# Emergent electrodynamics from the Nambu model for spontaneous Lorentz symmetry breaking

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After imposing the Gauss law constraint as an initial condition upon the Hilbert space of the Nambu model, in all its generic realizations, we recover QED in the corresponding nonlinear gauge $A_\mu A^\mu = n^2 M^2$. Our result is nonperturbative in the parameter $M$ for $n^2 \neq 0$ and can be extended to the $n^2 = 0$ case. This shows that, in the Nambu model, spontaneous Lorentz symmetry breaking dynamically generates gauge invariance, provided the Gauss law is imposed as an initial condition. In this way, electrodynamics is recovered, with the photon being realized as the Nambu-Goldstone modes of the spontaneously broken symmetry, which finally turns out to be nonobservable.

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## I. INTRODUCTION

The possible violation of Lorentz invariance has recently received a lot of attention, both from the experimental and from the theoretical perspective. In this latter case, the interest lies mainly in connection with possible effects arising from the drastic modifications of space-time at distances of the order of the Planck length, suggested by most of the current quantum gravity approaches. Experiments and astrophysical observations put severe bounds upon the parameters describing such violations, which are still being improved.

In this work we revisit a different and interesting idea related to spontaneous Lorentz symmetry breaking, which is that photons (and gravitons) could emerge as the corresponding Nambu-Goldstone bosons from that process. We focus on the case of electrodynamics. Normally, the null mass of both particles is explained by invoking some type of gauge invariance. Since this almost sacred principle has undoubtedly been fundamental in the development of physics, it is very interesting to explore the possibility that it could have a dynamical origin [1]. This idea goes back to the works of Nambu [2] and Bjorken [3], together with many other contributions [4]. Recently, it has been revived in Refs. [5–10].

One of the most explored approaches starts from a theory with a vector field $B^\mu$ endowed with the standard electrodynamic kinetic term plus a potential designed to break the Lorentz symmetry via a nonzero vacuum expectation value $\langle B^\mu \rangle$, which defines a preferred direction $n_\mu$ in space-time. This potential also breaks gauge invariance. The original model was proposed by Nambu [2] and recently has been generalized to incorporate the so-called bumblebee models [11]. In general, the subsequent symmetry breaking obtained from the nonzero minimum of the potential splits the original four degrees of freedom $B^\mu$ into three vectorial Nambu-Goldstone bosons $A^\mu$, to be identified with the photon satisfying the constraint $A_\mu A^\mu = n^2 M^2$, plus a massive scalar field $\sigma$, which is assumed to be excited at very high energies.

It is noteworthy to recall here that the constraint $A_\mu A^\mu = \pm M^2$ was originally proposed by Dirac as a way to derive the electromagnetic current from the additional excitations of the photon field, which now have ceased to be gauge degrees of freedom, avoiding, in this way, the problems arising from considering pointlike charges [12]. As reported in [13], this theory requires the existence of an aether emerging from quantum fluctuations, but does not necessarily imply the violation of Lorentz invariance, due to an averaging process of the quantum states producing such fluctuations.

In the framework of the Nambu model, calculations at tree level [2] and to one-loop level [7] have been carried out, showing that the possible Lorentz violating effects are not present in the physical observables and that the results are completely consistent with the standard gauge invariant electrodynamics. This is certainly a surprising result, which can be truly appreciated from the complexity of the calculation in Ref. [7] and that certainly deserves additional understanding. The above-mentioned calculations have been performed using an expansion of the nonlinear interaction up to order $1/M$ and $1/M^2$, respectively. The basic question that remains is to what extent the nonperturbative Nambu model describes the standard effects of full QED, which have been experimentally measured with astonishing precision.

In this work we recover and extend to the spacelike and lightlike cases the results regarding the equivalence of the timelike version ($n_\mu n^\mu > 0$) of the Nambu model and QED, given in Ref. [10]. Our proposal is to start with the full analysis of these theories at the classical level and then move on to the quantum regime. Even at the classical level the problem is a difficult one because of the presence of the nonlinear constraint $A_\mu A^\mu = n^2 M^2$, although nonlinear gauges have been successfully used in the quantization of Yang-Mills theories [14]. Despite this, and selecting an approach which does not require us to completely fix the gauge, we show that the Nambu model, with the Gauss law as an additional constraint, imposed as an initial condition, has exactly the same canonical structure as the electromagnetic field. This point becomes evident when both canonical structures are expressed in terms of the electric **E** and potential **A** fields. Since the quantum dynamics is determined by the canonical structure, this result warrants that both theories, in their nonperturbative form, are equivalent at the quantum level, thus yielding the same physical results.

The paper is organized as follows. In Sec. II we review the Nambu model, showing that its spacelike version arises from the spontaneous Lorentz symmetry breaking (SLSB) of a bumblebee model. The remaining sectors of the Nambu model are considered as a natural generalization of the spacelike case, without specifying the mechanism leading to the SLSB. In Sec. III we show that taking the phase space of the Nambu model and imposing the Gauss law constraint on it as an initial condition, the resulting canonical structure coincides with that of the electromagnetic field in the nonlinear gauge given by the particular SLSB considered. In Sec. IV we show that the separate procedures for dealing with the Nambu model, presented in Sec. III, can be given a unified description using a Lagrange multiplier to impose the resolution of the nonlinear constraint in each case. Finally, Sec. V contains the conclusions and final comments. Appendix A summarizes the canonical version of QED, which we use as a benchmark to state the equivalence with the different realizations of the Nambu model plus the Gauss law. Appendix B includes a specific example of the construction of the potentials in the nonrelativistic version of the nonlinear gauge, for the case of a constant magnetic field. Finally, in Appendix C we discuss the dynamics of the full SLSB bumblebee model in its spacelike version and show how electrodynamics is recovered in this case.

## II. SLSB AND THE NAMBU MODEL

As we will show in the following, the Nambu model can be motivated as the low energy limit of the bumblebee model [the metric is $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$],

$$\mathcal{L}(B_\mu) = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{m^2}{2}(B_\mu B^\mu) - \frac{\lambda}{4}(B_\mu B^\mu)^2 - J_\mu B^\mu, \qquad \partial_\mu J^\mu = 0, \tag{1}$$

with

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{2}$$

after the $SO(1, 3)$ symmetry is broken. Following the standard steps we obtain the broken symmetry Lagrangian

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\left(1+\frac{\omega}{\sigma_0}\right)^2 + \frac{1}{\sigma_0}\left(1+\frac{\omega}{\sigma_0}\right) \\ &\times F_{\mu\nu}(A^\mu\partial^\nu\omega) - \frac{1}{2}n^2(\partial_\mu\omega)(\partial^\mu\omega) \\ &+ \frac{1}{2\sigma_0^2}A^\nu A^\mu(\partial_\mu\omega)(\partial_\nu\omega) - \frac{1}{2}(-2m^2n^2)\omega^2 \\ &- \lambda\sigma_0\omega^3 - \frac{\lambda}{4}\omega^4 - J_\mu A^\mu\left(1+\frac{\omega}{\sigma_0}\right),\end{aligned} \tag{3}$$

where we have successively set $B^\mu = \Lambda^\mu_\nu(x)n^\nu\sigma(x) \equiv \frac{1}{\sigma_0}A^\mu(x)\sigma(x)$, $\sigma(x) = \sigma_0 + \omega(x)$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This parametrization requires

$$B_\mu B^\mu = \sigma^2 n^2, \qquad A_\mu A^\mu = \sigma_0^2 n^2. \tag{4}$$

Here $n^\nu$ is a constant vector characterizing the symmetry breaking vacuum according to

$$n^\alpha \text{ timelike: } SO(1,3) \to SO(3), \tag{5}$$

$$n^\alpha \text{ spacelike: } SO(1,3) \to SO(1,2). \tag{6}$$

Let us emphasize that the lightlike case ($n^2 = 0$) is forbidden because then $B_\mu B^\mu = 0$, so there is no spontaneous symmetry breaking since the potential turns out to be zero. The Goldstone modes are contained in the field $A^\mu$, which is restricted by the condition (4), so it contains only three degrees of freedom. The field $\omega(x)$ describes the massive fluctuations of $\sigma(x)$ around the minimum $\sigma_0$ of the potential, where

$$M_\omega = \sqrt{2m^2}. \tag{7}$$

In this way, the parameter $m^2$ in (1) must be positive. We also require $\lambda > 0$, in such a way that the potential is bounded at infinity. The minimum of the potential has to be real, and it satisfies

$$\sigma_0 = \sqrt{-\frac{m^2 n^2}{\lambda}} > 0, \tag{8}$$

which imposes $n^2 < 0$. Let us further consider the case where the effective theory described by (3) is in the energy range ($E \ll M_\omega$), where the field $\omega(x)$ is not excited. In this situation we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu, \qquad A_\mu A^\mu = -\frac{m^2}{\lambda} < 0. \tag{9}$$

It is interesting to remark that the choice $n^2 < 0$ in (3) also produces the right sign normalization for the kinetic term of the field $\omega$.

