In this case, this extra information would have no effect in determining the path of inflation. Announcements about x_{t+1} would shift agents' expectations about both a_{t+1} and x_{t+1} in the same way thus agents would still not be able to distinguish between the permanent and temporary component of the economy $E_t^i[a_{t+1}] = E_t^i[x_{t+1}]$ and hence inflation would still be arbitrarily indexed by ξ_2.

So far we have established that if the central bank offers communication about future realized productivity, a_{t+1}, it can uniquely pin down the stochastic path of inflation. In essence communication in our environment offers an extra constraint for the equilibrium path of inflation, allowing agents to distinguish between the permanent and idiosyncratic component of the economy. Next we consider whether the central Bank can convey the same information through indirect communication, via announcements about its expectation about future inflation $E_{t,1}^{cb}[\pi_{t+1}]$. Notice that the expectations of the central Bank are taken at the first stage during which the central bank has not yet observed a_t. Its information set will be the same as the producer's alongside with the extra signal it receives about future temporary shocks $I_{t,1}^{cb} = I_{t,1}^{p} \cup \{s_t\}$. This is important, since at the time of the announcement, the beliefs of the central bank in the case it had no further information are common knowledge between all agents in the economy. By hearing the announcement agents are able to decipher both that the central bank possesses superior information and back out its signal. Taking expectations of (3.26) at period $(t,1)$ we get the announcement of the central bank about its expectation on future inflation:

$$E^{cb}_{t,1}[\pi_{t+1}] = \Delta E^{cb}_{t,1}[a_{t+1} - a_t], \tag{3.27}$$

where $\Delta = \dfrac{\zeta}{(1+\zeta)(\phi_\pi - 1) + 1} + \dfrac{(1+\zeta)(\phi_\pi - 1) + \zeta}{(1+\zeta)^2(\phi_\pi - 1)^2 + 1}$.

Note that (3.27) is the expectation of the central bank about the determined value of inflation. All agents in the economy understand what the equilibrium value of ξ_2 would be in an economy with information about future productivity a_{t+1}. Upon hearing central bank's information about future inflation, agents are able to understand that the central bank has superior information since if the central bank had no further information $\{s_{t+1}\} = \emptyset$, its expectation about future inflation would be:

$$\tilde{E}^{cb}_{t,1}[\pi_{t+1}] = \frac{1-\zeta}{(\phi_\pi - 1)(1+\zeta)}\tilde{E}^{cb}_{t,1}[x_t] + (\tilde{\xi} + \frac{1-\tilde{\xi}}{(1+\zeta)(\phi_\pi - 1)})\tilde{E}^{cb}_{t,1}[x_{t+1}].$$

Since $\tilde{E}^{cb}_{t,1}[\pi_{t+1}] \neq E^{cb}_{t,1}[\pi_{t+1}]$, agents know that this signal contains information about the future productivity of the economy. Importantly,

$$E^{cb}_{t,1}[a_{t+1} - a_t] = (1-k)E^{cb}_t[x_{t+1}] + ks_{t+\tau} - E^{cb}_{t,1}[a_t]$$

[8] is linear to the central bank's signal s_{t+1}. Since both agents are Bayesian and understand the information structure, upon hearing $E^{cb}_{t,1}[\pi_{t+1}]$ are able to back out the information that the Central bank has about a_{t+1}. Then they update their beliefs about the permanent and idiosyncratic component of the economy and indeed they are able to determine the equilibrium value of ξ_2. By communicating its expectation about future inflation the central bank effectively communicates the extra information it has and is able to pin down the stochastic path of inflation. The same communication can be achieved by communicating information about expectations on future productivity. In particular, if the central bank announces:

$$E^{cb}_{t,1}[y_{t+1}] = \Omega E^{cb}_{t,1}[a_{t+1}], \tag{3.28}$$

where $\Omega = \dfrac{(\phi_\pi - 1)\zeta}{(1+\zeta)(\phi_\pi - 1) + 1} + \dfrac{(1+\zeta)(\phi_\pi - 1) + 1 + \zeta}{(1+\zeta)^2(\phi_\pi - 1) + 1}$.

Proposition 9 *The central bank can uniquely determine the stochastic path of inflation and output by announcing its expectation about future inflation $E^{cb}_{t,1}[\pi_{t+1}]$ or future productivity $E^{cb}_{t,1}[y_{t+1}]$.*

Proof: *Follows from the discussion above.*

[8] Where k is a collection of constants (see Appendix)

So far, we have constrained agents to include in their solution beliefs only about the relevant variables up to one period ahead. For this reason the degree of indeterminacy was one, since equilibrium depended arbitrarily only on the variable ξ_2. One could construct equilibria that depend on beliefs an arbitrary amount of periods ahead. In this case, the degree of indeterminacy would be equal to the number of realizations of uncertainty included. To demonstrate how communication would work in such general environment, start by considering an economy where inflation and output is indexed with both permanent and temporary shocks up to two periods ahead. To this end consider the conjectures:

$$\begin{aligned} y_t =& \theta_0 + \theta_1 a_t + \Theta_1 E_t^c[x_t] + \theta_2 E_t^c[a_{t+1}] + \Theta_2 E_t^c[x_{t+1}] + \kappa_1 E_t^p[a_t] + K_1 E_t^p[x_t] \\ &+ \kappa_2 E_t^p[a_{t+1}] + K_2 E_t^p[x_{t+1}] \end{aligned} \tag{3.29}$$

$$\begin{aligned} \pi_{t+1} =& \xi_0 + \xi_1 a_t + \Xi_1 E_t^c[x_t] + \xi_2 a_{t+1} + \Xi_2 E_{t+1}^c[x_{t+1}] + \xi_3 E_{t+1}^c[a_{t+2}] + \Xi_3 E_{t+1}^c[x_{t+1}] \\ & \omega_1 E_t^p[a_t] + \Omega_1 E_t^p[x_t] + \omega_2 E_{t+1}^p[a_{t+1}] + \Omega_2 E_{t+1}^p[x_{t+1}] + \omega_3 E_t^p[a_{t+2}] + \Omega_3 E_t^p[x_{t+2}] \end{aligned} \tag{3.30}$$

In this case, absent of any communication, from the perspective of time t cannot distinguish between the temporary and the permanent component of productivity both for periods $t+1$ and $t+2$. In this case, indeterminacy would be of degree 2 and indexed by the terms ξ_2 and ξ_3 as shown in the appendix. Moreover, if the cental bank provides direct communication about its information on future productivity shocks, a_{t+1}, a_{t+2} it can uniquely pin down the stochastic path of inflation with $\xi_2 = \dfrac{\zeta}{(\phi_\pi - 1)(1+\zeta)+1}$ and $\xi_3 = 0$.

Next we consider whether the central bank can communicate the same information via indirect communication about its expectation about future inflation. Suppose, similarly to before, that the central bank communicates its expectation about inflation one period ahead:

$$E_{t,1}^{cb}[\pi_{t+1}] = \Delta E_{t,1}^{cb}[a_{t+1} - a_t], \tag{3.31}$$

where $\Delta = \dfrac{\zeta}{(1+\zeta)(\phi_\pi - 1)+1} + \dfrac{(1+\zeta)(\phi_\pi - 1)+\zeta}{(1+\zeta)^2(\phi_\pi - 1)^2+1}$.

Notice that from equation (31) following the same logic as before, agents can back out the information that the central bank has about a_{t+1} but not a_{t+2}. Even though agents would be able to distinguish and put appropriate weight between temporary and permanent shocks of period $t+1$, they will still not be able to distinguish between the shocks that will occur in period $t+2$. That is, even though $E_t[a_{t+1}] \neq E_t[x_{t+1}]$, $E_t[a_{t+2}] = E_t[x_{t+2}]$. In this case, indeterminacy is pervasive and indexed by ξ_3. Further information is needed for agents to be able to determine the equilibrium path of inflation. This can be done if the central bank makes further announcements about inflation in the future.

In particular here by announcing its expectation about π_{t+2}:

$$E^{cb}_{t,1}[\pi_{t+2}] = \Delta E^{cb}_{t,1}[a_{t+2} - a_{t+1}]. \tag{3.32}$$

Equations (3.31 − 3.32) represent a system with two equations and two unknowns a_{t+1}, a_{t+2}. By solving this linear, invertible system of equations, agents are able to back out the extra information that the central bank has, and are thus able to determine the value of both ξ_2 and ξ_3. The equilibrium path of inflation is determinate.

