# Indefinite least squares with a quadratic constraint 

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#### Abstract

An abstract indefinite least squares problem with a quadratic constraint is considered. This is a quadratic programming problem with one quadratic equality constraint, where neither the objective nor the constraint are convex functions. Necessary and sufficient conditions are found for the existence of solutions.


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## 1. Introduction

Quadratic optimization is a fundamental problem in optimization theory and its applications. Economic equilibrium, combinatorial optimization and numerical partial differential equations are all sources of quadratic optimization problems. Quadratic programming (QP) with a convex objective function was shown to be polynomial-time solvable. However, QP with an indefinite quadratic term is NP-hard in general. Usually, duality concepts and variational methods are applied to characterize and compute global minimizers. The literature on quadratically constrained quadratic programming (QCQP) problems is abundant, specially in the finite dimensional setting [34, 35, 36, 37, 46]. In this case, these problems can be written in the following form:

$$
\begin{array}{ll}
\operatorname{minimize} & f_{0}(x)=x^{T} P_{0} x+q_{0}^{T} x+r_{0} \\
\text { subject to } & f_{i}(x)=x^{T} P_{i} x+q_{i}^{T} x+r_{i} \leq 0, \quad i=1,2, \ldots, m
\end{array}
$$

[^0]where $x \in \mathbb{R}^{n}$ is the optimization variable, and $P_{i} \in \mathbb{R}^{n \times n}, q_{i} \in \mathbb{R}^{n}, r_{i} \in \mathbb{R}$ are given problem data, for $i=0,1, \ldots, m$.

This kind of QCQP problems can also be posed in the infinite dimensional setting, in particular in reproducing kernel Hilbert spaces (RKHS), see e.g. [15, 27, 41]. There, these problems stand as

$$
\begin{array}{ll}
\operatorname{minimize} & f(x)=\left\langle T_{0} x, x\right\rangle+\langle c, x\rangle+\alpha_{0} \\
\text { subject to } & g_{i}(x)=\left\langle T_{i} x, x\right\rangle+\left\langle y_{i}, x\right\rangle+\alpha_{i} \leq 0, \quad i=1,2, \ldots, m
\end{array}
$$

where the optimization variable $x$ varies in a complex Hilbert space $(\mathcal{H},\langle\cdot, \cdot\rangle)$, and the data is composed of bounded operators $T_{i}: \mathcal{H} \rightarrow \mathcal{H}$, vectors $y_{i} \in \mathcal{H}$ and scalars $\alpha_{i} \in \mathbb{C}$, for $i=0,1, \ldots, m$.

On the one hand, if the operators $T_{i}$ are positive semidefinite, then the objetive and the restriction are convex functions and the problem can be solved using a generalized Lagrangian and a dual maximization problem, with the Karush-Kuhn-Tucker conditions, see e.g. 9, 10, 38, 43].

On the other hand, if the operators $T_{i}$ are neither positive nor negative semidefinite, then the objetive and the restrictions are not convex. Since the definiteness of the inner product in $\mathcal{H}$ plays no role at all, the aim of this work is to pose a similar QCQP problem with only one quadratic equality constraint (QP1QEC), but using indefinite inner product spaces as codomains of the operators involved. More precisely, this paper is devoted to studying the following abstract indefinite least squares problem (ILSP) with a quadratic constraint:

Problem 1. Given a Hilbert space $(\mathcal{H},\langle\cdot, \cdot\rangle)$, and Krein spaces $\left(\mathcal{K},[\cdot, \cdot]_{\mathcal{K}}\right)$ and $\left(\mathcal{E},[\cdot, \cdot]_{\mathcal{E}}\right)$, let $T: \mathcal{H} \rightarrow \mathcal{K}$ and $V: \mathcal{H} \rightarrow \mathcal{E}$ be bounded operators. Also, assume that $T$ has closed range and $V$ is surjective. Given $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$, analyze the existence of

$$
\min \left[T x-w_{0}, T x-w_{0}\right]_{\mathcal{K}}, \text { subject to }\left[V x-z_{0}, V x-z_{0}\right]_{\mathcal{E}}=0
$$

and if the minimum exists, find the set of arguments at which it is attained.
One motivation for studying this problem is related with practical issues derived from machine learning theory. The classical literature is formulated in RKHS, and the positive definiteness of the kernel implies that the objective functions involved in the QCQP are convex, see [18, 30, 45]. However, the main obstacle arising in the applications is to achieve the Mercer condition for the kernel, i.e. to verify that the kernel is positive definite. Numerically, this is a painful condition to verify. In [11, 12, 32, 33] different authors propose to use reproducing kernel Krein spaces (RKKS) instead of RKHS (avoiding the necessity of verifying the Mercer condition), which turns into a more efficient solving tool from the numerical point of view. The indefinite kernel techniques have been also applied to pattern recognition problems, see [25, 42].

Since $[\cdot, \cdot]_{\mathcal{K}}$ and $[\cdot, \cdot]_{\mathcal{E}}$ are indefinite inner products, the objective function $x \mapsto\left[T x-w_{0}, T x-w_{0}\right]_{\mathcal{K}}$ is not convex while the equality constraint $\left[V x-z_{0}, V x-z_{0}\right]_{\mathcal{E}}=0$ is sign indefinite.

If $\left(\mathcal{E},[\cdot, \cdot]_{\mathcal{E}}\right)$ is a Hilbert space，the above constrained ILSP consists in ana－ lyzing the existence of

$$
\min \left[T x-w_{0}, T x-w_{0}\right]_{\mathcal{K}}, \text { subject to } V x=z_{0} .
$$

In this case，the quadratic form $x \mapsto\left[T x-w_{0}, T x-w_{0}\right]_{\mathcal{K}}$ is minimized over the affine manifold $x_{0}+N(V)$ where $x_{0} \in \mathcal{H}$ is a solution to $V x=z_{0}$ ，see 21，22］．

In the general setting，the objective function is minimized over a set given by a quadratic constraint．Denote $\mathcal{C}_{V}$ the set of neutral elements of the quadratic form $x \mapsto[V x, V x]_{\mathcal{E}}$ ，i．e．

$$
\mathcal{C}_{V}=\left\{u \in \mathcal{H}:[V u, V u]_{\mathcal{E}}=0\right\} .
$$

Then，given any $x_{0} \in \mathcal{H}$ such that $V x_{0}=z_{0}$ ，Problem $\square$ can be restated in the following way：

Problem 1＇．Given $x_{0} \in \mathcal{H}$ and $w_{0} \in \mathcal{K}$ ，analyze the existence of

$$
\min _{y \in \mathcal{C}_{V}}\left[T\left(x_{0}+y\right)-w_{0}, T\left(x_{0}+y\right)-w_{0}\right]_{\mathcal{K}},
$$

and if the minimum exists，find the set of arguments at which it is attained．
A significant difficulty that arises is that $\mathcal{C}_{V}$ is not a convex set．Moreover， the convex hull of $\mathcal{C}_{V}$ is the complete Hilbert space $\mathcal{H}$ ，thus replacing $\mathcal{C}_{V}$ by its convex hull trivializes the problem．

