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On coderivatives and Lipschitzian properties of the dual pair in optimization

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ABSTRACT

In this paper, we apply the concept of coderivative and other tools from the generalized differentiation theory for set-valued mappings to study the stability of the feasible sets of both the primal and the dual problem in infinite-dimensional linear optimization with infinitely many explicit constraints and an additional conic constraint. After providing some specific duality results for our dual pair, we study the Lipschitz-like property of both mappings and also give bounds for the associated Lipschitz moduli. The situation for the dual shows much more involved than the case of the primal problem.

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1. Introduction

This paper deals with the following linear optimization problem

$$P: \quad \text{Sup} \quad \langle \bar{c}^*, x\rangle \\ \text{s.t.} \quad \langle a_t^*, x\rangle \le \bar{b}_t, \quad t \in T, \\ x \in Q, \tag{1}$$

where T is an arbitrary index set, possibly infinite, Q is a convex cone in a real Banach space X, $\bar{c}^*$ and a_t^*, $t \in T$, belong to the topological dual of X, denoted by X^*, and $\bar{b}_t$, $t \in T$, are real numbers. P is an infinite-dimensional optimization problem with possibly infinitely many linear inequality constraints (depending on the cardinality of T).

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Problems of this type have relevant applications in science and technology. A number of them are reported in [1,2], where the reader can find comprehensive overviews of *infinite-dimensional* and *semi-infinite optimization*, respectively. See also [3], which is confined to the so-called *continuous problem* (when the index set T is a compact Hausdorff space and the functions $t \mapsto a_t^*$ and $t \mapsto \overline{b}_t$ are continuous).

We assume that Q is *closed* and that the set $\{a_t^*,\ t \in T\} \subset X^*$ is fixed, arbitrary, and *bounded* for the dual norm in X^* defined by

$$\|x^*\| := \sup\left\{\langle x^*, x\rangle : \|x\| \leq 1\right\}.$$

(If no confusion arises, we use the same notation $\|\cdot\|$ for the given norm in X and the corresponding dual norm in X^*.)

As a consequence of the boundedness assumption and the generalized Cauchy–Schwarz inequality, we have that, for every $x \in X$,

$$\langle a_{(.)}^*, x\rangle \in \ell_\infty(T),$$

where $\ell_\infty(T)$ is the real Banach space of all bounded functions on T with the supremum norm

$$p \in \ell_\infty(T) \to \|p\|_\infty := \sup_{t\in T} |p_t|.$$

The subscript ∞ in the norm symbol will be omitted if no confusion arises. When the index set T is compact and the functions $a_{(.)}^*$ are continuous on T, we may substitute $\ell_\infty(T)$ by the space $\mathcal{C}(T)$ of continuous functions over a compact set.

By means of the linear mapping $A : X \to \ell_\infty(T)$ defined as $Ax := \langle a_{(.)}^*, x\rangle$, the problem P can be reformulated as

$$P: \quad \begin{array}{ll} \text{Sup} & \langle \overline{c}^*, x\rangle \\ \text{s.t.} & \begin{array}{l} Ax \leq \overline{b}, \\ x \in Q. \end{array} \end{array} \tag{2}$$

Here $\overline{b} = \left(\overline{b}_t\right)_{t\in T}$. Thanks to the boundedness of $\{a_t^*,\ t \in T\}$, the linear operator A is bounded, and so continuous, as

$$\|A\| = \sup_{\|x\|\leq 1} \|Ax\| = \sup_{\|x\|\leq 1} \sup_{t\in T} |\langle a_t^*, x\rangle| \leq \sup_{\|x\|\leq 1} \sup_{t\in T} \left\|a_t^*\right\| \ \|x\| = \sup_{t\in T} \left\|a_t^*\right\|.$$

If X is reflexive, associated with each $t \in T$, there exists some $x_t \in X$ such that $\|x_t\| = 1$ and satisfying $\langle a_t^*, x_t\rangle = \left\|a_t^*\right\|$; this fact leads to $\|A\| = \sup_{t\in T} \left\|a_t^*\right\|$.

The problem P is called *primal* as it has an associated *dual* problem D defined as follows:

$$D: \quad \begin{array}{ll} \text{Inf} & \langle \mu, \overline{b}\rangle \\ \text{s.t.} & \begin{array}{l} A^*\mu \in \overline{c}^* - Q^\circ, \\ \mu \geq 0, \end{array} \end{array}$$

where $\mu \in \ell_\infty(T)^*$, $A^* : \ell_\infty(T)^* \to X^*$ is the adjoint operator of A, i.e.

$$\left\langle A^*\mu, x\right\rangle = \langle \mu, Ax\rangle, \quad \text{for every } \mu \in \ell_\infty(T)^* \text{ and every } x \in X,$$

and Q° is the dual cone of Q

$$Q^\circ := \{q^* \in X^* : \langle q^*, q\rangle \leq 0 \text{ for all } q \in Q\}.$$

This dual problem falls in the duality model introduced by Kretschmer in [4] and it is developed here at an intermediate level of generality between the approaches in [5,6]. Anderson and Nash have given a detailed account of this theory in [1, Chapter 3]. In fact, our pair of dual problems P and D are particular instances of problems IP and IP^* in [1, pp. 38 and 39], respectively. Here, A is a continuous linear mapping between X and $\ell_\infty(T)$ with respect to the norm topologies, but Proposition 5 in [1, p. 37] applies to guarantee that our dual pair falls in the model studied in the book [1, Section 3.3]. Actually, the theory in [1, Section 3.3] is built on a *reflexive* context (dual pairs of vector spaces), but the reflexivity is required only to guarantee that the dual of the dual problem IP^*, i.e. IP^{**}, is identical to IP. Therefore, the reflexivity assumption has no influence in the arguments used in the proofs when this second dual IP^{**} is not involved.

The dual objects we study in the paper are the associated *feasible sets*

$$F_P := \left\{x \in X : Ax \leq \overline{b} \text{ and } x \in Q\right\},$$

and

$$F_D := \left\{\mu \in \ell_\infty(T)^* : A^*\mu \in \overline{c}^* - Q^\circ \text{ and } \mu \geq 0\right\},$$

the *optimal values*

$$v_P := \sup_{x\in F_P} \left\langle \overline{c}^*, x\right\rangle \quad \text{and} \quad v_D := \inf_{\mu\in F_D} \left\langle \mu, \overline{b}\right\rangle,$$

and the *optimal sets*

$$\{x \in F_P : \langle \overline{c}^*, x\rangle = v_P\} \quad \text{and} \quad \{\mu \in F_D : \langle \mu, \overline{b}\rangle = v_D\},$$

respectively.

P (D) is said to be *consistent* if its feasible set is nonempty; similarly, it is said to be *solvable* if its optimal set is nonempty.

The aim of this paper is to provide characterizations of the Lipschitzian stability of feasible solutions for both the primal and the dual problem in this infinite-dimensional setting. We do not require X to be reflexive. For the primal problem, we describe a formula for the associated Lipschitz modulus (Theorems 7 and 8) and for the dual problem, we give bounds for its Lipschitz exact bound (Theorems 18 and 19); the situation for the dual is much more involved than the case of the primal problem. In doing this, we use the standard tools from variational analysis as the notion of coderivative and its norm, and their relationship with the exact bound of the Lipschitzian moduli (see definitions in Sections 2 and 3).

Stability is a paradigmatic issue in optimization, and many users prefer to handle a *good-stable* solution instead of an *optimal-unstable* one. We refer the reader to [7] for the study of *qualitative* stability (formalized through certain semicontinuity properties of the feasible and the optimal set mappings) in semi-infinite optimization (i.e. X is the Euclidean space and T an infinite set), and to [8,9] for this type of analysis in infinite-dimensional programming. In relation to the *quantitative* perspective (via Lipschitzian properties), some relevant references are [10–14], etc. In some of these papers, special attention is paid to the case in which only continuous perturbations of certain particular coefficients are considered. The recent survey [15] provides a panorama of what has been done in the last fifteen years from both qualitative and quantitative perspectives. The closest references, from which this paper receives inspiration, are Cánovas et al. [14], and Ioffe and Sekiguchi [16], as well as the very recent preprint [17].

The paper is organized as follows. After Section 2, which is devoted to notation and basic definitions, Section 3 provides some duality results concerning our dual pair. Section 4 describes briefly the primal and dual feasible set mappings, whereas in the second part of the paper (Sections 5 and 6), we study the stability of the feasible set mappings associated with P and D when perturbations of the objective function $\langle \overline{c}^*, x\rangle$ and of the terms on the right-hand side of the constraints, i.e. of $\overline{b}_t$, $t \in T$, are considered. The stability analysis is done via the Lipschitz-like property of the involved mappings.

2. Notation and basic definitions

For a given subset $\Omega \subset Z$ of a Banach space Z, we denote by conv Ω and cone Ω the convex hull of Ω and the conical convex hull of Ω, respectively. From the topological side, we use the symbols w and w^* to indicate the weak and the weak* topology, respectively, and w- lim and w^*- lim represent the weak and the weak* topological limits, respectively. int Ω and cl Ω are the interior and the closure of Ω with respect to the norm topology, respectively; $\text{cl}^{\,w}\Omega$ stands for the closure in the weak topology; and $\text{cl}^{\,*}\Phi$ is the closure in the weak* topology of a given subset $\Phi \subset Z^*$ in the dual space. We also make use of the property that for convex sets, the norm and the weak closures coincide.

Furthermore, for any $\widehat{z} \in \Omega \subset Z$, with Ω convex, we denote by $N(\widehat{z}; \Omega)$ the normal cone to Ω at $\widehat{z}$ which is given by

$$N(\widehat{z}; \Omega) = \left\{z^* \in Z^* : \left\langle z^*, z - \widehat{z}\right\rangle \leq 0 \text{ for all } z \in \Omega\right\}. \tag{3}$$

In this paper, we shall consider different dual pairs $\{Z, Z^*\}$, where Z is endowed with the original topology of the norm, while Z^* is endowed with the weak* topology. In particular, it is well known (see, for instance, [18]) that there is an isometric isomorphism between $\ell_\infty(T)^*$ and the space

$$ba(T) = \left\{\mu: 2^T \to \mathbb{R}: \mu \text{ is bounded and additive}\right\},$$

satisfying the relationship

$$\langle \mu, p\rangle = \int_T p_t \mu(\mathrm{d}t) \quad \text{with } p = (p_t)_{t\in T}.$$

The dual norm on $\ell_\infty(T)^*$ is the total variation

$$\mu \in \ell_\infty(T)^* \to \|\mu\| := \sup_{A\subset T} \mu(A) - \inf_{B\subset T} \mu(B).$$

It is obvious that if $\mu \geq 0$, we have $\|\mu\| = \mu(T)$, since $\mu(\emptyset) = 0$.

Given an arbitrary set S, we denote by $\mathbb{R}_+^{(S)}$ the set of all $\lambda = (\lambda_s)_{s\in S}$, with $0 \leq \lambda_s \in \mathbb{R}$, for all $s \in S$, and such that $\lambda_s \neq 0$ for at most finitely many $s \in S$.

On the multivalued mapping side, given a set-valued mapping $\mathcal{M} : Z \rightrightarrows Y$, we denote its domain, graph and inverse mapping by

$$\text{dom}\,\mathcal{M} := \{z \in Z : \mathcal{M}(z) \neq \emptyset\},$$
$$\text{gph}\,\mathcal{M} := \{(z, y) \in Z \times Y : y \in \mathcal{M}(z)\},$$
$$\mathcal{M}^{-1}(y) := \{z \in Z : (z, y) \in \text{gph}\,\mathcal{M}\},$$

respectively.

If Z and Y are normed spaces, $\mathcal{M}$ is said to be *Lipschitz-like* around $(\widehat{z},\widehat{y}) \in \operatorname{gph}\mathcal{M}$ (*locally Lipschitz-like* in [19]) with *modulus* $\ell \geq 0$ if there exist neighborhoods U of $\widehat{z}$ and V of $\widehat{y}$ such that

$$\mathcal{M}(z) \cap V \subset \mathcal{M}(u) + \ell \, \|z-u\| \, \mathbb{B}_Y, \quad \text{for all } z, u \in U, \tag{4}$$

where $\mathbb{B}_Y$ is the closed unit ball in the space Y. The infimum of such moduli ℓ's over all possible combinations $\{\ell, U, V\}$ satisfying (4) is called the *exact Lipschitzian bound* of $\mathcal{M}$ around $(\widehat{z},\widehat{y})$ and is denoted by $\operatorname{lip}\mathcal{M}(\widehat{z},\widehat{y})$; it admits the following representation:

$$\operatorname{lip}\mathcal{M}(\widehat{z},\widehat{y}) = \limsup_{(z,y)\to(\widehat{z},\widehat{y})} \frac{\operatorname{dist}(y, \mathcal{M}(z))}{\operatorname{dist}(z, \mathcal{M}^{-1}(y))},$$

where $\inf\emptyset = \infty$ (and so, $\operatorname{dist}(x,\emptyset) = \infty$), and we adopt the convention $0/0 := 0$ and $\infty/\infty := \infty$. We put $\operatorname{lip}\mathcal{M}(\widehat{z},\widehat{y}) = \infty$ if $\mathcal{M}$ is not Lipschitz-like around $(\widehat{z},\widehat{y})$. Observe from (4) that if $\mathcal{M}$ is Lipschitz-like around $(\widehat{z},\widehat{y})$, then $\widehat{z} \in \operatorname{int}(\operatorname{dom}\mathcal{M})$.

This Lipschitz-like property of a mapping $\mathcal{M} : Z \rightrightarrows Y$ between Banach spaces is equivalent to the metric regularity property and also to the linear openness of the inverse mapping $\mathcal{M}^{-1} : Y \rightrightarrows Z$. (See [19] and references therein.)

The exact Lipschitzian bound of $\mathcal{M}$ around $(\widehat{z},\widehat{y})$ satisfies the following relation

$$\operatorname{lip}\mathcal{M}(\widehat{z},\widehat{y}) = \{\operatorname{sur}\mathcal{M}^{-1}(\widehat{y},\widehat{z})\}^{-1},$$

where $\operatorname{sur}\mathcal{M}^{-1}(\widehat{y},\widehat{z})$ is the so-called rate of surjection (openness) of $\mathcal{M}^{-1}$ around $(\widehat{y},\widehat{z})$, defined as follows (see, for instance, [16, p. 256])

$$\operatorname{sur}\mathcal{M}^{-1}(\widehat{y},\widehat{z}) = \liminf_{(y,z,\lambda)\to(\widehat{y},\widehat{z},0^+)} \frac{1}{\lambda} \sup\left\{ r \geq 0 : z + r\mathbb{B}_Z \subset \mathcal{M}^{-1}(y + \lambda\mathbb{B}_Y) \right\},$$

where $\mathbb{B}_Z$ and $\mathbb{B}_Y$ are the closed unit balls in Z and Y, respectively. When $\operatorname{sur}\mathcal{M}^{-1}(\widehat{y},\widehat{z}) = 0$, one gets $\operatorname{lip}\mathcal{M}(\widehat{z},\widehat{y}) = +\infty$, and $\mathcal{M}$ is not Lipschitz-like around $(\widehat{z},\widehat{y})$.