Following the same logic, by induction, we generalize the insight for the role of communication for conjectures that include shocks for an arbitrary τ periods ahead in the future.

$$y_t = \sum_{i=1,j=0}^{\tau,\tau-1} \xi_i E^c_t[a_{t+j}] + \sum_{i=1,j=0}^{\tau,\tau-1} \Xi_1 E^c_t[x_{t+j}] + \sum_{i=1,j=0}^{\tau,\tau-1} \kappa_i E^p_t[a_{t+j}] + \sum_{i=1,j=0}^{\tau,\tau-1} K_i E^p_t[x_{t+j}] \tag{3.33}$$

$$\begin{aligned}\pi_{t+1} =& \xi_0 + \xi_1 a_t + \Xi_1 E^c_t[x_t] + \omega_1 E^p_t[a_t] + \Omega_1 E^p_t[x_t] + \\ & \sum_{i=2,j=1}^{\tau+1,\tau} \xi_i E^c_{t+1}[a_{t+j}] + \sum_{i=2,j=1}^{\tau+1,\tau} \Xi_1 E^c_{t+1}[x_{t+j}] + + \sum_{i=2,j=1}^{\tau+1,\tau} \omega_i E^p_{t+1}[a_{t+j}] + \sum_{i=2,j=1}^{\tau+1,\tau} \Omega_i E^p_t[x_{t+j}]\end{aligned} \tag{3.34}$$

Proposition 10 *In a model that accounts for agents' beliefs up to τ periods ahead, the stochastic path of inflation is determinate if the Central Bank announces each period the vector of signals $< s_{t+1}, ..., s_{t+\tau} >$ or equivalently under indirect announcements about its expectation about future inflation: $\mathcal{A} = \{E^{cb}_{t,1}[\pi_{t+1}], .., E^{cb}_{t,1}[\pi_{t+\tau}]\}$.*

Remark : Allowing $\tau = \infty$ requires to supplement the analysis with a terminal condition on inflation which ensures that the stochastic path of inflation is bounded. This is not a transversality condition, deriving from optimality conditions, but is similar to the so-called "elusive terminal condition" that helps determine inflation in the canonical New Keynesian model.[9]

Lastly, notice that so far we have considered two cases. One in which the central bank does not possess information and inflation is indeterminate, and one in which the central bank possesses information and communicates it, determining inflation. One may want to consider the case in which the central bank possesses information but does not communicate it. In that case, the only change would be that instead of equating the

[9]For a discussion about terminal conditions see Castillo-Martinez and Reis (2019).

expectations of the consumer with those of the central bank in equation (3.16), we should have the beliefs of the central bank instead, since now the central bank knows everything that the consumer does and has some extra private information about the future state of productivity. In this case, and since we have assumed that if the central bank transmits no information, agents make no inferences about its information set, equilibrium would collapse in the one described before in the case of no communication. That is because future expected inflation depends on agents' beliefs about current and future output from the IS equation (3.16) since agents' information has remained unchanged, equilibrium would be indeterminate unless further information is provided.

3.3.5 Comprehensive communication

The insight that transparency allows the Central Bank to pin down the stochastic path of inflation extends in richer environments with multiple shocks, provided that the central bank has superior information than the agents about the particular shocks. In the case of multiple shocks though, the central bank must communicate further information to the agents about its expectation about future production. We dub this comprehensive communication.

To demonstrate this suppose that alongside with the shocks in the economy that affect productivity, now there exists an independently identically distributed demand shock $\eta_t \sim \mathcal{N}(0, \sigma_\eta^2)$. In this case, we further endow the central bank with an observation about η_{t+1}.[10] Agents' utility function is modified as follows:

$$U(C_t, N_t) = E_t\big[\sum_{t=0}^{\infty} e^{\eta_t}\beta^t(logC_t - \frac{N_t^{1+\zeta}}{1+\zeta})\big]. \tag{3.35}$$

The Philips curve remains unchanged while the adjusted Euler equation is:

$$E_t^c[y_{t+1}] - y_t = E_t^c[\eta_{t+1} - \eta_t] + (\phi_\pi - 1)E_t^c[\pi_{t+1}]. \tag{3.36}$$

And in place of (20) $-$ (21) we conjecture that output and inflation follows:

$$y_t = \theta_0 + \theta_1 a_t + \Theta_1 E_t^c[x_t] + \kappa_1 E_t^p[a_t] + K_1 E_t^p[x_t] \tag{3.37}$$

$$\begin{aligned}\pi_{t+1} =& \xi_1 a_t + \xi_2 a_{t+1} + \Xi_1 E_t^c[x_t] + \Xi_2 E_{t+1}^c[x_{t+1}] + \\ & \omega_1 E_t^p[a_t] + \omega_2 E_{t+1}^p[a_{t+1}] + \Omega_1 E_t^p[x_t] + \Omega_2 E_{t+1}^p[x_{t+1}] + \mu_1\eta_t + \mu_2\eta_{t+1}.\end{aligned} \tag{3.38}$$

Notice that (3.35) does not depend on the preference shocks (η_t) since both agents know this shock at the beginning of the period thus there exists no wedge in their beliefs. Taking expectations of (3.36) as of period t, substituting the conjectures in (3.36)$-$(3.15)

[10] Allowing the central bank to directly observe η_{t+1} is equivalent for our needs with giving it a noisy observation about η_{t+1}.

and matching coefficients, one acquires the equilibrium absent any communication is given by:

$$y_t = \frac{\zeta - \xi_2}{1+\zeta} a_t + \frac{\xi_2 - \tilde{\xi}}{1+\zeta} E_t^c[x_t] + \frac{1+\xi_2}{1+\zeta} E_t^p[a_t] + \frac{\tilde{\xi} - \xi_2}{1+\zeta} E_t^p[x_t] \tag{3.39}$$

$$\begin{aligned}\pi_{t+1} =& \frac{-\zeta + \xi_2}{(1+\zeta)(\phi_\pi - 1)} a_t + \xi_2 a_{t+1} + \frac{\tilde{\xi} - \xi_2}{(1+\zeta)(\phi_\pi - 1)} E_t^c[x_t] + (\tilde{\xi} - \xi_2) E_{t+1}^c[x_{t+1}] + \\ & \frac{-(1+\tilde{\xi})}{(1+\zeta)(\phi_\pi - 1)} E_t^p[x_t] + \frac{1-\tilde{\xi}}{(1+\zeta)(\phi_\pi - 1)} E_t^p[x_{t+1}] + \frac{1}{\phi_\pi - 1}\eta_t + \mu_2 \eta_{t+1}\end{aligned} \tag{3.40}$$

Where μ_2, ξ_2 remain undetermined and can take arbitrary values while $\tilde{\xi}$ has the same constant value as before. Similarly, if the central bank provides direct communication about its signals one period ahead, it can pin down the coefficients ξ_2, μ_2. In this case though, the central bank must communicate information about both shocks in the future $\{s_{t+1}, \eta_{t+1}\}$. The equilibrium path of output and inflation is given by:

$$y_t = \frac{(\phi_\pi - 1)\zeta}{(1+\zeta)(\phi_\pi - 1) + 1} a_t + \frac{(1+\zeta)(\phi_\pi - 1) + 1 + \zeta}{(1+\zeta)^2(\phi_\pi - 1) + 1} E_t^p[a_t] \tag{3.41}$$

$$\pi_{t+1} = \frac{\zeta}{(1+\zeta)(\phi_\pi - 1) + 1}(a_{t+1} - a_t) + \frac{(1+\zeta)(\phi_\pi - 1) + \zeta}{(1+\zeta)^2(\phi_\pi - 1)^2 + 1} E_{t+1}^p[(a_{t+1} - a_t)] + \frac{1}{(\phi_\pi - 1)}(\eta_t - \eta_{t+1}) \tag{3.42}$$

Proposition 11 *In the absence of communication, the stochastic path of inflation remains undetermined and is indexed with ξ_2, μ_2. Under communication $\mathcal{A}_t = \{s_{t+1}, \eta_{t+1}\}$, the stochastic path of inflation is determined.*

The key insight here is the same as before but because the economy is subject to two shocks, if the central bank were to announce only s_{t+1}, agents' expectations about the preference shock would be $E_t^i[\eta_{t+1}] = 0$ and the path of inflation would remain undetermined, while extra information shifts agents' expectations away from the priors determining the stochastic path of inflation and output.