The paper is organized as follows．Section 2 introduces the notation used along the work，as well as a brief exposition on Krein spaces and linear op－ erators on Krein spaces．Its main purpose is to present in Proposition 2.1 a version of Farkas＇Lemma（or $S$－procedure），and some of its consequences that are used repeatedly．Given linear operators $T: \mathcal{H} \rightarrow \mathcal{K}$ and $V: \mathcal{H} \rightarrow \mathcal{E}$ act－ ing between Krein spaces，let $T^{\#}$ and $V^{\#}$ denote the adjoints of $T$ and $V$ ， respectively，with respect to the indefinite inner products．If the quadratic form $x \mapsto[V x, V x]_{\mathcal{E}}$ is indefinite，Proposition［2．1］says that $T$ maps $\mathcal{C}_{V}$ into a non－ negative set of $\mathcal{K}$ if and only if there exists $\rho \in \mathbb{R}$ such that $T^{\#} T+\rho V^{\#} V$ is positive semidefinite．Moreover，if such $\rho$ exists，there is a closed interval ［ $\left.\rho_{-}, \rho_{+}\right]$of admissible values for $\rho$ ．If $\mathcal{P}^{ \pm}(V)$ denote the subsets of $\mathcal{H}$ where the quadratic form $x \mapsto[V x, V x]_{\mathcal{E}}$ takes positive and negative values，respectively， the extremal values $\rho_{ \pm}$are determined by

$$
\rho_{-}:=-\inf _{x \in \mathcal{P}^{+}(V)} \frac{[T x, T x]}{[V x, V x]} \quad \text { and } \quad \rho_{+}:=-\sup _{x \in \mathcal{P}^{-}(V)} \frac{[T x, T x]}{[V x, V x]},
$$

see Corollary 2.2
Section 3 starts describing under which conditions the objective function is bounded from below over the set $x_{0}+\mathcal{C}_{V}$ ，see Proposition 3．1．This implies that in order to have solutions to Problem $⿴ 囗 十$ it is necessary that $T\left(\mathcal{C}_{V}\right)$ is a
nonnegative set of $\mathcal{K}$. The rest of the section is devoted to presenting necessary and sufficient conditions for the existence of solutions to Problem 1 for a fixed initial data $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$, see Proposition 3.3 and Theorem 3.5.

Along Section 4 we find a set of necessary and sufficient conditions for the existence of solutions to Problem 1 for every initial data $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$. We start by showing that $T$ mapping $\mathcal{C}_{V}$ into a uniformly positive subset of $\mathcal{K}$ is a necessary condition. Although it is not enough for our purposes, it leads us into an extra necessary condition: the attainment of

$$
\sup _{x \in \mathcal{P}^{-}(V)} \frac{[T x, T x]}{[V x, V x]} \quad \text { and } \quad \inf _{x \in \mathcal{P}^{+}(V)} \frac{[T x, T x]}{[V x, V x]} .
$$

Finally, we show that the above condition together with $T\left(\mathcal{C}_{V}\right)$ being a uniformly positive set of $\mathcal{K}$ are not only necessary but sufficient for the existence of solutions for every initial data $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$. This result is stated in Theorem 4.10.

In Section 5 we present a full description of $\mathcal{Z}\left(w_{0}, z_{0}\right)$. By Theorem 3.5. given $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$ the set of solutions to Problem 1 is

$$
\mathcal{Z}\left(w_{0}, z_{0}\right)=x_{0}+\Omega
$$

where $x_{0} \in \mathcal{H}$ is any vector such that $V x_{0}=z_{0}$ and
$\Omega:=\left\{y \in \mathcal{C}_{V}:\left(T^{\#} T+\lambda V^{\#} V\right)\left(x_{0}+y\right)=T^{\#} w_{0}+\lambda V^{\#} z_{0}\right.$ for some $\left.\lambda \in\left[\rho_{-}, \rho_{+}\right]\right\}$.
We show how the structures of $\Omega$ and $\mathcal{Z}\left(w_{0}, z_{0}\right)$ depend on the location of $\lambda$ in the interval $\left[\rho_{-}, \rho_{+}\right]$. The main result of this section asserts that the set of solutions to Problem 1 is an affine manifold parallel to $N(T) \cap N(V)$ for every initial data $\left(w_{0}, z_{0}\right)$ belonging to an open and dense subset of the vector space $\mathcal{K} \times \mathcal{E}$.

As an application of the previous results, Section 6 presents a generalization of the abstract mixed splines problem.

## 2. Preliminaries

Along this work $\mathcal{H}$ denotes a complex (separable) Hilbert space. If $\mathcal{K}$ is another Hilbert space then $\mathcal{L}(\mathcal{H}, \mathcal{K})$ is the vector space of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $\mathcal{L}(\mathcal{H})=\mathcal{L}(\mathcal{H}, \mathcal{H})$ stands for the algebra of bounded linear operators in $\mathcal{H}$.

If $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ then $R(T)$ stands for the range of $T$ and $N(T)$ for its nullspace. The Moore-Penrose inverse of an operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is denoted by $T^{\dagger}$. Recall that $T^{\dagger} \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ if and only if $T$ has closed range. For detailed expositions on the Moore-Penrose inverse, see [6, 31].

The reduced minimum modulus $\gamma(T)$ of an operator $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$
\gamma(T)=\inf \left\{\|T x\|:\|x\|=1, x \in N(T)^{\perp}\right\}
$$

An operator $T \neq 0$ has closed range if and only if $\gamma(T)>0$. In this case, $\gamma(T)=\left\|T^{\dagger}\right\|^{-1}$.

An operator $A \in \mathcal{L}(\mathcal{H})$ is positive semidefinite if $\langle A x, x\rangle \geq 0$ for all $x \in \mathcal{H}$; and it is positive definite if there exists $\alpha>0$ such that $\langle A x, x\rangle \geq \alpha\|x\|^{2}$ for every $x \in \mathcal{H}$. The cone of positive semidefinite operators is denoted by $\mathcal{L}(\mathcal{H})^{+}$. We say that a selfadjoint operator $A \in \mathcal{L}(\mathcal{H})$ is indefinite if it is neither positive nor negative semidefinite, i.e. if there exist $x_{+}, x_{-} \in \mathcal{H}$ such that $\left\langle A x_{+}, x_{+}\right\rangle>0$ and $\left\langle A x_{-}, x_{-}\right\rangle<0$.

### 2.1. Krein spaces

In what follows we present the standard notation and some basic results on Krein spaces. For a complete exposition on the subject (and the proofs of the results below) see [2, 5, 8, 17, 39].

An indefinite inner product space $(\mathcal{F},[\cdot, \cdot])$ is a (complex) vector space $\mathcal{F}$ endowed with a Hermitian sesquilinear form $[\cdot, \cdot]: \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{C}$.

A vector $x \in \mathcal{F}$ is positive, negative, or neutral if $[x, x]>0,[x, x]<0$, or $[x, x]=0$, respectively. Likewise, a subspace $\mathcal{M}$ of $\mathcal{F}$ is positive if every $x \in \mathcal{M}, x \neq 0$ is a positive vector in $\mathcal{F}$; and it is nonnegative if $[x, x] \geq 0$ for every $x \in \mathcal{M}$. Negative, nonpositive and neutral subspaces are defined mutatis mutandis.

If $\mathcal{S}$ is a subset of an indefinite inner product space $\mathcal{F}$, the orthogonal companion to $\mathcal{S}$ is defined by

$$
\mathcal{S}^{[\perp]}=\{x \in \mathcal{F}:[x, s]=0 \text { for every } s \in \mathcal{S}\}
$$

It is easy to see that $\mathcal{S}^{[\perp]}$ is always a subspace of $\mathcal{F}$.
Definition. An indefinite inner product space $(\mathcal{H},[\cdot, \cdot])$ is a Krein space if it can be decomposed as a direct (orthogonal) sum of a Hilbert space and an anti Hilbert space, i.e. there exist subspaces $\mathcal{H}_{ \pm}$of $\mathcal{H}$ such that $\left(\mathcal{H}_{+},[\cdot, \cdot]\right)$ and $\left(\mathcal{H}_{-},-[\cdot, \cdot]\right)$ are Hilbert spaces,

$$
\begin{equation*}
\mathcal{H}=\mathcal{H}_{+} \dot{+} \mathcal{H}_{-} \tag{2.1}
\end{equation*}
$$

and $\mathcal{H}_{+}$is orthogonal to $\mathcal{H}_{-}$with respect to the indefinite inner product. Sometimes we use the notation $[\cdot, \cdot]_{\mathcal{H}}$ instead of $[\cdot, \cdot]$ to emphasize the Krein space considered.