Finally, given $\mathcal{M} : Z \rightrightarrows Y$ and $(\widehat{z},\widehat{y}) \in \operatorname{gph}\mathcal{M}$, the *coderivative* of $\mathcal{M}$ at $(\widehat{z},\widehat{y})$ (*normal coderivative* in [19]) is the positive homogeneous mapping $D^*\mathcal{M}(\widehat{z},\widehat{y}) : Y^* \rightrightarrows Z^*$ defined by:

$$D^*\mathcal{M}(\widehat{z},\widehat{y})\left(y^*\right) := \left\{ z^* \in Z^* : \left(z^*, -y^*\right) \in N((\widehat{z},\widehat{y}); \operatorname{gph}\mathcal{M}) \right\}, \quad y^* \in Y^*, \tag{5}$$

where $N((\widehat{z},\widehat{y}); \operatorname{gph}\mathcal{M})$ is the limiting normal cone to $\operatorname{gph}\mathcal{M}$ at $(\widehat{z},\widehat{y})$ defined in [19, p. 4], and that is given by (3) when $\operatorname{gph}\mathcal{M}$ is convex. The norm of this coderivative is defined as

$$\left\| D^*\mathcal{M}(\widehat{z},\widehat{y}) \right\| := \sup\left\{ \left\|z^*\right\| : z^* \in D^*\mathcal{M}(\widehat{z},\widehat{y})\left(y^*\right), \ \left\|y^*\right\| \leq 1 \right\}. \tag{6}$$

The notion of coderivative is recognized as a powerful tool of variational analysis when applied to problems of optimization and control (see [20,19,21], and the references therein). In [14] they were applied for the first time to analyze the stability of primal inequality systems in semi-infinite programming.

According to [16, Theorem 3], the convex set-valued mapping $\mathcal{M}^{-1}$ is *perfectly regular* at $(\widehat{y},\widehat{z}) \in \operatorname{gph}\mathcal{M}^{-1}$ if and only if

$$\operatorname{sur}\mathcal{M}^{-1}(\widehat{y},\widehat{z}) = \inf\left\{ \left\|y^*\right\| : y^* \in D^*\mathcal{M}^{-1}(\widehat{y},\widehat{z})\left(z^*\right), \ \left\|z^*\right\| = 1 \right\}.$$

If $\mathcal{M}^{-1}$ is perfectly regular at $(\widehat{y},\widehat{z}) \in \operatorname{gph}\mathcal{M}^{-1}$, the following equality holds:

$$\operatorname{lip}\mathcal{M}(\widehat{z},\widehat{y}) = \left\| D^*\mathcal{M}(\widehat{z},\widehat{y}) \right\|.$$

In fact, we have (see [16] for more details)

$$\begin{aligned}
\operatorname{lip}\mathcal{M}(\widehat{z},\widehat{y}) &= \left(\operatorname{sur}\mathcal{M}^{-1}(\widehat{y},\widehat{z})\right)^{-1} \\
&= \left(\inf\left\{ \left\|y^*\right\| : (z^*, y^*) \in \operatorname{gph} D^*\mathcal{M}^{-1}(\widehat{y},\widehat{z}), \ \left\|z^*\right\| = 1 \right\}\right)^{-1} \\
&= \sup\left\{ \left\|y^*\right\|^{-1} : (z^*, y^*) \in \operatorname{gph} D^*\mathcal{M}^{-1}(\widehat{y},\widehat{z}), \ \left\|z^*\right\| = 1 \right\} \\
&= \sup\left\{ \left\|z^*\right\| : (y^*, z^*) \in -\operatorname{gph} D^*\mathcal{M}(\widehat{z},\widehat{y}), \ \left\|y^*\right\| \leq 1 \right\} \\
&= \sup\left\{ \left\|z^*\right\| : z^* \in D^*\mathcal{M}(\widehat{z},\widehat{y})(y^*), \ \left\|y^*\right\| \leq 1 \right\}.
\end{aligned}$$

3. The feasible set mappings

If we allow for perturbations $c^* \in X^*$ and $b \in \ell_\infty(T)$ of the fixed $\overline{c}^*$ and $\overline{b}$, we may consider the *perturbed primal* and *dual* problems

$$P(b, c^*): \quad \begin{array}{ll} \text{Sup} & \langle \overline{c}^* + c^*, x\rangle \\ \text{s.t.} & \langle a_t^*, x\rangle \le \overline{b}_t + b_t, \quad t \in T, \\ & x \in Q, \end{array} \tag{7}$$

and

$$D(b, c^*): \quad \begin{array}{ll} \text{Inf} & \langle \mu, \overline{b} + b\rangle \\ \text{s.t.} & A^*\mu \in \overline{c}^* + c^* - Q^\circ, \\ & \mu \ge 0. \end{array} \tag{8}$$

In order to study the stability of this dual pair, we will consider the feasible set mappings $\mathcal{F}_P : \ell_\infty(T) \rightrightarrows X$ and $\mathcal{F}_D : X^* \rightrightarrows \ell_\infty(T)^*$ defined as follows:

$$\mathcal{F}_P(b) := \{x \in X : Ax \le \overline{b} + b \text{ and } x \in Q\},$$

and

$$\mathcal{F}_D(c^*) := \left\{\mu \in \ell_\infty(T)^* : A^*\mu \in \overline{c}^* + c^* - Q^\circ \text{ and } \mu \ge 0\right\}.$$

The corresponding inverse mappings are

$$\mathcal{F}_P^{-1}(x) := \begin{cases} Ax - \overline{b} + \ell_\infty(T)_+, & \text{if } x \in Q, \\ \emptyset, & \text{if } x \notin Q, \end{cases}$$

and

$$\mathcal{F}_D^{-1}(\mu) := \begin{cases} A^*\mu - \overline{c}^* + Q^\circ, & \text{if } \mu \ge 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Our main objective in this part of the paper is to characterize the Lipschitz-like property of $\mathcal{F}_P$ and $\mathcal{F}_D$, which is equivalent to studying the metric regularity of $\mathcal{F}_P^{-1}$ and $\mathcal{F}_D^{-1}$ at the respective points $(0, \overline{x}) \in \operatorname{gph} \mathcal{F}_P$ and $(0, \overline{\mu}) \in \operatorname{gph} \mathcal{F}_D$, and to determine both exact Lipschitzian bounds (or regularity modulus). Throughout the paper, the *nominal parameters* are the zero function $b = 0 \in \ell_\infty(T)$ for the primal problem, and the zero functional $c^* = 0 \in X^*$ for the dual problem.

It is well known that this property has important consequences in the overall stability of any constraint system, as well as in its sensitivity analysis, and it affects even the numerical complexity of the algorithms conceived for finding a solution of the system. In mathematical programming, many authors explored the relationship of this property with standard constraint qualifications as Mangasarian–Fromovitz, Slater, Robinson, etc. A deep study of this important property and of its consequences can be found in [22,23,16,24,25], etc.

The following reformulations of the perturbed primal and dual problems will allow us to apply known results in order to study their stability properties:

$$P(b, c^*): \quad \begin{array}{lll} \text{Sup} & \langle \overline{c}^*, x\rangle + \left\langle c^*, x\right\rangle & \\ \text{s.t.} & \langle a_t^*, x\rangle \le \overline{b}_t + b_t, & t \in T, \\ & \left\langle q^*, x\right\rangle \le 1, & q^* \in Q^\circ, \end{array} \tag{9}$$

and

$$D(b, c^*): \quad \begin{array}{lll} \text{Inf} & \left\langle \mu, \overline{b}\right\rangle + \langle \mu, b\rangle & \\ \text{s.t.} & \langle \mu, Aq\rangle \ge \langle \overline{c}^*, q\rangle + \langle c^*, q\rangle, & q \in \widetilde{Q}, \\ & \langle \mu, p\rangle \ge -1, & p \in \ell_\infty(T)_+, \end{array} \tag{10}$$

where $\widetilde{Q}$ is a convenient closed bounded set not containing the null vector and spanning the cone Q, which will be the general assumption on $\widetilde{Q}$ from now on.

As usual in convex optimization, the Slater condition is also very important in the study of the stability of the feasibility of these perturbed problems.

Definition 1. $\mathcal{F}_P$ satisfies the *strong Slater condition* at $b \in \ell_\infty(T)$ if there is some $\widehat{x} \in Q$ such that

$$\sup_{t\in T} \left\{\left\langle a_t^*, \widehat{x}\right\rangle - \overline{b}_t - b_t\right\} < 0. \tag{11}$$

Any point satisfying condition (11) is a *strong Slater point* of $\mathcal{F}_P$ at b.

Definition 2. $\mathcal{F}_D$ satisfies the *strong Slater condition* at $c^* \in X^*$ if there is some $\widehat{\mu} \in \ell_\infty(T)^*$, $\widehat{\mu} \geq 0$, such that

$$\inf_{q\in\widetilde{Q}} \left\{ \langle \widehat{\mu}, Aq \rangle - \left\langle \overline{c}^* + c^*, q \right\rangle \right\} > 0. \tag{12}$$

Such a $\widehat{\mu}$ is called a *strong Slater point* of $\mathcal{F}_D$ at c^*.

It is worth mentioning that this second definition is independent of the choice of any particular closed bounded set $\widetilde{Q}$ not containing the null vector and spanning the cone Q.

We also say that $P(b, c^*)$ satisfies the strong Slater condition at b if there exists some strong Slater point of $\mathcal{F}_P$ at b; similarly for $D(b, c^*)$.

Notice that $0 \notin \operatorname{cl}\operatorname{conv}\widetilde{Q}$ whenever $\mathcal{F}_D$ satisfies the strong Slater condition at a point $c^* \in X^*$, and this condition entails that the cone Q is pointed. Moreover, if $\widetilde{Q}$ is a compact base of Q, this condition is also implied by (15). In this case, also observe that if $\widehat{\mu}$ is a Slater point of $\mathcal{F}_D$ at c^* we have that, the infimum

$$\inf_{q\in\widetilde{Q}} \left\{ \langle \widehat{\mu}, Aq \rangle - \left\langle \overline{c}^* + c^*, q \right\rangle \right\} = \inf_{q\in\widetilde{Q}} \left\langle A^*\widehat{\mu} - \overline{c}^* - c^*, q \right\rangle$$

is attained at some point of $\widetilde{Q}$ and so, it is positive. This is a consequence of the weak compactness of $\widetilde{Q}$, and shows that in this case the typical Slater condition and the strong Slater condition are equivalent.

4. Some duality theory

In this section, we provide some specific duality results concerning the nominal problems $P \equiv P(0, 0)$ and $D \equiv D(0, 0)$. Before that we introduce two convex cones:

(a) The first one is

$$H := \{(Ax, \langle \overline{c}^*, x \rangle) : x \in Q\} + \ell_\infty(T)_+ \times (-\mathbb{R}_+), \tag{13}$$

where $\ell_\infty(T)_+$ is the positive cone in $\ell_\infty(T)$. It can be easily seen that P is equivalent to the problem

$$\begin{array}{ll} \text{Sup} & r \\ \text{s.t.} & (\overline{b}, r) \in H. \end{array}$$

(b) The second one is

$$K := (\mathbb{R}_+ \operatorname{conv}\{(a_t^*, \overline{b}_t) : t \in T\}) + Q^\circ \times \mathbb{R}_+. \tag{14}$$

In the paper, we use some convenient sets $\widetilde{Q}$ spanning the cone Q, i.e. sets such that $0 \notin \widetilde{Q}$ and $Q = \mathbb{R}_+\widetilde{Q}$. In particular, we say that Q has a *compact base* $\widetilde{Q}$ if there is some $\overline{x}^* \in X^*$, $\left\| \overline{x}^* \right\| = 1$, such that the set

$$\widetilde{Q} = \{q \in Q : \langle \overline{x}^*, q \rangle = 1\}, \tag{15}$$

is weakly compact and spans Q. According to the observation in the last paragraph of p. 85 in [26], $\widetilde{Q}$ is bounded as a consequence of the Banach–Steinhaus theorem [26, Th. 3.15]. Moreover, when X is reflexive, the fact of assuming that Q has a compact base, entails that Q° has a nonempty interior for the topology associated with the dual norm (Theorem 3.16 in [1], see also [26, Prop 4.36]).

Lemma 1. *Suppose that the closed convex cone Q has a compact base $\widetilde{Q}$. Then the following statements are equivalent:*

(i) *There is no $z \in \widetilde{Q}$ such that $Az \leq 0$ and $\left\langle \overline{c}^*, z \right\rangle \geq 0$.*
(ii) *There exists a strong Slater point of $\mathcal{F}_D$ at 0.*

Proof. (ii) $\Rightarrow$ (i) Reasoning by contradiction, suppose that there exists $\widehat{\mu} \in \ell_\infty(T)^*$, $\widehat{\mu} \geq 0$, such that

$$\inf_{q\in\widetilde{Q}} \left\{ \langle \widehat{\mu}, Aq \rangle - \left\langle \overline{c}^*, q \right\rangle \right\} > 0, \tag{16}$$

for a certain closed bounded set $\widetilde{Q}$ not containing the null vector and spanning the cone Q, and suppose that, at the same time, there is $z_0 \in \widetilde{Q}$ satisfying $Az_0 \leq 0$ and $\left\langle \overline{c}^*, z_0 \right\rangle \geq 0$. From that and $\widehat{\mu} \geq 0$, we get

$$\langle \widehat{\mu}, Az_0 \rangle - \left\langle \overline{c}^*, z_0 \right\rangle \leq 0,$$

contradicting (16).