The interesting observation in an economy with multiple shocks is that if the central bank wants to communicate its information with indirect signals, communicating its expectations about $E_{t,1}^{cb}[\pi_{t+1}]$ would not suffice to determine the path of inflation. That is because agents would not be able to infer its different signals about $\{a_{t+1}, \eta_{t+1}\}$ and they would face an identification issue. They would not be able to separate whether the changes in the central banks' expectations are due to a change in productivity or due to changes in future preference shock. To see this take expectations of (3.40) given the information set of the central bank at $(t, 1)$:

$$E^{cb}_{t,1}[\pi_{t+1}] = \Delta E^{cb}_{t,1}[a_{t+1} - a_t] + \frac{1}{\phi_\pi - 1} E^{cb}_{t,1}[\eta_t - \eta_{t+1}]. \tag{3.43}$$

By announcing the left hand side of (3.43) alone, agents are not able to back out the different shocks that will hit the economy at $(t+1)$. The central bank in this case, must provide more "comprehensive communication" that includes its expectations about future output that does not depend on demand shocks but only on future productivity:

$$E^{cb}_{t,1}[y_{t+1}] = \Big(\frac{(\phi_\pi - 1)\zeta}{(1+\zeta)(\phi_\pi - 1) + 1} + \frac{(1+\zeta)(\phi_\pi - 1) + 1 + \zeta}{(1+\zeta)^2(\phi_\pi - 1) + 1}\Big) E^{cb}_{t,1}[a_{t+1}] \tag{3.44}$$

From equation (3.44), agents are able to back out the signal that the central bank has about a_{t+1}. Then from equation (3.43) they can back out its information about η_{t+1} and thus the equilibrium path of inflation is determined.

Proposition 12 *Under comprehensive communication with announcements $E^{cb}_t[\pi_{t+1}]$, $E^{cb}_t[y_{t+1}]$ the central bank can uniquely pin down the stochastic path of inflation.*

Proof: *Follows from the discussion above.*

3.4 Conslusion

This paper studies the effect of transparency from a better than the agents informed central bank in controlling inflation. We study a cashless monetary economy which is subject to permanent and temporary productivity shocks. The central bank sets the prices of nominal bonds targeting expected inflation. We demonstrate that equilibria that depend both on realized productivity and the beliefs of the agents about permanent productivity exist but the distribution of inflation across date events is not unique and remains undetermined. The central bank with its rule cannot uniquely pin down inflation which is subject to sunspot equilibria. In the case that there exists some asymmetry in agents' information, when the classical dichotomy does not apply and nominal values affect realized output, the inability to control inflation can lead to inefficient output fluctuations. The inefficiency stems from agents being unable to distinguish between the permanent and temporary component of the economy in the future. Thus they might put positive weight on their beliefs about permanent productivity while optimal output should only depend on realized productivity.

On the contrary, if the central bank communicates noisy information about future realized productivity or the future temporary shocks of the economy, agents are able to distinguish between the permanent and idiosyncratic component of the economy. This allows the central bank to uniquely pin down the path of inflation and in turn determine output. We show that in an environment where all the agents are rational, the same can be achieved with indirect announcements about the central bank's beliefs about future inflation. This is similar to what has been dubbed in the literature as "Delphic" forward guidance since it represents non-committal announcements about the beliefs of the central bank on future inflation. We provide a novel argument in support of such announcements since we demonstrate that they can be useful in determining the path of inflation. We show that this extends to economies which have demand shocks as well.

3.5 Appendix

3.5.1 Kalman filters

No communication

We start by considering the agents' learning problems without any communication. At each period $t, 2$ the central bank has the same information as the consumer at $t, 1$, $I_{t,2}^{cb} = I_{t,1}^{c}$ thus we focus on the learning problem of the consumer. Let the consumer's prior about permanent productivity be given by:

$$x_t | I_{t-1}^c \sim \mathcal{N}(x_{t|t-1}, \sigma_{x|t-1}^2)$$

Where $x_{t|t-1} := E[x_t | I_{t-1}^c]$ and $\sigma_{x|t-1}^2 := Var[x_t | I_{t-1}^c]$. Upon observing their current productivity a_t, the consumers update their beliefs about x_t:

$$x_t | I_t^c \sim \mathcal{N}\Big((1-\mu_t) x_{t|t-1} + \mu_t a_t, \quad (\frac{1}{\sigma_{x,t-1}} + \frac{1}{\sigma_{\epsilon_t}})^{-1}\Big) \quad \text{where} \quad \mu_t = \frac{\frac{1}{\sigma_\epsilon^2}}{\frac{1}{\sigma_{x|t-1}^2} + \frac{1}{\sigma_\epsilon^2}}.$$

Their expectation about future permanent productivity is given by:

$$x_{t+1} | I_t^c \sim \mathcal{N}\Big(\rho x_{t|t}, \sigma_x^2\Big),$$

where

$$\sigma_{x,t}^2 = (\frac{1}{\sigma_{x|t-1}^2} + \frac{1}{\sigma_\epsilon^2})^{-1} + \sigma_e^2. \tag{3.45}$$

Let σ_x^2 denote the solution (fixed point) of the Riccati equation (3.45). A solution does not exist in the limit where $\sigma_e \to \infty$. We dismiss this case. We assume that at period 0 agents' learning problems are at their steady state $x_{0|-1} \sim \mathcal{N}(0, \sigma_x^2)$. The Kalman gain will thus be time invariant:

$$\mu = \frac{\frac{1}{\sigma_\epsilon^2}}{\frac{1}{\sigma_x^2} + \frac{1}{\sigma_\epsilon^2}}.$$

Moreover, in the absence of any communication:

$$a_{t+1} | I_t^c \sim \mathcal{N}(x_{t+1|t}, \sigma_x^2 + \sigma_\epsilon^2)$$

implying that agents cannot distinguish between the permanent and temporary component of productivity. Turning our attention to the producer, notice that at $t, 1$ when the producer makes decisions, they have only information about past realized productivity that is:

$$a_t|I^p_{t,1} \sim \mathcal{N}(x_{t|t-1}, \sigma_x^2 + \sigma_\epsilon^2)$$
$$x_t|I^p_{t,1} \sim \mathcal{N}(x_{t|t-1}, \sigma_x^2)$$

At period $t, 2$, the producer can back out productivity from the already incurred production and thus they share the same information with the consumer and the central bank, $I^p_{t,2} = I^c_{t,1}$, and they update their belief about the permanent component, this estimate forms their beliefs about the permanent and temporary component of the economy at period $t+1$.

$$x_t|I^p_{t,2} \sim \mathcal{N}\Big((1-\mu)x_{t|t-1} + \mu a_t, \quad (\frac{1}{\sigma_x} + \frac{1}{\sigma_{\epsilon_t}})^{-1}\Big)$$

$$E^p_{t,2}[x_{t+1}] = E^p_{t,2}[a_{t+1}] = \rho E^p_{t,2}[x_t]$$

All agents in the economy, from the perspective of period t cannot distinguish between the permanent and temporary component of productivity one period ahead. For any agent i: $E_t[a_{t+1}|I^i_t] = E_t[x_{t+1}|I^i_t]$. Lastly notice that the producer estimates consumer's beliefs about the permanent component of productivity as follows:

$$E^p_{t,1}\Big[E^c_{t,1}[x_t]\Big] = (1-\mu)x_{t|t-1} + \mu E^p_t[a_t] = E^p_{t,1}[x_t]$$

While the consumer since their information set strictly includes that of the producer can always estimate the producer's beliefs about the permanent and temporary component of productivity accurately.

Information with Communication

Now we consider agents' information when the central bank each period t communicates information about the temporary shocks of productivity up to τ periods ahead. Notice that this signal does not contain any information about the permanent component of productivity but only for the iid shock $\epsilon_{t+\tau}$. Thus agents update their beliefs about $a_{t+\tau}$ while their learning about permanent productivity remains unchanged. In particular for agent i using Bayes rule, one acquires:

$$a_{t+\tau} \sim \mathcal{N}\Big((1-k)E^i_t[x_{t+\tau}] + ks_{t+\tau}, (\sigma_x^2 + \sigma_\epsilon^2)^{-1} + (\sigma_u^2)^{-1}\Big), \tag{3.46}$$

where $k = \dfrac{(\sigma_u^2)^{-1}}{(\sigma_x^2 + \sigma_\epsilon^2)^{-1} + (\sigma_u^2)^{-1}}$.