In this way, we have shown that the spontaneous Lorentz symmetry breaking of the bumblebee model (1) leads to the Nambu model [2] only in its spacelike version. We consider the remaining sectors of the Nambu model as an extension of this result, without deriving them from the spontaneous symmetry breaking of a specific model. Thus we replace the second equation in (9) by the requirement

$$A_\mu A^\mu = M^2 n^2, \qquad M^2 > 0, \tag{10}$$

where $n^2$ can now be positive, negative, or zero. In each case we will show that $n^\mu$ indeed describes the vacuum around which the symmetry has been broken by the Nambu-Goldstone modes $A_\mu$.

The standard analysis of the model is made by explicitly solving the constraint $A_\mu A^\mu = M^2 n^2$ and subsequently substituting the result into the Lagrangian (9), thus providing a theory analogous to the standard nonlinear sigma model [15]. As we will show in Sec. IV, an alternative compact way of dealing with the model is to introduce the constraint via a Lagrange multiplier.

In the following, we examine the conditions under which this theory is equivalent to standard electrodynamics, in such a way that $A_\mu$ can be interpreted as the photon field in the nonlinear gauge $A_\mu A^\mu = M^2 n^2$. The choices of $n^2$ provide alternative forms in which the constraint $A_\mu A^\mu = M^2 n^2$ is to be solved so that the remaining symmetry is manifest. For the timelike case we have

$$A_0 = \sqrt{M^2 + \mathbf{A}^2}, \tag{11}$$

with $\mathbf{A} = (A^1, A^2, A^3)$ being three independent degrees of freedom, and where the variable $A_0$ is explicitly invariant under rotations.

The situation in the spacelike case is

$$A_3 = \sqrt{M^2 + (A^0)^2 - (A^1)^2 - (A^2)^2}, \tag{12}$$

where the independent variables are $(A^0, A^1, A^2)$ and the variable $A_3$ is explicitly invariant under the corresponding $SO(1,2)$ subgroup.

In the case $n^2 = 0$ we introduce the following parametrization [2],

$$\begin{aligned} A_0 &= c\left(1 + \frac{B}{c} + \frac{A_1^2 + A_2^2}{4c^2(1+\frac{B}{c})}\right), \\ A_3 &= c\left(1 + \frac{B}{c} - \frac{A_1^2 + A_2^2}{4c^2(1+\frac{B}{c})}\right), \end{aligned} \tag{13}$$

which provides the expansion parameter $c$ for the chosen dependent fields, which is analogous to $M$ in the remaining cases. Here the independent variables are $B(x)$, $A_1(x)$, and $A_2(x)$.

We identify the vacuum $(A_\mu)_0$ of each sector of the theory by setting equal to zero all space-time dependent fields in $A_\mu$. In this way, we obtain

$$\begin{aligned} n^2 > 0, &\qquad (A_\mu)_0 = (M, 0, 0, 0), \\ n^2 < 0, &\qquad (A_\mu)_0 = (0, 0, 0, M), \\ n^2 = 0, &\qquad (A_\mu)_0 = (c, 0, 0, c). \end{aligned} \tag{14}$$

## III. ELECTRODYNAMICS AS A CONSTRAINED NAMBU MODEL

The first hint of how the Nambu model is related to electrodynamics arises from the Lagrangian equations of motion. The variation of (9) produces

$$\delta\mathcal{L} = (\partial_\mu F^{\mu\nu} - J^\nu)\delta A_\nu, \tag{15}$$

and the equations of motion for each case are obtained by looking at the variation of the independent variables,

$$\text{timelike case:} \quad \delta A_0 = \frac{A_i}{A_0}\delta A_i, \tag{16}$$

$$\text{spacelike case:} \quad \delta A_3 = \frac{A_0}{A_3}\delta A_0 - \frac{A_a}{A_3}\delta A_a, \tag{17}$$

which produces

$$(\partial_k F^{k0} - J^0)\frac{A_i}{A_0} + (\partial_\mu F^{\mu i} - J^i) = 0, \qquad i = 1, 2, 3, \tag{18}$$

or

$$\begin{aligned} &(\partial_k F^{k0} - J^0) + (\partial_\mu F^{\mu 3} - J^3)\frac{A_0}{A_3} = 0, \\ &(\partial_\mu F^{\mu a} - J^a) - (\partial_\mu F^{\mu 3} - J^3)\frac{A_a}{A_3} = 0, \qquad a = 1, 2, \end{aligned} \tag{19}$$

respectively. Thus, we see that in order to recover electrodynamics from the Nambu model, we must further impose the Gauss law $\partial_i F^{i0} - J^0 = 0$ as an additional constraint. This cannot be done at the Lagrangian level because such a constraint involves the velocity $\dot{A}_i$.

Recalling that the quantum dynamics is determined by the canonical structure, our general strategy to prove the equivalence between the quantized Nambu model plus the Gauss law and QED will be to perform the Hamiltonian analysis of the former and to compare with the corresponding formulation of electrodynamics, summarized in Appendix A.

### A. The timelike case

We start from the Lagrangian density (9), where we have explicitly solved and substituted the nonlinear constraint in the form $A_0 = \sqrt{\mathbf{A}^2 + M^2}$,

$$\begin{aligned} \mathcal{L}_N(A_i) = {}& \tfrac{1}{2}((\partial_t A_i - \partial_i(\sqrt{\mathbf{A}^2 + M^2}))^2 - (\nabla \times \mathbf{A})^2) \\ & - J_0\sqrt{\mathbf{A}^2 + M^2} - J_i A^i. \end{aligned} \tag{20}$$

The canonically conjugated momenta $\Pi_i$ are

$$\Pi_i = \frac{\partial L}{\partial \dot{A}_i} = \partial_0 A_i - \partial_i\sqrt{M^2 + \mathbf{A}^2} = E_i, \tag{21}$$

leading to the canonical Hamiltonian density $\mathcal{H} = E_i \dot{A}_i - \mathcal{L}$,

$$\mathcal{H} = \tfrac{1}{2}\mathbf{E}^2 + \tfrac{1}{2}\mathbf{B}^2 + E_i\partial_i\sqrt{M^2 + \mathbf{A}^2} + J_0\sqrt{\mathbf{A}^2 + M^2} + J_iA^i, \tag{22}$$

with the basic nonzero Poisson brackets (PB)

$$\{A_i(\mathbf{x}, t), E_j(\mathbf{y}, t)\} = \delta_{ij}\delta^3(\mathbf{x} - \mathbf{y}). \tag{23}$$

The Hamiltonian equations of motion turn out to be

$$-\dot{E}_i + (\partial_k E_k - J_0)\frac{A_i}{\sqrt{M^2 + \mathbf{A}^2}} + [\nabla \times B]_i - J^i = 0, \tag{24}$$

where

$$E_i = \dot{A}_i - \partial_i\sqrt{M^2 + \mathbf{A}^2}, \qquad B_i = \epsilon_{ijk}\partial_j A^k. \tag{25}$$

In other words, we have recovered the Lagrangian equations of motion (18) for the Nambu model in this representation.