A pair of subspaces $\mathcal{H}_{ \pm}$as in (2.1) is called a fundamental decomposition of $\mathcal{H}$. Given a Krein space $\mathcal{H}$ and a fundamental decomposition $\mathcal{H}=\mathcal{H}_{+} \dot{+} \mathcal{H}_{-}$, the direct (orthogonal) sum of the Hilbert spaces $\left(\mathcal{H}_{+},[\cdot, \cdot]\right)$ and $\left(\mathcal{H}_{-},-[\cdot, \cdot]\right)$ is denoted by $(\mathcal{H},\langle\cdot, \cdot\rangle)$.

If $\mathcal{H}=\mathcal{H}_{+} \dot{+} \mathcal{H}_{-}$and $\mathcal{H}=\mathcal{H}_{+}^{\prime} \dot{+} \mathcal{H}_{-}^{\prime}$ are two different fundamental decompositions of $\mathcal{H}$, the corresponding associated inner products $\langle\cdot, \cdot\rangle$ and $\langle\cdot, \cdot\rangle^{\prime}$ turn out to be equivalent on $\mathcal{H}$. Therefore, the norm topology on $\mathcal{H}$ does not depend on the chosen fundamental decomposition.

A set $\mathcal{M}$ of a Krein space $(\mathcal{H},[\cdot, \cdot])$ is uniformly positive if there exists $\alpha>0$ such that

$$
[x, x] \geq \alpha\|x\|^{2} \quad \text { for every } x \in \mathcal{M}
$$

where $\|\cdot\|$ is the norm of any associated Hilbert space. Uniformly negative sets are defined mutatis mutandis.

If $\left(\mathcal{H},[\cdot, \cdot]_{\mathcal{H}}\right)$ and $\left(\mathcal{K},[\cdot, \cdot]_{\mathcal{K}}\right)$ are Krein spaces, $\mathcal{L}(\mathcal{H}, \mathcal{K})$ stands for the vector space of linear transformations which are bounded with respect to any of the associated Hilbert spaces $\left(\mathcal{H},\langle\cdot, \cdot\rangle_{\mathcal{H}}\right)$ and $\left(\mathcal{K},\langle\cdot, \cdot\rangle_{\mathcal{K}}\right)$. Given $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, the adjoint operator of $T$ (in the Krein spaces sense) is the unique operator $T^{\#} \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ such that

$$
[T x, y]_{\mathcal{K}}=\left[x, T^{\#} y\right]_{\mathcal{H}}, \quad x \in \mathcal{H}, y \in \mathcal{K}
$$

We frequently use that if $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $\mathcal{M}$ is a closed subspace of $\mathcal{K}$ then

$$
T^{\#}(\mathcal{M})^{[\perp]_{\mathcal{H}}}=T^{-1}\left(\mathcal{M}^{[\perp]_{\mathcal{K}}}\right)
$$

### 2.2. A version of Farkas' Lemma

Let $(\mathcal{H},\langle\cdot, \cdot\rangle)$ be a Hilbert space, and $\left(\mathcal{K},[\cdot, \cdot]_{\mathcal{K}}\right),\left(\mathcal{E},[\cdot, \cdot]_{\mathcal{E}}\right)$ be two Krein spaces. Let $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $V \in \mathcal{L}(\mathcal{H}, \mathcal{E})$. Recall that $\mathcal{C}_{V}$ denotes the set of neutral vectors of the quadratic form associated to $V^{\#} V$ :

$$
\mathcal{C}_{V}=\{y \in \mathcal{H}:[V y, V y]=0\}
$$

If $V^{\#} V$ is a positive (or negative) semidefinite operator in $\mathcal{H}$, then $\mathcal{C}_{V}$ coincides with $N(V)$. But, if $V^{\#} V$ is indefinite, the set $\mathcal{C}_{V}$ is strictly larger than $N(V)$. From now on $V^{\#} V$ is assumed to be indefinite; i.e. neither positive nor negative semidefinite.

The following result can be interpreted as another manifestation of the SLemma (or Farkas' lemma), see [36, 44]. It first appeared in [28]. For its proof, see Lemma 1.35 and Corollary 1.36 in [5, Chapter 1, $\S 1$ ].
Proposition 2.1. Given $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $V \in \mathcal{L}(\mathcal{H}, \mathcal{E})$, the following conditions are equivalent:
i) $T\left(\mathcal{C}_{V}\right)$ is a nonnegative set of $\mathcal{K}$;
ii) there exists $\rho \in \mathbb{R}$ such that $T^{\#} T+\rho V^{\#} V$ is positive semidefinite.

Let us also consider the subsets of $\mathcal{H}$ where the quadratic form associated to $V^{\#} V$ takes positive and negative values:
$\mathcal{P}^{+}(V):=\{x \in \mathcal{H}:[V x, V x]>0\} \quad$ and $\quad \mathcal{P}^{-}(V):=\{x \in \mathcal{H}:[V x, V x]<0\}$.
Corollary 2.2. If $T\left(\mathcal{C}_{V}\right)$ is a nonnegative set of $\mathcal{K}$, then

$$
\rho_{-}:=-\inf _{x \in \mathcal{P}^{+}(V)} \frac{[T x, T x]}{[V x, V x]}<+\infty \quad, \quad \rho_{+}:=-\sup _{x \in \mathcal{P}^{-}(V)} \frac{[T x, T x]}{[V x, V x]}>-\infty
$$

and $\rho_{-} \leq \rho_{+}$. In this case,

$$
T^{\#} T+\rho V^{\#} V \text { is positive semidefinite if and only if } \rho \in\left[\rho_{-}, \rho_{+}\right]
$$

If $\rho_{-} \neq \rho_{+}$, the positive operators $T^{\#} T+\rho V^{\#} V$ with $\rho \in\left(\rho_{-}, \rho_{+}\right)$share many properties. We collect here some of the results from 24], which are used along the paper.

Lemma 2.3. Assume that $T\left(\mathcal{C}_{V}\right)$ is a nonnegative set of $\mathcal{K}$ and that $\rho_{-} \neq \rho_{+}$. Then

$$
N\left(T^{\#} T+\rho V^{\#} V\right)=N\left(T^{\#} T\right) \cap N\left(V^{\#} V\right), \quad \text { for every } \quad \rho \in\left(\rho_{-}, \rho_{+}\right)
$$

Proposition 2.4. Assume that $T\left(\mathcal{C}_{V}\right)$ is a nonnegative set of $\mathcal{K}$ and that $\rho_{-} \neq$ $\rho_{+}$. Then
$R\left(\left(T^{\#} T+\rho V^{\#} V\right)^{1 / 2}\right)=R\left(\left(T^{\#} T+\rho^{\prime} V^{\#} V\right)^{1 / 2}\right), \quad$ for every $\rho, \rho^{\prime} \in\left(\rho_{-}, \rho_{+}\right)$.
Also, $R\left(\left(T^{\#} T+\rho_{ \pm} V^{\#} V\right)^{1 / 2}\right) \subseteq R\left(\left(T^{\#} T+\rho V^{\#} V\right)^{1 / 2}\right)$, for every $\rho \in\left(\rho_{-}, \rho_{+}\right)$.
Proposition 2.5. The following conditions are equivalent:
i) there exists $\alpha>0$ such that $[T y, T y] \geq \alpha\|y\|^{2}$ for every $y \in \mathcal{C}_{V}$;
ii) there exists $\rho \in \mathbb{R}$ such that $T^{\#} T+\rho V^{\#} V$ is a positive definite operator.

