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Magnetic vector potential and magnetic field intensity due to a finite current carrying cylinder considering a variable current density along its axial dimension

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Abstract. With the aim of introducing a computationally efficient solution for problems such as the fast computation of magnetic field magnitudes and forces in coils and windings, this paper presents analytical expressions for the magnetic vector potential and magnetic field intensity in radial and axial directions due to a finite cylinder with infinitesimal wall thickness carrying a linearly varying current density between the values at the lower and upper ends. All expressions have been derived in terms of complete elliptic integrals of first, second and third kind, whose evaluation is achieved by means of very fast algorithms. The formulas presented make possible the fast computation of magnetic field at any point in space at reduced computational cost. The formulation is not only specially suited for modeling the current distribution in foil windings of power transformers but also for representing the magnetization of transformer core legs. The present method is also useful for efficient modeling of cylinders with constant current density since it is a generalization of this especial case. Finally, an example is presented where the results achieved using the proposed method are compared with those obtained using the finite element method showing a very good agreement between them.

Keywords: Azimuthal current density, Biot-Savart law, elliptic integral, magnetic field

1. Introduction

In engineering applications involving axisymmetrical configurations, as in the case of the modeling of magnetic cores through a surface current density or in the case of foil conductors in transformers, it is necessary to determine the magnetic fields due to current densities that are distributed non-uniformly in the axial direction. For this type of applications where multiple field sources are modeled in space, the use of constant current densities can lead to significant numerical errors or to an excessive increase in the number of elements that are necessary for the accurate representation of the problem. Therefore, the authors of this work considered necessary to develop a new formulation for the magnetic field due to a linearly varying current density along the axial dimension of a current carrying cylinder. The most

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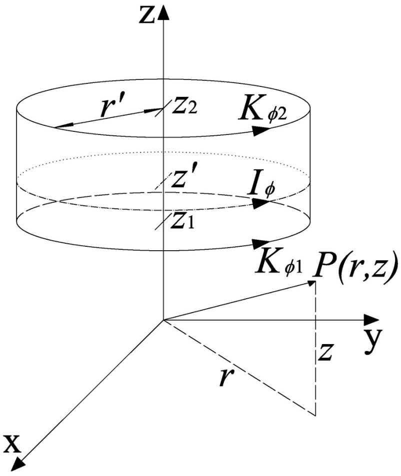

Fig. 1. Singular turn and finite cylinder with current densities in azimuthal direction.

popular method used for the magnetic field calculation in power transformers and reactors is the finite element method (FEM) [1,2]. However, this method normally requires excessive computational time to define the mesh, to assemble the equation system and to solve and post-processing the problem. This is the reason why analytical approaches are still very important in computational electromagnetics in many cases, since they can provide a faster, accurate and mesh-free methodology.

Mathematical expressions for the magnetic field originated by a cylindrical current source at a general point in space considering a constant current density has been presented in [3]. However, according to our knowledge, the expressions considering a linear variation of current density in axial direction have not been yet presented in the literature.

2. Basic expressions

From Biot-Savart law, the magnetic vector potential in an observation point $P(r, z)$ due to a filamentary loop carrying the current I_ϕ centered at the origin of the coordinate system and whose axis is the z axis is given by [4]:

$$A_\phi{}^{(st)}\left(P\right) = \frac{\mu_0 I_\phi}{2\pi} \int\limits_0^\pi \frac{\cos(\phi')r'}{\sqrt{r^2 + r'^2 + (z-z')^2 - 2rr'\cos(\phi')}} d\phi'. \quad (1)$$

where the primed coordinates are associated with a source point and the unprimed coordinates with the observation point; r' is the radius of the singular loop and z' is its z coordinate. As shown in Fig. 1, z_1 and z_2 are the z coordinates lower and upper limits of the cylinder having sheet current densities $K_{\phi 1}$ (A/m) and $K_{\phi 2}$ (A/m) respectively. The variation of the current density along the axial dimension of the cylinder is given by the linear equation

$$K_\phi\left(z'\right) = \frac{\left(K_{\phi_2} - K_{\phi_1}\right)\left(z' - z_1\right)}{z_2 - z_1} + K_{\phi_1}. \quad (2)$$

The magnetic vector potential due to the cylinder is found introducing the differential contribution of the current $I_\phi = K_\phi(z')dz'$ in Eq. (1) and integrating from z_1 to z_2 as follows

$$A_\phi{}^{(fc)}(P) = \frac{\mu_0}{2\pi}\int_{z_1}^{z_2}\int_{0}^{\pi}\frac{K_\phi(z')\cos(\phi')r'}{\sqrt{r^2 + r'^2 + (z - z')^2 - 2rr'\cos(\phi')}}d\phi' dz'. \tag{3}$$

The integrand of Eq. (3), which will be used in the following sections, is named as follows

$$f_A(\phi', z') = \frac{K_\phi(z')\cos(\phi')r'}{\sqrt{r^2 + r'^2 + (z - z')^2 - 2rr'\cos(\phi')}}. \tag{4}$$

Therefore, the magnetic vector potential can be written as follows

$$A_\phi{}^{(fc)}(P) = \frac{\mu_0}{2\pi}\int_{z_1}^{z_2}\int_{0}^{\pi} f_A(\phi', z')\, d\phi' dz'. \tag{5}$$

3. Solution for magnetic vector potential

This section aims to determine the values k_{A1} and k_{A2} so that the magnetic vector potential can be expressed as follows

$$A_\phi{}^{(fc)}(P) = k_{A1}K_{\phi 1} + k_{A2}K_{\phi 2}. \tag{6}$$

where k_{A1} and k_{A2} depend only on the coordinates of the source and field points. The advantage of the form in Eq. (6) is that the current densities can be considered either known or unknown. In the case that they are known, the value of magnetic vector potential can be determined directly; however, in some applications only the boundary condition is known on the cylinder surface or close to it. In this last case it is possible to determine the values of the unknown current densities with the aid of k_{A1} and k_{A2} and a suitable system of equations. Additionally, the current densities $K_{\phi 1}$ and $K_{\phi 2}$ can be considered as real or complex constant magnitudes, or time varying magnitudes, not affecting k_{A1} and k_{A2} which only depend on the geometry.