(i) $\Rightarrow$ (ii) Take $t_0 \notin T$ and define $\widetilde{T} := T \cup \{t_0\}$. Consider then the linear mapping $\widetilde{A} : X \to \ell_\infty(\widetilde{T})$ such that

$$\widetilde{A}(x) = (Ax, \langle -\overline{c}^*, x \rangle),$$

i.e.

$$\left(\widetilde{A}(x)\right)(t) := \left\langle a_t^*, x\right\rangle \quad \text{if } t \in T, \quad \text{and} \quad \left(\widetilde{A}(x)\right)(t_0) := \left\langle -\overline{c}^*, x\right\rangle.$$

The linear mapping $\widetilde{A}$ is bounded and then, continuous. Applying the statement Section 20.4(5) in [27] (see also Proposition 5 of Ch. 3 of [1]), we know that $\widetilde{A}$ is also continuous with respect to the weak topologies in X and $\ell_\infty(\widetilde{T})$, and therefore the image set $\widetilde{A}(\widetilde{Q})$ is convex and weakly compact. Since (i) can be formulated as

$$\widetilde{A}(\widetilde{Q}) \cap \left(-\ell_\infty(\widetilde{T})_+\right) = \emptyset,$$

we can apply the strong separation theorem (see, for instance, Theorem 3.17 in [26]) to establish the existence of $\widetilde{\mu} \in \ell_\infty(\widetilde{T})^*$ together with two scalars α and β such that

$$\left\langle \widetilde{\mu}, \widetilde{A}q\right\rangle \geq \beta > \alpha \geq \langle \widetilde{\mu}, \widetilde{p}\rangle, \quad \text{for all } q \in \widetilde{Q} \text{ and all } \widetilde{p} \in -\ell_\infty(\widetilde{T})_+. \tag{17}$$

Because $\widetilde{\mu}$ is finitely additive on $2^{\widetilde{T}}$, we decompose

$$\widetilde{\mu} = (\widehat{\mu}, \mu),$$

with $\widehat{\mu} : 2^T \to \mathbb{R}$ bounded and finitely additive and $\mu := \widetilde{\mu}(\{t_0\})$. In this way, if $S \subset T$,

$$\widetilde{\mu}(S \cup \{t_0\}) = \widehat{\mu}(S) + \mu.$$

Moreover, if $\widetilde{p} \in \ell_\infty(\widetilde{T})$, i.e. if $\widetilde{p} = (p, \rho) \in \ell_\infty(T) \times \mathbb{R}$, we have

$$\langle \widetilde{\mu}, \widetilde{p}\rangle = \int_{\widetilde{T}} \widetilde{p}_t \widetilde{\mu}(\mathrm{d}t) = \int_T p_t \widehat{\mu}(\mathrm{d}t) + \rho\mu = \langle \widehat{\mu}, p\rangle + \rho\mu. \tag{18}$$

Now (17) is written as follows

$$\langle \widehat{\mu}, Aq\rangle - \mu \left\langle \overline{c}^*, q\right\rangle \geq \beta > \alpha \geq \langle \widehat{\mu}, p\rangle + \rho\mu, \quad \forall q \in \widetilde{Q} \text{ and } \forall (p, \rho) \in (-\ell_\infty(T)_+) \times (-\mathbb{R}_+). \tag{19}$$

We proceed with the following discussion:

(a) Taking into account that $(-\ell_\infty(T)_+) \times (-\mathbb{R}_+)$ is a cone, we can take $\alpha = 0$.
(b) If $\mu < 0$, we can take $p = 0$ and $\rho > 0$, and the last inequality in (19) would fail; so $\mu \geq 0$.
(b_1) If $\mu = 0$, (19) gives rise to

$$\langle \widehat{\mu}, Aq\rangle \geq \beta > 0 \geq \langle \widehat{\mu}, p\rangle, \quad \forall q \in \widetilde{Q} \text{ and } \forall p \in -\ell_\infty(T)_+.$$

Then $\widehat{\mu} \geq 0$. Since $\widetilde{Q}$ is weakly compact, it will be bounded, which entails the existence of a constant M such that

$$\left|\left\langle \overline{c}^*, q\right\rangle\right| \leq M, \quad \forall q \in \widetilde{Q}.$$

By defining

$$\widehat{\mu}_0 := \frac{2M}{\beta}\widehat{\mu},$$

we get

$$\begin{aligned}\langle \widehat{\mu}_0, Aq\rangle - \left\langle \overline{c}^*, q\right\rangle &= \frac{2M}{\beta}\langle \widehat{\mu}, Aq\rangle - \left\langle \overline{c}^*, q\right\rangle \\ &\geq \frac{2M}{\beta}\beta - M = M > 0, \quad \forall q \in \widetilde{Q}.\end{aligned}$$

(b_2) If $\mu > 0$, we divide (19) by μ and defining $\widehat{\mu}_1 := \frac{1}{\mu}\widehat{\mu}$, one has

$$\langle \widehat{\mu}_1, Aq\rangle - \left\langle \overline{c}^*, q\right\rangle \geq \beta/\mu > 0, \quad \forall q \in \widetilde{Q}. \quad \square$$

To exclude the existence of the duality gap (i.e. that $v_P \neq v_D$) under certain assumptions we need the following lemma, which is an adaptation of Theorem 3.17 in [1] and concerns the notion of compact base of a cone.

Lemma 2. *If Q has a compact base $\widetilde{Q}$ and there exists a strong Slater $\widehat{\mu}$ point of $\mathcal{F}_D$ at 0, then the set H is closed.*

Now we establish our duality theorem.

Theorem 3. *If P is consistent, then the following statements hold:*

(i) *$v_P \leq v_D$ and, if D is also consistent, both values are finite.*
(ii) *If v_P is finite, Q has a compact base, and there exists a strong Slater point of $\mathcal{F}_D$ at 0, then there is no duality gap (i.e. $v_P = v_D$) and P is solvable.*
(iii) *If v_P is finite, and the cone K defined in (14) is w^*-closed, then there is no duality gap and D is solvable.*

Proof. (i) If P is consistent and D is inconsistent, then $v_P \leq +\infty = v_D$. If both problems are consistent, take $x \in F_P$ and $\mu \in F_D$. Since $\mu \in F_D$ then $\mu \geq 0$, and there must exist $q^* \in Q^\circ$ such that $\overline{c}^* = A^*\mu + q^*$. The result is a trivial consequence of the following observation

$$\left\langle \overline{c}^*, x\right\rangle = \left\langle A^*\mu, x\right\rangle + \left\langle q^*, x\right\rangle \leq \left\langle A^*\mu, x\right\rangle = \langle \mu, Ax\rangle \leq \left\langle \mu, \overline{b}\right\rangle.$$

(ii) By Lemma 2, H is closed, and the proof is a mere adaptation of the proofs of Theorems 3.9 and 3.22 in [1].
(iii) Obviously, the inequality $\left\langle \overline{c}^*, x\right\rangle \leq v_P$ is a consequence of the consistent system

$$\{\langle a_t^*, x\rangle \leq \overline{b}_t,\ t \in T;\ \langle q^*, x\rangle \leq 0,\ q^* \in Q^\circ\},$$

since $Q = Q^{\circ\circ}$ (see, for instance, Theorem 4.32 in [26]). Applying the asymptotic Farkas Lemma (Theorem 4.1 in [28]), we get

$$(\overline{c}^*, v_P) \in \text{cl}^*K = K,$$

and there exist $\lambda^0 \in \mathbb{R}_+^{(T)}$, $\overline{q}_0 \in Q^\circ$, and $\rho_0 \geq 0$ such that

$$(\overline{c}^*, v_P) = \sum_{t\in T} \lambda_t^0 \left(a_t^*, \overline{b}_t\right) + (\overline{q}_0, \rho_0). \tag{20}$$

If we consider

$$\mu_0 := \sum_{t\in T} \lambda_t^0 \delta_t \in \ell_\infty(T)^*,$$

where $\delta_t \in \ell_\infty(T)^*$ denotes the *Dirac measure* defined by $\langle \delta_t, p\rangle = p_t$, for any $p \in \ell_\infty(T)$, it is easy to see that $A^*\mu_0 = \sum_{t\in T} \lambda_t^0 a_t^*$, and therefore (20) yields

$$\overline{c}^* = A^*\mu_0 + \overline{q}_0 \quad \text{and} \quad v_P = \left\langle \mu_0, \overline{b}\right\rangle + \rho_0.$$

The first equality above shows that μ_0 is a feasible solution of D, and consequently

$$v_D \leq \left\langle \mu_0, \overline{b}\right\rangle \leq \left\langle \mu_0, \overline{b}\right\rangle + \rho_0 = v_P.$$

This inequality, together with the weak dual inequality established in (i), gives $v_D = v_P$ and shows that μ_0 is optimal for D. □

5. Lipschitzian stability of the primal feasible set mapping

In Cánovas et al. [14] the Lipschitz-like property and the calculus of the exact Lipschitzian bound for a certain primal feasible set mapping are related to some characteristic set. In the present setting, we need to take into account the conic constraint $x \in Q$ as well. Therefore, we define, as in [14], the *characteristic set* of $\mathcal{F}_P(b)$ as the convex subset of $X^* \times \mathbb{R}$ spanned by the set of coefficients of the constraint system:

$$C_P(b) := \text{conv}\left(\left\{\left(a_t^*, \overline{b}_t + b_t\right) : t \in T\right\} \cup (Q^\circ \times \{1\})\right). \tag{21}$$

We observe that

$$\text{cone}\, C_P(0) \equiv \mathbb{R}_+ C_P(0) = K,$$

where K is the cone defined in (14), and a closedness condition for the absence of the duality gap is related to this set according to Theorem 3(iii).

We will obtain characterizations of the consistency of the primal problem, and also of the Lipschitz-like property, the normal cone, the coderivative, and the exact Lipschitzian bound; all of these properties being expressed in terms of the given data, mainly through this characteristic set $C_P(b)$. Indeed, these properties follow by taking into account the expression (9) of the primal problem and by an almost straightforward application of the results established in the aforementioned paper [14], with the difference that now we need to pay special attention to the unboundedness of the coefficients of the constraints $\langle q^*, x\rangle \leq 1$, and observe that any $x \in Q$ satisfies $\langle q^*, x\rangle \leq 0$, for all $q^* \in Q^\circ$, since $Q = Q^{\circ\circ}$. In this way we can easily adapt the proof of Lemma 2.3 in [14] to prove that the consistency of $P(b, c^*)$ is equivalent to $(0, -1) \notin \text{cl}^*\text{cone}\, C_P(b)$, and that a consistent problem $P(b, c^*)$ satisfies the strong Slater condition if and only if $(0, 0) \notin \text{cl}^* C_P(b)$, if and only if $b \in \text{int}\,(\text{dom}\, \mathcal{F}_P)$, if and only if $\mathcal{F}_P$ is Lipschitz-like around (b, x) for all $x \in \mathcal{F}_P(b)$. In the following proposition, we show that if $\text{int}\, Q \neq \emptyset$, we can add a new condition to the previous list. If X is reflexive, $\text{int}\, Q \neq \emptyset$ if and only if Q° has a compact base [1, Theorem 3.16].

Proposition 4. *Let $\mathcal{F}_P(b) \neq \emptyset$. If $\operatorname{int} Q \neq \emptyset$, then $\mathcal{F}_P$ is Lipschitz-like around (b, x) for all $x \in \mathcal{F}_P(b)$ if and only if there exists some $\overline{x} \in X$ such that $(b, \overline{x}) \in \operatorname{int}(\operatorname{gph} \mathcal{F}_P)$.*

Proof. $(\Leftarrow)$ If $(b, \overline{x}) \in \operatorname{int}(\operatorname{gph} \mathcal{F}_P)$, then $b \in \operatorname{int}(\operatorname{dom} \mathcal{F}_P)$ which is equivalent to $\mathcal{F}_P$ being Lipschitz-like around (b, x), for all $x \in \mathcal{F}_P(b)$.

$(\Rightarrow)$ Let $\widehat{q} \in \operatorname{int} Q$ be fixed. Now $\mathcal{F}_P(b) \neq \emptyset$ and $\mathcal{F}_P$ being Lipschitz-like around (b, x) for all $x \in \mathcal{F}_P(b)$ gives that $\mathcal{F}_P$ satisfies the strong Slater condition at b. If $\widehat{x} \in Q$ is any strong Slater point for $\mathcal{F}_P$ at b, let

$$\sup_{t \in T} \left(\left\langle a_t^*, \widehat{x} \right\rangle - \overline{b}_t - b_t \right) \leq -\vartheta < 0,$$

for some $\vartheta > 0$. Put

$$M := \sup_{t \in T} \left\| a_t^* \right\| < +\infty$$

and

$$\lambda := \frac{\vartheta}{6\,(1 + M)\,(\|\widehat{q}\| + 1)} > 0,$$

and observe that $\widehat{x} + \lambda \widehat{q} \in \operatorname{int} Q$. Finally, let $r > 0$ be such that $\widehat{x} + \lambda \widehat{q} + u \in Q$ for $\|u\| < r$. Hence, if $u \in X$ and $b' \in \ell_\infty(T)$ are such that $\|u\| < \min\{r, \frac{\vartheta}{6(1+M)}\}$ and $\left\| b' \right\|_\infty < \frac{\vartheta}{6}$, then

$$\begin{aligned}
\left\langle a_t^*, \widehat{x} + \lambda \widehat{q} + u \right\rangle - \overline{b}_t - b_t - b_t' &\leq \left\langle a_t^*, \widehat{x} \right\rangle + \left\| a_t^* \right\| (\lambda \|\widehat{q}\| + \|u\|) - \overline{b}_t - b_t - b_t' \\
&\leq -\vartheta + M \left(\frac{\vartheta \|\widehat{q}\|}{6\,(1 + M)\,(\|\widehat{q}\| + 1)} + \frac{\vartheta}{6\,(1 + M)} \right) + \frac{\vartheta}{6} \\
&\leq -\frac{\vartheta}{2} < 0,
\end{aligned}$$

for all $t \in T$. Therefore $(b, \overline{x}) \in \operatorname{int}(\operatorname{gph} \mathcal{F}_P)$ for $\overline{x} = \widehat{x} + \lambda \widehat{q}$. $\square$

Next, we state the most remarkable results about the coderivative and the Lipschitz-like property. The proofs are omitted for the sake of brevity. The reader can also find an alternative approach in the recent preprint [17]. With the aid of Dirac measures, we can characterize the coderivative $D^* \mathcal{F}_P\,(0, \widehat{x})\,(x^*)$ as the following result shows.

Proposition 5. *Let $\widehat{x} \in \mathcal{F}_P(0)$, $\mu \in \ell_\infty(T)^*$, and $x^* \in X^*$. Then $\mu \in D^* \mathcal{F}_P\,(0, \widehat{x})\,(x^*)$ if and only if*

$$\left(\mu, -x^*, -\left\langle x^*, \widehat{x} \right\rangle \right) \in \operatorname{cl}^* \left(\operatorname{cone} \left\{ \left(-\delta_t, a_t^*, \overline{b}_t \right),\ t \in T \right\} + \{0\} \times Q^\circ \times \{0\} \right).$$

Theorem 6. *Let $\widehat{x} \in \mathcal{F}_P(0)$. Then $\mathcal{F}_P$ is Lipschitz-like around $(0, \widehat{x})$ if and only if*

$$D^* \mathcal{F}_P(0, \widehat{x})(0) = \{0\}.$$

Theorem 7. *Let $\widehat{x} \in \mathcal{F}_P(0)$. Then,*

(i) *If $\widehat{x}$ is a strong Slater point of $\mathcal{F}_P$ at $b = 0$, then $\|D^* \mathcal{F}_P(0, \widehat{x})\| = 0$.*
(ii) *If $\widehat{x}$ is not a strong Slater point of $\mathcal{F}_P$ at $b = 0$, then $\|D^* \mathcal{F}_P(0, \widehat{x})\| > 0$ and it can be calculated as*

$$\left\| D^* \mathcal{F}_P(0, \widehat{x}) \right\| = \sup \left\{ \left\| x^* \right\|^{-1} : \left(x^*, \left\langle x^*, \widehat{x} \right\rangle \right) \in \operatorname{cl}^* C_P(0) \right\}. \tag{22}$$

Remark 1. In case (ii), if the strong Slater condition is not satisfied at $b = 0$, then $(0, 0) \in \operatorname{cl}^* C_P\,(0)$ and according to (6), we get $\|D^* \mathcal{F}_P(0, \widehat{x})\| = \infty$.

Theorem 8. *Let $\widehat{x} \in \mathcal{F}_P(0)$. Then*

$$\operatorname{lip} \mathcal{F}_P(0, \widehat{x}) = \left\| D^* \mathcal{F}_P(0, \widehat{x}) \right\|. \tag{23}$$

Remark 2. This result is also a consequence of the perfect regularity of $\mathcal{F}_P^{-1}$ at $(\widehat{x}, 0)$ which can be proved following an argument similar to the one used in [17, Proposition 5]. If the strong Slater condition is not satisfied, then both terms in (23) are $+\infty$.

6. Lipschitzian stability of the dual feasible set mapping

From the dual point of view, we will show similar properties as in the previous section in relation to the Lipschitzian stability of the dual feasible set mapping $\mathcal{F}_D$. Here, special care is required since the perturbations $\langle c^*, q\rangle$ on the right-hand side of the dual constraints have some special structure and the theory developed in [14] does not apply in general.

In this section, we will consider a bounded closed set $\widetilde{Q}$, not containing the null vector and spanning the cone Q.

