Journal of Sound and Vibration (2001) **244**(4), 738–745
doi:10.1006/jsvi.2000.3492, available online at http://www.idealibrary.com on IDEAL®

VIBRATIONS OF SIMPLY SUPPORTED RECTANGULAR PLATES WITH VARYING THICKNESS AND SAME ASPECT RATIO CUTOUTS

H. A. LARRONDO AND D. R. AVALOS

Department of Physics, School of Engineering, Universidad Nacional de Mar del Plata, 7600 – Mar del Plata, Argentina

AND

P. A. A. LAURA AND R. E. ROSSI

Department of Engineering, Universidad Nacional del Sur and Institute of Applied Mechanics (CONICET), 8000 – Bahía Blanca, Argentina

(*Received* 3 *January* 2000, *and in final form* 13 *October* 2000)

1. INTRODUCTION

Several analytical solutions have been published on vibrating rectangular plates with free edge cutouts [1–3]. On the other hand, no solutions are available in the case of plates of non-uniform thickness.

The present study deals with an approximate solution using a truncated double Fourier series which satisfies identically the outer boundary conditions and the Rayleigh–Ritz method. Obviously, the natural boundary conditions at the hole edges are not satisfied but this is admissible when using the R–R method. The numerical experiments performed show that good rate of convergence is achieved when going from a (400×400) to a (900×900) determinantal equation.

In several instances an independent solution has been obtained using a finite element algorithm [4] and good agreement with the analytical solution is shown to exist.

When the plate is simply connected the values of fundamental frequency coefficients are in excellent agreement with the results obtained by Appl and Byers [1, 4].

2. APPROXIMATE ANALYTICAL SOLUTION

For the rectangular plate under study, depicted in Figure 1, the Rayleigh–Ritz variational approach requires minimization of the functional

$$J[W'] = U[W'] - T[W'], \tag{1}$$

where W' is the true displacement amplitude of the plate, $U[W']$ is its maximum strain energy and $T[W']$ is the maximum kinetic energy for the displacement amplitude of the plate.

0022-460X/01/290738 + 08 $35.00/0

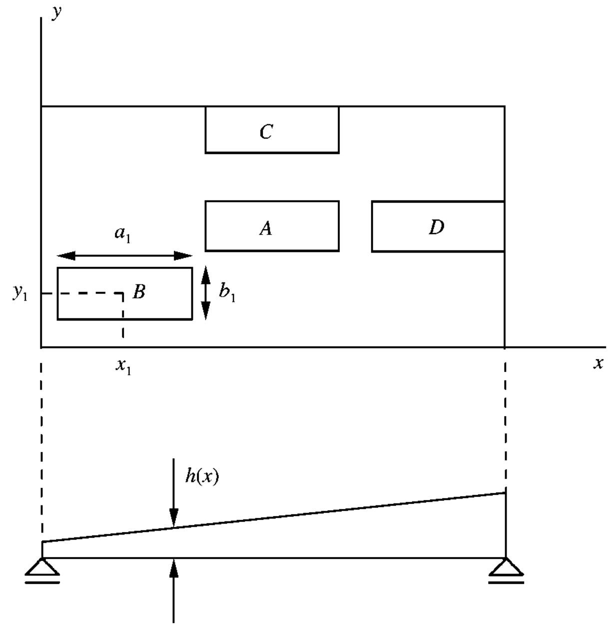

Figure 1. Mechanical system under analysis for different positions of the cutout: (a) $x'_1 = a/2$, $y'_1 = b/2$; (b) $x'_1 = a/5$, $y'_1 = b/5$; (c) $x'_1 = a/2$, $y'_1 = b - b_1/2$; and (d) $x'_1 = a - a_1/2$, $y'_1 = b/2$.