Substituting Eq. (2) in Eq. (3) and changing the order of integration, the value of k_{Ai} can be derived, which is a generalization of k_{A1} and k_{A2} so that

$$k_{Ai} = \frac{\mu_0}{2\pi}\int_{0}^{\pi}\int_{z_1}^{z_2}\frac{\left(\frac{z'-z_1}{z_2-z_1} - k_i\right)\cos(\phi')r'}{\sqrt{r^2 + r'^2 + (z - z')^2 - 2rr'\cos(\phi')}}dz' d\phi' \tag{7}$$

where if $k_i = 1$ then $k_{Ai} = -k_{A1}$ and if $k_i = 0$ then $k_{Ai} = k_{A2}$

The integral ζ_{Aiz} is defined as the inner indefinite integral in Eq. (7) as follows,

$$\zeta_{Aiz} = \int \frac{\left(\frac{z'-z_1}{z_2-z_1} - k_i\right)\cos(\phi')r'}{\sqrt{r^2 + r'^2 + (z - z')^2 - 2rr'\cos(\phi')}}dz' \tag{8}$$

The integration of ζ_{Aiz} yields

$$\zeta_{Aiz} = -\frac{r'\cos(\phi')\sqrt{r^2 + r'^2 + (z-z')^2 - 2rr'\cos(\phi')}}{z_1 - z_2} + \frac{r'\cos(\phi')\log\left(z' - z + \sqrt{r^2 + r'^2 + (z-z')^2 - 2rr'\cos(\phi')}\right)}{(z_1 - z + k_i z_2 - k_i z_1)^{-1}(z_1 - z_2)} \tag{9}$$

Rewriting ζ_{Aiz} in terms of two new variables ζ_{Aiz1} and ζ_{Aiz2} the following expression is obtained,

$$\zeta_{Aiz} = -\frac{r'\left(\zeta_{Aiz1} - \zeta_{Aiz2}\left(z_1 - z + k_i z_2 - k_i z_1\right)\right)}{z_1 - z_2} \tag{10}$$

where,

$$\zeta_{Aiz1} = \cos\left(\phi'\right)\sqrt{r^2 + r'^2 + (z-z')^2 - 2rr'\cos(\phi')} \tag{11}$$

$$\zeta_{Aiz2} = \cos\left(\phi'\right)\log\left(z' - z + \sqrt{r^2 + r'^2 + (z-z')^2 - 2rr'\cos(\phi')}\right) \tag{12}$$

For the sake of simplicity, the integration limits z_1 and z_2 will not be evaluated until the end of the derivation. Thus, the definite integral with respect to ϕ' of k_{Ai} is defined as

$$\zeta_{Ai\phi} = -\frac{\mu_0 r'\left(\zeta_{Ai\phi1} - \zeta_{Ai\phi2}\left(z_1 - z + k_i z_2 - k_i z_1\right)\right)}{2\pi\left(z_1 - z_2\right)} \tag{13}$$

where $\zeta_{Ai\phi1}$ and $\zeta_{Ai\phi2}$ are the definited integrals of ζ_{Aiz1} and ζ_{Aiz2} respectively with respect to ϕ',

$$\zeta_{Ai\phi1} = \int_0^\pi \zeta_{Aiz1} d\phi' \tag{14}$$

$$\zeta_{Ai\phi2} = \int_0^\pi \zeta_{Aiz2} d\phi' \tag{15}$$

The solution for the integrals $\zeta_{Ai\phi1}$ and $\zeta_{Ai\phi2}$ is found by applying the method of integration by parts and the appropriate formulas from [5], leading to the following solutions,

$$\zeta_{Ai\phi1} = \frac{1}{3rr'}\sqrt{(r+r')^2 + (z-z')^2}\Bigg(K(k)\left(\left(r-r'\right)^2 + \left(z-z'\right)^2\right) - E(k)\left(r^2 + r'^2 + \left(z-z'\right)^2\right)\Bigg) \tag{16}$$

$$\zeta_{Ai\phi2} = \frac{1}{2rr'}\left(\frac{\pi}{2}\left(\left|r^2 - r'^2\right| - \left(r^2 + r'^2\right)\right) + \frac{(z-z')}{\sqrt{(r+r')^2 + (z-z')^2}}\begin{pmatrix} 2rr'\left(E(k) - \Pi\left(\alpha^2, k\right)\right) + \\ \left(r^2 + r'^2\right)\left(E(k) - 2K(k) + \Pi\left(\alpha^2, k\right)\right) + \\ \left(E(k) - K(k)\right)\left(z^2 + z'\left(z' - 2z\right)\right) \end{pmatrix}\right) \tag{17}$$