Now the characteristic set of $\mathcal{F}_D(c^*)$, relative to $\widetilde{Q}$, is defined as the following convex subset of $\ell_\infty(T) \times \mathbb{R}$:

$$C_D\left(c^*\right) := \operatorname{conv}\left(\left\{\left(Aq, \left\langle \overline{c}^* + c^*, q\right\rangle\right) : q \in \widetilde{Q}\right\} \cup \{(p, -1) : p \in \ell_\infty(T)_+\}\right). \tag{24}$$

Observe that

$$\operatorname{cone} C_D(0) \equiv \mathbb{R}_+ C_D(0) = H,$$

where H is the cone defined in (13), and a closedness condition for the absence of the duality gap is related to this set according to Theorem 3(ii) and Lemma 2. (H does not depend on the choice of $\widetilde{Q}$.)

6.1. Characterization of stably consistent dual problems

The stability with respect to the consistency of the dual problems will be analyzed through the mapping $\mathcal{F}_D$ by noting that a dual problem $D(b, c^*)$ is *stably consistent* if and only if $c^* \in \operatorname{int}\ (\operatorname{dom} \mathcal{F}_D)$. Observe that an application of the classical Robinson–Ursescu theorem [23,19] implies that this condition is equivalent to $\mathcal{F}_D$ being Lipschitz-like around (c^*, μ) for all $\mu \in \mathcal{F}_D\,(c^*)$, because the graph of $\mathcal{F}_D^{-1} : \ell_\infty(T)^* \rightrightarrows X^*$ is closed and convex, and $\ell_\infty(T)^*$ and X^* are Banach spaces.

Lemma 9. *Given $c^* \in X^*$ and the following linear system posed in $\ell_\infty(T)^*$*

$$\sigma_D(c^*) := \left\{ \begin{array}{ll} \langle \mu, Aq\rangle \geq \left\langle \overline{c}^* + c^*, q\right\rangle, & q \in \widetilde{Q}, \\ \langle \mu, p\rangle \geq -1, & p \in \ell_\infty(T)_+ \end{array} \right\}, \tag{25}$$

then

$$\sigma_D(c^*) \text{ is consistent (i.e. } c^* \in \operatorname{dom} \mathcal{F}_D) \iff (0, 1) \notin \operatorname{cl} H(c^*),$$

where

$$H(c^*) = \{(Ax, \langle \overline{c}^* + c^*, x\rangle) : x \in Q\} + \ell_\infty(T)_+ \times (-\mathbb{R}_+).$$

(Observe that $H(0) = H$.)

Proof. It follows directly from Theorem 3.1 in [28]. $\square$

Proposition 10. *Let $c^* \in \operatorname{dom} \mathcal{F}_D$. If we suppose that $0 \notin \operatorname{cl}\operatorname{conv} \widetilde{Q}$, then the following statements are equivalent:*

(i) *There is some $\widehat{\mu} \geq 0$ that is a strong Slater point for $\mathcal{F}_D$ at c^*.*
(ii) $(0, 0) \notin \operatorname{cl}^* C_D\,(c^*)$.
(iii) $c^* \in \operatorname{int}\ (\operatorname{dom} \mathcal{F}_D)$.

Proof. (i) $\Leftrightarrow$ (ii) and (i) $\Rightarrow$ (iii) are proved following the same arguments of the proof of Lemma 2.3 in [14] to the system (25) posed in $\ell_\infty(T)^*$.

(iii) $\Rightarrow$ (i) If $c^* \in \operatorname{int}\ (\operatorname{dom} \mathcal{F}_D)$ then $c^* + c'^* \in \operatorname{dom} \mathcal{F}_D$ whenever $\left\|c'^*\right\| \leq \varepsilon$ for $\varepsilon > 0$ small enough. On the other hand, $0 \notin \operatorname{cl}\operatorname{conv} \widetilde{Q}$ and the strong separation property gives the existence of $x^* \in X^*$, $\|x^*\| = 1$, and $\beta \in \mathbb{R}$ such that

$$\left\langle x^*, q\right\rangle \geq \beta > 0 \quad \text{for all } q \in \widetilde{Q}.$$

Now, for $c'^* = \varepsilon x^*$, there exists $\widehat{\mu} \geq 0$ such that

$$\langle \widehat{\mu}, Aq\rangle - \left\langle \overline{c}^* + c^* + c'^*, q\right\rangle \geq 0 \quad \text{for all } q \in \widetilde{Q}.$$

Take any such $q \in \widetilde{Q}$. From

$$\left\langle c'^*, q\right\rangle = \varepsilon \left\langle x^*, q\right\rangle \geq \varepsilon\beta > 0$$

it follows that

$$\langle \widehat{\mu}, Aq\rangle - \left\langle \overline{c}^* + c^*, q\right\rangle \geq \left\langle c'^*, q\right\rangle \geq \varepsilon\beta > 0.$$

Hence $\widehat{\mu} \geq 0$ is a strong Slater point of $\mathcal{F}_D$ at c^*. $\square$

Remark 3. Observe that the hypothesis $0 \notin \operatorname{cl}\operatorname{conv} \widetilde{Q}$ is only needed for the implication (iii) $\Rightarrow$ (i). Also recall that the existence of a strong Slater point implies the condition $0 \notin \operatorname{cl}\operatorname{conv} \widetilde{Q}$. Another important observation is that indeed condition (ii) $(0, 0) \notin \operatorname{cl}^* C_D\,(c^*)$ is equivalent to (ii)′ $(0, 0) \notin \operatorname{cl} C_D\,(c^*)$, but we prefer to keep the cl^* notation because we will always be considering the w^*-topology on $\ell_\infty(T)^{**}$.

6.2. Characterization of coderivatives

Let us note that when $x^{**} \in X$ and $b^{**} \in \ell_\infty(T)$ in Proposition 11 and Theorem 13, we may replace the weak*-closure by the norm closure because we are considering the closures of convex sets.

Now we will give a characterization of the normal cone to gph $\mathcal{F}_D$ at $(\widehat{c}^*, \widehat{\mu})$.

Proposition 11. *Let $(\widehat{c}^*, \widehat{\mu}) \in \operatorname{gph} \mathcal{F}_D$ and let $(x^{**}, b^{**}) \in X^{**} \times \ell_\infty(T)^{**}$. Then*

$$\left(x^{**}, b^{**}\right) \in N\left(\left(\widehat{c}^*, \widehat{\mu}\right); \operatorname{gph} \mathcal{F}_D\right)$$

if and only if $-(x^{**}, b^{**}, \langle \widehat{c}^*, x^{**}\rangle + \langle \widehat{\mu}, b^{**}\rangle)$ *belongs to*

$$\operatorname{cl}^*\{\{(-q, Aq, \langle \overline{c}^*, q\rangle) : q \in Q\} + \{0\} \times \ell_\infty(T)_+ \times (-\mathbb{R}_+)\}. \tag{26}$$

Proof. The proof follows from the definition of $N\left((\widehat{c}^*, \widehat{\mu}); \operatorname{gph} \mathcal{F}_D\right)$, by taking into account that the set gph $\mathcal{F}_D$ can be expressed as

$$\operatorname{gph} \mathcal{F}_D = \left\{\left(c^*, \mu\right) \in X^* \times \ell_\infty(T)^* : \begin{matrix}\left\langle\left(c^*, \mu\right), (-q, Aq)\right\rangle \geq \left\langle\overline{c}^*, q\right\rangle, & q \in \widetilde{Q} \\ \left\langle\left(c^*, \mu\right), (0, p)\right\rangle \geq -1, & p \in \ell_\infty(T)_+\end{matrix}\right\},$$

and applying the asymptotic Farkas Lemma (Theorem 4.1 in [28]). □

The following lemma will be used in the proof of the next theorem which provides a characterization of the coderivative of $\mathcal{F}_D$ at any given $(0, \widehat{\mu}) \in \operatorname{gph} \mathcal{F}_D$.

Lemma 12. *Let $\widehat{\mu} \in \mathcal{F}_D(0), x \in X^{**}$ and $b^{**} \in \ell_\infty(T)^{**}$. If $x^{**} \in D^*\mathcal{F}_D(0, \widehat{\mu})(b^{**})$ then there exists a net*

$$\{(q_\nu, p_\nu)\}_{\nu \in \mathcal{N}} \subset Q \times \ell_\infty(T)_+$$

such that

$$x^{**} = w^*\text{-}\lim_{\nu \in \mathcal{N}} q_\nu,$$
$$b^{**} = w^*\text{-}\lim_{\nu \in \mathcal{N}} (Aq_\nu + p_\nu),$$
$$\left\langle\widehat{\mu}, b^{**}\right\rangle = \lim_{\nu \in \mathcal{N}} \left\langle\overline{c}^*, q_\nu\right\rangle.$$

Moreover, if $\widehat{\mu}$ is a strong Slater point for $\mathcal{F}_D$ at 0, then $x^{**} = 0$.

Proof. Let $\widehat{\mu} \in \mathcal{F}_D(0), x \in X^{**}$, and $b^{**} \in \ell_\infty(T)^{**}$ be such that $x^{**} \in D^*\mathcal{F}_D(0, \widehat{\mu})(b^{**})$. From the definition of the coderivative given in (5), $x^{**} \in D^*\mathcal{F}_D(0, \widehat{\mu})(b^{**})$ if and only if $(x^{**}, -b^{**}) \in N((0, \widehat{\mu}); \operatorname{gph} \mathcal{F}_D)$, if and only if (by Proposition 11) $(-x^{**}, b^{**}, \langle \widehat{\mu}, b^{**}\rangle)$ belongs to the set in (26).

Then there exists a net $(q_\nu, p_\nu, -\gamma_\nu)_{\nu \in \mathcal{N}} \subset Q \times \ell_\infty(T)_+ \times (-\mathbb{R}_+)$ such that

$$(-x^{**}, b^{**}, \langle \widehat{\mu}, b^{**}\rangle) = w^*\text{-}\lim_{\nu \in \mathcal{N}}\{(-q_\nu, Aq_\nu, \langle \overline{c}^*, q_\nu\rangle) + (0, p_\nu, -\gamma_\nu)\}. \tag{27}$$

Clearly

$$x^{**} = w^*\text{-}\lim_{\nu \in \mathcal{N}} q_\nu,$$
$$b^{**} = w^*\text{-}\lim_{\nu \in \mathcal{N}} (Aq_\nu + p_\nu), \tag{28}$$

and

$$\langle \widehat{\mu}, b^{**}\rangle = \lim_{\nu \in \mathcal{N}} (\langle \overline{c}^*, q_\nu\rangle - \gamma_\nu). \tag{29}$$

Applying expression (27) to $(0, \widehat{\mu}, -1)$, and from $\langle \widehat{\mu}, p_\nu\rangle \geq 0$ and $\widehat{\mu} \in \mathcal{F}_D(0)$, one gets

$$\begin{aligned} 0 &= \lim_{\nu \in \mathcal{N}} \left(\langle \widehat{\mu}, Aq_\nu + p_\nu\rangle - \left\langle\overline{c}^*, q_\nu\right\rangle + \gamma_\nu\right) \\ &\geq \limsup_{\nu \in \mathcal{N}} \left(\left\langle A^*\widehat{\mu}, q_\nu\right\rangle - \left\langle\overline{c}^*, q_\nu\right\rangle + \gamma_\nu\right) \\ &\geq \limsup_{\nu \in \mathcal{N}} \gamma_\nu \geq 0. \end{aligned}$$

Hence

$$\lim_{\nu \in \mathcal{N}} \gamma_\nu = 0.$$

Finally, assume that $\widehat{\mu}$ is a strong Slater point for $\mathcal{F}_D$ at 0 and let $\vartheta > 0$ be such that

$$\inf_{q\in\widetilde{Q}} \left\{ \langle \widehat{\mu}, Aq\rangle - \left\langle \overline{c}^*, q\right\rangle \right\} \geq \vartheta > 0.$$

Then, from (28) and (29),

$$\lim_{\nu\in\mathcal{N}} \langle \widehat{\mu}, Aq_\nu + p_\nu\rangle = \left\langle \widehat{\mu}, b^{**}\right\rangle = \lim_{\nu\in\mathcal{N}} \left\langle \overline{c}^*, q_\nu\right\rangle. \tag{30}$$

Moreover, we can express $q_\nu = \rho_\nu \widetilde{q}_\nu$ with $\widetilde{q}_\nu \in \widetilde{Q}$ and $\rho_\nu > 0$, for all $\nu \in \mathcal{N}$, and from (30)

$$\begin{aligned} 0 &= \lim_{\nu\in\mathcal{N}} \left\{ \langle \widehat{\mu}, \rho_\nu A\widetilde{q}_\nu + p_\nu\rangle - \left\langle \overline{c}^*, \rho_\nu \widetilde{q}_\nu\right\rangle \right\} \\ &\geq \limsup_{\nu\in\mathcal{N}} \rho_\nu \left(\left\langle A^*\widehat{\mu}, \widetilde{q}_\nu\right\rangle - \left\langle \overline{c}^*, \widetilde{q}_\nu\right\rangle \right) \\ &\geq \vartheta \limsup_{\nu\in\mathcal{N}} \rho_\nu. \end{aligned}$$

The fact that $\vartheta > 0$ yields $\lim_{\nu\in\mathcal{N}} \rho_\nu = 0$. Since $x^{**} = w^*\text{-}\lim_{\nu\in\mathcal{N}} q_\nu$ we obtain, as a consequence of the w^*-lower semicontinuity of the norm, and the boundedness of $\widetilde{Q}$,

$$\left\| x^{**}\right\| \leq \liminf_{\nu\in\mathcal{N}} \|\rho_\nu \widetilde{q}_\nu\| \leq \left(\sup_{q\in\widetilde{Q}} \|q\| \right) \left(\lim_{\nu\in\mathcal{N}} \rho_\nu \right) = 0,$$

which gives $x^{**} = 0$. □

Theorem 13. *Let $\widehat{\mu} \in \mathcal{F}_D(0)$. If $x^{**} \in X^{**}$ and $b^{**} \in \ell_\infty(T)^{**}$, then $x^{**} \in D^*\mathcal{F}_D(0, \widehat{\mu})(b^{**})$ if and only if $(x^{**}, b^{**}, \langle \widehat{\mu}, b^{**}\rangle)$ belongs to*

$$\text{cl}^*\{\{(q, Aq, \langle \overline{c}^*, q\rangle) : q \in Q\} + \{0\} \times \ell_\infty(T)_+ \times \{0\}\}.$$

Proof. ($\Rightarrow$) It follows readily from the previous lemma.