In this case, $\mathcal{C}_{T} \cap \mathcal{C}_{V}=\{0\}$.

## 3. Indefinite least squares problems with a quadratic constraint

From now on $(\mathcal{H},\langle\cdot, \cdot\rangle)$ denotes a Hilbert space, $\left(\mathcal{K},[\cdot, \cdot]_{\mathcal{K}}\right)$ and $\left(\mathcal{E},[\cdot, \cdot]_{\mathcal{E}}\right)$ denote Krein spaces; and $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ has closed range and $V \in \mathcal{L}(\mathcal{H}, \mathcal{E})$ is surjective. The quadratically constrained ILSP under consideration is the following:

Problem 1'. Given $x_{0} \in \mathcal{H}$ and $w_{0} \in \mathcal{K}$, analyze the existence of

$$
\min _{y \in \mathcal{C}_{V}}\left[T\left(x_{0}+y\right)-w_{0}, T\left(x_{0}+y\right)-w_{0}\right]_{\mathcal{K}}
$$

and if the minimum exists, find the set of arguments at which it is attained.
Problem 1 is equivalent to Problem 1'. In fact, Problem 1' with initial data $\left(w_{0}, x_{0}\right)$ is the same as Problem 1 with the initial data $\left(w_{0}, z_{0}\right)$ where $z_{0}:=V x_{0}$. Conversely, Problem 1 with initial data $\left(w_{0}, z_{0}\right)$ can be rephrased as Problem $1^{\prime}$ with initial data $\left(w_{0}, x_{0}\right)$ where $x_{0} \in \mathcal{H}$ is any vector such that $V x_{0}=z_{0}$.

Moreover, the set of solutions to both problems is the same and we refer to them indistinctly as $\mathcal{Z}\left(w_{0}, z_{0}\right)$.

We begin by studying under which conditions the infimum among the values of the objective function $x \mapsto\left[T x-w_{0}, T x-w_{0}\right]$ over the set $x_{0}+\mathcal{C}_{V}$ is finite.

Proposition 3.1. Given $x_{0} \in \mathcal{H}$ and $w_{0} \in \mathcal{K}$, the following conditions are equivalent:

$$
\begin{equation*}
\text { i) } \inf _{y \in \mathcal{C}_{V}}\left[T\left(x_{0}+y\right)-w_{0}, T\left(x_{0}+y\right)-w_{0}\right]>-\infty \tag{3.1}
\end{equation*}
$$

ii) there exists a constant $c \geq 0$ such that

$$
\begin{equation*}
\left|\left[T x_{0}-w_{0}, T y\right]\right|^{2} \leq c[T y, T y], \quad \text { for every } y \in \mathcal{C}_{V} \tag{3.2}
\end{equation*}
$$

Proof. Suppose that $\inf _{y \in \mathcal{C}_{V}}\left[T\left(x_{0}+y\right)-w_{0}, T\left(x_{0}+y\right)-w_{0}\right]=k>-\infty$.
Then, for every $y \in \mathcal{C}_{V}$,

$$
\begin{equation*}
[T y, T y]+2 \operatorname{Re}\left[T x_{0}-w_{0}, T y\right]+\left[T x_{0}-w_{0}, T x_{0}-w_{0}\right]-k \geq 0 \tag{3.3}
\end{equation*}
$$

Replacing $y$ by $t y$ for a fixed $y \in \mathcal{C}_{V}$ and $t \in \mathbb{R}$, (3.3) gives

$$
\begin{equation*}
a t^{2}+b t+c \geq 0 \quad \text { for every } t \in \mathbb{R} \tag{3.4}
\end{equation*}
$$

where $a=[T y, T y], b=2 \operatorname{Re}\left[T x_{0}-w_{0}, T y\right]$ and $c=\left[T x_{0}-w_{0}, T x_{0}-w_{0}\right]-$ $k \geq 0$. But (3.4) holds if and only if $a \geq 0$ and $b^{2}-4 a c \leq 0$, i.e.

$$
\left(\operatorname{Re}\left[T x_{0}-w_{0}, T y\right]\right)^{2} \leq c[T y, T y], \quad \text { for every } y \in \mathcal{C}_{V} .
$$

Now, if $\left[T x_{0}-w_{0}, T y\right]=e^{i \theta}\left|\left[T x_{0}-w_{0}, T y\right]\right|$, with $\theta \in[0,2 \pi)$, set $v:=e^{i \theta} y \in$ $\mathcal{C}_{V}$, then $[T v, T v]=[T y, T y]$ and $\operatorname{Re}\left[T x_{0}-w_{0}, T v\right]=\left|\left[T x_{0}-w_{0}, T y\right]\right|$. Therefore,

$$
\left|\left[T x_{0}-w_{0}, T y\right]\right|^{2} \leq c[T y, T y], \quad \text { for every } y \in \mathcal{C}_{V}
$$

Conversely, let $c \geq 0$ be such that (3.2) holds. Then $[T y, T y] \geq 0$ for every $y \in \mathcal{C}_{V}$ and

$$
\left(\operatorname{Re}\left[T x_{0}-w_{0}, T y\right]\right)^{2} \leq\left|\left[T x_{0}-w_{0}, T y\right]\right|^{2} \leq c[T y, T y]
$$

For an arbitrary (fixed) vector $y \in \mathcal{C}_{V}$ define $a$ and $b$ as above. Therefore, $a \geq 0$, $b^{2}-4 a c \leq 0$, and (3.4) follows. Or equivalently,

$$
\left[T\left(x_{0}+t y\right)-w_{0}, T\left(x_{0}+t y\right)-w_{0}\right] \geq\left[T x_{0}, T x_{0}\right]-c,
$$

where $y \in \mathcal{C}_{V}$ and $t \in \mathbb{R}$. Since $y \in \mathcal{C}_{V}$ is arbitrary, (3.1) holds.
In view of Proposition 3.1, we assume that the following hypotheses hold for the rest of this section.
Hypotheses 3.2. $T^{\# T}$ and $V^{\#} V$ are indefinite operators on $\mathcal{H}$ and

$$
T\left(\mathcal{C}_{V}\right) \text { is a nonnegative set of } \mathcal{K} \text {. }
$$

If $T^{\#} T$ is a semidefinite operator then Problem turns out to be a leastsquares problem with a quadratic constraint instead of an indefinite leastsquares problem, and the results below also hold in this case with some minor adjustments.

The existence of solutions to Problem 1 is equivalent to the existence of a vector $y_{0} \in \mathcal{C}_{V}$ such that $c:=\left[T y_{0}, T y_{0}\right]$ satisfies (3.2). This is expressed in the next proposition; the proof follows the lines of the proof of [23, Proposition 3.1].

Proposition 3.3. Given $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$, let $x_{0} \in \mathcal{H}$ be such that $V x_{0}=z_{0}$. Then, $\mathcal{Z}\left(w_{0}, z_{0}\right) \neq \varnothing$ if and only if there exists $y_{0} \in \mathcal{C}_{V}$ such that

$$
\begin{equation*}
\left|\left[T x_{0}-w_{0}, T y\right]\right|^{2} \leq\left[T y_{0}, T y_{0}\right][T y, T y], \quad \text { for every } y \in \mathcal{C}_{V} \tag{3.5}
\end{equation*}
$$

with equality when $y=y_{0}$.
In this case, $x_{0}+y_{0} \in \mathcal{Z}\left(w_{0}, z_{0}\right)$ if and only if $y_{0} \in \mathcal{C}_{V}$ satisfies (3.5) and

$$
\begin{equation*}
\left[T\left(x_{0}+y_{0}\right)-w_{0}, T y_{0}\right]=0 \tag{3.6}
\end{equation*}
$$

Another characterization of the existence of solutions to Problem 1 can be given by means of a normal equation. We study first the case of solutions $\widetilde{x}$ satisfying the stronger constraint $V \widetilde{x}=z_{0}$.