where $K(k)$, $E(k)$ and $\Pi(\alpha^2, k)$ are complete elliptic integrals of first, second and third kind respectively [6], which have the following arguments,

$$k = \sqrt{\frac{4rr'}{(r+r')^2 + (z-z')^2}} \tag{18}$$

$$\alpha^2 = \frac{4rr'}{(r+r')^2} \tag{19}$$

The solution of the integral Eq. (14) can be found in Appendix A. The strategy to solve the remaining elliptic integrals presented in this paper is similar to that presented for integral Eq. (14). The solution of the integrals $\zeta_{Ai\phi 1}$ and $\zeta_{Ai\phi 2}$ makes possible the determination of the magnetic vector potential due to the cylinder by evaluation of the following expressions

$$k_{A1} = -\zeta_{Ai\phi}\big|_{\substack{k_i\,=\,1 \\ z'\,=\,z_2}} + \zeta_{Ai\phi}\big|_{\substack{k_i\,=\,1 \\ z'\,=\,z_1}} \tag{20}$$

$$k_{A2} = \zeta_{Ai\phi}\big|_{\substack{k_i\,=\,0 \\ z'\,=\,z_2}} - \zeta_{Ai\phi}\big|_{\substack{k_i\,=\,0 \\ z'\,=\,z_1}} \tag{21}$$

4. Expressions for the magnetic field intensity

The magnetic flux density is related to the magnetic vector potential by

$$\vec{B} = \nabla \times \vec{A} \tag{22}$$

Recalling that the problem has been formulated using cylindrical coordinates, it follows from Eq. (22) that,

$$B_r = -\frac{\partial A_\phi}{\partial z} \tag{23}$$

$$B_z = \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r} \tag{24}$$

Assuming that the cylinder is located in a homogeneous medium with permeability equal to that of the vacuum, the constitutive relation

$$\vec{B} = \mu_0 \vec{H} \tag{25}$$

holds. Therefore, substituting Eqs (23) and (24) in Eq. (25) and solving for H_r and H_z respectively, the expressions of the magnetic field intensity in radial and axial directions at the evaluation point P are obtained,

$$H_r{}^{(fc)}(P) = -\frac{1}{\mu_0}\frac{\partial A_\phi{}^{(fc)}(P)}{\partial z} \tag{26}$$

$$H_z{}^{(fc)}(P) = \frac{1}{\mu_0}\frac{1}{r}\frac{\partial\left(rA_\phi{}^{(fc)}(P)\right)}{\partial r} \tag{27}$$

5. Solution for the magnetic field intensity in radial direction

Introducing Eq. (5) in Eq. (26) the expression for the magnetic field intensity in the radial direction can be obtained

$$H_r{}^{(fc)}(P) = -\frac{1}{2\pi}\int_{z_1}^{z_2}\int_0^{\pi}\frac{\partial}{\partial z}f_A(\phi', z')\,d\phi' dz' \tag{28}$$

Recalling that $K_\phi(z')$ is not a function of z, the derivative in the innermost integral in Eq. (28) becomes,

$$\frac{\partial}{\partial z}f_A(\phi', z') = -\frac{K_\phi(z')\,r'\cos(\phi')(z-z')}{\left(r^2 + r'^2 + (z-z')^2 - 2rr'\cos(\phi')\right)^{3/2}} \tag{29}$$

As in the case of the magnetic vector potential, it is assumed that the expression for the magnetic flux density has the form,

$$H_r{}^{(fc)}(P) = k_{r1}K_{\phi 1} + k_{r2}K_{\phi 2} \tag{30}$$

Introducing Eq. (2) in Eq. (29), replacing it in Eq. (28) and performing some algebraic manipulations, the following expression for the geometric dependant part associated with the magnetic field intensity in radial direction is obtained,

$$k_{ri} = \frac{1}{2\pi}\int_0^{\pi}\int_{z_1}^{z_2}\frac{\left(\frac{z'-z_1}{z_2-z_1} - k_i\right)r'\cos(\phi')(z-z')}{\left(r^2 + r'^2 + (z-z')^2 - 2rr'\cos(\phi')\right)^{3/2}}dz' d\phi' \tag{31}$$

where if $k_i = 1$ then $k_{ri} = -k_{r1}$ and if $k_i = 0$ then $k_{ri} = k_{r2}$. The derivation of Eq. (31) can be found in Appendix B.

The inner indefinite integral in Eq. (31) is named as,

$$\zeta_{Hriz} = \int \frac{\left(\frac{z'-z_1}{z_2-z_1} - k_i\right)r'\cos(\phi')(z-z')}{\left(r^2 + r'^2 + (z-z')^2 - 2rr'\cos(\phi')\right)^{3/2}}dz' \tag{32}$$

Solving the integral ζ_{Hriz} the following expression can be obtained

$$\zeta_{Hriz} = \frac{r'\cos(\phi')(z_1 - z' + k_i z_2 - k_i z_1)}{(z_1 - z_2)\sqrt{r^2 + r'^2 + (z-z')^2 - 2rr'\cos(\phi')}} + \frac{r'\cos(\phi')\log\left(z' - z + \sqrt{r^2 + r'^2 + (z-z')^2 - 2rr'\cos(\phi')}\right)}{z_1 - z_2} \tag{33}$$