Importantly in this case, for all agents beliefs about the permanent future component of productivity are different from their beliefs about future realized productivity $E^i_t[a_{t+1}] \neq$

$E^i_t[x_{t+1}]$. Lastly, in the case with communication, since now $E^p_{t,1}[a_t] \neq E^p_t[x_t]$, the second order belief of the producer, which we denote by $\tilde{E}^p_{t,1}[x_t]$, is given by:

$$\tilde{E}^p_{t,1}[x_{t+\tau}] = E^p_{t,1}\Big[E^c_{t,1}[x_{t+\tau}]\Big] = (1-\mu)x_{t+\tau|t-1} + \mu E^p_t[a_{t+\tau}] \neq E^p_t[x_{t+\tau}]$$

This differs form their beliefs in the case without communication since it now incorporates the information about the temporary component of productivity at period τ. One could consider more complicate signals for example the central bank could communicate noisy signals about productivity $\tilde{s}_{t+\tau} = a_{t+\tau} + u_{t+\tau}$. Notice that this signal contains information both about the permanent and temporary component of the economy. In that case, agents would first update their beliefs about the permanent component of productivity before estimating future realized productivity. Even though the derivation of the learning problem would be more complex, the role of information for the purposes of the problem would remain the same, shifting expectations away from the priors and allowing agents to distinguish between the permanent and temporary shock of the economy.

3.5.2 Equilibria without approximations

Symmetric Information

Here we consider equilibria without approximations starting with the benchmark symmetric information case. In this case no agents observes the signal $s_{t+\tau}$. Next notice that absent of any frictions, $A_t = Y_t = C_t$ and that at the moment that the central bank steps in to set inflation, its information set is the same as the consumer's. Next combining the central bank's interest rate rule (3.5) with the Euler equation (3.7) that comes from the consumer's optimization problem one acquires:

$$E^c_{t,1}[\Pi_{t+1}]^{-\phi_\pi} = E^c_{t,1}\big[\frac{1}{\Pi_{t+1}}\frac{A_t}{A_{t+1}}\big]. \tag{3.47}$$

Conjecture that future inflation follows:

$$\pi_{t+1} = \xi_0 + \xi_1 a_t + \xi_2 a_{t+1} + \Xi_2 E_{t+1}[x_{t+1}]$$

Taking natural logs and substituting the conjecture into the LHS of (3.47) one acquires:

$$e^{-\phi_\pi(\xi_0+\xi_1 a_t+\xi_2 E^c_t[a_{t+1}]+\Xi_2 E_{t+1}[x_{t+1}])} = e^{-\phi_\pi(\xi_0+\xi_1 a_t+(\xi_2+\Xi_2)E_t[x_{t+1}])}. \tag{3.48}$$

where the second inequality stems from the fact that in the absence of any communication, $E^c_t[x_{t+1}] = E^c_t[a_{t+1}]$. Turning to the RHS of (3.47) notice that:

$$E^c_t[\pi_{t+1} + a_{t+1}] = \xi_0 + \xi_1 a_t + (1+\xi_2)E^c_t[a_{t+1}] + \Xi_2 E_t[x_{t+1}]$$
$$Var^c[\pi_{t+1} + a_{t+1}] = (1+\xi_2+\Xi_2)^2\sigma^2_x + (1+\xi_2)^2\sigma^2_\epsilon$$

Thus the RHS becomes:

$$e^{-(\xi_0+\xi_1 a_t+(1+\xi_2)E^c_t[a_{t+1}]+\Xi_2 E_t[x_{t+1}])+a_t-1/2Var[\pi_{t+1}+a_{t+1}]} \tag{3.49}$$

Matching coefficients with (47) – (48) one acquires:

$$-\phi_\pi(\xi_2+\Xi_2) = -(1+\xi_2+\Xi_2)$$

$$-\phi_\pi\xi_1 = (1-\xi_1)$$

$$(1-\phi_\pi)\xi_0 = \frac{1}{2}Var[\pi_{t+1}+a_{t+1}]$$

From the system of the above equations, one acquires that $\xi_1 = \dfrac{1}{1-\phi_\pi}$, $(\xi_2+\Xi_2) = \dfrac{1}{\phi_\pi - 1}$ while $\xi_0 = \dfrac{1}{2(1-\phi_\pi)}Var[\pi_{t+1}+a_{t+1}]$ as in the main text. Importantly, the coefficients ξ_2, Ξ_2 can take arbitrary values. We have one more unknowns than equations thus the system is undetermined.

Assymetric information

Proof of Proposition 7: Next we consider equilibria without approximations for the assymetric information case and we start with the benchmark of no communication. Start with combining the consumer's optimal labour supply condition (3.6) and the producer's optimization problem (3.8). Add and subtract P_{t-1} and note that $\lambda_t = \dfrac{1}{P_tC_t}$. Next confirm that they can be rewritten as:

$$N_t^\zeta = \frac{1}{\Pi_t C_t}\frac{E^p_{t,1}[\frac{A_t}{C_t}]}{E^p_{t,1}[\frac{1}{\Pi_t C_t}]} \tag{3.50}$$

Next, turning to the Euler equation, it can be rewritten as:

$$E^c_{t,1}[\Pi_{t+1}]^{-\phi_\pi} = E^c_{t,1}\Big[\frac{1}{\Pi_{t+1}}\frac{A_t}{A_{t+1}}\Big]. \tag{3.51}$$

Equations (3.50)-(3.51) correspond to (3.15) – (3.16) in the main text. Next start with the conjectures about output and inflation:

$$y_t = \theta_0 + \theta_1 a_t + \Theta_1 E^c_t[x_t] + \kappa_1 E^p_t[a_t] + K_1 E^p_t[x_t] \tag{3.52}$$

$$\begin{aligned}\pi_{t+1} =& \xi_0 + \xi_1 a_t + \Xi_1 E^c_t[x_t] + \omega_1 E^p_t[a_t] + \Omega_1 E^p_t[x_t] + \\ &+ \xi_2 a_{t+1} + \Xi_2 E^c_{t+1}[x_{t+1}] + \omega_2 E^p_{t+1}[a_{t+1}] + \Omega_2 E^p_{t+1}[x_{t+1}].\end{aligned} \tag{3.53}$$

We will show that given all shocks being normally distributed, (3.52) – (3.53) imply that C_t and Π_t are log- normally distributed.

Start with the optimality condition (3.50), take logs and substitute the technology ($y_t = a_t + n_t$) and the market clearing condition $c_t = y_t$ and get:

$$e^{\zeta(y_t - a_t)} = e^{-(\pi_t + y_t)} \frac{E_t^p[e^{(a_t - y_t)}]}{E_t^p[e^{-(\pi_t + y_t)}]} \tag{3.54}$$

Substituting conjectures (3.51) – (3.52) in the LHS of (3.54) one acquires:

$$e^{\zeta(\theta_0 + \theta_1 a_t + \Theta_1 E_t^c[x_t] + \kappa_1 E_t^p[a_t] + K_1 E_t^p[x_t] - a_t)}. \tag{3.55}$$

Turning our attention to the RHS we start by showing that the numerator term $E_t^p[e^{(a_t - y_t)}]$ of (54) is normally distributed. Substituting the conjectures one acquires:

$$E_t^p[e^{(a_t - y_t)}] = E_t^p[e^{(a_t - \theta_0 - \theta_1 a_t - \Theta_1 E_t^c[x_t] - \kappa_1 E_t^p[a_t] - K_1 E_t^p[x_t])}].$$

Conditional on the producer's information, and using the fact that without any information, $E_t^p\big[E_t^c[x_t]\big] = E_t^p[x_t]$ the exponent is normally distributed with mean:

$$E_t^p[a_t - y_t] = -\theta_0 + (1 - \theta_1 - \kappa_1)E_t^p[a_t] - (\Theta_1 + K_1)E_t^p[x_t].$$

For the variance notice that $E_t^p\big[E_t^c[x_t]\big] = E_t^p[(1-\mu)E_t^c[x_{t-1}] + \mu a_t]$ with the first term of the equation being known to the producer while the second one being a source of variation. Thus variance is given by:

$$Var_t^p[a_t - y_t] = (1 - \theta_1 - \mu\Theta_1)^2 \sigma_{p,a}^2 \quad \text{where} \quad \sigma_{a,p}^2 = \sigma_x^2 + \sigma_\epsilon^2.$$