Lemma 3.4. Given $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$, let $\widetilde{x} \in \mathcal{H}$ such that $V \widetilde{x}=z_{0}$. Then,

$$
\widetilde{x} \in \mathcal{Z}\left(w_{0}, z_{0}\right) \text { if and only if } T^{\#} T \widetilde{x}=T^{\#} w_{0} .
$$

In this case, $\mathcal{Z}\left(w_{0}, z_{0}\right)=\widetilde{x}+\mathcal{C}_{T} \cap \mathcal{C}_{V}$.
Proof. Suppose that $\widetilde{x} \in \mathcal{Z}\left(w_{0}, z_{0}\right)$, and let $y \in \mathcal{C}_{V}$. By Proposition 3.3,

$$
\left|\left[T \widetilde{x}-w_{0}, T y\right]\right|^{2} \leq\left[T \widetilde{y}_{0}, T \widetilde{y}_{0}\right][T y, T y], \quad \text { for all } y \in \mathcal{C}_{V}
$$

where $\widetilde{y}_{0}$ is any vector in $\mathcal{C}_{V}$ such that

$$
\min _{y \in \mathcal{C}_{V}}\left[T(\widetilde{x}+y)-w_{0}, T(\widetilde{x}+y)-w_{0}\right]=\left[T\left(\widetilde{x}+\widetilde{y}_{0}\right)-w_{0}, T\left(\widetilde{x}+\widetilde{y}_{0}\right)-w_{0}\right] .
$$

But since $\widetilde{x} \in \mathcal{Z}\left(w_{0}, z_{0}\right)$, we can take $\widetilde{y}_{0}=0$ and thus $\left[T \widetilde{x}-w_{0}, T y\right]=0$ for all $y \in \mathcal{C}_{V}$. Hence, $T^{\#}\left(T \widetilde{x}-w_{0}\right) \in \mathcal{C}_{V}^{\perp}=\{0\}$.

Conversely, assume that $T^{\#} T \widetilde{x}=T^{\#} w_{0}$. Let $y \in \mathcal{C}_{V}$, then

$$
\begin{aligned}
& {[T(\widetilde{x}+y}\left.-w_{0}, T(\widetilde{x}+y)-w_{0}\right] \\
&=\left[T \widetilde{x}-w_{0}, T \widetilde{x}-w_{0}\right]+[T y, T y]+2 \operatorname{Re}\left[T \widetilde{x}-w_{0}, T y\right] \\
& \quad=\left[T \widetilde{x}-w_{0}, T \widetilde{x}-w_{0}\right]+[T y, T y] \\
& \quad \geq\left[T \widetilde{x}-w_{0}, T \widetilde{x}-w_{0}\right]
\end{aligned}
$$

because $T\left(\mathcal{C}_{V}\right)$ is a nonnegative set. Then $\widetilde{x} \in \mathcal{Z}\left(w_{0}, z_{0}\right)$. Moreover, the minimum is attained if and only if $y \in \mathcal{C}_{T} \cap \mathcal{C}_{V}$.

The following theorem establishes the normal equation that characterizes the solutions to Problem 1 in the general case. According to Hypothesis 3.2 the parameters $\rho_{ \pm}$introduced in Corollary 2.2 are well-defined.

Theorem 3.5. Given $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$, let $\widetilde{x} \in \mathcal{H}$. Then, $\widetilde{x} \in \mathcal{Z}\left(w_{0}, z_{0}\right)$ if and only if there exists $\lambda \in\left[\rho_{-}, \rho_{+}\right]$such that

$$
\begin{equation*}
\left(T^{\#} T+\lambda V^{\#} V\right) \widetilde{x}=T^{\#} w_{0}+\lambda V^{\#} z_{0} \tag{3.7}
\end{equation*}
$$

and

$$
\left[V \widetilde{x}-z_{0}, V \widetilde{x}-z_{0}\right]=0
$$

Proof. Consider the function $F: \mathcal{H} \rightarrow \mathbb{R}$ given by $F(x)=\left[T x-w_{0}, T x-w_{0}\right]$. This function is Fréchet differentiable at every $x \in \mathcal{H}$ and its Fréchet derivative at $x$ is given by:

$$
D F(x) \Delta x=2 \operatorname{Re}\left(\left[T x-w_{0}, T \Delta x\right]\right), \quad \Delta x \in \mathcal{H}
$$

Indeed, given $x \in \mathcal{H}$,

$$
\begin{aligned}
& \frac{|F(x+\Delta x)-F(x)-D F(x) \Delta x|}{\|\Delta x\|} \\
& \quad=\frac{\left|2 \operatorname{Re}\left(\left[T x-w_{0}, T \Delta x\right]\right)+[T \Delta x, T \Delta x]-2 \operatorname{Re}\left(\left[T x-w_{0}, T \Delta x\right]\right)\right|}{\|\Delta x\|} \\
& \quad=\frac{|[T \Delta x, T \Delta x]|}{\|\Delta x\|} \leq\|T\|^{2}\|\Delta x\| \rightarrow 0,
\end{aligned}
$$

as $\|\Delta x\| \rightarrow 0$. Analogously, the function $G: \mathcal{H} \rightarrow \mathbb{R}$ given by $G(x)=$ $\left[V x-z_{0}, V x-z_{0}\right]$ is Fréchet differentiable at every $x \in \mathcal{H}$ and its Fréchet derivative at $x$ is given by:

$$
D G(x) \Delta x=2 \operatorname{Re}\left(\left[V x-z_{0}, V \Delta x\right]\right), \quad \Delta x \in \mathcal{H}
$$

In the same fashion, the second order Fréchet derivatives at $x \in \mathcal{H}$ are given by

$$
\begin{array}{ll}
D^{2} F(x)\left(\Delta x_{1}, \Delta x_{2}\right)=2 \operatorname{Re}\left(\left[T \Delta x_{1}, T \Delta x_{2}\right]\right), & \Delta x_{1}, \Delta x_{2} \in \mathcal{H} \\
D^{2} G(x)\left(\Delta x_{1}, \Delta x_{2}\right)=2 \operatorname{Re}\left(\left[V \Delta x_{1}, V \Delta x_{2}\right]\right), & \Delta x_{1}, \Delta x_{2} \in \mathcal{H}
\end{array}
$$

Now, assume that $\widetilde{x} \in \mathcal{Z}\left(w_{0}, z_{0}\right)$. If $V \widetilde{x}=z_{0}$, then the result follows from Lemma 3.4 choosing an arbitrary $\lambda \in\left[\rho_{-}, \rho_{+}\right]$. On the other hand, if $V \widetilde{x} \neq z_{0}$ then $D G(\widetilde{x}) \neq 0$. Hence, by [29, §7.7 Thm. 2] there exists $\lambda \in \mathbb{R}$ such that $D F(\widetilde{x})+\lambda D G(\widetilde{x})=0$, i.e.

$$
\operatorname{Re}\left(\left[T \widetilde{x}-w_{0}, T \Delta x\right]+\lambda\left[V \widetilde{x}-z_{0}, V \Delta x\right]\right)=0, \quad \text { for every } \Delta x \in \mathcal{H}
$$