($\Leftarrow$) The definition of $D^*\mathcal{F}_D(0, \widehat{\mu})(b^{**})$ and Proposition 11 applied to $(x^{**}, -b^{**})$ give the result. □

Lemma 14. *Given $\widehat{\mu} \in \mathcal{F}_D(0)$, the following statements hold:*

(i) *If $\widetilde{Q}$ is bounded and $\widehat{\mu}$ is not a strong Slater point of $\mathcal{F}_D$ at 0, then the set*

$$S_D := \{b^{**} \in \ell_\infty(T)^{**} | (b^{**}, \langle \widehat{\mu}, b^{**}\rangle) \in \text{cl}^* C_D(0)\} \tag{31}$$

is nonempty and w^*-closed. Moreover, if $\widetilde{Q}$ is a compact base of Q, then

$$\{b \in \ell_\infty(T) | (b, \langle \widehat{\mu}, b\rangle) \in C_D(0)\} \neq \emptyset.$$

(ii) *If $\widehat{\mu}$ is a strong Slater point of $\mathcal{F}_D$ at 0, then $S_D = \emptyset$.*

Proof. (i) If $\widehat{\mu}$ is not a Slater point of $\mathcal{F}_D$ at 0, there exists a sequence $(q_k)_{k=1}^\infty \subset \widetilde{Q}$ such that

$$\lim_{k\to\infty} \left\{ -\langle \widehat{\mu}, Aq_k\rangle + \left\langle \overline{c}^*, q_k\right\rangle \right\} = 0. \tag{32}$$

First, suppose that $\|q\| \leq R$ for any $q \in \widetilde{Q}$. Since $Aq \in \ell_\infty(T)$, one has

$$\begin{aligned} \|Aq\|_{**} &= \sup\left\{ |\langle \mu, Aq\rangle| : \mu \in \ell_\infty(T)^*, \|\mu\| \leq 1 \right\} \\ &= \|Aq\|_\infty \\ &= \sup\left\{ \left|\left\langle a_t^*, q\right\rangle\right| : t \in T \right\} \\ &\leq R \sup\left\{ \left\| a_t^*\right\| : t \in T \right\}, \end{aligned}$$

for any $q \in \widetilde{Q}$. Thus, the set $\{Aq,\ q \in \widetilde{Q}\}$ is bounded in $\ell_\infty(T)^{**}$, and then $\text{cl}^*\{Aq,\ q \in \widetilde{Q}\}$ is w^*-compact by the Alaoglu theorem. Thus there are a subnet $(q_{k_\nu})_{\nu\in\mathcal{N}}$ from the sequence $(q_k)_{k\in\mathbb{N}}$, and some $b^{**} \in \text{cl}^*\{Aq, q \in \widetilde{Q}\}$ such that

$$\lim_{\nu\in\mathcal{N}} \left\langle \mu, Aq_{k_\nu}\right\rangle = \left\langle \mu, b^{**}\right\rangle$$

for all $\mu \in \ell_\infty(T)^*$. Hence

$$\left\langle \widehat{\mu}, b^{**}\right\rangle = \lim_{\nu\in\mathcal{N}} \left\langle \widehat{\mu}, Aq_{k_\nu}\right\rangle = \lim_{\nu\in\mathcal{N}} \left\langle \overline{c}^*, q_{k_\nu}\right\rangle,$$

which implies

$$\begin{aligned}\left(b^{**},\left\langle\widehat{\mu},b^{**}\right\rangle\right) &= w^{*}\text{-}\lim_{\nu\in\mathcal{N}}\left(Aq_{k_\nu},\left\langle\overline{c}^{*},q_{k_\nu}\right\rangle\right)\\ &\in \operatorname{cl}^{*}\left\{\left(Aq,\left\langle\overline{c}^{*},q\right\rangle\right),q\in\widetilde{Q}\right\}\\ &\subset \operatorname{cl}^{*}C_D(0).\end{aligned}$$

Hence S_D is not empty. Furthermore, S_D is the preimage of $\operatorname{cl}^{*}C_D(0)$ under the w^*-continuous application $b^{**}\mapsto (b^{**},\langle\widehat{\mu},b^{**}\rangle)$ on $\ell_\infty(T)^{**}$, so S_D is w^*-closed in $\ell_\infty(T)^{**}$.

Now, assume that $\widetilde{Q}$ is a compact base and put $S':=\{b\in\ell_\infty(T)|(b,\langle\widehat{\mu},b\rangle)\in C_D(0)\}$. There exists a subnet $(q_{k_\nu})_{\nu\in\mathcal{N}}$ that weakly converges to some element $\overline{q}\in\widetilde{Q}$. Then, (32) gives

$$\langle\widehat{\mu},A\overline{q}\rangle=\lim_{\nu\in\mathcal{N}}\left\langle A^{*}\widehat{\mu},q_{k_\nu}\right\rangle=\lim_{\nu\in\mathcal{N}}\left\langle\overline{c}^{*},q_{k_\nu}\right\rangle=\left\langle\overline{c}^{*},\overline{q}\right\rangle,$$

and therefore

$$(A\overline{q},\langle\widehat{\mu},A\overline{q}\rangle)=(A\overline{q},\langle\overline{c}^{*},\overline{q}\rangle)\in C_D(0).$$

Hence,

$$A\overline{q}\in S'$$

and this set is nonempty.

(ii) If $\widehat{\mu}$ is a strong Slater point of $\mathcal{F}_D$ at 0, take $\vartheta>0$ such that

$$\inf_{q\in\widetilde{Q}}\left\{\langle\widehat{\mu},Aq\rangle-\left\langle\overline{c}^{*},q\right\rangle\right\}\geq\vartheta>0.$$

For any $(b^{**},\alpha)\in\operatorname{cl}^{*}C_D(0)$ consider nets $(\lambda^\nu)_{\nu\in\mathcal{N}}\subset\mathbb{R}_+^{(\widetilde{Q})}$, $(\gamma^\nu)_{\nu\in\mathcal{N}}\subset\mathbb{R}_+^{(\ell_\infty(T)_+)}$ such that $\sum_{q\in\widetilde{Q}}\lambda_q^\nu+\sum_{p\in\ell_\infty(T)_+}\gamma_p^\nu=1$ and

$$\left(b^{**},\alpha\right)=\lim_{\nu\in\mathcal{N}}\left(\sum_{q\in\widetilde{Q}}\lambda_q^\nu\left(Aq,\left\langle\overline{c}^{*},q\right\rangle\right)+\sum_{p\in\ell_\infty(T)_+}\gamma_p^\nu\,(p,-1)\right).\tag{33}$$

Without loss of generality, suppose that both nets $\left(\sum_{q\in\widetilde{Q}}\lambda_q^\nu\right)_{\nu\in\mathcal{N}}$ and $\left(\sum_{p\in\ell_\infty(T)_+}\gamma_p^\nu\right)_{\nu\in\mathcal{N}}$ are convergent. By applying (33) to $(\widehat{\mu},-1)$ we obtain

$$\begin{aligned}\left\langle\widehat{\mu},b^{**}\right\rangle-\alpha &= \lim_{\nu\in\mathcal{N}}\left\{\sum_{q\in\widetilde{Q}}\lambda_q^\nu\left(\langle\widehat{\mu},Aq\rangle-\left\langle\overline{c}^{*},q\right\rangle\right)+\sum_{p\in\ell_\infty(T)_+}\gamma_p^\nu\left(\langle\widehat{\mu},p\rangle+1\right)\right\}\\ &\geq \vartheta\lim_{\nu\in\mathcal{N}}\sum_{q\in\widetilde{Q}}\lambda_q^\nu+\lim_{\nu\in\mathcal{N}}\sum_{p\in\ell_\infty(T)_+}\gamma_p^\nu\\ &> 0,\end{aligned}$$

which gives that $(b^{**},\langle\widehat{\mu},b^{**}\rangle)\notin\operatorname{cl}^{*}C_D(0)$. □

Now we will make use of the following condition on the cone Q:

(A): *$\widetilde{Q}$ is a closed spanning subset of Q such that there are two positive real numbers r and R, and some $\overline{x}^{*}\in X^{*}$, $\left\|\overline{x}^{*}\right\|=1$, satisfying*

$$r\leq\left\langle\overline{x}^{*},q\right\rangle\leq\|q\|\leq R\tag{34}$$

for all $q\in\widetilde{Q}$.

Notice that, in this case, the cone Q is pointed. As an example, we may take any compact base $\widetilde{Q}$ of Q, since $\widetilde{Q}$ is bounded.

The next theorem gives an estimate of the norm $\|D^{*}\mathcal{F}_D(0,\widehat{\mu})\|$, which will be useful for providing an estimate of the exact Lipschitzian bound of the dual feasible set mapping.

Theorem 15. *Suppose that Q satisfies condition* (A) *and that $\widehat{\mu}\in\mathcal{F}_D(0)$. Then,*

(i) *If $\widehat{\mu}$ is a strong Slater point of $\mathcal{F}_D$ at 0, then $\|D^{*}\mathcal{F}_D(0,\widehat{\mu})\|=0$.*
(ii) *If $\widehat{\mu}$ is not a strong Slater point of $\mathcal{F}_D$ at 0, then $\|D^{*}\mathcal{F}_D(0,\widehat{\mu})\|>0$ and*

$$r\Delta\leq\left\|D^{*}\mathcal{F}_D(0,\widehat{\mu})\right\|\leq R\Delta,$$

where

$$\Delta:=\sup\left\{\left\|b^{**}\right\|^{-1}:\left(b^{**},\left\langle\widehat{\mu},b^{**}\right\rangle\right)\in\operatorname{cl}^{*}C_D(0)\right\}.\tag{35}$$

Proof. (i) Assume that $\widehat{\mu}$ is a strong Slater point of $\mathcal{F}_D$ at 0. By Lemma 12, $x^{**} = 0$ when $x^{**} \in D^*\mathcal{F}_P(0, \widehat{\mu})(b^{**})$, thus

$$\left\|D^*\mathcal{F}_D(0, \widehat{\mu})\right\| = \sup\left\{\left\|x^{**}\right\| : x^{**} \in D^*\mathcal{F}_D(0, \widehat{\mu})(b^{**}),\ \left\|b^{**}\right\| \leq 1\right\} = 0.$$

(ii) If $\widehat{\mu}$ is not a strong Slater point for $\mathcal{F}_D$ at 0, S_D is not empty and we can take some $b^{**} \in S_D$ (31). Then, from (24) and (31), since $(b^{**}, \langle\widehat{\mu}, b^{**}\rangle) \in \mathrm{cl}^*C_D(0)$ there exist nets $\{\lambda^\nu\}_{\nu\in\mathcal{N}} \subset \mathbb{R}_+^{(\widetilde{Q})}$, $\{\gamma^\nu\}_{\nu\in\mathcal{N}} \subset \mathbb{R}_+^{(\ell_\infty(T)_+)}$ such that

$$\sum_{q\in\widetilde{Q}} \lambda_q^\nu + \sum_{p\in\ell_\infty(T)_+} \gamma_p^\nu = 1,\ \text{for all } \nu \in \mathcal{N},$$

and

$$\left(b^{**}, \left\langle\widehat{\mu}, b^{**}\right\rangle\right) = w^*\text{-}\lim_{\nu\in\mathcal{N}} \left(\sum_{q\in\widetilde{Q}} \lambda_q^\nu \left(Aq, \left\langle\overline{c}^*, q\right\rangle\right) + \sum_{p\in\ell_\infty(T)_+} \gamma_p^\nu\,(p, -1)\right).$$

As in the proof of Lemma 12, it follows that

$$0 = \lim_{\nu\in\mathcal{N}} \sum_{p\in\ell_\infty(T)_+} \gamma_p^\nu,$$

$$b^{**} = w^*\text{-}\lim_{\nu\in\mathcal{N}} \left\{\sum_{q\in\widetilde{Q}} \lambda_q^\nu Aq + \sum_{p\in\ell_\infty(T)_+} \gamma_p^\nu p\right\},$$

and

$$\left\langle\widehat{\mu}, b^{**}\right\rangle = \lim_{\nu\in\mathcal{N}} \sum_{q\in\widetilde{Q}} \lambda_q^\nu \left\langle\overline{c}^*, q\right\rangle.$$

Now, take any fixed $q_1 \in \widetilde{Q}$, and for each $\nu \in \mathcal{N}$, define

$$z_\nu := \sum_{q\in\widetilde{Q}} \lambda_q^\nu q + \left(1 - \sum_{q\in\widetilde{Q}} \lambda_q^\nu\right) q_1 \in \operatorname{conv}\widetilde{Q},$$

then

$$\|z_\nu\| \leq \sum_{q\in\widetilde{Q}} \lambda_q^\nu \|q\| + \left(1 - \sum_{q\in\widetilde{Q}} \lambda_q^\nu\right) \|q_1\| \leq R.$$

By the Alaoglu theorem, we may consider, without loss of generality, that the net $\{z_\nu\}_{\nu\in\mathcal{N}}$ is w^*-convergent to some $z^{**} \in X^{**}$. Observe that $z^{**} = w^*\text{-}\lim_{\nu\in\mathcal{N}} \sum_{q\in\widetilde{Q}} \lambda_q^\nu q$ because $\lim_{\nu\in\mathcal{N}} \sum_{q\in\widetilde{Q}} \lambda_q^\nu = 1$, and so $z^{**} \in \mathrm{cl}^*(\operatorname{conv}\widetilde{Q})$. Then, by (34), $\|z^{**}\| \geq r$ and so, $z^{**} \neq 0$.

Now we have

$$\left(z^{**}, b^{**}, \left\langle\widehat{\mu}, b^{**}\right\rangle\right) = w^*\text{-}\lim_{\nu\in\mathcal{N}} \left(\sum_{q\in\widetilde{Q}} \lambda_q^\nu \left(q, Aq, \left\langle\overline{c}^*, q\right\rangle\right) + \sum_{p\in\ell_\infty(T)_+} \gamma_p^\nu\,(0, p, 0)\right),$$

and so $(z^{**}, b^{**}, \langle\widehat{\mu}, b^{**}\rangle)$ belongs to

$$\mathrm{cl}^*\left\{\left\{\left(q, Aq, \left\langle\overline{c}^*, q\right\rangle\right) : q \in Q\right\} \cup \left\{(0, p, 0), p \in \ell_\infty(T)_+\right\}\right\}.$$

Hence Theorem 13 gives

$$z^{**} \in D^*\mathcal{F}_D(0, \widehat{\mu})(b^{**}).$$

Next we consider two cases:

(a) $\mathcal{F}_D$ does not satisfy the strong Slater condition at 0. In this case, $(0, 0) \in \mathrm{cl}^*C_D(0)$ and we may take $b^{**} = 0$; since $0 \neq \lambda z^{**} \in D^*\mathcal{F}_D(0, \widehat{\mu})(0)$ for all $\lambda > 0$, it follows that

$$\left\|D^*\mathcal{F}_D(0, \widehat{\mu})\right\| = +\infty = \sup\left\{\left\|b^{**}\right\|^{-1} : \left(b^{**}, \left\langle\widehat{\mu}, b^{**}\right\rangle\right) \in \mathrm{cl}^*C_D(0)\right\}.$$

(b) $\mathcal{F}_D$ satisfies the strong Slater condition at 0. If $b^{**} = 0$ then $(0, 0) \in \mathrm{cl}^* C_D(0)$ which contradicts Proposition 10. Hence $b^{**} \neq 0$ and

$$\left\|b^{**}\right\|^{-1} z^{**} \in D^*\mathcal{F}_D(0, \widehat{\mu})(\|b^{**}\|^{-1}b^{**}),$$

so

$$\begin{aligned}\left\|D^*\mathcal{F}_D(0, \widehat{\mu})\right\| &= \sup\left\{\left\|x^{**}\right\| : x^{**} \in D^*\mathcal{F}_D(0, \widehat{\mu})(b_1^{**}),\ \|b_1^{**}\| \leq 1\right\} \\ &\geq \|\ \|b^{**}\|^{-1} z^{**}\|. \end{aligned} \tag{36}$$

Moreover, from condition (A),

$$\left\langle \overline{x}^*, z_\nu \right\rangle \geq r > 0 \quad \text{for all } \nu \in \mathcal{N},$$

which gives that

$$\left\langle \overline{x}^*, z^{**} \right\rangle \geq r > 0,$$

and hence

$$\left\|z^{**}\right\| = \sup_{\|x^*\|\leq 1} \left|\left\langle x^*, z^{**}\right\rangle\right| \geq \left\langle \overline{x}^*, z^{**}\right\rangle \geq r > 0.$$

From (36) we obtain

$$\left\|D^*\mathcal{F}_D(0, \widehat{\mu})\right\| \geq r \left\|b^{**}\right\|^{-1} > 0,$$

which holds for every $b^{**} \in S_D$, thus

$$\left\|D^*\mathcal{F}_D(0, \widehat{\mu})\right\| \geq r \max\left\{\left\|b^{**}\right\|^{-1} : \left(b^{**}, \left\langle \widehat{\mu}, b^{**}\right\rangle\right) \in \mathrm{cl}^* C_D(0)\right\}.$$

We can put "max" above because $0 \notin S_D$ and S_D is w^*-closed, so the w^*-upper semicontinuous function $b^{**} \to \|b^{**}\|^{-1}$, restricted to S_D, attains a maximum on it, taking into account that, for any $b_0^{**} \in S_D$, the set

$$\left\{b^{**} \in S_D : \left\|b^{**}\right\|^{-1} \geq \left\|b_0^{**}\right\|^{-1}\right\}$$

is obviously bounded in $\ell_\infty(T)^{**}$.