Thus the above term is normally distributed and can be rewritten as:

$$e^{-\theta_0 + (1-\theta_1-\kappa_1)E_t^p[a_t] - (\Theta_1 + K_1)E_t^p[x_t] + \dfrac{1}{2} Var^p[a_t - y_t]}$$

Next notice that the denominator of (5), $E_t^p[e^{-(\pi_t + y_t)}]$ is equal with:

$$E_t^p\Big[e^\wedge - \Big(\xi_0 + \xi_1 a_{t-1} + \Xi_1 E_t^c[x_{t-1}] + \omega_1 E_t^p[a_{t-1}] + \Omega_1 E_t^p[x_{t-1}] + \xi_2 a_t + \Xi_2 E_t^c[x_t] + \omega_2 E_t^p[a_t] + \Omega_2 E_t^p[x_t]$$

$$+\theta_0 + \theta_1 a_t + \Theta_1 E_t^c[x_t] + \kappa_1 E_t^p[a_t] + K_1 E_t^p[x_t]\Big)\Big] (3.56)$$

Again here use the fact that $E_t^p\big[E_t^c[x_t]\big] = E_t^p[x_t]$ thus the above term is normally distributed with

$$E_t^p[-(\pi_t - y_t)] = -(\xi_0 + \theta_0) - (\xi_1 + \omega_1)a_{t-1} - (\Xi_1 + \Omega_1)E_t^p[x_{t-1}] - (\xi_2 + \omega_2 + \kappa_1 + \theta_1)E_t^p[a_t]$$

$$\text{-}(\Xi_2 + \Omega_2 + \Theta_1 + K_1)E_t^p[x_t]$$

$$Var_t^p(\pi_t + y_t) = (\xi_2 + \theta_1 + \mu\Xi_2 + \mu\Theta_1)^2\sigma_{a,p}^2$$

Thus the fraction can be expressed as:

$$\frac{E_t^p[e^{(a_t - y_t)}]}{E_t^p[e^{-(\pi_t + y_t)}]} =$$

$$\frac{e^{-G/2+(1-\theta_1-\kappa_1)E_t^p[a_t]-\theta_0-(\Theta_1+K_1)E_t^p[x_t]}}{e^{-F/2-(\xi_0+\theta_0)-(\xi_1+\omega_1)a_{t-1}-(\Xi_1+\Omega_1)E_t^p[x_{t-1}]-(\xi_2+\omega_2+\kappa_1+\theta_1)E_t^p[a_t]-(\Xi_2+K_2)E_t^p[x_t]}}$$

$$= e\hat{}\{E_t^p[a_t]+\xi_0+(\omega_1+\xi_1)E_t^p[a_{t-1}]+(\Xi_1+\Omega_1)E_t^p[x_{t-1}]+(\xi_2+\omega_2)E_t^p[a_t]+(\Xi_2+\Omega_2)E_t^p[x_t]+(F-G)/2\}$$

$$= e^{E_t^p[a_t]-E_t^p[\pi_t]+(F-G)/2}$$

Where $F = Var_t^p[\pi_t - y_t]/2$ and $G = Var_t^p[a_t - y_t]/2$ which are constants. Turning our attention back to (54) we now have:

$$\begin{aligned} e^{\zeta(y_t - a_t)} &= e^{-(\pi_t+y_t)+E_t^p[a_t]+E_t^p[\pi_t]} \iff \\ e^{(1+\zeta)y_t} &= e^{\zeta a_t+E_t^p[a_t]+E_t^p[\pi_t]-\pi_t+F-G} \end{aligned} \tag{3.57}$$

And confirm that (56) corresponds to the equilibrium condition (15) ignoring constants.

The exponent of the LHS and RHS of (57) can be rewritten respectively as:

$$(1+\zeta)\Big(\theta_0 + \theta_1 a_t + \Theta_1 E_t^c[x_t] + \kappa_1 E_t^p[a_t] + K_1 E_t^p[x_t]\Big) \tag{3.58}$$

$$\begin{aligned} E_t^p[a_t] + \zeta a_t + E_t^p\Big(\xi_0 + \xi_1 a_{t-1} + \xi_2 a_t + \Xi_1 E_t^c[x_t] + \omega_1 E_t^p[a_t] + \Omega_1 E_t^p[x_t]\Big) - \\ \Big(\xi_0 + \xi_1 a_{t-1} + \xi_2 a_t + \Xi_1 E_t^c[x_t] + \omega_1 E_t^p[a_t] + \Omega_1 E_t^p[x_t]\Big) = \end{aligned}$$

$$(1+\xi_2)E_t^p[a_t] + \Xi_2 E_t^p[x_t] + (\zeta - \xi_2)a_t - \Xi_2 E_t^c[x_t] \tag{3.59}$$

Matching coefficients between (58) and (59) and taking into account that $E_t^p[a_t] = E_t^p[x_t]$, one acquires:

$$\theta_0 = \frac{F-G}{1+\zeta} \quad \theta_1 = \frac{\zeta - \xi_2}{1+\zeta}, \quad \Theta_1 = \frac{-\Xi_2}{1+\zeta}, \quad (\kappa_1 + K_1) = \frac{1+\xi_2+\Xi_2}{1+\zeta}$$

Next we turn our attention to equation (3.51) Take natural logs and substitute the conjecture into the LHS of (3.51) one acquires:

$$e^{-\phi_\pi(\xi_0+\xi_1 a_t+\Xi_1 E^c_t[x_t]+\omega_1 E^p_t[a_t]+\Omega_1 E^p_t[x_t]+\xi_2 a_{t+1}+\Xi_2 E^c_t[x_{t+1}]+\omega_2 E^p_{t+1}[a_{t+1}]+\Omega_2 E^p_{t+1}[x_{t+1}])} \tag{3.61}$$

Turning to the RHS of (3.51), the denominator $E[\pi_{t+1}+y_{t+1}]$ is normally distributed with mean and variance:

$$E^c_t[\pi_{t+1}+y_{t+1}] = \theta_0+\xi_0+\xi_1 a_t+\Xi_1 E^c_t[x_t]+(1+\xi_2+\theta_1)E^c_t[a_{t+1}]+(\Xi_2+\Theta_1)E^c_t[x_{t+1}]+$$

$$+(\omega_2+\kappa_1)E^p_{t+1}[a_{t+1}]+(\Omega_2+K_2)E^p_{t+1}[x_{t+1}]$$

$$Var^c[\pi_{t+1}+y_{t+1}] = [\Theta_1(1-\mu)+\Xi_1+\Xi_2(1-\mu)]^2\sigma^2_x+(\theta_1+\mu\Theta_1+\xi_2+\mu\Xi_2)^2\sigma^2_\alpha$$

Thus the RHS becomes:

$$e^{-(\xi_0+\xi_1 a_t+(1+\xi_2)E^c_t[a_t]+\Xi_2 E^c_t[x_{t+1}]+\omega_2 E^p_t[a_{t+1}]+\Omega_2 E^p_t[x_{t+1}])+E^c_t[y_{t+1}]-y_t-1/2Var[\pi_{t+1}+y_{t+1}]} \tag{3.62}$$

Tanking into account that $E^c_t[x_{t+1}] = E^c_t[a_{t+1}]$ and $E^p_{t+1}[x_{t+1}] = E^p_{t+1}[a_{t+1}]$ and matching coefficients of $(3.61)-(3.62)$, one acquires:

$$\frac{-\theta_1}{\phi_\pi-1}=\xi_1 \quad \frac{-\Theta_1}{\phi_\pi-1}=\Xi_1 \quad \frac{\theta_1+\Theta_1}{\phi_\pi-1}=(\xi_2+\Xi_2)$$
$$\frac{-K_1-\kappa_1}{\phi_\pi-1}=(\omega_1+\Omega_1) \quad \frac{K_1+\kappa_1}{\phi_\pi-1}=(\omega_2+\Omega_2) \quad \xi_0=\frac{Var[\pi_{t+1}+y_{t+1}]}{2(\phi_\pi-1)} \tag{3.63}$$

Matching the set of equilibrium conditions $(3.59)-(3.63)$ one arrives at the conclusion of proposition 1 since the coefficients ξ_2, Ξ_2 remain undetermined. The equilibrium output and inflation are given by $(3.23), (3.24)$ concluding the proof of Proposition 7. ∎