In the case of rectangular plates with varying thickness its functional can be written (see, e.g., reference [1]) as

$$U[W'] = \frac{1}{2}\iint D(x')\left\{\left(\frac{\partial^2 W'}{\partial x'^2} + \frac{\partial^2 W'}{\partial y'^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 W'}{\partial x'^2}\frac{\partial^2 W'}{\partial y'^2} - \left(\frac{\partial^2 W'}{\partial x'\partial y'}\right)^2\right]\right\}\mathrm{d}x'\,\mathrm{d}y' \tag{2}$$

and

$$T[W'] = \frac{1}{2}\rho\omega^2\iint h(x')W'^2\,\mathrm{d}x'\,\mathrm{d}y'. \tag{3}$$

In equations (2) and (3) the integrals are to be taken over the actual area A of the plate surface, i.e., not including the cutouts.

In equation (2) above, $D(x)$ is the flexural rigidity of the plate, which in the case under study takes the form

$$D(x') = \frac{Eh^3(x')}{12(1-\nu^2)} = \frac{Eh_0^3(1+\alpha x'/a)^3}{12(1-\nu^2)} = D_0(1+\alpha x'/a)^3. \tag{4}$$

Taking the lengths of the sides of the rectangular plate to be a and b in the x and y directions respectively, and introducing

$$W = W'/a, \qquad x = x'/a, \qquad y = y'/b \quad \text{and} \quad r = b/a. \tag{5}$$

equation (2) can be cast in a non-dimensional form. One gets, for the functional for the whole system of Figure 1:

$$J_{nd} = \frac{2J}{D_0 r} = \iint (1 + \alpha x)^3 \left\{ \left(W_{xx} + \frac{W_{yy}}{r^2} \right)^2 - \frac{2(1-\nu)}{r^2} (W_{xx} W_{yy} - W_{xy}^2) \right\} \mathrm{d}x\, \mathrm{d}y$$
$$- \Omega^2 \iint (1 + \alpha x) W^2 \,\mathrm{d}x\, \mathrm{d}y, \tag{6}$$

where as usual

$$\Omega^2 = \rho \omega^2 h_0 a^4 / D_0, \qquad \Omega = \sqrt{\rho h_0 / D_0}\, \omega\, a^2 \tag{7}$$

is the non-dimensional frequency coefficient.

Following previous works [2], the displacement amplitude $W(x, y)$ of the plate is expressed in an approximated way by means of the following double Fourier series expression:

$$W(x, y) = \sum_{m,n=1}^{M,N} b_{mn} \sin(m\pi x) \sin(n\pi y). \tag{8}$$

Needless to say, the presence of the factor $(1 + \alpha x/a)^3$ in equation (6) makes the resulting analytical expressions particularly lengthy when the cutouts are present. The calculus, however, is straightforward.

In order to minimize the functional in equation (6), one has to take its partial derivatives with respect to the coefficients b_{nm} of expression (8) and equate these derivatives to zero. That is to say,

$$\frac{\partial J_{nd}}{\partial b_{mn}} = 0, \quad m, n = 1, 2, \ldots . \tag{9}$$

System (9) yields an MN homogeneous linear system of equations in the b_{mn}'s. A secular determinant in the natural frequency coefficients of the system results from the non-triviality condition.

The present study is concerned with the determination of the first four frequency coefficients, Ω_1–Ω_4, in the case of isotropic rectangular plates of linear varying thickness for different locations and sizes of the rectangular holes.

The analytical procedure is not valid in the case of slits [3].

An independent solution has been obtained for several cases using the very efficient algorithm developed by Bogner *et al.* [5]. The domain has been subdivided into 800 square elements of uniform thickness taken equal to the average value of the plate element thickness. It has been found that, in general, this procedure yields results which approximate the exact eigenvalue from below.