The next step is to implement the Gauss law constraint via an arbitrary function $\chi$ so that the modified Hamiltonian density is

$$\tilde{\mathcal{H}} = \tfrac{1}{2}\mathbf{E}^2 + \tfrac{1}{2}\mathbf{B}^2 - (\partial_i E_i - J_0)(\sqrt{M^2 + \mathbf{A}^2} + \chi) + J_iA^i. \tag{26}$$

Redefining the arbitrary function

$$\Theta = \chi + \sqrt{M^2 + \mathbf{A}^2}, \tag{27}$$

we finally obtain

$$\tilde{\mathcal{H}} = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - \Theta(\partial_i E_i - J_0) + J_iA^i, \tag{28}$$

with the corresponding first order action

$$S = \int d^3y(E_i\dot{A}_i - \tilde{\mathcal{H}}). \tag{29}$$

In this way, the Hamiltonian density adopts the form (A13) of Appendix A. The Poisson bracket relations also coincide with those of (A14). We have thus shown that the imposition of the Gauss law in the Nambu Hamiltonian density (22), which is performed in (26), does in fact lead to classical electrodynamics. Let us remark that, as will be shown in Sec. IV, the dynamics of the Nambu model, for any choice of $n^2$, preserves the Gauss law in time, so that it is sufficient to impose it as an initial condition.

### B. The spacelike case

Here we start from (9) but make the substitution $A_3 = \sqrt{M^2 + A_0^2 - A_aA_a}$,

$$\mathcal{L}_N(A^0, A^a) = \tfrac{1}{2}(\mathbf{E}_T^2 + E_3^2 - \mathbf{B}^2) - J_0A^0 - J^aA_a - J^3\sqrt{M^2 + A_0^2 - A_aA_a}, \quad a = 1, 2, \tag{30}$$

where the independent degrees of freedom are now $A_0, A_a$. Since $A_3$ is just shorthand for $\sqrt{M^2 + A_0^2 - A_T^2}$, we have

$$\dot{A}_3 = \frac{A_0}{A_3}\dot{A}_0 - \frac{A_a}{A_3}\dot{A}_a, \tag{31}$$

which leads to

$$E_3 = \frac{A_0}{A_3}\dot{A}_0 - \frac{A_a}{A_3}\dot{A}_a - \partial_3 A_0, \tag{32}$$

$$E_a = \dot{A}_a - \partial_a A_0. \tag{33}$$

The canonically conjugated momenta are

$$\Pi_0 = \frac{\partial L_N}{\partial \dot{A}_0} = E_3\frac{A_0}{A_3} = \left(\frac{A_0}{A_3}\dot{A}_0 - \frac{A_a}{A_3}\dot{A}_a - \partial_3 A_0\right)\frac{A_0}{A_3}, \tag{34}$$

$$\Pi_a = \frac{\partial L_N}{\partial \dot{A}_a} = E_a - E_3\frac{A_a}{A_3} = \dot{A}_a - \partial_a A_0 - \left(\frac{A_0}{A_3}\dot{A}_0 - \frac{A_b}{A_3}\dot{A}_b - \partial_3 A_0\right)\frac{A_a}{A_3}, \tag{35}$$

with the nonzero Poisson brackets now being

$$\{A_0(\mathbf{x}, t), \Pi_0(\mathbf{y}, t)\} = \delta^3(\mathbf{x} - \mathbf{y}), \quad \{A_a(\mathbf{x}, t), \Pi_b(\mathbf{y}, t)\} = \delta_{ab}\delta^3(\mathbf{x} - \mathbf{y}). \tag{36}$$

Solving for the velocities in (34) and (35) yields

$$\dot{A}_0 = \left(\frac{\mathbf{A}_T^2}{A_0^2}\Pi_0 + \frac{1}{A_0}(A_a\Pi_a + A_a\partial_a A_0 + A_3\partial_3 A_0)\right), \tag{37}$$

$$\dot{A}_a = \Pi_a + \partial_a A_0 + \frac{A_a}{A_0^2}A_0\Pi_0. \tag{38}$$

We can now write the electric field in terms of the momenta as follows:

$$E_a = \Pi_a + A_a\frac{\Pi_0}{A_0}, \qquad E_3 = A_3\frac{\Pi_0}{A_0}. \tag{39}$$

The Hamiltonian density

$$\mathcal{H} = \Pi_0\dot{A}_0 + \Pi_a\dot{A}_a - \tfrac{1}{2}(\mathbf{E}_T^2 + E_3^2 - \mathbf{B}^2) + J^0A_0 + J^aA_a + J^3\sqrt{M^2 + A_0^2 - A_aA_a} \tag{40}$$

can be written as

$$\mathcal{H} = \frac{1}{2}\frac{\Pi_0^2}{A_0^2}(A_aA_a + A_3^2) + \frac{1}{2}\Pi_a\Pi_a + \frac{\Pi_0}{A_0}A_a\Pi_a - A_0\left(\partial_a\left(\frac{\Pi_0}{A_0}A_a\right) + \partial_3\left(\frac{\Pi_0}{A_0}A_3\right) + \partial_a\Pi_a\right) + \frac{1}{2}\mathbf{B}^2(A_0, A_a) + J^0A_0 + J^aA_a + J^3\sqrt{M^2 + A_0^2 - A_aA_a}. \tag{41}$$

To make contact with the standard form of the Hamiltonian in QED, we rewrite

$$\frac{1}{2}\Pi_a\Pi_a + \frac{\Pi_0}{A_0}A_a\Pi_a = \frac{1}{2}E_a^2 - \frac{1}{2}\left(\frac{\Pi_0}{A_0}\right)^2 A_aA_a, \tag{42}$$

using (39), and realize that the Gauss law adopts the following form in terms of the independent variables:

$$\partial_i E_i = \partial_a E_a + \partial_3 E_3$$
$$= \partial_a \Pi_a + \partial_a\left(A_a \frac{\Pi_0}{A_0}\right) + \partial_3\left(A_3 \frac{\Pi_0}{A_0}\right). \quad (43)$$

Substituting (42) and (43) in (41) we obtain

$$\mathcal{H} = \tfrac{1}{2}E_3^2 + \tfrac{1}{2}E_a^2 + \tfrac{1}{2}\mathbf{B}^2(A_0, A_a) - A_0(\partial_i E_i - J_0) + J^a A_a + J^3 A_3. \quad (44)$$

The above Hamiltonian density corresponds to the Nambu model, and the resulting equations of motion coincide with those obtained in (19). Equation (44) has just the form of the standard Hamiltonian density for electrodynamics, except that $A_3$, $E_3$ are notations for the corresponding functions in terms of the dynamical coordinates and momenta. Notice that in this case $A_0$ is not a Lagrange multiplier, so we are still missing the Gauss law to recover the electrodynamics.

Next, we follow the established procedure by imposing the Gauss law on the Nambu Hamiltonian (44) via the arbitrary function $\chi$. The result is

$$\tilde{\mathcal{H}} = \tfrac{1}{2}E_3^2 + \tfrac{1}{2}E_a^2 + \tfrac{1}{2}\mathbf{B}^2(A_0, A_a) - (A_0 + \chi)(\partial_i E_i - J_0) + J^a A_a + J^3 A_3. \quad (45)$$

The redefinition $\Theta = A_0 + \chi$ leaves

$$\tilde{\mathcal{H}} = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - \Theta(\partial_i E_i - J_0) + J^a A_a + J^3 A_3, \quad (46)$$

which is precisely of the form (A13). As emphasized at the end of Sec. III A, it is enough to impose the Gauss law as an initial condition. Now we use the transformations from the original spacelike variables $A_0$, $A_a$, $\Pi_0$, $\Pi_a$ to the set $A_i$, $E_j (i, j = 1, 2, 3)$:

$$A_a = A_a, \qquad A_3 = \sqrt{M^2 + A_0^2 - A_a A_a}, \quad (47)$$

$$E_a = \Pi_a + A_a \frac{\Pi_0}{A_0}, \qquad E_3 = \frac{\Pi_0}{A_0}\sqrt{M^2 + A_0^2 - A_a A_a}, \quad (48)$$

together with the brackets (36) for $A_0$, $A_a$, $\Pi_0$, $\Pi_a$, to verify that we recover the brackets among $A_i$, $E_j$ given in (A14), which define classical electrodynamics. In other words, we have shown that the change of variables (47) and (48) is a canonical transformation. We also have

$$\Pi_0 \dot{A}_0 + \Pi_a \dot{A}_a = E_i \dot{A}_i, \quad (49)$$

as expected.