The second expression in Eq. (33) has already been defined in Eq. (12), therefore ζ_{Hriz} can be rewritten as follows,

$$\zeta_{Hriz} = \frac{r'\left(\zeta_{Hriz1}(z_1 - z' + k_i z_2 - k_i z_1) + \zeta_{Aiz2}\right)}{z_1 - z_2} \tag{34}$$

where

$$\zeta_{Hriz1} = \frac{\cos(\phi')}{\sqrt{r^2 + r'^2 + (z - z')^2 - 2rr'\cos(\phi')}} \tag{35}$$

The indefinite integral with respect to ϕ' of k_{ri} is defined as,

$$\zeta_{Hri\phi} = \frac{r'\left(\zeta_{Hri\phi1}\left(z_1 - z' + k_i z_2 - k_i z_1\right) + \zeta_{Ai\phi2}\right)}{2\pi\left(z_1 - z_2\right)} \tag{36}$$

where $\zeta_{Hri\phi1}$ is the definite integral of ζ_{Hriz1} with respect to ϕ'

$$\zeta_{Hri\phi1} = \int_0^\pi \zeta_{Hriz1} d\phi' \tag{37}$$

which has the following solution,

$$\zeta_{Hri\phi1} = \frac{K(k)\left(r^2 + r'^2 + (z - z')^2\right) - E(k)\left((r + r')^2 + (z - z')^2\right)}{rr'\sqrt{(r + r')^2 + (z - z')^2}} \tag{38}$$

With the solution of the integrals $\zeta_{Ai\phi2}$ and $\zeta_{Hri\phi1}$ it is possible to determine the magnetic field intensity due to the cylinder evaluating the following expressions,

$$k_{r1} = -\zeta_{Hri\phi}\Big|_{\substack{k_i = 1 \\ z' = z_2}} + \zeta_{Hri\phi}\Big|_{\substack{k_i = 1 \\ z' = z_1}} \tag{39}$$

$$k_{r2} = \zeta_{Hri\phi}\Big|_{\substack{k_i = 0 \\ z' = z_2}} - \zeta_{Hri\phi}\Big|_{\substack{k_i = 0 \\ z' = z_1}} \tag{40}$$

6. Solution of the magnetic field intensity in axial direction

As in the case of the magnetic field intensity in the radial direction, the equation of the field intensity in the axial direction can be written as follows,

$$H_z^{(fc)}(P) = k_{z1}K_{\phi1} + k_{z2}K_{\phi2} \tag{41}$$

Introducing Eq. (5) in Eq. (27) gives the expression for the magnetic field intensity in the axial direction,

$$H_z^{(fc)}(P) = \frac{1}{2\pi r}\int_{z_1}^{z_2}\int_0^\pi \frac{\partial}{\partial r}\left(r f_A\left(\phi', z'\right)\right) d\phi' dz' \tag{42}$$

Performing the derivative in the inner integral of Eq. (42) and after a few algebraic manipulations the expression of the geometric constant k_{zi} is obtained,

$$k_{zi} = \frac{1}{2\pi r}\int_0^\pi\int_{z_1}^{z_2} \frac{\left(\frac{z' - z_1}{z_2 - z_1} - k_i\right) r'\cos(\phi')}{\sqrt{r^2 + r'^2 + (z - z')^2 - 2rr'\cos(\phi')}} dz' d\phi'$$

$$+\frac{1}{2\pi r}\int_{0}^{\pi}\int_{z_1}^{z_2}-\frac{\left(\frac{z'-z_1}{z_2-z_1}-k_i\right)rr'\cos(\phi')\left(r-r'\cos\left(\phi'\right)\right)}{\left(r^2+r'^2+(z-z')^2-2rr'\cos(\phi')\right)^{3/2}}dz'd\phi' \tag{43}$$

In which, where $k_i = 1$ then $k_{zi} = -k_{z1}$ and if $k_i = 0$ then $k_{zi} = k_{z2}$
The indefinite inner integral of Eq. (44) is defined as,

$$\zeta_{Hziz} = \int \frac{\left(\frac{z'-z_1}{z_2-z_1}-k_i\right)r'\cos(\phi')}{\sqrt{r^2+r'^2+(z-z')^2-2rr'\cos(\phi')}}dz' + \int -\frac{\left(\frac{z'-z_1}{z_2-z_1}-k_i\right)rr'\cos(\phi')\left(r-r'\cos\left(\phi'\right)\right)}{\left(r^2+r'^2+(z-z')^2-2rr'\cos(\phi')\right)^{3/2}}dz' \tag{44}$$

Solving the integral Eq. (44) the following expression is found,

$$\zeta_{Hziz} = \frac{r'\begin{pmatrix}\zeta_{Hziz1}rr'-\zeta_{Aiz1}-\zeta_{Hriz1}r^2\\ -\left(z+\left(k_i-1\right)z_1-k_iz_2\right)\left(\zeta_{Aiz2}+4r\left(\zeta_{Hziz2}r-\zeta_{Hziz3}r'\right)\left(z-z'\right)\right)\end{pmatrix}}{z_1-z_2} \tag{45}$$

where,

$$\zeta_{Hziz1} = \frac{\cos\left(\phi'\right)^2}{\sqrt{r^2+r'^2+(z-z')^2-2rr'\cos\left(\phi'\right)}} \tag{46}$$

$$\zeta_{Hziz2} = \frac{\cos\left(\phi'\right)\left(r^2+r'^2-2rr'\cos\left(\phi'\right)\right)^{-1}}{4\sqrt{r^2+r'^2+(z-z')^2-2rr'\cos\left(\phi'\right)}} \tag{47}$$

$$\zeta_{Hziz3} = \frac{\cos\left(\phi'\right)^2\left(r^2+r'^2-2rr'\cos\left(\phi'\right)\right)^{-1}}{4\sqrt{r^2+r'^2+(z-z')^2-2rr'\cos\left(\phi'\right)}} \tag{48}$$