To get the other estimate, observe that from the definitions of the coderivative and the normal cone, we have

$$\begin{aligned} x^{**} \in\ & D^*\mathcal{F}_D(0, \widehat{\mu})(0) \Leftrightarrow \left(x^{**}, 0\right) \in N\left((0, \widehat{\mu})\,; \mathrm{gph}\, \mathcal{F}_D\right) \\ &\Leftrightarrow \left\langle \left(x^{**}, 0\right), \left(c^*, \mu\right) - (0, \widehat{\mu})\right\rangle \leq 0 \quad \text{for all } \left(c^*, \mu\right) \in \mathrm{gph}\, \mathcal{F}_D \\ &\Leftrightarrow \left\langle x^{**}, c^*\right\rangle \leq 0 \quad \text{for all } c^* \in \mathrm{dom}\, \mathcal{F}_D. \end{aligned}$$

Now, since we are assuming that $\mathcal{F}_D$ satisfies the strong Slater condition at 0, Proposition 10(iii) gives that $0 \in \mathrm{int}\ (\mathrm{dom}\, \mathcal{F}_D)$, hence $x^{**} = 0$. Thus, we have

$$x^{**} \in D^*\mathcal{F}_D(0, \widehat{\mu})(0) \Leftrightarrow x^{**} = 0.$$

Therefore $\|D^*\mathcal{F}_D(0, \widehat{\mu})\|$ is equal to

$$\max\left\{0; \sup\left\{\left\|x^{**}\right\| : x^{**} \neq 0, x^{**} \in D^*\mathcal{F}_D(0, \widehat{\mu})(b^{**}), 0 < \left\|b^{**}\right\| \leq 1\right\}\right\}. \tag{37}$$

For any $x^{**} \in D^*\mathcal{F}_D(0, \widehat{\mu})(b^{**})$, $x^{**} \neq 0$, with $0 < \|b^{**}\| \leq 1$, $b^{**} \in \ell_\infty(T)^{**}$, we can apply Lemma 12 to get the existence of a net

$$\{(\widetilde{q}_\nu, p_\nu, \lambda_\nu)\}_{\nu\in\mathcal{N}} \subset \widetilde{Q} \times \ell_\infty(T)_+ \times (\mathbb{R}_+ \setminus \{0\})$$

such that

$$\begin{aligned} x^{**} &= w^*\text{-}\lim_{\nu\in\mathcal{N}} (\lambda_\nu \widetilde{q}_\nu), \\ b^{**} &= w^*\text{-}\lim_{\nu\in\mathcal{N}} \left(\lambda_\nu A\widetilde{q}_\nu + p_\nu\right), \\ \left\langle \widehat{\mu}, b^{**}\right\rangle &= \lim_{\nu\in\mathcal{N}} \left\langle \overline{c}^*, \lambda_\nu \widetilde{q}_\nu\right\rangle. \end{aligned} \tag{38}$$

Observe that

$$0 < \left\|x^{**}\right\| \leq \liminf_{\nu\in\mathcal{N}} \|\lambda_\nu \widetilde{q}_\nu\| \leq R \liminf_{\nu\in\mathcal{N}} \lambda_\nu. \tag{39}$$

If $\lim_{\nu\in\mathcal{N}}\lambda_\nu = +\infty$, (38) gives rise to $(0,0)\in \mathrm{cl}^*C_D(0)$, which contradicts the current assumption that $\mathcal{F}_D$ satisfies the strong Slater condition at 0. So,

$$\alpha := \liminf_{\nu\in\mathcal{N}}\lambda_\nu < +\infty,$$

it follows from (39) that $\alpha > 0$ and, if we suppose that the own net $\{\lambda_\nu\}_{\nu\in\mathcal{N}}$ converges to α, and from (38), we conclude that

$$(\alpha^{-1}b^{**}, \langle\widehat{\mu}, \alpha^{-1}b^{**}\rangle) \in \mathrm{cl}^*C_D(0),$$

because for each $\varepsilon > 0$ we have

$$\left((\alpha+\varepsilon)^{-1}b^{**}, \left\langle\widehat{\mu}, (\alpha+\varepsilon)^{-1}b^{**}\right\rangle\right) = w^*\text{-}\lim_{\nu\in\mathcal{N}}\left\{\frac{\lambda_\nu}{\lambda_\nu+\varepsilon}\left(A\widetilde{q}_\nu, \left\langle\overline{c}^*, \widetilde{q}_\nu\right\rangle\right) + \frac{\varepsilon}{\lambda_\nu+\varepsilon}(\varepsilon^{-1}p_\nu, 0)\right\},$$

and so

$$\begin{aligned}\mathrm{cl}^*C_D(0) &\ni \lim_{\varepsilon\to 0^+}\left[w^*\text{-}\lim_{\nu\in\mathcal{N}}\left\{\frac{\lambda_\nu}{\lambda_\nu+\varepsilon}\left(A\widetilde{q}_\nu, \left\langle\overline{c}^*, \widetilde{q}_\nu\right\rangle\right) + \frac{\varepsilon}{\lambda_\nu+\varepsilon}(\varepsilon^{-1}p_\nu, -1)\right\}\right]\\ &= (\alpha^{-1}b^{**}, \langle\widehat{\mu}, \alpha^{-1}b^{**}\rangle) + \lim_{\varepsilon\to 0^+}\left[w^*\text{-}\lim_{\nu\in\mathcal{N}}\left(0, \frac{\varepsilon}{\lambda_\nu+\varepsilon}\right)\right]\\ &= (\alpha^{-1}b^{**}, \langle\widehat{\mu}, \alpha^{-1}b^{**}\rangle).\end{aligned}$$

From $0 < \|b^{**}\| \le 1$, and $\|x^{**}\| \le R\alpha$ it follows that

$$\begin{aligned}\left\|x^{**}\right\| &\le R\alpha\left\|b^{**}\right\|^{-1} = R\left\|\alpha^{-1}b^{**}\right\|^{-1}\\ &\le R\max\left\{\left\|b_1^{**}\right\|^{-1} : \left(b_1^{**}, \left\langle\widehat{\mu}, b_1^{**}\right\rangle\right) \in \mathrm{cl}^*C_D(0)\right\}.\end{aligned}$$

Finally, we conclude from (37) that

$$\left\|D^*\mathcal{F}_D(0,\widehat{\mu})\right\| \le R\max\left\{\left\|b^{**}\right\|^{-1} : \left(b^{**}, \left\langle\widehat{\mu}, b^{**}\right\rangle\right) \in \mathrm{cl}^*C_D(0)\right\}. \quad\square$$

Remark 4. Notice that, from this proof, if $\widetilde{Q} = \{q \in Q : \|q\| = 1\}$ (the normalized cone) the constant r in this theorem can be obtained from the strong separation property: since $0 \notin \mathrm{cl\,conv}\,Q_2$, take any $\overline{x}^* \in X^*$, $\left\|\overline{x}^*\right\| = 1$, and $r \in \mathbb{R}$ such that $\left\langle\overline{x}^*, q\right\rangle \ge r > 0$ for all $q \in \widetilde{Q}$. If we could choose $r = R = 1$ then, in case (b) of the previous proof, we would have the equality

$$\left\|D^*\mathcal{F}_D(0,\widehat{\mu})\right\| = \max\left\{\left\|b^{**}\right\|^{-1} : \left(b^{**}, \left\langle\widehat{\mu}, b^{**}\right\rangle\right) \in \mathrm{cl}^*C_D(0)\right\}. \tag{40}$$

We do not know if any equality holds in (35) in general cases with $r \ne R$.

Example 1 (*Constants* $r = R = 1$)**.** Consider $T = \mathbb{N}$, $X = \ell_1$ and the closed convex cone

$$Q = \{(x_n) \in \ell_1 : x_n \ge 0 \text{ for all } n \in \mathbb{N}\}.$$

Let $\mathbf{1}^*$ be the sequence in $\ell_\infty = X^*$ whose terms are all equal to 1, and let e_n^*, $n = 1, 2, \ldots$, be the standard canonical vectors in ℓ_∞. Then $\langle\mathbf{1}^*, x\rangle = 1$ for all $x \in \widetilde{Q} := \{q \in Q : \|q\|_1 = 1\}$, and since $\|\mathbf{1}^*\|_\infty = 1$, it follows that condition (A) holds for $\widetilde{Q}$ with $r = R = 1$. If $a_n^* := -e_n^* \in \ell_\infty$, and $\overline{c}^* := -\mathbf{1}^* \in \ell_\infty$ be given, then $\overline{\mu} = 0$ is a strong Slater point of $\mathcal{F}_D$ at 0, while the Dirac measure $\widehat{\mu} = \delta_1 \in \mathcal{F}_D(0)$ is not, hence the equality (40) takes place at $(0,\widehat{\mu}) = (0,\delta_1)$.

Example 2. Let $X = \mathbb{R}^2$ be endowed with the Euclidean norm, $T = \mathbb{N}$, $\overline{c}^* = (0,1)$, $a_n := (1,2)$ for all n, $\widehat{\mu} = \delta_1$. Consider

$$Q = \left\{(x,y) \in \mathbb{R}^2 : y \ge |x|\right\},$$

and take

$$\widetilde{Q}_1 = \{(x,y) \in Q : |x| + |y| = 1\}.$$

Then $\langle(0,1), (x,y)\rangle = y \ge \frac{1}{2}$ and $\|(x,y)\| \le 1$ for all $(x,y) \in \widetilde{Q}_1$, so we can take $r_1 = \frac{1}{2}$ and $R_1 = 1$. (These are the largest r and smallest R we can choose for $\widetilde{Q}_1$.)

Another possibility is to take

$$\widetilde{Q}_2 = \{(x,y) \in Q : y = 2\}.$$

Now $r_2 = 2$ ($\langle(0,1), (x,y)\rangle = y \ge 2$) and $R_2 = 2\sqrt{2}$.

We have, for $i = 1, 2$,

$$C_D^i(0,0) := \operatorname{conv}\left(\left\{\left((x+2y)_{n\in\mathbb{N}}, y\right) : (x,y) \in \widetilde{Q}_i\right\} \cup \left\{(p,-1) : p \in \ell_\infty(\mathbb{N})_+\right\}\right).$$

In particular,

$$C_D^2(0,0) = \operatorname{conv}\left(\left\{\left((x+4)_{n\in\mathbb{N}}, 2\right) : |x| \le 2\right\} \cup \left\{(p,-1) : p \in \ell_\infty(\mathbb{N})_+\right\}\right).$$

Then:

(i) $\max\left\{\|b^{**}\|^{-1} : (b^{**}, \langle\widehat{\mu}, b^{**}\rangle) \in \operatorname{cl}^* C_D^1(0,0)\right\} = 2$,

(ii) $\max\left\{\|b^{**}\|^{-1} : (b^{**}, \langle\widehat{\mu}, b^{**}\rangle) \in \operatorname{cl}^* C_D^2(0,0)\right\} = \frac{1}{2}$ and
$R_2 \max\left\{\|b^{**}\|^{-1} : (b^{**}, \langle\widehat{\mu}, b^{**}\rangle) \in \operatorname{cl}^* C_D^2(0,0)\right\} = \sqrt{2}$,

(iii) $\|D^*\mathcal{F}_D(0,\widehat{\mu})\| = \sqrt{2}$,
so

$$\begin{aligned}\left\|D^*\mathcal{F}_D(0,\widehat{\mu})\right\| &= R_2 \max\left\{\left\|b^{**}\right\|^{-1} : \left(b^{**}, \left\langle\widehat{\mu}, b^{**}\right\rangle\right) \in \operatorname{cl}^* C_D^2(0,0)\right\} \\ &< R_1 \max\left\{\left\|b^{**}\right\|^{-1} : (b^{**}, \left\langle\widehat{\mu}, b^{**}\right\rangle) \in \operatorname{cl}^* C_D^1(0,0)\right\},\end{aligned}$$

which implies that sometimes we can have the equality

$$\left\|D^*\mathcal{F}_D(0,\widehat{\mu})\right\| = R\left\{\left\|b^{**}\right\|^{-1} : \left(b^{**}, \left\langle\widehat{\mu}, b^{**}\right\rangle\right) \in \operatorname{cl}^* C_D(0)\right\},$$

but not always.

6.3. Lipschitzian bound for the dual feasible set mapping

Theorem 16. *Let $\widehat{\mu} \in \mathcal{F}_D(0)$ and suppose that $0 \notin \operatorname{cl}\operatorname{conv}\widetilde{Q}$. Then, $\mathcal{F}_D$ is Lipschitz-like around $(0,\widehat{\mu})$ if and only if*

$$D^*\mathcal{F}_D(0,\widehat{\mu})(0) = \{0\}.$$

Proof. ($\Rightarrow$) It follows directly from Theorem 1.44 in [19] by taking into account that gph $\mathcal{F}_D$ is convex.