As a final comment in this section we observe that the possible singularities envisaged in Sec. 4 of Ref. [10], arising in the spacelike case due to the possibility of $A_0 = 0$ and which are not present in the timelike case, do in fact appear in our construction via the factor $1/A_0$ which occurs in many of the intermediate steps in this section, and also in the corresponding equations in Sec. IV. Nevertheless, such a singularity is absent both in the final Hamiltonian and in the final Dirac brackets.

### C. The lightlike case

In this case the degrees of freedom are the fields $B$ and $A^a$, $a = 1, 2$, according to (13). To simplify the expressions we redefine $c + B \to B$, and thus we get

$$A_0 = \frac{1}{4B}(4B^2 + A_a A_a), \quad (50)$$

$$A_3 = \frac{1}{4B}(4B^2 - A_a A_a). \quad (51)$$

From here on, $A_0$ and $A_3$ are just the notation for the functions given by the above equations, and their time derivatives are

$$\dot{A}_0 = \frac{1}{B}\left(A_3 \dot{B} + \frac{1}{2} A_a \dot{A}_a\right), \quad (52)$$

$$\dot{A}_3 = \frac{1}{B}\left(A_0 \dot{B} - \frac{1}{2} A_a \dot{A}_a\right). \quad (53)$$

In terms of these degrees of freedom, the components of the electric field are

$$E_3 = \dot{A}_3 - \partial_3 A_0 = \frac{1}{B}\left(A_0 \dot{B} - \frac{1}{2} A_a \dot{A}_a - B\partial_3 A_0\right), \quad (54)$$

$$E_a = \dot{A}_a - \partial_a A_0. \quad (55)$$

As in the preceding cases, the starting point is the Lagrangian density (9), which takes the form

$$\mathcal{L}_N(A^0, A^a) = \frac{1}{2}(\mathbf{E}_a^2 + E_3^2 - \mathbf{B}^2) - J_0\left(B + \frac{A_a^2}{4B}\right) - J^3\left(B - \frac{A_a^2}{4B}\right) - J^a A_a. \quad (56)$$

The canonical momenta of $A_a$ and $B$ can be written

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial \dot{A}_a} = E_a - \frac{A_a}{2B} E_3, \qquad \Pi_B = \frac{\partial \mathcal{L}}{\partial \dot{B}} = \frac{A_0}{B} E_3, \quad (57)$$

and they satisfy the nonzero fundamental Poisson brackets

$$\{B(\mathbf{x}, t), \Pi_B(\mathbf{y}, t)\} = \delta^3(\mathbf{x} - \mathbf{y}),$$
$$\{A_a(\mathbf{x}, t), \Pi_b(\mathbf{y}, t)\} = \delta_{ab}\delta^3(\mathbf{x} - \mathbf{y}). \quad (58)$$

From (57), the electric field in terms of the momenta is

$$E_a = \Pi_a + \frac{A_a}{2A_0}\Pi_B, \qquad E_3 = \frac{B}{A_0}\Pi_B. \quad (59)$$

and the velocities can be expressed as

$$\dot{A}_a = \Pi_a + \partial_a A_0 + \frac{A_a}{2A_0}\Pi_B, \qquad (60)$$

$$\dot{B} = \frac{1}{A_0}\left(B\Pi_B + B\partial_3 A_0 + \frac{1}{2}A_a(\Pi_a + \partial_a A_0)\right). \qquad (61)$$

The Hamiltonian density

$$\mathcal{H} = \Pi_a\dot{A}_a + \Pi_B\dot{B} - \frac{1}{2}(\mathbf{E}_a^2 + E_3^2 - \mathbf{B}^2) + J_0\left(B + \frac{A_a^2}{4B}\right) + J^3\left(B - \frac{A_a^2}{4B}\right) + J^a A_a \qquad (62)$$

now takes the form

$$\mathcal{H} = \frac{1}{2}\Pi_a^2 + \frac{B}{2A_0}\Pi_B^2 + \frac{A_a}{2A_0}\Pi_B\Pi_a + \frac{1}{2}\mathbf{B}^2 + \left[\partial_a\Pi_a + \partial_3\left(\frac{\Pi_B B}{A_0}\right) + \partial_a\left(\frac{A_a}{2A_0}\Pi_B\right) + J^0\right]A_0 + J^3A_3 + J^aA_a. \qquad (63)$$

Introducing the electric and magnetic fields according to (59), we have

$$\mathbf{E}^2 = \frac{B}{A_0}(\Pi_B)^2 + (\Pi_a)^2 + \frac{A_a}{A_0}\Pi_B\Pi_{A_a}, \qquad (64)$$

and the Gauss law becomes

$$\partial_i F^{i0} - J^0 = \partial_3 E^3 + \partial_a E^a - J^0 = -\left[\partial_3\left(\frac{B}{A_0}\Pi_B\right) + \partial_a\left(\Pi_{A_a} + \frac{A_a}{2A_0}\Pi_B\right) + J^0\right]. \qquad (65)$$

These relations imply that the Hamiltonian can be expressed, using the electric and magnetic fields and the Gauss law, in the compact form

$$\mathcal{H} = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - (\partial_i F^{i0} - J^0)A^0 + J_aA^a + J_3A^3, \qquad (66)$$

which corresponds again to the Nambu model. As in the previous cases, imposing the Gauss law as an initial constraint, we obtain the Hamiltonian of the electromagnetic field in the nonlinear gauge $A_\mu A^\mu = 0$,

$$\mathcal{H} = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - (\partial_i F^{i0} - J^0)\Theta + J_iA^i, \qquad (67)$$

where $\Theta$ is an arbitrary function, corresponding to the primary constraint $(\partial_i F^{i0} - J^0)$. Similarly to the spacelike case, in the intermediate steps of this construction it is possible for singularities to appear because of the factor $1/B$. However, as in the spacelike case, such singularities are not present in the final Hamiltonian density or in the Dirac brackets. Once more, starting from the definitions (50) and (51) together with (54) and (55) in terms of the canonical variables, and using the brackets (58), it is a direct matter to recover the Poisson algebra of electrodynamics (A14) in terms of the fields $A_i$ and $E_i$.

## IV. UNIFIED DISCUSSION OF THE NAMBU MODEL

The alternative possibilities to realize the spontaneous symmetry breaking can be jointly discussed by incorporating the nonlinear constraint via a Lagrange multiplier $\lambda$,

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu + \lambda(A_\mu A^\mu - M^2n^2), \qquad (68)$$

which yields the equations of motion

$$\partial_\mu F^{\mu\nu} - J^\nu + 2\lambda A^\nu = 0, \qquad (69)$$

$$A_\mu A^\mu - M^2n^2 = 0. \qquad (70)$$

From Eq. (68) we obtain the relation

$$\partial_0(\partial_i F^{i0} - J^0) = -\partial_k\left(\frac{A^k}{A^0}(\partial_i F^{i0} - J^0)\right), \qquad (71)$$

which implies that the imposition of the Gauss law as an initial condition guarantees that $\partial_i F^{i0} - J^0 = 0$ remains valid for arbitrary times. It is also important to emphasize that the choice $\lambda(x, t = 0) = 0$ as an initial condition leads, according to Eq. (68), to the requirement that all four Maxwell equations must be satisfied at $t = 0$, in particular, the Gauss law.