3. NUMERICAL RESULTS

All calculations were performed for isotropic simply supported rectangular plates of linearly varying thickness taking the Poisson ratio equal to 0·3. Table 1 illustrates the case of rectangular plates with linearly varying thickness and no cutouts present. The results are

TABLE 1

Values of the first frequency coefficient Ω_1 in the case of rectangular plates with linearly varying thickness and no cutouts. Comparison with results from the open technical literature and with the finite element method

$r = b/a$	α	This work	[1, 5]†	FE results
4	0·0	—	—	10·487
	0·2	11·508	11·508	11·508
	0·5	12·958	12·957	12·956
	0·8	14·334	14·335	14·329
2	0·0	—	—	12·337
	0·2	13·548	13·549	13·549
	0·5	15·301	15·304	15·301
	0·8	16·995	16·993	16·993
1	0·0	—	—	19·739
	0·2	21·692	21·691	21·692
	0·5	24·556	24·556	24·556
	0·8	27·362	27·353	27·362
1/2	0·0	—	—	49·348
	0·2	54·160	54·162	54·161
	0·5	61·010	60·980	61·011
	0·8	67·537	67·499	67·539

†The mean values of upper and lower bounds have been tabulated.

TABLE 2

Values of Ω_1–Ω_4, in the case of rectangular plates with different positions and sizes of the equal aspect ratio cutouts when $\alpha = 0{\cdot}2$ and $b/a = \frac{1}{2}$: analytical and finite element results

Cutout position	Cutout size a_1/a	Ω_1	Ω_2	Ω_3	Ω_4	
A	0·1	53·070	86·695	140·21	183·03	(1) Analytical
		52·749	86·631	139·74	181·80	(2) Finite element
	0·2	50.897	86·003	141·85	168·06	(1)
		50.623	85·737	141·61	163·90	(2)
B	0·1	53·872	86·414	140·54	181·33	(1)
		53·785	86·198	140·27	180·65	(2)
	0·2	52·721	85·031	138·93	179·69	(1)
		51·584	83·574	136·96	176·97	(2)
C	0·1	54·068	86·057	140·70	180·23	(1)
		54·009	86·012	140·60	181·51	(2)
	0·2	52·556	84·067	138·27	173·19	(1)
		52·330	83·951	137·92	171·88	(2)
D	0·1	54·135	86·662	140·69	183·12	(1)
		54·126	86·592	140·45	183·08	(2)
	0·2	53·772	85·848	140·30	180·71	(1)
		53·741	85·657	139·96	180·06	(2)

TABLE 3

Values of Ω_1–Ω_4, in the case of rectangular plates with different positions and sizes of the equal aspect ratio cutouts when $\alpha = 0{\cdot}2$ and $b/a = \frac{2}{3}$, 1 and $\frac{3}{2}$: analytical results

b/a	Cutout position	Cutout size a_1/a	Ω_1	Ω_2	Ω_3	Ω_4
2/3	A	0·1	34·746	67·718	107·97	121·32
		0·2	33·854	66·869	102·58	122·90
	B	0·1	35·058	67·551	107·36	121·66
		0·2	34·439	66·932	106·64	120·89
	C	0·1	35·195	67·261	107·66	121·81
		0·2	34·583	65·780	103·73	120·79
	D	0·1	35·219	67·696	107·87	121·69
		0·2	35·014	67·034	106·43	121·10
1	A	0·1	21·474	54·119	54·151	86·064
		0·2	21·124	52·734	52·768	84·105
	B	0·1	21·583	53·843	54·088	86·424
		0·2	21·225	53·541	53·748	87·435
	C	0·1	21·679	53·913	54·048	86·121
		0·2	21·476	52·831	53·039	84·194
		0·1	21·684	53·898	54·112	86·078
	D	0·2	21·532	52·949	53·116	84·012
3/2	A	0·1	15·447	30·106	48·111	53·886
		0·2	15·054	29·729	45·684	54·595
	B	0·1	15·581	29·993	47·920	53·999
		0·2	15·298	29·669	47·654	53·734
	C	0·1	15·654	30·079	48·036	54·026
		0·2	15·541	29·754	47·173	53·833
	D	0·1	15·652	29·884	48·022	54·123
		0·2	15·443	29·195	46·354	53·720

compared with determinations available in the open literature and with fundamental eigenvalues obtained by means of the finite element method. The agreement is excellent for all the situations considered. Tables 2–4 depict values of Ω_1–Ω_4 for rectangular plates of linearly varying thickness and for different positions and sizes of the cutouts when the parameter α in expression (4) is equal to 0·2. In the case of Tables 2 and 4 the frequency coefficients have also been determined using the finite element method and good agreement is observed. Tables 5–7 deal with the situation where $\alpha = 0{\cdot}5$. The comparison with finite element results is presented in Tables 5 and 7. Good engineering agreement is observed for the first four natural frequency coefficients. For all the situations the parameter b/a has been taken equal to $\frac{1}{2}$, $\frac{2}{3}$, 1, $\frac{3}{2}$ and 2.