Proof of Proposition 8: Next for proposition 8 we consider the case where the central bank transmits information about the productivity shocks one period ahead. In this case, for all agents, $E^i_t[a_{t+1}] \neq E^i_t[x_{t+1}]$ and for the producer $E^p_t[a_t] \neq E^p_t[x_t]$. This further implies the $E^p_t[E^c_{t-1}[x_t]] = (1-\mu)E^p_{t-1}[x_t]+\mu E^p_t[a_t]) \neq E^p_t[x_t]$ since even though $E^p_t[x_t] = E^p_{t-1}[x_t]$, $E^p_t[a_t] \neq E^p_t[x_t]$. Taking these observations into account and working similarly as before, notice that the fraction on the RHS of (54) now becomes:

$$\frac{E^p_t[e^{(a_t-y_t)}]}{E^p_t[e^{-(\pi_t+y_t)}]} =$$

$$\frac{e^{-G'+(1-\theta_1-\kappa_1)E_t^p[a_t]-\theta_0-\Theta_1((1-\mu)E_{t-1}^p[x_t]+\mu E_t^p[a_t])+K_1E_t^p[x_t]}}{e^{-F'-(\xi_0+\theta_0)-(\xi_1+\omega_1)a_{t-1}-(\Xi_1+\Omega_1)E_t^p[x_{t-1}]-(\xi_2+\omega_2+\kappa_1+\theta_1)E_t^p[a_t]-(\Xi_2+\Theta_1)(1-\mu)E_{t-1}^p[x_t]+\mu E_t^p[a_t])}},$$

With $G' = Var^p[a_t - y_t]/2 = (1-\theta_1-\mu\Theta_1)\sigma'^2_{\alpha,p}/2$ and $\sigma'_{\alpha,p}$ is the variance of a_t with information given by (46), while $F' = Var^p[\pi_t + y_t]/2 = (\xi_2 + \mu\Xi_2 + \theta_1 + \mu\Theta_1)\sigma'^2_{a,p}/2$.

Thus (54) can be rewritten as:

$$e^{(1+\zeta)\left(\theta_0+\theta_1a_t+\Theta_1E_t^c[x_t]+\kappa_1E_t^p[a_t]+K_1E_t^p[x_t]\right)} =$$

$$e^{(1+\xi_2)E_t^p[a_t]+\Xi_2E_t^p[x_t]+(\zeta-\xi_2)a_t-\Xi_2E_t^c[x_t]+\Xi_2(\mu E_t^p[x_t]+(1-\mu)E_t^p[a_t])+F'-G'}.$$

Matching coefficients one acquires:

$$\theta_1 = \frac{\zeta-\xi_2}{1+\zeta}, \quad \Theta_1 = \frac{-\Xi_2}{1+\zeta}, \quad \kappa_1 = \frac{1+\xi_2+\Xi_2(1-\mu)}{1+\zeta}, \quad K_1 = \frac{\Xi_2\mu}{1+\zeta}, \quad \theta_0 = \frac{F'-G'}{1+\zeta}$$

The analysis of (51) remains the same as above thus:

$$e^{-\phi_\pi(\xi_0+\xi_1a_t+\Xi_1E_t^c[x_t]+\omega_1E_t^p[a_t]+\Omega_1E_t^p[x_t]+\xi_2a_{t+1}+\Xi_2E_t^c[x_{t+1}]+\omega_2E_{t+1}^p[a_{t+1}]+\Omega_2E_{t+1}^p[x_{t+1}])} =$$

$$e^{-(\xi_0+\xi_1a_t+(1+\xi_2)E_t^c[a_t]+\Xi_2E_t^c[x_{t+1}]+\omega_2E_t^p[a_{t+1}]+\Omega_2E_t^p[x_{t+1}])+E_t^c[y_{t+1}]-y_t-1/2Var[\pi_{t+1}+y_{t+1}]}$$

And matching coefficients one acquires:

$$\frac{-\theta_1}{\phi_\pi-1} = \xi_1 \quad \frac{-\Theta_1}{\phi_\pi-1} = \Xi_1 \quad \frac{\theta_1}{\phi_\pi-1} = \xi_2 \quad \frac{\Theta_1}{\phi_\pi-1} = \Xi_2$$

$$\frac{-\kappa_1}{\phi_\pi-1} = \omega_1 \quad \frac{-K_1}{\phi_\pi-1} = \Omega_1 \quad \frac{\kappa_1}{\phi_\pi-1} = \omega_2 \quad \frac{K_1}{\phi_\pi-1} = \Omega_2 \quad \xi_0 = \frac{Var[\pi_{t+1}+y_{t+1}]}{\phi_\pi-1}$$

Solving the system of equations one is able to pin down the coefficients and conclude the proof of Proposition 2:

$$\theta_1 = \frac{(\phi_\pi-1)\zeta}{(1+\zeta)(\phi_\pi-1)+1} \quad \kappa_1 = \frac{(1+\zeta)(\phi_\pi-1)+1+\zeta}{(1+\zeta)^2(\phi_\pi-1)+1}, \quad \Xi_2 = \Theta_2 = 0$$

$$\xi_2 = -\xi_1 = \frac{\zeta}{(1+\zeta)(\phi_\pi-1)+1}, \quad \omega_2 = -\omega_1 = \frac{(1+\zeta)(\phi_\pi-1)+\zeta}{(1+\zeta)^2(\phi_\pi-1)^2+1}$$

∎

3.5.3 Multiple shocks

Proof of Proposition 10: Next we consider conjectures for inflation that depend on the temporary and permanent shocks for more than one period in the future. Remember that the key point here is to show that more signals are required to pin down inflation in that case. To this end suppose that the central bank transmits information one period ahead for ϵ_{t+1} but no information about ϵ_{t+2} and consider the conjectures:

$$\begin{aligned} y_t =& \theta_0 + \theta_1 a_t + \Theta_1 E^c_t[x_t] + \theta_2 E^c_t[a_{t+1}] + \Theta_2 E^c_t[x_{t+1}] + \kappa_1 E^p_t[a_t] + K_1 E^p_t[x_t] \\ &+ \kappa_2 E^p_t[a_{t+1}] + K_2 E^p_t[x_{t+1}] \end{aligned} \tag{3.64}$$

$$\begin{aligned} \pi_{t+1} =& \xi_0 + \xi_1 a_t + \Xi_1 E^c_t[x_t] + \xi_2 a_{t+1} + \Xi_2 E^c_{t+1}[x_{t+1}] + \xi_3 E^c_{t+1}[a_{t+2}] + \Xi_3 E^c_{t+1}[x_{t+1}] \\ & \omega_1 E^p_t[a_t] + \Omega_1 E^p_t[x_t] + \omega_2 E^p_{t+1}[a_{t+1}] + \Omega_2 E^p_{t+1}[x_{t+1}] + \omega_3 E^p_t[a_{t+2}] + \Omega_3 E^p_t[x_{t+2}] \end{aligned} \tag{3.65}$$

Substituting the new conjecture about output in the RHS of (54) one acquires:

$$e^{\theta_0+\theta_1 a_t+\Theta_1 E^c_t[x_t]+\theta_2 E^c_t[a_{t+1}]+\Theta_2 E^c_t[x_{t+1}]+\kappa_1 E^p_t[a_t]+K_1 E^p_t[x_t]+\kappa_2 E^p_t[a_{t+1}]+K_2 E^p_t[x_{t+1}]-a_t}$$

The nominator on the LHS of (54) can be rewritten as:

$$E^p_t[e^{(a_t-\theta_0+\theta_1 a_t+\Theta_1 E^c_t[x_t]+\theta_2 E^c_t[a_{t+1}]+\Theta_2 E^c_t[x_{t+1}]+\kappa_1 E^p_t[a_t]+K_1 E^p_t[x_t]+\kappa_2 E^p_t[a_{t+1}]+K_2 E^p_t[x_{t+1}])}] =$$

To see that this is normally distributed, start with $E^p_t[a_t - y_t]$ which can be rewritten as:

$$E^p_t[(a_t - \theta_0 - \theta_1 a_t - \Theta_1 E^c_t[x_t] - \theta_2 E^c_t[a_{t+1}] - \Theta_2 E^c_t[x_{t+1}] - \kappa_1 E^p_t[a_t] - K_1 E^p_t[x_t] - \kappa_2 E^p_t[a_{t+1}] - K_2 E^p_t[x_{t+1}])] \tag{3.66}$$