There are several possibilities to work out $\lambda(x, t)$. If we take a particular value for the free index in Eq. (69), $\nu = \sigma$, to solve the Lagrange multiplier, it yields

$$\lambda = -\frac{1}{2A^\sigma}(\partial_\mu F^{\mu\sigma} - J^\sigma) \qquad (72)$$

(no sum over $\sigma$), and thus we obtain

$$A^\sigma(\partial_\mu F^{\mu\nu} - J^\nu) - (\partial_i F^{i\sigma} - J^\sigma)A^\nu = 0, \qquad (73)$$

$$A_\mu A^\mu - M^2n^2 = 0. \qquad (74)$$

For $\sigma = 0$ these equations correspond to the timelike case (18), and for $\sigma = 3$ they produce the spacelike equations of motion (19). This shows that the Lagrange multiplier formulation of the Nambu model is equivalent to the direct-substitution approach developed in Sec. III. If we introduce the definitions

$$E_i = \dot{A}_i - \partial_i A_0, \qquad B_i = \epsilon_{ijk}\partial_j A^k, \qquad (75)$$

and impose an additional constraint, the Gauss law $\nabla \cdot \mathbf{E} - J^0 = 0$, the above equations become Maxwell equations in the nonlinear gauge (74). Let us emphasize that (73) contains the two cases corresponding to (11) and (12), together with the lightlike choice $n^2 = 0$.

The quantum dynamics of this model is determined by the canonical formalism. For the Nambu model such a formulation can be obtained in a very compact way, directly from the Lagrangian density (68), using the Dirac approach for singular Lagrangians. The coordinate fields are $A^\mu$ and $\lambda$, and the definitions of their canonical momenta lead to

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = F^{0i}, \qquad \pi_0 = \frac{\partial \mathcal{L}}{\partial \dot{A}^0} = 0, \qquad \pi_\lambda = \frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = 0. \tag{76}$$

From here we can make explicit the time derivative of the spatial components of the vector field,

$$\dot{A}^i = -\pi^i + \partial^i A^0, \tag{77}$$

and two primary constraints emerge,

$$\phi_1 = \pi_0, \qquad \phi_2 = \pi_\lambda. \tag{78}$$

Thus, the extended Hamiltonian density is

$$\mathcal{H} = -\tfrac{1}{2}\pi_i\pi^i + \tfrac{1}{4}F_{ij}F^{ij} + J_iA^i + (J_0 - \partial_i\pi^i)A^0 - \lambda(A_\mu A^\mu - M^2n^2) + u\pi_0 + v\pi_\lambda. \tag{79}$$

The consistency conditions of the primary constraints lead to the secondary ones,

$$\phi_3 = \{\pi_\lambda, H\} = A_\mu A^\mu - M^2n^2 = 0, \tag{80}$$

$$\phi_4 = \{\pi_0, H\} = 2\lambda A_0 + (\partial_i\pi^i - J_0) = 0. \tag{81}$$

In turn, the equations of motion of these last constraints fix the arbitrary functions $u$ and $v$. Thus, we have the set of second-class constraints $\phi_1$, $\phi_2$, $\phi_3$, and $\phi_4$, which determine a subspace where the Hamiltonian density becomes

$$H = -\tfrac{1}{2}\pi_i\pi^i + \tfrac{1}{4}F_{ij}F^{ij} + (J_0 - \partial_i\pi^i)A^0 + j_iA^i. \tag{82}$$

To make explicit the dynamics it is also necessary to construct the Dirac brackets that define the symplectic structure of the model. The constraint $\phi_3$ admits several realizations. One of these expresses the time component of $A^\mu$ in terms of the spatial ones (the timelike case),

$$A_0 = \sqrt{M^2 - A^iA_i}. \tag{83}$$

Here the spatial components of $A^\mu$ are the degrees of freedom, and the resulting Dirac brackets are

$$\{A^j(\mathbf{x}, t), \pi_i(\mathbf{y}, t)\}_D = \eta_i^j\delta^3(\mathbf{x} - \mathbf{y}). \tag{84}$$

Then, the canonical momenta conjugated to the $A^j$ fields are directly the $\pi_j$. Another realization is obtained when we take a spatial component as the dependent field (the spacelike case), such as

$$A_3 = (M^2 + A^0A_0 + A^aA_a)^{1/2}, \tag{85}$$

where $a = 1, 2$. Now the Dirac brackets among the degrees of freedom $(A^0, A^1, A^2)$ and the momenta $\pi_j$ become

$$\{A_0(\mathbf{x}, t), \pi_3(\mathbf{y}, t)\}_D = \frac{1}{A_0(\mathbf{x}, t)}(M^2 + A_0^2(\mathbf{x}, t) + A^a(\mathbf{x}, t)A_a(\mathbf{x}, t))^{1/2}\delta^3(\mathbf{x} - \mathbf{y}), \tag{86}$$

$$\{A_0(\mathbf{x}, t), \pi_a(\mathbf{y}, t)\}_D = -\frac{A_a(\mathbf{x}, t)}{A_0(\mathbf{x}, t)}\delta^3(\mathbf{x} - \mathbf{y}), \tag{87}$$

$$\{A^a(\mathbf{x}, t), \pi_3(\mathbf{y}, t)\}_D = 0, \tag{88}$$

$$\{A^a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)\}_D = \eta_b^a\delta^3(\mathbf{x} - \mathbf{y}). \tag{89}$$

The momenta $\pi_j$ are not the canonical conjugate momenta of the degrees of freedom $(A^0, A^1, A^2)$, but from the above Dirac brackets we obtain

$$\{A^3(\mathbf{x}, t), \pi_3(\mathbf{y}, t)\}_D = \{(M^2 + A^0(\mathbf{x}, t)A_0(\mathbf{x}, t) + A^a(\mathbf{x}, t)A_a(\mathbf{x}, t))^{1/2}, \pi_3(\mathbf{y}, t)\}_D = \delta^3(\mathbf{x} - \mathbf{y}), \tag{90}$$

and thus the $A^i$ and $\pi_j$ fields satisfy the same Dirac brackets as in the timelike case

$$\{A^i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)\}_D = \eta_j^i\delta^3(\mathbf{x} - \mathbf{y}). \tag{91}$$

From the first equation in (76) we have $\pi_i = -E_i$ in both cases. Imposing the Gauss law

$$(\partial_iE_i - J^0) = 0, \tag{92}$$

as an initial constraint on the phase space of the Nambu model, we obtain the extended Hamiltonian density for the electromagnetic field,

$$H = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - \mathbf{J}\cdot\mathbf{A} - \Theta(\nabla\cdot\mathbf{E} - J^0), \tag{93}$$

where $\Theta$ is an arbitrary function, with the usual brackets between the potential and the electric field,

$$\{A_i(\mathbf{x}, t), E_j(\mathbf{y}, t)\} = \delta_{ij}\delta^3(\mathbf{x} - \mathbf{y}). \tag{94}$$

The constrained phase space of the Hamiltonian for the free Nambu model includes subspaces where (82) becomes negative, because of the term $A^0\partial_i\pi^i$. In general, the existence of these subspaces leads to instabilities and spoils unitarity. When the Gauss law is imposed as an additional constraint to obtain electrodynamics in the nonlinear gauge, this term disappears from the constrained Hamiltonian, which becomes positive definite and yields a unitary theory.

## V. FINAL COMMENTS

The Nambu model can be motivated by the spontaneous Lorentz symmetry breaking in the spacelike sector of the bumblebee model given by Eq. (1). It is defined by the Lagrangian density of electrodynamics plus the nonlinear constraint $A_\mu A^\mu = n^2M^2$ and, in this sense, it is similar to the nonlinear sigma model. The standard way of dealing with the Nambu model is by solving the constraints in their different choices according to the character of $n_\mu$, which is now arbitrary, and then substituting the solution into the Lagrangian. In this way, it is clear that the Nambu model contains three massless degrees of freedom, corresponding to the Nambu-Goldstone modes of the SLSB, and manifestly breaks gauge invariance. As shown by Nambu, at the classical and tree levels [2], this model, plus the Gauss law as an additional constraint, is equivalent to standard electrodynamics in the nonlinear gauge $A_\mu A^\mu = n^2 M^2$. Further calculations including one-loop effects [7] corroborate this conjecture. The above-mentioned calculations are carried only up to order $1/M$ and $1/M^2$, respectively. On the other hand, Ref. [10] contains a discussion of the nonperturbative equivalence in the timelike case.