The remaining expressions have been defined previously. The definite integral of k_{zi} with respect to ϕ' is given by the following expression,

$$\zeta_{Hzi\phi} = \frac{r'\begin{pmatrix}\zeta_{Hzi\phi1}rr'-\zeta_{Ai\phi1}-\zeta_{Hri\phi1}r^2\\ -\left(z+\left(k_i-1\right)z_1-k_iz_2\right)\left(\zeta_{Ai\phi2}+4r\left(\zeta_{Hzi\phi2}r-\zeta_{Hzi\phi3}r'\right)\left(z-z'\right)\right)\end{pmatrix}}{2\pi r\left(z_1-z_2\right)} \tag{49}$$

where,

$$\zeta_{Hzi\phi1} = \int_{0}^{\pi}\zeta_{Hziz1}d\phi' \tag{50}$$

$$\zeta_{Hzi\phi 2} = \int_0^{\pi} \zeta_{Hziz2} d\phi' \tag{51}$$

$$\zeta_{Hzi\phi 3} = \int_0^{\pi} \zeta_{Hziz3} d\phi' \tag{52}$$

The solutions of Eqs (50), (51) and (52) are respectively,

$$\zeta_{Hzi\phi 1} = \frac{\begin{pmatrix} K(k)\left(r^4 + \left(r'^2 + (z-z')^2\right)^2 + 2r^2\left(2r'^2 + (z-z')^2\right)\right) \\ -E(k)\left(r^2 + r'^2 + (z-z')^2\right)\left((r+r')^2 + (z-z')^2\right) \end{pmatrix}}{3r^2 r'^2 \sqrt{(r+r')^2 + (z-z')^2}} \tag{53}$$

$$\zeta_{Hzi\phi 2} = -\frac{K(k)(r+r')^2 - \Pi\left(\alpha^2, k\right)\left(r^2 + r'^2\right)}{4rr'(r+r')^2 \sqrt{(r+r')^2 + (z-z')^2}} \tag{54}$$

$$\zeta_{Hzi\phi 3} = -\frac{\begin{pmatrix} K(k)(r+r')^2\left(2r^2 + 2r'^2 + (z-z')^2\right) \\ -E(k)(r+r')^2\left((r+r')^2 + (z-z')^2\right) \\ -\Pi\left(\alpha^2, k\right)\left(r^2 + r'^2\right)^2 \end{pmatrix}}{8r^2 r'^2 (r+r')^2 \sqrt{(r+r')^2 + (z-z')^2}} \tag{55}$$

The previous equations make possible to formulate the solution for the magnetic field intensity in the axial direction by means of Eq. (41) and the following expressions,

$$k_{z1} = -\zeta_{Hzi\phi}\Big|_{\substack{k_i = 1 \\ z' = z_2}} + \zeta_{Hzi\phi}\Big|_{\substack{k_i = 1 \\ z' = z_1}} \tag{56}$$

$$k_{z2} = \zeta_{Hzi\phi}\Big|_{\substack{k_i = 0 \\ z' = z_2}} - \zeta_{Hzi\phi}\Big|_{\substack{k_i = 0 \\ z' = z_1}} \tag{57}$$

7. Numerical implementation

It can be said that engineering applications often require the determination of interactions between multiple field sources and many observation points, where the determination of the three field magnitudes A_ϕ, H_r and H_z is usually of interest. The term *interaction* is used here to refer to the influence of the field produced by a source at an observation point. The proposed methodology is particularly efficient in this type of situation, since the elliptic integrals $K(k)$, $E(k)$ and $\Pi(\alpha^2, k)$ are calculated once for each interaction, and then they are reused in the analytical expressions of $\zeta_{Ai\phi 1}$, $\zeta_{Ai\phi 2}$, $\zeta_{Hri\phi 1}$, $\zeta_{Hzi\phi 1}$, $\zeta_{Hzi\phi 2}$ and $\zeta_{Hzi\phi 3}$ whose evaluation is straightforward.

Elliptic integrals of the first and second kind have been implemented using the method of arithmetic geometric mean (AGM) proposed in [7] and the elliptic integral of the third kind by using 10 point Gauss

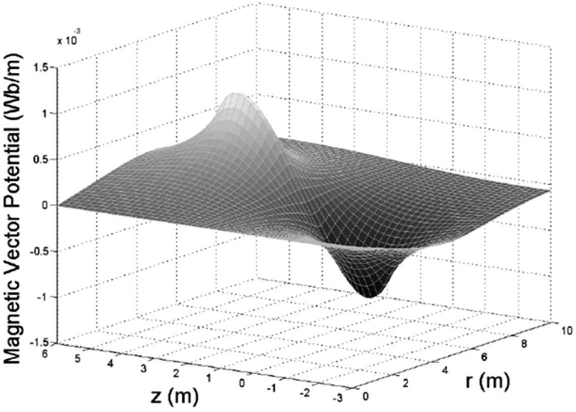

Fig. 2. 3D representation of magnetic vector potential for the example case.