Next remember that we denote with $\tilde{E}$ the second order beliefs of the producer that are calculated as follows:

$$\tilde{E}^p_t[x_t] = E^p_t[E^c_t[x_t]] = (1-\mu)E^p_{t-1}[x_t] + \mu E^p_t[a_t]$$

$$\begin{aligned} \tilde{E}^p_t[a_{t+1}] = E^p_t[E^c_t[a_{t+1}]] =& E^p_t[\frac{(\sigma^2_x+\sigma^2_\epsilon)^{-1} E^c_t[x_{t+1}] + (\sigma^2_u)^{-1} s_{t+1}}{(\sigma^2_x+\sigma^2_\epsilon)^{-1} + (\sigma^2_u)^{-1}}] \\ =& \frac{(\sigma^2_x+\sigma^2_\epsilon)^{-1} \rho E^p_t[E^c_t[x_t]] + (\sigma^2_u)^{-1} s_{t+1}}{(\sigma^2_x+\sigma^2_\epsilon)^{-1} + (\sigma^2_u)^{-1}} \end{aligned}$$

$$\tilde{E}^p_t[x_{t+1}] = E^p_t[E^x_t[x_{t+1}]] = \rho E^p_t[E^c_t[x_t]]$$

With this in mind, (64) is normally distributed with mean and variance

$$E_t^p[a_t - y_t] =$$

$$-(\theta_0+(1+\theta_1)E_t^p[a_t]+\Theta_1\tilde{E}_t^p[x_t]+\theta_2\tilde{E}_t^p[a_{t+1}]+\Theta_2\tilde{E}_t^p[x_{t+1}]+\kappa_1 E_t^p[a_t]-K_1E_t^p[x_t]-\kappa_2E_t^p[a_{t+1}]-K_2E_t^p[x_{t+1}]))]$$

$$G'' = Var^p[a_t - y_t] = (1-\theta_1-\Theta_1\mu+\theta_2 k\rho\mu-\Theta_2\rho(1-\mu))^2\sigma'^2_{\alpha,p}.$$

where $k = \dfrac{(\sigma_x^2+\sigma_\epsilon^2)^{-1}}{(\sigma_x^2+\sigma_\epsilon^2)^{-1}+(\sigma_u^2)^{-1}}$. Thus the nominator can be rewritten as:

$$e^{-(\theta_0+(1+\theta_1)E_t^p[a_t]+\Theta_1\tilde{E}_t^p[x_t]+\theta_2\tilde{E}_t^p[a_{t+1}]+\Theta_2\tilde{E}_t^p[x_{t+1}]+\kappa_1E_t^p[a_t]-K_1E_t^p[x_t]-\kappa_2E_t^p[a_{t+1}]-K_2E_t^p[x_{t+1}])+1/2G'']}$$

Working in the same manner for the denominator, confirm that $(\pi_t + y_t)$ follows a normal distribution with mean:

$$\begin{aligned}E_t^p(\pi_t+y_t) =&(\theta_0+\xi_0)+(\xi_1+\omega_1)a_{t-1}+(\Xi_1+\Omega_1)E_t^p[x_{t-1}]+(\xi_2+\theta_1+\kappa_1+\omega_2)E_t^p[a_t]+\\&(\xi_2+\theta_1)\tilde{E}_t^p[x_t]+(\Omega_2+K_1)E_t^p[x_t]+(\xi_3+\theta_2)\tilde{E}_t^p[a_{t+1}]+(\Xi_3+\Theta_2)\tilde{E}_t^p[x_{t+1}]+\\&(\omega_3+\kappa_2)E_t^p[x_{t+1}]\end{aligned}$$

$$Var^p(\pi_t+y_t) = F'' = (\xi_1+\omega_1 k\rho\mu+(\xi_2+\theta_1)+\mu(\xi_2+\Theta_1)+(\Xi_3+\Theta_2k\rho(1-\mu))^2\sigma^2_{a,p}$$

Thus the fraction can be rewritten as:

$$e^{E_t^p[a_t]+(\xi_0)+(\xi_1+\omega_1)a_{t-1}+(\Xi_1+\Omega_1)E_t^p[x_{t-1}]+(\xi_2+\omega_2)E_t^p[a_t]+(\xi_2)\tilde{E}_t^p[x_t]+(\Omega_2)E_t^p[x_t]+(\xi_3)\tilde{E}_t^p[a_{t+1}]+(\Xi_3)\tilde{E}_t^p[x_{t+1}]+(\omega_3)E_t^p[x_{t+1}]-G''+F''}$$

Or equivalently:

$$e^{E_t^p[a_t]+E_t^p[\pi_t]+const}$$

Thus (54) can be rewritten as:

$$e^{(1+\zeta)y_t} = e^{\zeta a_t+E_t^p[a_t]+E_t^p[\pi_t]-\pi_t+F''-G''}$$

Substituting in for the conjectures and cancelling out the common terms between $E_t^p[\pi_t] - \pi_t$ one acquires:

$$e^{(1+\zeta)(\theta_0+\theta_1a_t+\Theta_1E_t^c[x_t]+\theta_2E_t^c[a_{t+1}]+\Theta_2E_t^c[x_{t+1}]+\kappa_1E_t^p[a_t]+K_1E_t^p[x_t]+\kappa_2E_t^p[a_{t+1}]+K_2E_t^p[x_{t+1}])} = \tag{3.67}$$

$$e^{\zeta a_t+E_t^p[a_t]+(\xi_2)E_t^p[a_t]+(\Xi_2)\tilde{E}_t^p[x_t]+(\xi_3)\tilde{E}_t^p[a_{t+1}]+(\Xi_3)\tilde{E}_t^p[x_{t+1}]-G''+F''-\xi_2a_t-\Xi_2E_t^c[x_t]-\xi_3E_t^c[a_{t+1}]-\Xi_3E_t^c[x_{t+1}]} \tag{3.68}$$

Lastly notice that the second order beliefs of the producer can be rewritten as follows:

$$\tilde{E}_t^p[x_t] = (1-\mu)E_t^p[x_t] + \mu E_t^p[a_t]$$

$$\tilde{E}_t^p[x_{t+1}] = E_t^p\Big[\rho((1-\mu)E_{t-1}^c[x_t]+\mu[a_t])\Big] = \rho((1-\mu)E_t^p[x_t]+\mu E_t^p[a_t]) = E_t^p[x_{t+1}]+\rho\mu(E_t^p[a_t]-E_t^p[x_t])$$

$$\tilde{E}_t^p[a_{t+1}] = E_t^p\Big[E_t^c[a_{t+1}]\Big] = E_t^p\Big[(1-k)E_t^c[x_{t+1}] + ks_{t+1}\Big] =$$

$$(1-k)\big(E_t^p[x_{t+1}] + \rho\mu(E_t^p[a_t] - E_t^p[x_t])\big) + ks_{t+1} = E_t^p[a_{t+1}] + (1-k)\rho\mu(E_t^p[a_t] - E_t^p[x_t])$$