Replacing $\Delta x$ by $-i \Delta x$, the imaginary part is also zero. Thus,

$$
\left[T \widetilde{x}-w_{0}, T \Delta x\right]+\lambda\left[V \widetilde{x}-z_{0}, V \Delta x\right]=0, \quad \text { for every } \Delta x \in \mathcal{H}
$$

Therefore,

$$
\left(T^{\#} T+\lambda V^{\#} V\right) \widetilde{x}=T^{\#} w_{0}+\lambda V^{\#} z_{0}
$$

Moreover, by [1, Prop. 2.4.19], for every $\Delta x \in \mathcal{H}$,

$$
\begin{aligned}
0 \leq D^{2}(F+\lambda G)(\widetilde{x}) \cdot(\Delta x, \Delta x) & =2 \operatorname{Re}[T \Delta x, T \Delta x]+\lambda 2 \operatorname{Re}[V \Delta x, V \Delta x] \\
& =2\left\langle\left(T^{\#} T+\lambda V^{\#} V\right) \Delta x, \Delta x\right\rangle
\end{aligned}
$$

Hence $T^{\#} T+\lambda V^{\#} V \in \mathcal{L}(\mathcal{H})^{+}$, or equivalently, $\lambda \in\left[\rho_{-}, \rho_{+}\right]$(see Corollary 2.2).

Conversely, assume that $\left[V \widetilde{x}-z_{0}, V \widetilde{x}-z_{0}\right]=0$ and that there exists $\lambda \in$ $\left[\rho_{-}, \rho_{+}\right]$such that $\left(T^{\#} T+\lambda V^{\#} V\right) \widetilde{x}=T^{\#} w_{0}+\lambda V^{\#} z_{0}$. Given $x_{0} \in \mathcal{H}$ such that $V x_{0}=z_{0}$ there exists $y_{0} \in \mathcal{C}_{V}$ such that $\widetilde{x}=x_{0}+y_{0}$. Then,

$$
\begin{equation*}
\left(T^{\#} T+\lambda V^{\#} V\right) y_{0}=-T^{\#}\left(T x_{0}-w_{0}\right) \tag{3.8}
\end{equation*}
$$

and $\left[T x_{0}-w_{0}, T y_{0}\right]=-\left\langle\left(T^{\#} T+\lambda V^{\#} V\right) y_{0}, y_{0}\right\rangle=-\left[T y_{0}, T y_{0}\right]$. Hence, $\widetilde{x}=$ $x_{0}+y_{0}$ satisfies (3.6).

Since $T^{\#} T+\lambda V^{\#} V$ is positive semidefinite for $\lambda \in\left[\rho_{-}, \rho_{+}\right]$,

$$
\begin{aligned}
\left|\left[T x_{0}-w_{0}, T y\right]\right|^{2} & =\left|\left\langle-\left(T^{\#} T+\lambda V^{\#} V\right) y_{0}, y\right\rangle\right|^{2} \\
& \leq\left\langle\left(T^{\#} T+\lambda V^{\#} V\right) y_{0}, y_{0}\right\rangle\left\langle\left(T^{\#} T+\lambda V^{\#} V\right) y, y\right\rangle \\
& =\left[T y_{0}, T y_{0}\right][T y, T y]
\end{aligned}
$$

because $\left\langle\left(T^{\#} T+\lambda V^{\#} V\right) y, y\right\rangle=[T y, T y]$, for all $y \in \mathcal{C}_{V}$. Then, the result follows from Proposition 3.3

The next example shows how $\mathcal{Z}\left(w_{0}, z_{0}\right)$ depends on the initial data $\left(w_{0}, z_{0}\right)$, even in the situation in which the spectral decompositions determined by $T^{\#} T$ and $V^{\#} V$ are very simple.

Example 1. Assume that $\mathcal{H}$ is decomposed as $\mathcal{H}=\mathcal{H}_{1} \oplus \mathcal{H}_{2} \oplus \mathcal{H}_{3}$ and consider operators $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $V \in \mathcal{L}(\mathcal{H}, \mathcal{E})$ such that $T^{\#} T$ and $V^{\#} V$ can be represented by

$$
T^{\#} T=\left[\begin{array}{ccc}
I & 0 & 0 \\
0 & -\frac{1}{2} I & 0 \\
0 & 0 & I
\end{array}\right] \quad \text { and } \quad V^{\#} V=\left[\begin{array}{ccc}
4 I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & -I
\end{array}\right]
$$

The operator $T^{\#} T+\rho V^{\#} V$ is positive semidefinite if and only if $1+4 \rho \geq 0$, $\rho-\frac{1}{2} \geq 0$ and $1-\rho \geq 0$. Hence, it is readily seen that

$$
\rho_{-}=\frac{1}{2} \quad \text { and } \quad \rho_{+}=1
$$

In the following we show that $\mathcal{Z}\left(w_{0}, z_{0}\right) \neq \varnothing$ for every $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$, and we describe $\mathcal{Z}\left(w_{0}, z_{0}\right)$ in each case. Since $[V x, V x]=4\left\|x_{1}\right\|^{2}+\left\|x_{2}\right\|^{2}-\left\|x_{3}\right\|^{2}$, the set $\mathcal{C}_{V}$ can be described as

$$
\mathcal{C}_{V}=\left\{y_{1}+y_{2}+\left(4\left\|y_{1}\right\|^{2}+\left\|y_{2}\right\|^{2}\right)^{1 / 2} y_{3}: y_{1} \in \mathcal{H}_{1}, y_{2} \in \mathcal{H}_{2}, y_{3} \in \mathcal{S}_{3}\right\}
$$

where $\mathcal{S}_{i}$ stands for the unit sphere in $\mathcal{H}_{i}$ for $i=1,2,3$.
Given $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$, let $x_{0} \in \mathcal{H}$ be such that Vx$x_{0}=z_{0}$. If $y \in \mathcal{C}_{V}$, Theorem 3.5 assures that $\widetilde{x}=x_{0}+y \in \mathcal{Z}\left(w_{0}, z_{0}\right)$ if and only if there exists $\lambda \in\left[\frac{1}{2}, 1\right]$ such that (3.7) holds, or equivalently, if

$$
\begin{equation*}
\left(T^{\#} T+\lambda V^{\#} V\right) y=-T^{\#}\left(T x_{0}-w_{0}\right) \tag{3.9}
\end{equation*}
$$

Writing $-T^{\#}\left(T x_{0}-w_{0}\right)=x_{1}+x_{2}+x_{3}$, with $x_{i} \in \mathcal{H}_{i}$, and decomposing $y \in \mathcal{C}_{V}$ as

$$
y=y_{1}+y_{2}+\left(4\left\|y_{1}\right\|^{2}+\left\|y_{2}\right\|^{2}\right)^{1 / 2} y_{3}
$$

with $y_{1} \in \mathcal{H}_{1}, y_{2} \in \mathcal{H}_{2}$, and $y_{3} \in \mathcal{S}_{3}$, (3.9) reads as

$$
\begin{align*}
(1+4 \lambda) y_{1} & =x_{1}  \tag{3.10}\\
\left(\lambda-\frac{1}{2}\right) y_{2} & =x_{2}  \tag{3.11}\\
(1-\lambda)\left(4\left\|y_{1}\right\|^{2}+\left\|y_{2}\right\|^{2}\right)^{1 / 2} y_{3} & =x_{3} \tag{3.12}
\end{align*}
$$

Since $\lambda \in\left[\frac{1}{2}, 1\right]$, (3.10) says that $y_{1}=\frac{1}{1+4 \lambda} x_{1}$. If $x_{1}+x_{2}+x_{3}=0$, it is easy to see that $\mathcal{Z}\left(w_{0}, z_{0}\right)=\left\{x_{0}\right\}$. In the following, we study the situations where this is not the case.