($\Leftarrow$) Let $D^*\mathcal{F}_D(0,\widehat{\mu})(0) = \{0\}$ and suppose that $\mathcal{F}_D$ is not Lipschitz-like around $(0,\widehat{\mu})$. Then, by the Robinson–Ursescu theorem and Proposition 10, $(0,0) \in \operatorname{cl}^* C_D(0)$ and so there are nets $\{\lambda^\nu\}_{\nu\in\mathcal{N}} \subset \mathbb{R}_+^{(\widetilde{Q})}$, $\{\gamma^\nu\}_{\nu\in\mathcal{N}} \subset \mathbb{R}_+^{(\ell_\infty(T)_+)}$ such that

$$\sum_{q\in\widetilde{Q}} \lambda_q^\nu + \sum_{p\in\ell_\infty(T)_+} \gamma_p^\nu = 1, \quad \text{for all } \nu \in \mathcal{N},$$

and

$$(0,0) = w^*\text{-}\lim_\nu \left(\sum_{q\in\widetilde{Q}} \lambda_q^\nu \left(Aq, \left\langle\overline{c}^*, q\right\rangle\right) + \sum_{p\in\ell_\infty(T)_+} \gamma_p^\nu\,(p,-1)\right).$$

By setting $b^{**} = 0$ and by following the same steps as in the proof of (ii) in the previous Theorem 15, one can obtain $z^{**} \in X^{**}$ such that $0 \neq z^{**} \in D^*\mathcal{F}_D(0,\widehat{\mu})(0) = \{0\}$, which constitutes a contradiction. Therefore $\mathcal{F}_D$ is Lipschitz-like around $(0,\widehat{\mu})$. □

Remark 5. We may have $D^*\mathcal{F}_D(0,\widehat{\mu})(0) = \{0\}$ and $\mathcal{F}_D$ not Lipschitz-like around $(0,\widehat{\mu})$, if the condition $0 \notin \operatorname{cl}\operatorname{conv}\widetilde{Q}$ does not hold. For instance, consider the case of $T = \{t_0\}$ and $X = c_0$ which is the Banach space of bounded real sequences converging to 0, with the supremum norm. Then $X^* = \ell_1 = \ell_1(\mathbb{N})$ and $X^{**} = \ell_\infty = \ell_\infty(\mathbb{N})$. Let $Q = \{q \in c_0 : q_n \ge 0,\ n \in \mathbb{N}\}$ and $\widetilde{Q} = \{q \in Q : \|q\|_\infty = 1\}$. If $a_{t_0}^* = \overline{c}^* = \left(\frac{1}{n!}\right)_{n=1}^\infty \in \ell_1$, and by observing that each $e^k = (0,\ldots,0,1,0,\ldots)$, with $e_k^k = 1$ and all other $e_n^k = 0$, is in $\widetilde{Q}$, it is easy to see that $0 \in \operatorname{cl}\operatorname{conv}\widetilde{Q}$, $\widehat{\mu} = 1 \in \mathcal{F}_D(0)$, and $c^* = 0 \notin \operatorname{int}(\operatorname{dom}\mathcal{F}_D)$, which implies that $\mathcal{F}_D$ is not Lipschitz-like around $(0,1)$. Nonetheless, we can show that $D^*\mathcal{F}_D(0,\widehat{\mu})(0) = \{0\}$. Indeed, if $x^{**} \in D^*\mathcal{F}_D(0,\widehat{\mu})(0)$, then by Lemma 12 there exists a net $\{(q_\nu, p_\nu)\}_{\nu\in\mathcal{N}} \subset Q \times \ell_\infty(T)_+$ such that

$$\begin{aligned} x^{**} &= w^*\text{-}\lim_{\nu\in\mathcal{N}} q_\nu, \\ 0 &= w^*\text{-}\lim_{\nu\in\mathcal{N}} (Aq_\nu + p_\nu), \\ 0 &= \lim_{\nu\in\mathcal{N}} \left\langle\overline{c}^*, q_\nu\right\rangle. \end{aligned} \tag{41}$$

From $0 = \lim_{\nu\in\mathcal{N}} \langle \overline{c}^*, q_\nu \rangle = \lim_{\nu\in\mathcal{N}} \sum_{m=1}^{\infty} \frac{q_{\nu,m}}{m!}$, it follows that $\lim_{\nu\in\mathcal{N}} q_{\nu,k} = 0$ for any positive integer k. Now, if $x^{**} = (x_k^{**})_{k=1}^{\infty}$, then (41) gives that $x_k^{**} = \langle e^k, x^{**} \rangle = \lim_{\nu\in\mathcal{N}} \langle e^k, q_\nu \rangle = \lim_{\nu\in\mathcal{N}} q_{\nu,k} = 0$. Therefore $x^{**} = 0$, and hence $D^*\mathcal{F}_D(0, \widehat{\mu})(0) = \{0\}$.

In order to get an estimate of the exact Lipschitzian bound for $\mathcal{F}_D$ around $(0, \widehat{\mu})$, $\operatorname{lip} \mathcal{F}_D(0, \widehat{\mu})$, recall that

$$\operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) = \limsup_{(c^*,\mu)\to(0,\widehat{\mu})} \frac{\operatorname{dist}\,(\mu, \mathcal{F}_D(c^*))}{\operatorname{dist}\,\left(c^*, \mathcal{F}_D^{-1}(\mu)\right)}.$$

The extended Ascoli distance formula

$$\operatorname{dist}\,\left(\mu, \mathcal{F}_D(c^*)\right) = \sup_{(b^{**},\alpha)\in \operatorname{cl}^* C_D(c^*)} \frac{[\alpha - \langle \mu, b^{**} \rangle]_+}{\|b^{**}\|} \tag{42}$$

holds true when $\mathcal{F}_D$ satisfies the strong Slater condition at c^*, by a straightforward application of Lemma 4.3 in [14]. On the other hand we can show a lower bound for $\operatorname{dist}\,\left(c^*, \mathcal{F}_D^{-1}(\mu)\right)$.

Lemma 17. *Suppose that $\|q\| \le R$ for all $q \in \widetilde{Q}$. Let $c^* \in X^*$ and $\mu \in \ell_\infty(T)^*$ be such that $(c^*, \mu) \notin \operatorname{gph} \mathcal{F}_D$ and $\mathcal{F}_D^{-1}(\mu) \neq \emptyset$. Then*

$$\operatorname{dist}\,\left(c^*, \mathcal{F}_D^{-1}(\mu)\right) \ge R^{-1} \sup_{q\in\widetilde{Q}} \left[- \langle \mu, Aq \rangle + \langle \overline{c}^* + c^*, q \rangle\right]_+ > 0. \tag{43}$$

Proof. Since $\mathcal{F}_D^{-1}(\mu) \neq \emptyset$, we have that $\langle -p, \mu \rangle \le 1$, for all $p \in \ell_\infty(T)_+$ and

$$\mathcal{F}_D^{-1}(\mu) = \left\{ d^* \in X^* \mid \langle \mu, Aq \rangle - \langle \overline{c}^*, q \rangle - \langle d^*, q \rangle \ge 0, \text{ for all } q \in \widetilde{Q} \right\}.$$

Take any $d^* \in \mathcal{F}_D^{-1}(\mu)$, then (remember that $0 < \|q\| \le R$ for all $q \in \widetilde{Q}$)

$$\begin{aligned} \left\| c^* - d^* \right\| &= \sup_{\|x\|\le 1} \left| \langle c^*, x \rangle - \langle d^*, x \rangle \right| \\ &\ge \sup_{q\in\widetilde{Q}} \left(\langle c^*, R^{-1}q \rangle - \langle d^*, R^{-1}q \rangle \right) \\ &\ge R^{-1} \sup_{q\in\widetilde{Q}} \left(\langle c^*, q \rangle - \langle \mu, Aq \rangle + \langle \overline{c}^*, q \rangle \right). \end{aligned}$$

Furthermore, since $c^* \notin \mathcal{F}_D(\mu)$, there exists $q_0 \in \widetilde{Q}$ such that

$$- \langle \mu, Aq_0 \rangle + \langle \overline{c}^* + c^*, q_0 \rangle > 0.$$

Therefore

$$\begin{aligned} \operatorname{dist}\,\left(c^*, \mathcal{F}_D^{-1}(\mu)\right) &= \inf_{d^*\in\mathcal{F}_D^{-1}(\mu)} \left\| c^* - d^* \right\| \\ &\ge R^{-1} \sup_{q\in\widetilde{Q}} \left(- \langle \mu, Aq \rangle + \langle \overline{c}^* + c^*, q \rangle \right) > 0, \end{aligned}$$

which implies (43). □

The next result shows that indeed the situation of the exact Lipschitzian bound for $\mathcal{F}_D$ around $(0, \widehat{\mu})$ is similar to that of $\|D^*\mathcal{F}_D(0, \widehat{\mu})\|$ in Theorem 15.

Theorem 18. *Assume condition* (A) *and let $\widehat{\mu} \in \mathcal{F}_D(0)$. Then:*

(i) *If $\widehat{\mu}$ is a strong Slater point of $\mathcal{F}_D$ at 0, then $\operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) = 0$.*
(ii) *If $\widehat{\mu}$ is not a strong Slater point of $\mathcal{F}_D$ at 0, then*

$$r\Delta \le \operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) \le R\Delta,$$

where

$$\Delta := \sup \left\{ \left\| b^{**} \right\|^{-1} : \left(b^{**}, \langle \widehat{\mu}, b^{**} \rangle\right) \in \operatorname{cl}^* C_D(0) \right\}.$$

Proof. We will split the proof in two cases:

• $\mathcal{F}_D$ does not satisfy the strong Slater condition at $c^* = 0$. Then, (i) cannot occur, and with respect to (ii), observe that condition (A) and Proposition 10 implies that $\mathcal{F}_D$ is not Lipschitz-like around $(0, \widehat{\mu})$ and $(0, 0) \in \text{cl}^*C_D(0)$. Hence

$$\text{lip}\,\mathcal{F}_D(0, \widehat{\mu}) = \infty = \sup\left\{\left\|b^{**}\right\|^{-1} : \left(b^{**}, \left\langle\widehat{\mu}, b^{**}\right\rangle\right) \in \text{cl}^*C_D(0)\right\}.$$

• $\mathcal{F}_D$ satisfies the strong Slater condition at $c^* = 0$. In this case, another application of Proposition 10 gives that $0 \in \text{int}\,(\text{dom}\,\mathcal{F}_D)$ and hence $\mathcal{F}_D$ is Lipschitz-like around $(0, \widehat{\mu})$, so $\text{lip}\,\mathcal{F}_D(0, \widehat{\mu}) < \infty$. Now choose any sequence $\left\{\left(c_j^*, \mu_j\right)\right\}$ that converges to $(0, \widehat{\mu})$, such that $c_j^* \in \text{int}\,(\text{dom}\,\mathcal{F}_D)$ for all j (thus $\mathcal{F}_D$ satisfies the strong Slater condition at c_j^* as well), and

$$\text{lip}\,\mathcal{F}_D(0, \widehat{\mu}) = \lim_{j\to\infty} \frac{\text{dist}\left(\mu_j, \mathcal{F}_D(c_j^*)\right)}{\text{dist}\left(c_j^*, \mathcal{F}_D^{-1}(\mu_j)\right)}.$$

(i) Suppose that $\widehat{\mu}$ is a strong Slater point of $\mathcal{F}_D$ at 0, and let $\vartheta > 0$ be such that

$$\inf_{q\in\widetilde{Q}} \left\{\langle\widehat{\mu}, Aq\rangle - \left\langle\overline{c}^*, q\right\rangle\right\} \geq \vartheta > 0.$$

Let j be large enough such that $\left\|c_j^*\right\| < \frac{\vartheta}{4R}$, and $\left\|\widehat{\mu} - \mu_j\right\| < \frac{\vartheta}{4(1+M)R}$, where $M = \sup_{t\in T}\left\|a_t^*\right\| < \infty$.

If μ_j does not satisfy the condition $\mu_j \geq 0$, then $\mathcal{F}_D^{-1}(\mu_j) = \emptyset$ and so

$$\frac{\text{dist}\left(\mu_j, \mathcal{F}_D(c_j^*)\right)}{\text{dist}\left(c_j^*, \mathcal{F}_D^{-1}(\mu_j)\right)} = 0.$$

In the case when $\mu_j \geq 0$, then for any $q \in \widetilde{Q}$

$$\begin{aligned}\left\langle\mu_j, Aq\right\rangle - \left\langle\overline{c}^* + c_j^*, q\right\rangle &= \langle\widehat{\mu}, Aq\rangle - \left\langle\overline{c}^*, q\right\rangle - \left\langle\widehat{\mu} - \mu_j, Aq\right\rangle - \left\langle c_j^*, q\right\rangle \\ &\geq \vartheta - MR\left\|\widehat{\mu} - \mu_j\right\| - \left\|c_j^*\right\| R \\ &> \frac{\vartheta}{2},\end{aligned}$$

implying that $\mu_j \in \mathcal{F}_D(c_j^*)$ and so $\text{dist}\left(\mu_j, \mathcal{F}_D(c_j^*)\right) = 0$. Thus $\text{lip}\,\mathcal{F}_D(0, \widehat{\mu}) = 0$.

(ii) Suppose that $\widehat{\mu}$ is not a strong Slater point for $\mathcal{F}_D$ at 0. Since gph $\mathcal{F}_D$ is convex, we may apply Proposition 1.37 and Theorem 1.44 in [19] to get that

$$\left\|D^*\mathcal{F}_D(0, \widehat{\mu})\right\| \leq \text{lip}\,\mathcal{F}_D(0, \widehat{\mu}). \tag{44}$$

Hence from Theorem 15, we obtain

$$0 < r\ \max\left\{\left\|b^{**}\right\|^{-1} : \left(b^{**}, \left\langle\widehat{\mu}, b^{**}\right\rangle\right) \in \text{cl}^*C_D(0)\right\} \leq \text{lip}\,\mathcal{F}_D(0, \widehat{\mu}), \tag{45}$$

where r is the constant in (34).

Consider a sequence $\left\{\left(c_j^*, \mu_j\right)\right\}$ as above with $\mu_j \notin \mathcal{F}_D(c_j^*)$ and $\mathcal{F}_D^{-1}(\mu_j) \neq \emptyset$ (which gives $\mu_j \geq 0$). From now on, the proof follows as the proof of Theorem 4.6 in [14]; actually we need to consider any $(b^{**}, \alpha) \in \text{cl}^*C_D\left(c_j^*\right)$ and choose any net in $C_D\left(c_j^*\right)$ that w^*-converges to (b^{**}, α) to show, after some algebra together with (43), that

$$\frac{\alpha - \left\langle\mu_j, b^{**}\right\rangle}{\text{dist}\left(c_j^*, \mathcal{F}_D^{-1}(\mu_j)\right)} \leq R.$$

Then use this inequality and the Ascoli distance formula (42), to get

$$\frac{\text{dist}\left(\mu_j, \mathcal{F}_D(c_j^*)\right)}{\text{dist}\left(c_j^*, \mathcal{F}_D^{-1}(\mu_j)\right)} \leq R\ \sup\left\{\left\|b^{**}\right\|^{-1} : \left(b^{**}, \alpha\right) \in \text{cl}^*C_D^+(c_j^*, \mu_j)\right\},$$

where

$$C_D^+(c_j^*, \mu_j) := \left\{\left(b^{**}, \alpha\right) \in \text{cl}^*C_D\left(c_j^*\right) : \alpha - \left\langle\mu_j, b^{**}\right\rangle > 0\right\},$$

and, by letting $j \to \infty$, we obtain

$$\text{lip}\,\mathcal{F}_D(0, \widehat{\mu}) \leq \limsup_{j\to\infty} R\ \sup\left\{\left\|b^{**}\right\|^{-1} : \left(b^{**}, \alpha\right) \in \text{cl}^*C_D^+(c_j^*, \mu_j)\right\}.$$

Finally, we follow exactly the steps in the cited Theorem 4.6 in [14] to get the desired estimate

$$\operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) \leq R \ \max\left\{ \left\| b^{**} \right\|^{-1} : \left(b^{**}, \left\langle \widehat{\mu}, b^{**} \right\rangle\right) \in \operatorname{cl}^* C_D(0) \right\},$$

which completes the proof in view of (45). □

Example 3. An application of this theorem and (44) to the data in Example 2 by taking $\widetilde{Q} = \widetilde{Q}^2$, gives $\operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) = \sqrt{2}$ because $\sqrt{2} = \|D^* \mathcal{F}_D\,(0, \widehat{\mu})\| \leq \operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) \leq R_2 \ \max\left\{ \|b^{**}\|_\infty^{-1} : (b^{**}, \langle \widehat{\mu}, b^{**} \rangle) \in \operatorname{cl} C_D^2 \right\} = \sqrt{2}$.