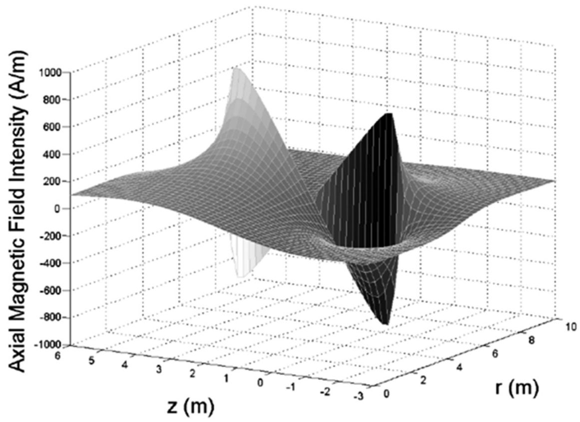

Fig. 3. 3D representation of magnetic field intensity in axial direction for the example case.

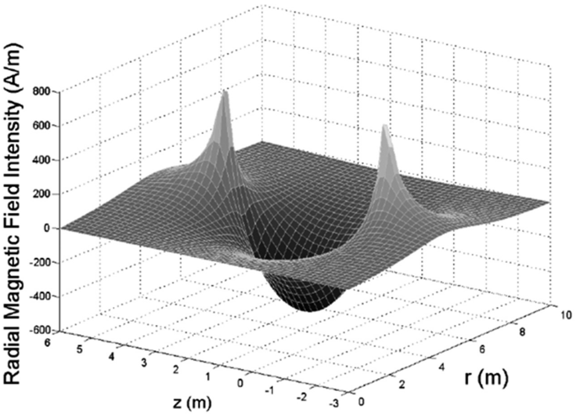

Fig. 4. 3D representation of magnetic field intensity in radial direction for the example case.

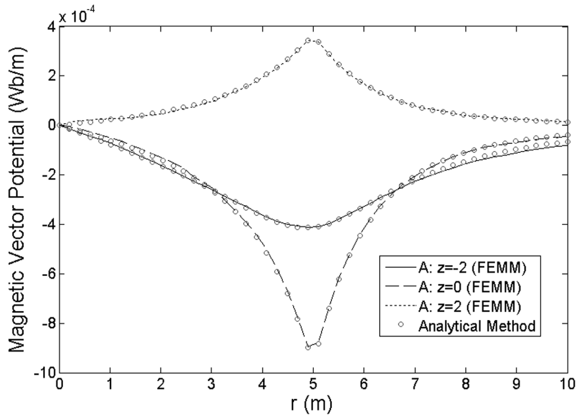

Fig. 5. 2D representation of magnetic vector potential for the example case.

Legendre quadrature as proposed in [8]. These methods present superior performance compared with traditional adaptive integration methods.

A test case was implemented with 700 random cylinders and 700 random evaluation points for a total of 490,000 interactions, for which were determined A_ϕ, H_r and H_z obtaining a total of 1,470,000 outputs from the routine. The average time to solve this case was 3.2 seconds; the same case took approximately 13 times using traditional techniques of integration by double quadrature on a personal computer with Intel Core i7® 2.8-GHz.

8. Example case and validation

Figures 2–4 show the results of a test case considering a current carrying cylinder whose dimensions are $r' = 5$ m, $z_1 = -1$ m, $z_2 = 4$ m and current densities $K_1 = -2000$ A/m and $K_2 = 2000$ A/m.

It can be seen from Fig. 2 that in the upper half cylinder the magnetic vector potential is positive,

Table 1
Calculation times of example case

Classification	FEMM	Analytical method
Geometry construction (s)	10.6685	–
Mesh construction (s)	3.3918	–
Building and solving equation system (s)	18.3712	–
Calculation time of fields[1] (s)	3.2426	0.053718
Total number of elements	303,616	1
Overall computation (s)	35.6741	0.053718

Calculations performed in a PC with Intel Core i7® 2.8-GHz.
[1]The vector potential and the magnetic field intensities in radial and axial direction were calculated on a grid of (51 × 51) 2601 evaluation points.

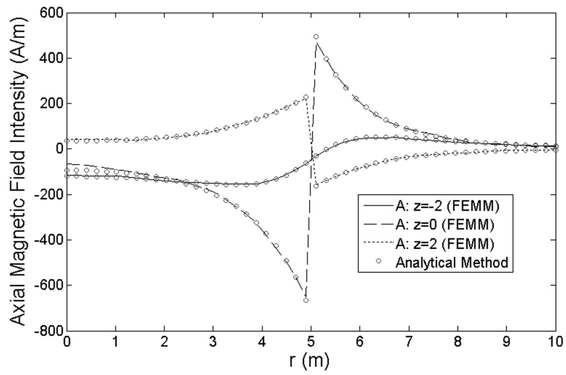

Fig. 6. 2D representation of magnetic field intensity in axial direction for the example case.

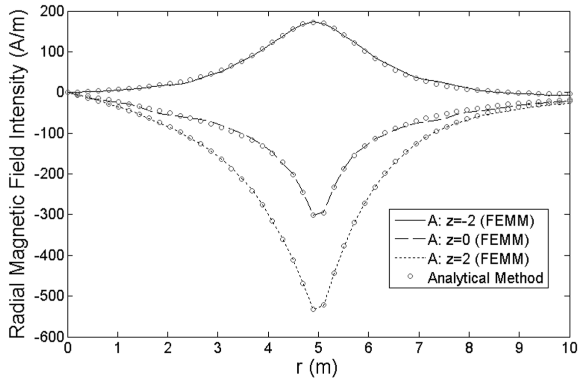

Fig. 7. 2D representation of magnetic field intensity in radial direction for the example case.

while it is negative in the lower half cylinder, which is caused by the change of the current sheet density direction in the middle of the cylinder.