Comparing with quantum electrodynamics *à la* Dirac, as described in Appendix A, where the Gauss law is imposed as a first-class constraint upon the Hilbert space of the problem, we are able to show that the quantized Nambu model in the timelike and spacelike cases, plus the quantized Gauss law imposed on the corresponding Hilbert space, does indeed reproduce QED in a nonperturbative way with respect to the parameter $M$. This result is also extended to the lightlike case ($n^2 = 0$). It is important to emphasize that the dynamics of the Nambu model guarantees the conservation of the Gauss law in such a way that it is sufficient to impose it as an initial condition. This is the main result of this work.

The proof goes as follows. For each case (timelike, spacelike, or lightlike) we start from the corresponding canonical degrees of freedom, arising after solving the constraint $A_\mu A^\mu = n^2 M^2$, and obtain their Dirac brackets. After rewriting the standard electrodynamic variables $A_i$, $E_i$ in terms of the canonical degrees of freedom and with the use of the related Dirac brackets, we prove that they satisfy the symplectic algebra of electrodynamics, given by the brackets (A14). Moreover, each of the original Hamiltonian densities (22), (41), and (63) adopts the form (A13) of electrodynamics, after the Gauss law is imposed via an arbitrary function upon them. Since the quantum dynamics is determined by the canonical structure, we can now construct the corresponding quantum formalism. The fields $A_i$, $E_i$ will satisfy the commutation relations of QED, arising from the brackets (A14), and the respective Hamiltonian densities will become the QED Hamiltonian after the Gauss law is imposed. Since this constraint is first class, we adopt here Dirac's point of view that the quantum theory is defined in the Hilbert subspace of physical states that are annihilated by the constraint. The fact that QED is recovered in the gauge corresponding to each solution of the nonlinear constraint can be traced back to the particular initial form of the Nambu model used in each case.

It is appropriate to remark that the imposition of the Gauss law in the otherwise nongauge invariant Nambu model is equivalent to restoring such invariance upon the full system, because this first-class constraint is the generator of gauge transformations. For this reason a more accurate statement is that gauge invariance is indeed dynamically recovered from the Nambu model in order to reproduce QED, provided it is imposed as an initial condition. This statement coincides with the results in Refs. [2,10]. The results of Ref. [7] also support this point of view because the perturbative calculation performed there is carried on a Hilbert space where the photons are transverse to begin with, and therefore the Gauss law is imposed from the outset.

As shown in Sec. IV, the separate procedures described in Sec. III can be given a unified description using a Lagrange multiplier to impose the resolution of the nonlinear constraint in each case. In this way, it is straightforward to show that for every realization of a spontaneous Lorentz symmetry breaking consistent with $A_\mu A^\mu = n^2 M^2$, and after imposing the Gauss law, we obtain electrodynamics in the corresponding nonlinear gauge. Furthermore, using the Dirac method, we explicitly show that in the canonical formalism for the timelike and spacelike cases, the canonical brackets among the electric and the potential fields are the same as in electrodynamics.

Nambu has also proposed that the solution of the associated Hamilton-Jacobi equation for a particle in a given electromagnetic field should provide, via the principal Hamilton-Jacobi function, the function that allows us to make the change of gauge from an initial one (here taken as the Coulomb gauge) to the nonlinear gauge $A_\mu A^\mu = n^2 M^2$ [2]. In Appendix B we have explicitly constructed an example of this procedure for the case of a constant magnetic field. We have considered the $1/M$ (with $M \to \infty$) limit of the nonlinear constraint in the timelike case, which is consistent with the nonrelativistic limit of the Hamilton-Jacobi equation.

Finally, we discuss the full SLSB model, corresponding to the spacelike case, arising from the bumblebee model (1), where the massive field becomes dynamical instead of being automatically frozen as is done in the Nambu model. The equations of motion again preserve the time evolution of the Gauss law. Its imposition as an initial condition produces a consistent solution for the massive field which leads to electrodynamics in the related spacelike nonlinear gauge.

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## APPENDIX A: ELECTRODYNAMICS

Let us briefly review the standard Hamiltonian formulation of electrodynamics. We start from the second order action

$$S_2(A_i, A_0) = \int d^3y \mathcal{L}_2 = \int d^3y \left[ \frac{1}{2}((\dot{A}_i - \partial_i A_0)^2 - (\nabla \times \mathbf{A})^2) - J^0 A_0 - J^i A_i \right]. \tag{A1}$$

The canonical momenta are

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \dot{A}_i - \partial_i A_0 = E_i, \qquad \Pi_0 = 0, \tag{A2}$$

which produce the Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2}\mathbf{\Pi}^2 + \tfrac{1}{2}(\nabla \times \mathbf{A})^2 - (\partial_i E_i - J^0)A_0 + J^i A_i. \tag{A3}$$

We also have the PB

$$\{A_i(\mathbf{x}, t), A_j(\mathbf{y}, t)\} = 0, \qquad \{E_i(\mathbf{x}, t), E_i(\mathbf{y}, t)\} = 0, \tag{A4}$$

$$\begin{aligned} \{A_i(\mathbf{x}, t), \Pi_j(\mathbf{y}, t)\} &= \delta_{ij}\delta^3(\mathbf{x} - \mathbf{y}), \\ \{A_0(\mathbf{x}, t), \Pi_0(\mathbf{y}, t)\} &= \delta^3(\mathbf{x} - \mathbf{y}). \end{aligned} \tag{A5}$$

The canonical theory is constructed via Dirac's method, due to the fact that the constraint $\Sigma = \Pi_0 \simeq 0$ is present.

The extended Hamiltonian density is

$$\begin{aligned} \mathcal{H}_E = \tfrac{1}{2}\mathbf{E}^2 + \tfrac{1}{2}(\nabla \times \mathbf{A})^2 - A_0(\partial_i E_i - J^0) \\ + J^i A_i + u\Pi_0. \end{aligned} \tag{A6}$$

The condition

$$0 = \dot{\Sigma}(x) = \left\{\Sigma(x), \int d^3y \mathcal{H}_E(y)\right\} \tag{A7}$$

leads to the Gauss law constraint

$$\Omega = (\partial_i E_i - J^0). \tag{A8}$$

Finally, we obtain $\dot{\Omega} = 0$, by virtue of current conservation. In this way, the final first order action for electrodynamics is

$$S_1(A_i, E_i, A_0) = \int d^3y \mathcal{L}_1 = \int d^3y(E_i\dot{A}_i - \mathcal{H}), \tag{A9}$$

where

$$\mathcal{H} = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + J^i A_i - A_0(\partial_i E_i - J^0) + u\Pi_0, \tag{A10}$$

with the two first-class constraints

$$\Pi_0 \simeq 0, \qquad \partial_i E_i - J^0 \simeq 0. \tag{A11}$$

Normally, one fixes

$$\Pi_0 \simeq 0, \qquad A_0 \simeq \Theta, \tag{A12}$$

with $\Theta$ an arbitrary function to be consistently determined after the remaining first-class constraint is fixed, which yields

$$\mathcal{H} = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + J^i A_i - \Theta(\partial_i E_i - J^0). \tag{A13}$$

The remaining PB are

$$\begin{aligned} &\{A_i(\mathbf{x}, t), A_j(\mathbf{y}, t)\} = 0, \qquad \{E_i(\mathbf{x}, t), E_i(\mathbf{y}, t)\} = 0, \\ &\{A_i(\mathbf{x}, t), E_j(\mathbf{y}, t)\} = \delta_{ij}\delta^3(\mathbf{x} - \mathbf{y}). \end{aligned} \tag{A14}$$