For the double Fourier series in equation (8), $M = N = 20$ have been used, that is to say a secular determinant of order 400 was generated for all the situations. For these values of M and N satisfactory convergence is achieved for all situations as has been checked by incrementing M and N to 30 (i.e., a determinant of order 900). As usual, special care has been taken to manipulate such large determinants and 80 bit floating point variables (IEEE—standard temporary reals) have been used in order to obtain reliable results.

In general, no dynamic stiffening effect has been found.

TABLE 4

Values of Ω_1–Ω_4, in the case of rectangular plates with different positions and sizes of the equal aspect ratio cutouts when $\alpha = 0{\cdot}2$ and $b/a = 2$: analytical and finite element results

Cutout position	Cutout size a_1/a	Ω_1	Ω_2	Ω_3	Ω_4	
A	0·1	13·277	21·685	35·055	45·993	(1) Analytical
		13·197	21·671	34·936	45·663	(2) Finite element
	0·2	12.736	21·515	35·474	42·108	(1)
		12.668	21·448	35·416	41·050	(2)
B	0·1	13·473	21·570	35·105	45·739	(1)
		13·453	21·511	35·025	45·632	(2)
	0·2	13·138	21·163	34·728	45·450	(1)
		13·102	21·080	34·577	45·258	(2)
C	0·1	13·540	21·664	35·167	45·984	(1)
		13·537	21·643	35·098	45·958	(2)
	0·2	13·432	21·455	35·113	44·788	(1)
		13·421	21·404	35·034	44·475	(2)
D	0·1	13·533	21·499	35·195	45·841	(1)
		13·521	21·487	35·176	45·666	(2)
	0·2	13·230	20·963	34·667	43·531	(1)
		13·179	20·934	34·589	43·165	(2)

TABLE 5

Values of Ω_1–Ω_4, in the case of rectangular plates with different positions and sizes of the equal aspect ratio cutouts when $\alpha = 0{\cdot}5$ and $b/a = \frac{1}{2}$: analytical and finite element results

Cutout position	Cutout size a_1/a	Ω_1	Ω_2	Ω_3	Ω_4	
A	0·1	59·845	97·939	158·53	202·65	(1) Analytical
		59·500	97·838	158·03	201·57	(2) Finite element
	0·2	57.500	97·083	160·09	188·14	(1)
		57.196	96·799	159·72	183·88	(2)
B	0·1	60·662	97·789	158·78	200·05	(1)
		60·547	97·547	158·48	198·98	(2)
	0·2	59·290	96·377	156·90	198·52	(1)
		57·928	94·785	154·69	195·13	(2)
C	0·1	60·882	97·400	158·85	201·95	(1)
		60·816	97·349	158·74	201·31	(2)
	0·2	59·154	95·182	156·04	193·13	(1)
		58·904	95·046	155·64	191·86	(2)
D	0·1	60·987	98·055	158·97	202·98	(1)
		60·982	97·991	158·73	202·97	(2)
	0·2	60·663	97·203	158·35	202·03	(1)
		60·639	97·011	157·93	201·81	(2)