Additionally, Fig. 3 shows the change of sign in the magnetic field intensity in the axial direction at both sides of the cylinder.

Finally, Fig. 4 shows a strong radial component at the central region of the cylinder caused by the contributions in the same direction of both cylinder halves. The field intensities in the radial direction at the ends of the cylinder are in the opposite direction of that of at the central region.

It is worth mentioning that a simulation such as the one performed in this example is difficult to implement using traditional methods like FEM, since the variation of current density in the axial direction could only be considered by means discrete steps. However, an approximation of the cylinder of the example has been modeled in the program FEMM [9] by means an arrangement of 500 sub-cylinders of 1mm of thickness with constant current density. The current of each sub-cylinder has been imposed so that the current density distribution is given by Eq. (2). The finite element simulation was performed by means the default mesh provided by the program. The arrangement of sub-cylinders was enclosed in a sphere with a radius of 11 m specified with a mixed boundary condition for the open boundary representation. In Table 1 a summary of the performance between finite element simulation and analytical simulation is presented. It is worth mentioning that the finite element method is at a serious disadvantage in this particular example compared with the analytical approach, since the cylinder has a very small thickness and it must be subdivided into many parts to represent the linear distribution of current density. This causes a substantial increase in the time needed for the construction of the geometry and the number of elements necessary to discretize the problem.

In Figs 5–7 the values of magnetic vector potential and magnetic field intensity in axial and radial direction are plotted for different cutting planes $z = -2, z = 0$ and $z = 2$ respectively. The formulation developed in this paper shows a good agreement with the simulation using FEMM, and has the main advantage of a considerable time saving.

9. Conclusion

The present paper presents the solution of the magnetic field due to a cylindrical source with linearly variable sheet current density. The corresponding equations have been analytically derived by direct application of the Biot Savart's law without introducing any restrictions or approximations. The advantage of the solution found is that it is expressed in terms of complete elliptic integrals, which can be evaluated in a very short computing time and with high accuracy.

The equations presented here can be used as the core of a robust and efficient tool in applications such as the modeling of magnetic materials and foil sheet windings in transformers. The use of a variable current density leads to a more accurate modeling of the magnetic field due to transformer coils and windings and it involves a relatively low level of discretization.

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Appendix A

Consider the integral of ζ_{Aiz1},

$$s = \int_0^\pi \cos\left(\phi'\right) \sqrt{r^2 + r'^2 + (z - z')^2 - 2rr' \cos(\phi')} d\phi' \tag{A.1}$$

The objective of this Appendix is to express Eq. (A.1) as a function of elliptic integrals. Applying the method of integration by parts, defining

$$u = \sqrt{r^2 + r'^2 + (z - z')^2 - 2rr' \cos(\phi')}$$

$$dv = \cos\left(\phi'\right) d\phi'$$

therefore,

$$du = \frac{rr' \sin(\phi')}{\sqrt{r^2 + r'^2 + (z - z')^2 - 2rr' \cos(\phi')}} d\phi'$$

$$v = \sin\left(\phi'\right)$$

Consequently the integral Eq. (A.1) can be rewritten as,

$$s = uv|_0^\pi - \int_0^\pi v du = -rr' \int_0^\pi \frac{\sin\left(\phi'\right)^2}{\sqrt{r^2 + r'^2 + \left(z - z'\right)^2 - 2rr'\cos\left(\phi'\right)}} d\phi'$$

It can be proved that,

$$s = -rr' \int_0^\pi \frac{\sin\left(\phi'\right)^2}{\sqrt{r^2 + r'^2 + \left(z - z'\right)^2 + 2rr'\cos\left(\phi'\right)}} d\phi'$$

$$= -rr' \int_0^\pi \frac{\sin\left(\phi'\right)^2}{\sqrt{r^2 + r'^2 + \left(z - z'\right)^2 - 2rr'\cos\left(\phi'\right)}} d\phi' \quad \text{(A.2)}$$

Defining by convenience,

$$a = r^2 + r'^2 + \left(z - z'\right)^2 \quad \text{(A.3)}$$

$$b = 2rr' \quad \text{(A.4)}$$

It is possible to rewrite Eq. (A.2) in simplified form as follows,

$$s = -rr' \int_0^\pi \frac{\sin\left(\phi'\right)^2}{\sqrt{a + b\cos\left(\phi'\right)}} d\phi' \quad \text{(A.5)}$$

Rewriting Eq. (A.5) in the following way,

$$s = -rr'\zeta \quad \text{(A.6)}$$

where

$$\zeta = \int_0^\pi \frac{\sin\left(\phi'\right)^2}{\sqrt{a + b\cos\left(\phi'\right)}} d\phi' \quad \text{(A.7)}$$