The equations of motion are

$$\dot{A}_i = \{A_i, H\}, \rightarrow \mathbf{E} = -\dot{\mathbf{A}} - \nabla\Theta. \tag{A15}$$

Taking $\nabla\times$ from the above equation we recover Faraday's law,

$$\nabla \times \mathbf{E} = -\nabla \times \dot{\mathbf{A}} = -\dot{\mathbf{B}}. \tag{A16}$$

The remaining equation arises from

$$\dot{E}_i = \{E_i, H\}, \rightarrow \dot{\mathbf{E}} - \nabla \times \mathbf{B} = \mathbf{J}. \tag{A17}$$

## APPENDIX B: EXPLICIT GAUGE TRANSFORMATION IN THE NONRELATIVISTIC CASE

In general, it proves to be very difficult to go from a known gauge to the nonlinear gauge $A_\mu A^\mu = M^2$. Let us assume that we start from the Coulomb gauge with potentials $\tilde{A}_\mu$; then the required potentials in the nonlinear gauge are obtained by the transformation

$$A_\mu = \tilde{A}_\mu + \partial_\mu \Lambda, \tag{B1}$$

together with the equation

$$(\tilde{A}_\mu + \partial_\mu \Lambda)^2 = M^2, \tag{B2}$$

which determines the required gauge function $\Lambda$.

As suggested by Nambu, $\Lambda$ can be identified with the principal Hamilton-Jacobi function $S$ corresponding to a particle with charge $q$ and mass $m$ moving in the corresponding electromagnetic fields. In fact, the related Hamilton-Jacobi equation is

$$\left(\partial_\mu S - \frac{q}{c}A_\mu\right)^2 = m^2c^2, \tag{B3}$$

in such a way that the relation between the two problems is

$$\Lambda = -\frac{c}{q}S, \qquad M = \frac{c^2 m}{q}. \tag{B4}$$

Here we give an example of such a construction for the timelike case, in the nonrelativistic approximation. The nonrelativistic Hamilton-Jacobi equation is

$$(\partial_t \bar{S} - q\Phi) = -\frac{1}{2m}\left(\partial_i \bar{S} + \frac{q}{c}A^i\right)^2, \tag{B5}$$

where

$$\bar{S} = S \mp mc^2 t. \tag{B6}$$

The minus sign in the square root arises from the corresponding form of the Hamilton-Jacobi equation for the free particle,

$$\partial_t \bar{S} + \frac{1}{2m}(\partial_i \bar{S})^2 = 0. \tag{B7}$$

On the other hand, the $M \to \infty$ limit of the nonlinear gauge condition $A_\mu A^\mu = M^2$ is

$$A_0 \mp M = \pm \frac{A_i^2}{2M}. \tag{B8}$$

The constant $M$ can be reabsorbed in $A_0$, so that $A_0 \to A_0 \mp M$ and the gauge condition is

$$A_0 = \pm \frac{A_i^2}{2M}. \tag{B9}$$

In this way, the nonrelativistic gauge function $\lambda$ is determined by

$$\left(\tilde{\Phi} + \frac{1}{c}\partial_t \lambda\right) = -\frac{(\tilde{A}^i - \partial_i \lambda)^2}{2M}, \qquad \tilde{A}^\mu = (\tilde{\Phi}, \tilde{\mathbf{A}}), \tag{B10}$$

which is rewritten as

$$\left(q\tilde{\Phi} + \frac{q}{c}\partial_t \lambda\right) = -\frac{(\frac{q}{c}\tilde{A}^i - \frac{q}{c}\partial_i \lambda)^2}{2(\frac{qM}{c^2})}. \tag{B11}$$

Comparing (B11) with (B5) we obtain

$$-\frac{q}{c}\lambda = \bar{S}, \qquad m = \frac{qM}{c^2}, \tag{B12}$$

which reproduces the relativistic relation between $\Lambda$ and $S$.

Let us consider the simple case of a constant magnetic field $\mathbf{B} = B\hat{\mathbf{k}}$. In the Coulomb gauge we have

$$\tilde{\Phi} = 0, \qquad \tilde{\mathbf{A}} = \tfrac{1}{2}\mathbf{B} \times \mathbf{r}. \tag{B13}$$

Following Nambu's suggestion we use the Hamilton-Jacobi approach to determine the gauge transformation function $\lambda$. To this end, we calculate the Hamilton-Jacobi principal function $\bar{S}$. As is well known, this function corresponds to the action of the system evaluated at fixed end points. We calculate

$$\bar{S}(\mathbf{x}_0, t_0; \mathbf{x}_1, t_1) = \int_{t_0}^{t_1} dt L(\mathbf{x}(t), \dot{\mathbf{x}}(t)), \tag{B14}$$

just by substituting the solution of the equations of motion into the Lagrangian and integrating. We take the end point conditions as

$$\begin{aligned} &x(t_0) = 0, \quad y(t_0) = 0, \quad z(t_0) = 0, \\ &x(t_1) = x_1, \quad y(t_1) = y_1, \quad z(t_1) = z_1. \end{aligned} \tag{B15}$$

The Lagrangian is

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{m}{2}\dot{\mathbf{x}}^2 + \frac{q}{c}\tilde{\mathbf{A}} \cdot \dot{\mathbf{x}} = \frac{m}{2}\dot{\mathbf{x}}^2 + \frac{qB}{2c}(x\dot{y} - y\dot{x}), \tag{B16}$$

together with the equations of motion

$$\ddot{x} - \frac{qB}{mc}\dot{y} = 0, \qquad \ddot{y} + \frac{qB}{mc}\dot{x} = 0, \qquad \ddot{z} = 0. \tag{B17}$$

The solutions are

$$\begin{aligned} x(t) = &-\frac{1}{2}\left(x_1 + y_1 \cot\frac{\theta_1}{2}\right)(\cos\omega t - 1) \\ &+ \frac{1}{2}\left(x_1 \cot\frac{\theta_1}{2} - y_1\right)\sin\omega t, \end{aligned} \tag{B18}$$

$$\begin{aligned} y(t) = &\frac{1}{2}\left(x_1 + y_1 \cot\frac{\theta_1}{2}\right)\sin\omega t \\ &+ \frac{1}{2}\left(x_1 \cot\frac{\theta_1}{2} - y_1\right)(\cos\omega t - 1), \end{aligned} \tag{B19}$$

$$z(t) = \frac{z_1}{t_1} t, \tag{B20}$$

with $\omega = qB/mc$ and $\theta_1 = \omega t_1$. Substituting them into the Lagrangian we obtain

$$L(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = \frac{m z_1^2}{2t_1^2} + \frac{m\omega^2}{8}(x_1^2 + y_1^2)\csc^2\left(\frac{\theta_1}{2}\right)\cos\omega t, \tag{B21}$$

which produces the final result

$$\bar{S}(\mathbf{x}_0, t_0; \mathbf{x}_1, t_1) = \frac{m}{2}\frac{z_1^2}{t_1} + \frac{m\omega}{4}(x_1^2 + y_1^2)\cot\left(\frac{\omega t_1}{2}\right). \tag{B22}$$

A direct calculation verifies that $\bar{S}$ satisfies the Hamilton-Jacobi equation:

$$\frac{1}{2m}\left[\left(\frac{\partial \bar{S}}{\partial x} + \frac{qBy}{2c}\right)^2 + \left(\frac{\partial \bar{S}}{\partial y} - \frac{qBx}{2c}\right)^2 + \left(\frac{\partial \bar{S}}{\partial z}\right)^2\right] + \frac{\partial \bar{S}}{\partial t} = 0. \tag{B23}$$

From Eq. (B12) the gauge function is

$$\lambda(\mathbf{x}, t) = -\frac{1}{2}\frac{M}{c}\frac{z^2}{t} - \frac{B}{4}(x^2 + y^2)\cot\left(\frac{Bc}{2M}t\right). \tag{B24}$$