Taking this into account and matching coefficients between (66) – (67) one acquires:

$$\theta_0 = \frac{(F''-G'')/2}{1+\zeta}, \quad \theta_1 = \frac{\zeta-\xi_2}{1+\zeta}, \quad \Theta_1 = \frac{-\Xi_2}{1+\zeta}, \quad \theta_2 = \frac{-\xi_3}{1+\zeta}, \quad \Theta_2 = \frac{-\Xi_3}{1+\zeta}$$

$$\kappa_1 = \frac{1+\xi_2+\mu(\Xi_2+\xi_3(1-k)\rho+\rho\Xi_3)}{1+\zeta}, \quad K_1 = \frac{\Xi_2(1-\mu)-\rho\mu(\Xi_3+(1-k)\xi_3)}{1+\zeta}, \quad \kappa_2 = \frac{\xi_3}{1+\zeta},$$

$$K_2 = \frac{\Xi_3}{1+\zeta} \tag{3.69}$$

Next we turn our attention to (3.51) ans we start with the denominator which in logs can be rewritten as $E_t^c[e^{(\pi_{t+1}+y_{t+1})}]$ This is normally distributes with mean and variance:

$$\begin{aligned} E_t^c[\pi_{t+1}+y_{t+1}] =&(\xi_0+\theta_0) + \xi_1 a_t + \Xi_1 E_t^c[x_t] + (\xi_2+\theta_1)E_t^c[a_{t+1}] + (\Xi_2+\Theta_1)E_{t+1}^c[x_{t+1}]+ \\ &(\xi_3+\theta_2)E_{t+1}^c[a_{t+2}] + (\Xi_3+\Theta_2)E_{t+1}^c[x_{t+1}] + \omega_1 E_t^p[a_t] + \Omega_1 E_t^p[x_t]+ \\ &(\omega_2+\kappa_1)E_{t+1}^p[a_{t+1}] + (\Omega_2+K_1)E_{t+1}^p[x_{t+1}] + (\omega_3+\kappa_2)E_t^p[a_{t+2}]+ \\ &(\Omega_3+K_3)E_{t+1}^p[x_{t+2}] \end{aligned}$$

$$(Var)^c[\pi_{t+1}+y_{t+1}] = [\Xi_1+(\Xi_2+\Theta_1+\Xi_2+\Theta_2)(1-\mu)]^2\sigma_x^2+(\theta_1+\mu(\Theta_1+\Xi_2+\Theta_2+\Xi_3)+\xi_2+2+\theta_2+\xi_3)^2\sigma_\alpha^2$$

Thus (3.51) can be rewritten as

$$e^{1/(1-\phi_\pi)(E_t^c[y_[t+1]-y_t))} = e^{E_t^c[\pi_{t+1}]}$$

Moreover notice that these expessions hold regardless on whether the central bank communicates information about a_{t+2} if the central bank offers no new information besides a_{t+1}, then we have the extra condition that $E_t^i[a_{t+2}] = E_t^i[x_{t+2}]$ having this in mind we match coefficients with the above equation and we end up with two sets of equilibrium conditions:

$$\frac{-\theta_1}{\phi_\pi - 1} = \xi_1 \quad \frac{-\Theta_1}{\phi_\pi - 1} = \Xi_1 \quad \frac{\theta_1}{\phi_\pi - 1} = \xi_2 \quad \frac{\Theta_1}{\phi_\pi - 1} = \Xi_2 \quad (\xi_3 + \Xi_3) = \frac{(\theta_2 + \Theta_2)}{\phi_\pi - 1}$$

$$\frac{-\kappa_1}{\phi_\pi - 1} = \omega_1 \quad \frac{-K_1}{\phi_\pi - 1} = \Omega_1 \quad \frac{\kappa_1}{\phi_\pi - 1} = \omega_2 \quad \frac{K_1}{\phi_\pi - 1} = \Omega_2 \quad \xi_0 = \frac{Var[\pi_{t+1} + y_{t+1}]}{\phi_\pi - 1}$$

$$(\omega_3 + \Omega_3) = \frac{\kappa_2 + K_2}{\phi_\pi - 1} \tag{3.70}$$

While with announcements the equations for the coefficients become:

$$\frac{-\theta_1}{\phi_\pi - 1} = \xi_1 \quad \frac{-\Theta_1}{\phi_\pi - 1} = \Xi_1 \quad \frac{\theta_1}{\phi_\pi - 1} = \xi_2 \quad \frac{\Theta_1}{\phi_\pi - 1} = \Xi_2 \quad \Xi_3 = \frac{(\Theta_2)}{\phi_\pi - 1}$$

$$\xi_3 = \frac{(\theta_2)}{\phi_\pi - 1} \quad \frac{-\kappa_1}{\phi_\pi - 1} = \omega_1 \quad \frac{-K_1}{\phi_\pi - 1} = \Omega_1 \quad \frac{\kappa_1}{\phi_\pi - 1} = \omega_2 \quad \frac{K_1}{\phi_\pi - 1} = \Omega_2$$

$$\xi_0 = \frac{Var[\pi_{t+1} + y_{t+1}]}{\phi_\pi - 1}, \quad \omega_3 = \frac{\kappa_2}{\phi_\pi - 1}, \quad \Omega_3 = \frac{K_2}{\phi_\pi - 1} \tag{3.71}$$

Notice that in both cases the coefficients related with the producer are uniquely pinned down by the coefficients of the consumer. This implies that (3.69) – (3.70) are a system of (9) equations with (10) unknowns thus the system is undetermined. On the contrary, (3.69) – (3.71) is a system of (9) equations with (9) unknowns thus the system is determined as claimed in the main text. In the same manner, by induction this result generalizes for τ periods since in order to have determinacy in our economy signals for the whole horizon are required, concluding the proof of proposition 10. ∎

Proof of Proposition 11: Lastly, for Proposition 11, notice that the Philips curve is the same as described in equation (3.59). Thus :

$$e^{E_t^c[y_{t+1}] - y_y - E_t^c[\eta_{t+1} - \eta_t]} =$$

$$e^{\xi_1 a_t + \xi_2 a_{t+1} + \Xi_1 E_t^c[x_t] + \Xi_2 E_t^c[x_{t+1}] + \omega_1 E_t^p[a_t] + \omega_2 E_{t+1}^p[a_{t+1}] + \Omega_1 E_t^p[x_t] + \Omega_2 E_{t+1}^p[x_{t+1}] + \mu_1 \eta_t + \mu_2 E_t^c[\eta_{t+1}]} \tag{3.72}$$

Notice that if the central bank offers no information about η_{t+1}, $E_t^c[\eta_{t+1}] = 0$ thus the coefficient μ_2 remains undetermined. On the contrary, if the central bank offers information and $E_t^c[\eta_{t+1}] \neq 0$ then $\mu_2 = \frac{1}{\phi_\pi - 1}$. Which concludes the proof of Proposition 5.

3.5.4 Central bank with private information

Lastly we examine the case that the central bank possesses superior information but does not communicate it with the agents. We will argue that such information does not play any role in equilibrium in our setup, allowing us to focus only in the case that the central bank does not have any information. Since we have assumed that agents in the absence of any information do not update their priors, we show that the case where the central bank withholds information collapses to the one where the central bank has no information for the purposes of determining inflation. To see this start with the case of symmetric information where $y_t = a_t$ and notice that the only change would be in our IS equation (16) which would instead be written as:

$$E_t^c[y_{t+1}] - y_t = \phi_\pi E_t^{cb}[\pi_{t+1}] - E_t^c[\pi_{t+1}], \tag{3.73}$$

since now the central bank has some private information.

Next conjecture for inflation:

$$\pi_{t+1} = \xi_1 a_t + \xi_2[a_{t+1}] + \Xi_2 E_{t+1}^c[x_{t+1}] + \xi_3 E_t^{cb}[a_{t+1}]. \tag{3.74}$$

Here we include in the conjecture the beliefs of the central bank from the perspective of period t about a_{t+1} since this is the only way that the agents' information differs.

Next notice that:

$$E_t^{cb}[a_{t+1}] = (1-k)E_{t-1}^{cb}[x_{t+1}] + ks_{t+1}$$
$$E_t^c[E_t^{cb}[a_{t+1}]] = (1-k)E_{t-1}^c[x_{t+1}] + kE_t^c[s_{t+1}] = (1-k)E_{t-1}^c[x_{t+1}]$$

Where the second equation uses the fact that agents have the same information about the permanent component of productivity and the fact that $E_t^c[s_{t+1}] = 0$.

Substitution output and the conjecture in (3.73) we have:

$$\begin{aligned} E_t^c[a_{t+1}] - a_t =& (\phi_\pi - 1)\xi_1 a_t + (\phi_\pi - 1)(\xi_2 + \Xi_2)E_t^c[x_{t+1}] + (\Xi_3 + \xi_3)(1-k)E_{t-1}^c[x_{t+1}] \\ &+ \phi_\pi \xi_3 E_t^{cb}[a_{t+1}]. \end{aligned} \tag{3.75}$$

The last line uses the fact that $E_t^c[a_{t+1}] = E_t^c[x_{t+1}]$. Matching coefficients with the LHS one acquires:

$$\xi_1 = \frac{-1}{\phi_\pi - 1}, \quad (\xi_2 + \Xi_2) = \frac{1}{\phi_\pi - 1}, \quad \xi_3 = 0$$

and notice that this is the same equilibrium with the one where the central banks has no information and it makes no announcements. The same logic applies and in the case with asymmetric information since any private information that the central bank has, agents would estimate using the available information thus any private information would not play a role in equilibrium.

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