Table 6

Values of Ω_1–Ω_4, in the case of rectangular plates with different positions and sizes of the equal aspect ratio cutouts when $\alpha = 0{\cdot}5$ and $b/a = \frac{2}{3}$, 1 and $\frac{3}{2}$: analytical results

b/a	Cutout position	Cutout size a_1/a	Ω_1	Ω_2	Ω_3	Ω_4
2/3	A	0·1	39·301	76·473	120·61	137·11
		0·2	38·362	75·477	114·86	138·60
	B	0·1	39·611	76·384	119·78	137·40
		0·2	38·916	75·785	119·16	136·46
	C	0·1	39·760	76·062	120·30	137·50
		0·2	30·049	74·418	116·06	136·29
	D	0·1	39·803	76·557	120·66	137·46
		0·2	39·621	75·920	119·62	136·65
1	A	0·1	24·336	60·937	61·113	97·412
		0·2	23·994	59·351	59·534	95·235
	B	0·1	24·430	60·625	61·088	97·802
		0·2	24·029	60·357	60·781	99·187
	C	0·1	24·537	60·800	60·991	97·459
		0·2	24·305	59·445	59·988	95·354
		0·1	24·551	60·745	61·173	97·337
	D	0·2	24·435	59·765	60·189	94·854
3/2	A	0·1	17·494	34·040	54·267	60·739
		0·2	17·095	33·601	51·543	61·448
	B	0·1	17·613	33·876	54·131	60·784
		0·2	17·281	33·498	53·937	60·613
	C	0·1	17·704	34·014	54·263	60·850
		0·2	17·579	33·652	53·286	60·565
	D	0·1	17·708	33·787	54·285	60·978
		0·2	17·543	32·976	52·559	60·583

The approach presented here can be extended in a straightforward manner to the case of rectangular plates of general anisotropy and also to other combinations of boundary conditions, using the appropriate complete set of eigenfunctions.

ACKNOWLEDGMENTS

The present study has been sponsored by Secretaría General de Ciencia y Tecnología (Universidad Nacional de Mar del Plata and Universidad Nacional del Sur) and by CONICET Research and Development Program at the School of Engineering Physics Department (UNMDP) and at the Institute of Applied Mechanics (Bahía Blanca). The authors are indebted to Prof. W. Soedel and to an anonymous referee for their constructive criticism.

REFERENCES

1. A. W. Leissa 1969 Vibration of plates. *NASA SP* 160.
2. P. A. A. Laura, E. Romanelli and R. E. Rossi 1997 *Journal of Sound and Vibration* **202**, 275–283. Transverse vibrations of simply supported rectangular plates with rectangular cutouts.

TABLE 7

Values of Ω_1–Ω_4, in the case of rectangular plates with different positions and sizes of the equal aspect ratio cutouts when $\alpha = 0{\cdot}5$ and $b/a = 2$: analytical and finite element results

Cutout position	Cutout size a_1/a	Ω_1	Ω_2	Ω_3	Ω_4	
A	0·1	15·016	24·544	39·645	51·866	(1) Analytical
		14·926	24·528	39·513	51·497	(2) Finite element
	0·2	14·444	24·349	40·084	47·551	(1)
		14·365	24·272	40·000	46·397	(2)
B	0·1	15·202	24·370	39·623	51·664	(1)
		15·176	24·297	39·524	51·546	(2)
	0·2	14·849	23·876	39·252	51·479	(1)
		14·742	23·777	39·094	51·305	(2)
C	0·1	15·292	24·524	39·738	51.937	(1)
		15·287	24·500	39·661	51·904	(2)
	0·2	15·173	24·302	39·655	50·598	(1)
		15·160	24·242	39·556	50·252	(2)
D	0·1	15·290	24·316	39·780	51·826	(1)
		15·279	24·302	39·767	51·651	(2)
	0·2	15·035	23·659	39·273	49·336	(1)
		14·985	23·625	39·204	48·896	(2)

3. D. R. AVALOS, H. A. LARRONDO and P. A. A. LAURA 1999 *Journal of Sound and Vibration* **222**, 691–695. Analysis of vibrating rectangular anisotropic plates with free-edge holes.
4. F. C. APPL and N. R. BYERS 1965 *Journal of Applied Mechanics* **32**, 163–167. Fundamental frequency of simply supported rectangular plates with linearly varying thickness.
5. F. K. BOGNER, R. L. FOX and L. A. SCHMIT 1966 *Matrix Methods in Structural Mechanics AFFDL-TR* **66–80**, 397–443. The generation of inter-element-compatible stiffnes and mass matrices by the use of interpolation formulas.