Hence, the potentials in the nonlinear gauge are

$$\begin{aligned} A_0 &= \frac{1}{c}\frac{\partial \lambda}{\partial t} = \frac{1}{2}\frac{M}{c^2}\frac{z^2}{t^2} + \frac{B^2}{8M}(x^2 + y^2)\csc\left(\frac{Bc}{2M}t\right), \\ A_x &= \tilde{A}_x + \frac{\partial \lambda}{\partial x} = -\frac{B}{2}\left(y + x\cot\left(\frac{Bc}{2M}t\right)\right), \\ A_y &= \tilde{A}_y + \frac{\partial \lambda}{\partial y} = +\frac{B}{2}\left(x - y\cot\left(\frac{Bc}{2M}t\right)\right), \\ A_z &= \frac{\partial \lambda}{\partial z} = -\frac{M}{c}\frac{z}{t}. \end{aligned} \tag{B25}$$

It is straightforward to verify that, indeed, we have

$$A_0 = \frac{1}{2M}\mathbf{A}^2. \tag{B26}$$

## APPENDIX C: THE FULL SLSB MODEL

In this appendix we explore the relation between electrodynamics and the full SLSB model arising from the bumblebee model given in Eq. (1). We emphasize again that this Lagrangian describes a physical system only after the spontaneous symmetry breaking is realized. For example, the mass parameter in Eq. (1) has the wrong sign for describing a massive vector field $B_\mu$. As we have shown in Sec. II, a correct realization is obtained only in the case $n^2 < 0$. We take $n^2 = -1$, and the corresponding Lagrangian reads

$$\begin{aligned}\mathcal{L}(A_\mu, \omega, \Theta) = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\left(1+\frac{\omega}{\sigma_0}\right)^2 + \frac{1}{\sigma_0}\left(1+\frac{\omega}{\sigma_0}\right) \\ &\times F_{\mu\nu}(A^\mu\partial^\nu\omega) + \frac{1}{2}(\partial_\mu\omega)(\partial^\mu\omega) \\ &+ \frac{1}{2\sigma_0^2}A^\nu A^\mu(\partial_\mu\omega)(\partial_\nu\omega) - \frac{1}{2}(2m^2)\omega^2 \\ &- \lambda\sigma_0\omega^3 - \frac{\lambda}{4}\omega^4 - J_\mu A^\mu\left(1+\frac{\omega}{\sigma_0}\right) \\ &+ \Theta(A_\mu A^\mu + \sigma_0^2), \end{aligned} \tag{C1}$$

where we have introduced the nonlinear condition via the Lagrange multiplier $\Theta$. Here $\sigma_0^2 = m^2/\lambda$, and $m^2$, $\lambda$ are positive.

Next we examine the equations of motion in order to search for dynamical conditions that could reproduce electrodynamics. In order to simplify the dynamics it is convenient to adopt the following parametrization of the Nambu-Goldstone modes $A_\mu$:

$$C_\mu = \left(1+\frac{\omega}{\sigma_0}\right)A_\mu = \frac{\sigma}{\sigma_0}A_\mu, \qquad C_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu, \tag{C2}$$

where we also recall the notation $\sigma = \sigma_0 + \omega$. This is motivated precisely by the way in which the symmetry breaking was introduced in Sec. II. As expected, we have the identity

$$\begin{aligned}-\frac{1}{4}C_{\mu\nu}C^{\mu\nu} = &-\frac{1}{4}\frac{\sigma^2}{\sigma_0^2}F_{\mu\nu}F^{\mu\nu} + \frac{\sigma}{\sigma_0^2}F_{\nu\mu}A^\nu\partial^\mu\omega \\ &+ \frac{1}{2}\partial_\mu\omega\partial^\mu\omega + \frac{1}{2\sigma_0^2}A^\nu A^\mu\partial_\mu\omega\partial_\nu\omega, \end{aligned} \tag{C3}$$

which reproduces the first four terms on the right-hand side of (C1). The result is

$$\begin{aligned}\mathcal{L}(C_\mu, \omega, \Omega) = &-\frac{1}{4}C_{\mu\nu}C^{\mu\nu} - m^2\omega^2 - \lambda\sigma_0\omega^3 - \frac{\lambda}{4}\omega^4 \\ &- J_\mu C^\mu + \frac{1}{2}\Omega(C_\mu C^\mu + (\sigma_0+\omega)^2), \end{aligned} \tag{C4}$$

with a redefinition of the Lagrange multiplier $\Theta$. The equations of motion are

$$\partial_\mu C^{\mu\nu} - J^\nu + \Omega C^\nu = 0, \tag{C5}$$

$$C_\mu C^\mu + (\sigma_0 + \omega)^2 = 0, \tag{C6}$$

$$-2m^2\omega - 3\lambda\sigma_0\omega^2 - \lambda\omega^3 + \Omega(\sigma_0+\omega) = 0, \tag{C7}$$

where the last one can be rewritten as

$$(\sigma_0 + \omega)[\Omega - \lambda\omega(\omega + 2\sigma_0)] = 0. \tag{C8}$$

This equation is satisfied for arbitrary $\Omega$ when $\omega = -\sigma_0$. In this case we have

$$C_\mu = \left(1+\frac{\omega}{\sigma_0}\right)A_\mu = 0, \tag{C9}$$

and the remaining equations consistently lead to $J^\nu = 0$, so the Goldstone field has no dynamics. The physically relevant configurations arise when $\omega \neq -\sigma_0$. Then Eq. (C8) reads

$$\Omega - \lambda\omega(\omega + 2\sigma_0) = 0. \tag{C10}$$

Following the discussion in Sec. C-1 in Ref. [10], we could explore the possibility of setting the initial condition $\Omega(x, t=0) = 0$. Notice that this would require us to set the four Maxwell equations, including the Gauss law, as initial conditions, according to Eq. (C5). We prefer to think the other way around, and we impose $(\partial_i C^{i0} - J^0) = 0$ as an initial condition. Since Eq. (C5) has the same form as Eq. (69), the dynamics guarantees that the Gauss law with respect to the field $C^{i0}$ is valid for all times, which in turn leads to $\Omega(x, t) = 0$, from the zero component of Eq. (C5). That is to say, we now have

$$(\partial_i C^{i0} - J^0)(\mathbf{x}, t) = 0 \rightarrow \Omega(\mathbf{x}, t) = 0. \tag{C11}$$

In this way, we also recover Maxwell equations for $C^{\mu\nu}(x, t)$:$\partial_\mu C^{\mu\nu} - J^\nu = 0$. Nevertheless, let us observe that we are interested in the Gauss law $(\partial_i F^{i0} - J^0) = 0$ together with the Maxwell equations for $F_{\mu\nu}$. To this end, we consider the equation of motion (C10), which yields the solutions

$$\omega = 0, \rightarrow C_\mu = A_\mu, \qquad \partial_\mu F^{\mu\nu} = J^\nu. \tag{C12}$$

$$\omega = -2\sigma_0, \rightarrow C_\mu = -A_\mu, \qquad \partial_\mu F^{\mu\nu} = -J^\nu. \tag{C13}$$

In both situations the gauge fixing condition

$$A_\mu A^\mu + \sigma_0^2 = 0 \tag{C14}$$

is satisfied. The case $\omega = 0$ describes the electromagnetic field, in the nonlinear gauge (C14), coupled to the current $J^\nu$, while $\omega = -2\sigma_0$ corresponds to an electromagnetic field in the same gauge, but coupled to the current $-J^\nu$.

Summarizing, we have shown that the imposition of the Gauss law as an initial condition in the full spontaneously broken spacelike bumblebee model leads to electrodynamics in the nonlinear gauge $A_\mu A^\mu + \sigma_0^2 = 0$. The situation is analogous to the Nambu model, except that here we have obtained the solutions $\omega(x, t) = 0$ and $\omega(x, t) = -2\sigma_0$ dynamically as a consequence of imposing the Gauss law as an initial condition, together with satisfying the corresponding equations of motion.

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