Therefore, it is valid to apply the formula 289.04 from [5], hence Eq. (A.7) can be expressed as follows

$$\zeta = 4g \int_0^{u_1} \mathrm{sn}(u\,|k)^2 \mathrm{cn}(u\,|k)^2 du \quad \text{(A.8)}$$

where,

$$g = \frac{2}{\sqrt{a + b}} \quad \text{(A.9)}$$

$$m = k^2 = \frac{2b}{a + b} \quad \text{(A.10)}$$

$$\mathrm{am}(u_1|\,k) = \frac{\pi}{2} \to u_1 = \mathrm{am}^{-1}\left(\frac{\pi}{2}\middle|\,k\right) = F\left(\frac{\pi}{2}\middle|\,k\right) = K\left(k\right) \quad \text{(A.11)}$$

where sn and cn are jacobian elliptic functions, and am is the Jacobi amplitude. Using the formula 361.01 from [5], the solution of the integral Eq. (A.8) is

$$\zeta = 4g\left(\frac{1}{3m^2}\left(\left(2-m\right)E\left(u\right)-2u\left(1-m\right)-m\,\mathrm{sn}\left(u|\,k\right)\mathrm{cn}\left(u|\,k\right)\mathrm{dn}\left(u|\,k\right)\right)\right)\Bigg|_{u=0}^{u=u_1=\mathrm{am}^{-1}\left(\frac{\pi}{2}|k\right)} \tag{A.12}$$

Performing the evaluation of the integration limits of equation Eq. (A.12)) to obtain,

$$\zeta = 4g\left(\frac{\left(2-m\right)E\left(k\right)-2\left(1-m\right)K\left(k\right)}{3m^2}\right) \tag{A.13}$$

Substituting Eqs (A.3) and (A.4) in Eqs (A.9) and (A.10), and introducing Eqs (A.9) and (A.10) in Eq. (A.13) the following expression is obtained,

$$\begin{aligned}\zeta &= \frac{1}{3r^2r'^2}\sqrt{\left(r+r'\right)^2+\left(z-z'\right)^2}\,\cdot\\ &\left(E\left(k\right)\left(r^2+r'^2+\left(z-z'\right)^2\right)-K\left(k\right)\left(\left(r-r'\right)^2+\left(z-z'\right)^2\right)\right)\end{aligned} \tag{A.14}$$

Finally, substituting Eq. (A.14) in Eq. (A.6) the solution of the integral Eq. (A.1) is found

$$s = \frac{1}{3rr'}\sqrt{\left(r+r'\right)^2+\left(z-z'\right)^2}\left(K\left(k\right)\left(\left(r-r'\right)^2+\left(z-z'\right)^2\right)-E\left(k\right)\left(r^2+r'^2+\left(z-z'\right)^2\right)\right)$$

Appendix B

The objective of this appendix is to present the derivation of Eq. (31). Introducing Eq. (2) in Eq. (29), replacing it in Eq. (28) the following equation is obtained

$$H_r^{(fc)}\left(P\right) = \frac{1}{2\pi}\int_{z_1}^{z_2}\int_{0}^{\pi}\frac{\left(\frac{\left(K_{\phi 2}-K_{\phi 1}\right)\left(z'-z_1\right)}{z_2-z_1}+K_{\phi_1}\right)r'\cos\left(\phi'\right)\left(z-z'\right)}{\left(r^2+r'^2+\left(z-z'\right)^2-2rr'\cos\left(\phi'\right)\right)^{3/2}}d\phi' dz' \tag{B.1}$$

Collecting $K_{\phi 1}$ and $K_{\phi 2}$ in Eq. (B.1) the following expression is obtained,

$$\begin{aligned}H_r^{(fc)}\left(P\right) = k_{r1}K_{\phi_1}+k_{r2}K_{\phi_2} = &\left(\frac{1}{2\pi}\int_{z_1}^{z_2}\int_{0}^{\pi}\frac{\left(-\frac{z'-z_1}{z_2-z_1}+1\right)r'\cos(\phi')\left(z-z'\right)}{\left(r^2+r'^2+\left(z-z'\right)^2-2rr'\cos\left(\phi'\right)\right)^{3/2}}d\phi' dz'\right)K_{\phi_1}\\ &+\left(\frac{1}{2\pi}\int_{z_1}^{z_2}\int_{0}^{\pi}\frac{\left(\frac{z'-z_1}{z_2-z_1}\right)r'\cos(\phi')\left(z-z'\right)}{\left(r^2+r'^2+\left(z-z'\right)^2-2rr'\cos\left(\phi'\right)\right)^{3/2}}d\phi' dz'\right)K_{\phi_2}\end{aligned}$$

therefore,

$$k_{r1} = \frac{1}{2\pi}\int_{z_1}^{z_2}\int_{0}^{\pi}\frac{\left(-\frac{z'-z_1}{z_2-z_1}+1\right)r'\cos(\phi')\left(z-z'\right)}{\left(r^2+r'^2+\left(z-z'\right)^2-2rr'\cos\left(\phi'\right)\right)^{3/2}}d\phi' dz'$$

$$k_{r2} = \frac{1}{2\pi}\int_{z_1}^{z_2}\int_{0}^{\pi}\frac{\left(\frac{z'-z_1}{z_2-z_1}\right) r'\cos(\phi')\,(z-z')}{\left(r^2+r'^2+(z-z')^2-2rr'\cos(\phi')\right)^{3/2}}d\phi' dz'$$

Introducing the new variable k_i, it is possible to generalize k_{r1} and k_{r2} as follows,

$$k_{ri} = \frac{1}{2\pi}\int_{0}^{\pi}\int_{z_1}^{z_2}\frac{\left(\frac{z'-z_1}{z_2-z_1}-k_i\right) r'\cos(\phi')\,(z-z')}{\left(r^2+r'^2+(z-z')^2-2rr'\cos(\phi')\right)^{3/2}}dz' d\phi'$$

it can be seen that if $k_i = 1$ then $k_{ri} = -k_{r1}$ and if $k_i = 0$ then $k_{ri} = k_{r2}$. The derivation of k_{Ai} and k_{zi} is similar to that presented in this Appendix.

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