- Case 1: $x_{3}=0$.

Since $y_{3} \neq 0$, (3.12) yields $\lambda=1$. Moreover,

$$
\mathcal{Z}\left(w_{0}, z_{0}\right)=x_{0}+\frac{1}{5} x_{1}+2 x_{2}+\left(\frac{4}{25}\left\|x_{1}\right\|^{2}+4\left\|x_{2}\right\|^{2}\right)^{1 / 2} \mathcal{S}_{3}
$$

- Case 2: $x_{3} \neq 0$ and $x_{2} \neq 0$.

In this case, (3.11) and (3.12) yield $\lambda \in\left(\frac{1}{2}, 1\right)$. Therefore,

$$
\mathcal{Z}\left(w_{0}, z_{0}\right)=\left\{x_{0}+\frac{1}{1+4 \lambda} x_{1}+\frac{1}{\lambda-\frac{1}{2}} x_{2}+\frac{1}{1-\lambda} x_{3}\right\}
$$

- Case 3: $x_{3} \neq 0$ and $x_{2}=0$.

In this case, two different situations have to be considered. Indeed, (3.11) implies that either $y_{2}=0$ or $\lambda=\frac{1}{2}$. Also, (3.12) says that

$$
\frac{4}{(1+4 \lambda)^{2}}\left\|x_{1}\right\|^{2}+\left\|y_{2}\right\|^{2}=\frac{1}{(1-\lambda)^{2}}\left\|x_{3}\right\|^{2}
$$

Denoting $\gamma=\frac{\left\|x_{1}\right\|}{\left\|x_{3}\right\|}$, we can distinguish between two different cases:
i) if $\gamma>3$ then $\lambda:=\frac{2 \gamma-1}{2 \gamma+4}$ is contained in the interval $\left(\frac{1}{2}, 1\right)$, hence $y_{2}=0$ and

$$
\mathcal{Z}\left(w_{0}, z_{0}\right)=\left\{x_{0}+\frac{1}{1+4 \lambda} x_{1}+\frac{1}{1-\lambda} x_{3}\right\}
$$

ii) if $\gamma \leq 3$ then $\lambda=\frac{1}{2}$ and

$$
\mathcal{Z}\left(w_{0}, z_{0}\right)=x_{0}+\frac{1}{3} x_{1}+2\left\|x_{3}\right\|\left(1-\frac{\gamma^{2}}{9}\right)^{1 / 2} \mathcal{S}_{2}+2 x_{3}
$$

Remark 3.6. The above example can be easily generalized, replacing the constants 4 and $\frac{1}{2}$ appearing in the block matrix representations of $V^{\#} V$ and $T^{\#} T$ by arbitrary reals $\alpha$ and $\beta$ such that $\alpha>1$ and $0<\beta<1$, respectively.

Given $\alpha$ and $\beta$ such that $\alpha>1$ and $0<\beta<1$, the parameter $\lambda$ varies between $\rho_{-}=\beta$ and $\rho_{+}=1$. Then, Case 3 splits into two according to $\frac{\left\|x_{1}\right\|}{\left\|x_{3}\right\|} \geq \gamma_{\alpha}$ or $\frac{\left\|x_{1}\right\|}{\left\|x_{3}\right\|}<\gamma_{\alpha}$, where $\gamma_{\alpha}$ is a constant depending of $\alpha$. If $\frac{\left\|x_{1}\right\|}{\left\|x_{3}\right\|} \geq \gamma_{\alpha}$ then $\lambda \in(\beta, 1)$, and if $\frac{\left\|x_{1}\right\|}{\left\|x_{3}\right\|}<\gamma_{\alpha}$ then $\lambda=\beta$.

In the example above, Problem 1 admits solution for every $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$, mainly due to the invertibility of the operator $T^{\#} T+\rho V^{\#} V$ for $\rho \in\left(\rho_{-}, \rho_{+}\right)$. The next section presents necessary and sufficient conditions for the existence of solutions to Problem 1 for arbitrary initial data.

## 4. Necessary and sufficient conditions for the existence of solutions for arbitrary initial data

The aim of this section is to characterize under which conditions Problem 1 admits a solution for every $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$. To do so we suppose that $N(T) \cap N(V)=\{0\}$. Later on we express the results for the general case. We assume the following:

Hypothesis 4.1. $T^{\#} T$ and $V^{\#} V$ are indefinite operators on $\mathcal{H}$, such that

$$
N(T) \cap N(V)=\{0\}
$$

We first show some necessary conditions.
Lemma 4.2. Assume that $\mathcal{Z}(w, z) \neq \varnothing$ for every $(w, z) \in \mathcal{K} \times \mathcal{E}$. Then:
i) $\rho_{-} \neq \rho_{+}$;
ii) $N\left(T^{\#} T\right) \cap N(V)=\{0\}$;
iii) $\mathcal{H}=N\left(T^{\#} T\right)^{\perp}+N(V)^{\perp}$.

Proof. i) Assume that $\rho_{-}=\rho_{+}=\rho$. Then, by Theorem 3.5 for any $(w, z) \in$ $\mathcal{K} \times \mathcal{E}$ there exists $\widetilde{x} \in \mathcal{H}$ such that

$$
\left(T^{\#} T+\rho V^{\#} V\right) \widetilde{x}=T^{\#} w+\rho V^{\#} z
$$

Since $(w, z)$ is arbitrary, $R\left(T^{\#}\right) \subseteq R\left(T^{\#} T+\rho V^{\#} V\right)$. But this is a contradiction to [24, Thm. 4.17]. Therefore, $\rho_{-} \neq \rho_{+}$.
ii) By Lemma 2.3, $\rho_{-} \neq \rho_{+}$implies that $\mathcal{C}_{T} \cap \mathcal{C}_{V}=N\left(T^{\#} T\right) \cap N(V)$. Given $x_{0} \in \mathcal{H}$ and $w_{0} \in \mathcal{K}$, by Proposition 3.3 there exists $y_{0} \in \mathcal{C}_{V}$ such that

$$
\left|\left[T x_{0}-w_{0}, T y\right]\right|^{2} \leq\left[T y_{0}, T y_{0}\right][T y, T y], \quad \text { for all } y \in \mathcal{C}_{V}
$$

Hence, $T x_{0}-w_{0} \in T\left(\mathcal{C}_{T} \cap \mathcal{C}_{V}\right)^{[\perp]}$. Since $x_{0}$ and $w_{0}$ are arbitrary,

$$
T\left(N\left(T^{\#} T\right) \cap N(V)\right)^{[\perp]}=T\left(\mathcal{C}_{T} \cap \mathcal{C}_{V}\right)^{[\perp]}=\mathcal{K}
$$

and thus $N\left(T^{\#} T\right) \cap N(V) \subseteq N(T)$. Consequently, $N\left(T^{\#} T\right) \cap N(V)=N(T) \cap$ $N(V)=\{0\}$.
iii) By item $i$, we only need to show that $N\left(T^{\#} T\right)+N(V)$ is closed. Assume that $\left(x_{n}\right)_{n \in \mathbb{N}}$ is a sequence in $N\left(T^{\#} T\right)$ and $\left(u_{n}\right)_{n \in \mathbb{N}}$ is a sequence in $N(V)$ such that $x_{n}+u_{n} \rightarrow x_{0} \in \overline{N\left(T^{\#} T\right)+N(V)}$. Since $\mathcal{Z}\left(0, V x_{0}\right) \neq \varnothing$, by Theorem 3.5 there exist $\lambda \in\left[\rho_{-}, \rho_{+}\right]$and $y_{0} \in \mathcal{C}_{V}$ such that

$$
\begin{equation*}
\left(T^{\#} T+\lambda V^{\#} V\right) y_{0}=-T^{\#} T x_{0} \tag{4.1}
\end{equation*}
$$

In what follows we prove that in this case, $y_{0} \in N(V)$; then, by (4.1) $x_{0}+y_{0} \in$ $N\left(T^{\#} T\right)$, or equivalently, $x_{0} \in N\left(T^{\#} T\right)+N(V)$.