The last theorem in the paper provides an estimate of the difference between $\operatorname{lip} \mathcal{F}_D(0, \widehat{\mu})$ and $\|D^* \mathcal{F}_D(0, \widehat{\mu})\|$ which is an immediate consequence of the results above, as well as a technical assumption guaranteeing the equality between both. So, the situation for the dual is much more involved than in the case of the primal problem where the equality between both constants always holds (Theorem 8) as a consequence of the fact that $\mathcal{F}_P^{-1}$ is a perfectly regular mapping (see Proposition 5 in [17]). The upper bound for $\operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) - \|D^* \mathcal{F}_D(0, \widehat{\mu})\|$ given next depends on the cone constraint provided by Q, and the characteristic set $C_D(0)$ corresponding to some special spanning closed set $\widetilde{Q}$. This estimate is described in the following corollary.

Theorem 19. *In relation to the dual feasible set mapping $\mathcal{F}_D$ the following two statements hold:*

(i) *Assume condition* (A) *and suppose that $\mathcal{F}_D$ satisfies the strong Slater condition at $c^* = 0$. Let $\widehat{\mu} \in \mathcal{F}_D(0)$, then there are constants r and R, $0 < r \leq R$, which only depends on the cone Q, such that*

$$\begin{aligned} 0 &\leq \operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) - \left\| D^* \mathcal{F}_D(0, \widehat{\mu}) \right\| \\ &\leq (R - r) \max\left\{ \left\| b^{**} \right\|^{-1} : \left(b^{**}, \left\langle \widehat{\mu}, b^{**} \right\rangle\right) \in \operatorname{cl}^* C_D(0) \right\}. \end{aligned}$$

(ii) *If $\mathcal{F}_D^{-1}(\ell_\infty(T)_+^*)$ has nonempty interior for the norm topology in X^* and*

$$\{q \in Q : \|q\| = 1\} \tag{46}$$

is w^*-closed in X^{**}, then

$$\operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) = \left\| D^* \mathcal{F}_D(0, \widehat{\mu}) \right\|. \tag{47}$$

Proof. (i) It is a straightforward consequence of Theorems 15 and 18, and the fact that $\|D^* \mathcal{F}_D(0, \widehat{\mu})\| \leq \operatorname{lip} \mathcal{F}_D(0, \widehat{\mu})$ by (44).

(ii) Remember that $\mathcal{F}_D^{-1} : \ell_\infty(T)^* \rightrightarrows X^*$ is defined through

$$\mathcal{F}_D^{-1}(\mu) := \begin{cases} A^*\mu - \overline{c}^* + Q^\circ, & \text{if } \mu \geq 0, \\ \emptyset, & \text{otherwise}. \end{cases}$$

Since $\operatorname{gph} \mathcal{F}_D^{-1}$ is convex and we are assuming that $\mathcal{F}_D^{-1}(\ell_\infty(T)_+^*)$ has nonempty interior for the norm topology in X^*, we may apply Proposition 5 in [16] to see that this mapping is perfectly regular and so, (47) holds.

Observe that the set $\{q \in Q : \|q\| = 1\} = Q \cap \{x^{**} \in X^{**} : \|x^{**}\| = 1\}$ is w^*-compact in X^{**} by the Alaoglu theorem.

Suppose that $\widehat{\mu} \in \mathcal{F}_D(0)$, and take $(b^{**}, x^{**}) \in \ell_\infty(T)^{**} \times X^{**}$ such that

$$S_{(\operatorname{gph} \mathcal{F}_D^{-1}) - (\widehat{\mu}, 0)}(b^{**}, x^{**}) =: M < +\infty \quad \text{and} \quad \left\| x^{**} \right\| = 1, \tag{48}$$

where $S_{(\operatorname{gph} \mathcal{F}_D^{-1}) - (\widehat{\mu}, 0)}$ is the support function of the convex set $(\operatorname{gph} \mathcal{F}_D^{-1}) - (\widehat{\mu}, 0)$.

Since $0 \in \mathcal{F}_D^{-1}(\widehat{\mu})$ there must exist $\widehat{q}^* \in Q^\circ$ such that

$$0 = A^*\widehat{\mu} - \overline{c}^* + \widehat{q}^*,$$

and the first condition in (48) reads

$$\left\langle \mu - \widehat{\mu}, b^{**} \right\rangle + \left\langle A^*(\mu - \widehat{\mu}) + q^* - \widehat{q}^*, x^{**} \right\rangle \leq M \quad \text{for all } \mu \geq 0 \text{ and all } q^* \in Q^\circ.$$

Taking $\mu = \widehat{\mu}$ yields

$$\left\langle q^*, x^{**} \right\rangle \leq \left\langle \widehat{q}^*, x^{**} \right\rangle \quad \text{for all } q^* \in Q^\circ,$$

i.e.,

$$x^{**} \in Q^{\circ\circ} \cap \{x^{**} \in X^{**} : \left\| x^{**} \right\| = 1\} = Q \cap \{x^{**} \in X^{**} : \left\| x^{**} \right\| = 1\},$$

and Proposition 5 in [16] applies. □

Remark 6. Condition (46) is automatically satisfied if X is the Euclidean space. In the infinite-dimensional setting, (46) also holds, for instance, when X is reflexive and Q is finite dimensional (for instance, finitely generated). Also in this case, $\mathcal{F}_D^{-1}(\ell_\infty(T)_+^*)$ has nonempty interior for the norm topology in X^* when $\operatorname{int} Q^\circ$ is nonempty, and this is implied by the existence of a compact base for Q.

Example 4. Revisiting Example 1 where $r = R = 1$, we can conclude from part (i) of this last theorem that $\operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) = \|D^*\mathcal{F}_D(0, \widehat{\mu})\|$.

Example 5. This example shows that the equality (47) may hold even when $r \neq R$ and the set (46) is not w^*-closed in X^{**}. Consider $X = c_0$, the Banach space of bounded real sequences converging to 0, with the supremum norm. Then $X^* = \ell_1 = \ell_1(\mathbb{N})$ and $X^{**} = \ell_\infty = \ell_\infty(\mathbb{N})$. Let

$$Q = \{q \in c_0 : 2q_1 \geq q_n \geq 0,\ n \in \mathbb{N}\} \quad \text{and} \quad \widetilde{Q} = \{q \in Q : \|q\|_\infty = 1\};$$

$\mathbf{1}^*=(1)_{n=1}^\infty \in \operatorname{cl}^*\widetilde{Q}$, so this set $\widetilde{Q}$ is not w^*-closed in ℓ_∞. Furthermore, whenever $q \in \widetilde{Q}$, then $q_n = 1$ for at least one $n \in \mathbb{N}$, and $\frac{1}{2} \leq q_1 \leq 1$. If $x^* = e_1 := (1, 0, \ldots, 0, \ldots) \in \ell_1$, then $\langle x^*, q\rangle = q_1 \geq \frac{1}{2}$ for any $q \in \widetilde{Q}$. Indeed, $r = \frac{1}{2}$ is the largest r we can choose to satisfy $\langle z^*, q\rangle \geq r$ for some $z^* \in \ell_1$, $\|z^*\|_1 = 1$, and for all $q \in \widetilde{Q}$, because, for $q^k \in \widetilde{Q},\ k \in \mathbb{N},\ k \neq 1$, defined by $q_1^k = \frac{1}{2}$, $q_k^k = 1$, and $q_n^k = 0$ otherwise, it holds that $\langle z^*, q^k\rangle = z_1^*\frac{1}{2} + z_k^* \to z_1^*\frac{1}{2} \leq \frac{1}{2}$. Now consider $T = \{t_0\}$ and put $a_{t_0}^* := a^* = \left(\frac{-1}{2^{n-1}}\right)_{n=1}^\infty \in \ell_1$, also fix $\overline{c}^* = a^*$. Observe that $\ell_\infty(T) = \ell_\infty(T)^* = \ell_\infty(T)^{**} = \mathbb{R}$. Then $\mu = 0$ is a strong Slater point for $\mathcal{F}_D$ at $c^* = 0$, while $\widehat{\mu} = 1 \in \mathcal{F}_D(0)$ is not a strong Slater point. The characteristic set is given by

$$C_D(0) = \operatorname{conv}\left(\left\{(\langle a^*, q\rangle, \langle a^*, q\rangle) : q \in \widetilde{Q}\right\} \cup \{(p, -1) : p \in \mathbb{R},\ p \geq 0\}\right) \subset \mathbb{R}^2;$$

and by taking into account that $-2 \leq -q_1 - 1 \leq \langle a^*, q\rangle \leq -q_1 \leq -\frac{1}{2}$ for any $q \in \widetilde{Q}$, an application of Theorem 15 gives

$$\left\|D^*\mathcal{F}_D(0, \widehat{\mu})\right\| \leq \sup\left\{\left\|b^{**}\right\|^{-1} : \left(b^{**}, \langle\widehat{\mu}, b^{**}\rangle\right) = \left(b^{**}, b^{**}\right) \in \operatorname{cl}^*C_D(0)\right\} = 2. \tag{49}$$

(Here $R = 1$.) Also $\operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) \leq 2$ because of Theorem 18. On the other hand, from Theorem 13, we can see that for each $k \in \mathbb{N},\ k \neq 1$, $\overline{q}^k$ defined by

$$\overline{q}_1^k = 1 - \frac{1}{2^{k-1}}, \qquad \overline{q}_{k+1}^k = 2\overline{q}_1^k, \quad \text{and} \quad q_n^k = 0 \text{ otherwise},$$

belongs to $D^*\mathcal{F}_D(0, \widehat{\mu})\left(b^k\right)$, where $b^k := \langle a^*, \overline{q}^k\rangle$. Since $\left|b^k\right| = \left|\langle a^*, \overline{q}^k\rangle\right| = 1 - \frac{1}{2^{k(k-2)}} < 1$, we obtain that

$$\begin{aligned}\left\|D^*\mathcal{F}_D(0, \widehat{\mu})\right\| &= \sup\left\{\left\|z^*\right\| : z^* \in D^*\mathcal{F}_D(0, \widehat{\mu})\left(b^{**}\right), \left\|b^{**}\right\| \leq 1\right\} \\ &\geq \left\|\overline{q}^k\right\| = \left\|\overline{q}^k\right\|_\infty = 2\left(1 - \frac{1}{2^{k-1}}\right).\end{aligned}$$

By letting $k \to \infty$ and in view of (49), it follows that $\|D^*\mathcal{F}_D(0, \widehat{\mu})\| = 2$. Finally, by (44) $\|D^*\mathcal{F}_D(0, \widehat{\mu})\| \leq \operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) \leq 2$, therefore we obtain the equalities $\|D^*\mathcal{F}_D(0, \widehat{\mu})\| = \operatorname{lip} \mathcal{F}_D(0, \widehat{\mu}) = 2$.

References

[1] E. Anderson, P. Nash, Linear Programming and Infinite-Dimensional Spaces, John Wiley and Sons, Chichester, 1987.
[2] M.A. Goberna, M.A. López, Linear Semi-Infinite Optimization, Wiley, Chichester, 1998.
[3] B. Brosowski, Parametric Semi-Infinite Optimization, Peter Lang, Frankfurt, 1982.
[4] K.S. Kretschmer, Programmes in paired spaces, Canad. J. Math. 13 (1961) 221–238.
[5] R.J. Duffin, Infinite programs. Linear inequalities and related systems, Ann. of Math. Stud. 38 (1956) 157–170.
[6] J.M. Borwein, Semi-infinite programming duality: how special is it? in: A.V. Fiacco, K.O. Kortanek (Eds.), Semi-Infinite Programming and Applications, Springer-Verlag, Berlin, 1983.
[7] M.J. Cánovas, M.A. López, J. Parra, M.I. Todorov, Stability and well-posedness in linear semi-infinite programming, SIAM J. Optim. 10 (1999) 82–99.
[8] N. Dinh, M.A. Goberna, M.A. López, On the stability of the feasible set in optimization problems, SIAM J. Optim. 20 (2010) 2254–2280.
[9] N. Dinh, M.A. Goberna, M.A. López, On the stability of the optimal value and the optimal set in optimization problems, Preprint, Dpt, of Statistics and Operations Research, Alicante University, 2010.
[10] M.J. Cánovas, A.L. Dontchev, M.A. López, J. Parra, Metric regularity of semi-infinite constraint systems, Math. Program. 104 (2005) 329–346.
[11] M.J. Cánovas, F.J. Gómez-Senent, J. Parra, Regularity modulus of arbitrarily perturbed linear inequality systems, J. Math. Anal. Appl. 343 (2008) 315–327.
[12] M.J. Cánovas, A. Hantoute, M.A. López, J. Parra, Lipschitz modulus in convex semi-infinite optimization via d.c. functions, ESAIM Control Optim. Calc. Var. 15 (2009) 763–781.
[13] M.J. Cánovas, D. Klatte, M.A. López, J. Parra, Metric regularity in convex semi-infinite optimization under canonical perturbations, SIAM J. Optim. 18 (2007) 717–732.
[14] M.J. Cánovas, M.A. López, B.S. Mordukhovich, J. Parra, Variational analysis in semi-infinite and infinite programming, I: stability of linear inequality systems of feasible solutions, SIAM J. Optim. 20 (3) (2009) 1504–1526.
[15] M.A. López, Stability in infinitely constrained optimization: a personal tour, Preprint, Dpt, of Statistics and Operations Research, Alicante University, 2010.
[16] A.D. Ioffe, Y. Sekiguchy, Regularity estimates for convex multifunctions, Math. Program., Ser. B 117 (2009) 255–270.
[17] A.D. Ioffe, On stability of solutions to systems of convex inequalities, CRM Preprints, 2010, p. 984.

[18] N. Dunford, J.T. Schwartz, Linear Operators Part I: General Theory, Wiley, New York, 1988.
[19] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation I: Basic Theory, Springer, Berlin, 2006.
[20] D. Klatte, B. Kummer, Nonsmooth equations in optimization, in: Regularity, Calculus, Methods and Applications, Kluwer, Dordrecht, 2002.
[21] B.S. Mordukhovich, J.V. Outrata, Coderivative analysis of quasi-variational inequalities with applications to stability and optimization, SIAM J. Optim. 18 (2007) 389–412.
[22] A.V. Dmitruk, A.Y. Kruger, Metric regularity and systems of generalized equations, J. Math. Anal. Appl. 342 (2008) 864–873.
[23] A.D. Ioffe, Metric regularity and subdifferential calculus, Russian Math. Surveys 55 (2000) 501–558.
[24] B.S. Mordukhovich, Complete characterizations of openness, metric regularity, and lipschitzian properties of multifunctions, Trans. Amer. Math. Soc. 340 (1993) 1–35.
[25] B.S. Mordukhovich, Coderivatives of set-valued mappings: calculus and applications, Nonlinear Anal. 30 (1997) 3059–3070.
[26] A. Fabian, P. Habala, P. Hájek, V. Montesinos-Santalucía, J. Pelant, V. Zizler, Functional Analysis and Infinite-Dimensional Geometry, Springer-Verlag, New York, 2001.
[27] G. Köthe, Topological Vector Spaces. I, in: Die Grundlehren der Mathematischen Wissenschaften, Band, vol. 159, Springer-Verlag New York Inc., New York, 1969, Translated from the German by D. J. H. Garling.
[28] N. Dinh, M.A. Goberna, M.A. López, From linear to convex systems: consistency, Farkas' lemma and applications, J. Convex Anal. 13 (2006) 279–290.