On the one hand, since $x_{n} \in N\left(T^{\#} T\right)$,

$$
\begin{equation*}
T^{\#} T u_{n}=T^{\#} T\left(u_{n}+x_{n}\right) \rightarrow T^{\#} T x_{0}=-\left(T^{\#} T+\lambda V^{\#} V\right) y_{0} \tag{4.2}
\end{equation*}
$$

On the other hand, since $u_{n} \in N(V)$ and $y_{0} \in \mathcal{C}_{V},\left\langle V^{\#} V\left(u_{n}+y_{0}\right), u_{n}+y_{0}\right\rangle=$ 0 for every $n \in \mathbb{N}$. Then for any $\rho \in\left(\rho_{-}, \rho_{+}\right), \rho \neq \lambda$, it holds that

$$
\left(T^{\#} T+\rho V^{\#} V\right)^{1 / 2}\left(u_{n}+y_{0}\right) \rightarrow 0
$$

In fact,

$$
\begin{aligned}
\left\|\left(T^{\#} T+\rho V^{\#} V\right)^{1 / 2}\left(u_{n}+y_{0}\right)\right\|^{2} & =\left\langle T^{\#} T\left(u_{n}+y_{0}\right), u_{n}+y_{0}\right\rangle \\
& =\left\langle T^{\#} T\left(u_{n}+y_{0}\right), u_{n}\right\rangle+\left\langle T^{\#} T\left(u_{n}+y_{0}\right), y_{0}\right\rangle
\end{aligned}
$$

Also, since $y_{0} \in \mathcal{C}_{V}$,

$$
\left\langle T^{\#} T\left(u_{n}+y_{0}\right), y_{0}\right\rangle=\left\langle T^{\#} T u_{n}+\left(T^{\#} T+\lambda V^{\#} V\right) y_{0}, y_{0}\right\rangle \rightarrow 0
$$

and

$$
\begin{aligned}
\left\langle T^{\#} T\left(u_{n}+y_{0}\right), u_{n}\right\rangle & =\left\langle T^{\#} T\left(x_{n}+u_{n}+y_{0}\right), x_{n}+u_{n}\right\rangle \\
& \rightarrow\left\langle T^{\#} T\left(x_{0}+y_{0}\right), x_{0}\right\rangle=0,
\end{aligned}
$$

because $T^{\#} T\left(x_{0}+y_{0}\right)=-\lambda V^{\#} V y_{0} \in N\left(T^{\#} T\right)^{\perp} \cap N(V)^{\perp}$, see (4.1).
Therefore $\left(T^{\#} T+\rho V^{\#} V\right)\left(u_{n}+y_{0}\right) \rightarrow 0$, or equivalently, $T^{\#} T u_{n} \rightarrow-\left(T^{\#} T+\right.$ $\left.\rho V^{\#} V\right) y_{0}$, for every $\rho \in\left(\rho_{-}, \rho_{+}\right)$But, by (4.2), $T^{\#} T u_{n} \rightarrow-\left(T^{\#} T+\lambda V^{\#} V\right) y_{0}$. Hence, $\left(T^{\#} T+\rho V^{\#} V\right) y_{0}=\left(T^{\#} T+\lambda V^{\#} V\right) y_{0}$. So that $V^{\#} V y_{0}=0$, and thus $y_{0} \in N(V)$ and

$$
T^{\#} T y_{0}=\left(T^{\#} T+\lambda V^{\#} V\right) y_{0}=-T^{\#} T x_{0}
$$

Then $x_{0}+y_{0} \in N\left(T^{\#} T\right)$, or $x_{0} \in N\left(T^{\#} T\right)+N(V)$ as claimed.

Lemma 4.2 allows us to prove the following necessary condition for Problem 1 admitting a solution for every initial data point.

Proposition 4.3. Assume that $\mathcal{Z}(w, z) \neq \varnothing$ for every $(w, z) \in \mathcal{K} \times \mathcal{E}$. Then there exists $\alpha>0$ such that $[T y, T y] \geq \alpha\|y\|^{2}$ for every $y \in \mathcal{C}_{V}$
Proof. By Lemma4.2 $\rho_{-} \neq \rho_{+}, N\left(T^{\#} T\right) \cap N(V)=\{0\}$ and $\mathcal{H}=N\left(T^{\#} T\right)^{\perp}+$ $N(V)^{\perp} \subseteq N(T)^{\perp}+N(V)^{\perp}$. Given $\rho^{\prime} \in\left(\rho_{-}, \rho_{+}\right)$, we claim that $N(T)^{\perp} \subseteq$ $R\left(T^{\#} T+\rho^{\prime} V^{\#} V\right)^{1 / 2}$. In fact, let $x_{0} \in N(T)^{\perp}=R\left(T^{\#}\right)$, and let $w_{0} \in \mathcal{K}$ be such that $T^{\#} w_{0}=x_{0}$. Since $\mathcal{Z}\left(w_{0}, 0\right) \neq \varnothing$, by Theorem 3.5 there exist $\lambda \in\left[\rho_{-}, \rho_{+}\right]$and $y_{0} \in \mathcal{C}_{V}$ such that

$$
\left(T^{\#} T+\lambda V^{\#} V\right) y_{0}=T^{\#} w_{0}
$$

By Proposition 2.4,

$$
x_{0}=T^{\#} w_{0}=\left(T^{\#} T+\lambda V^{\#} V\right) y_{0} \subseteq R\left(\left(T^{\#} T+\rho^{\prime} V^{\#} V\right)^{1 / 2}\right)
$$

Since $x_{0}$ is arbitrary, we have that $N(T)^{\perp} \subseteq R\left(\left(T^{\#} T+\rho^{\prime} V^{\#} V\right)^{1 / 2}\right)$. Using this fact, it holds that

$$
N(V)^{\perp}=R\left(V^{\#} V\right) \subseteq R\left(T^{\#}\right)+R\left(T^{\#} T+\rho^{\prime} V^{\#} V\right) \subseteq R\left(\left(T^{\#} T+\rho^{\prime} V^{\#} V\right)^{1 / 2}\right)
$$

which implies

$$
\mathcal{H}=N(T)^{\perp}+N(V)^{\perp}=R\left(\left(T^{\#} T+\rho^{\prime} V^{\#} V\right)^{1 / 2}\right)
$$

and thus $\mathcal{H}=R\left(T^{\#} T+\rho^{\prime} V^{\#} V\right)$, see [16]. Hence, $T^{\#} T+\rho^{\prime} V^{\#} V$ is a positive definite operator, or equivalently, by Proposition 2.5, there exists $\alpha>0$ such that $[T y, T y] \geq \alpha\|y\|^{2}$ for every $y \in \mathcal{C}_{V}$.

Next we establish the conditions that guarantee the existence of solutions for arbitrary initial data. For the rest of this section we assume the following hypothesis:

Hypothesis 4.4. Given $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $V \in \mathcal{L}(\mathcal{H}, \mathcal{E})$ such that $T^{\#} T$ and $V^{\#} V$ are indefinite operators on $\mathcal{H}$, assume that there exists $\alpha>0$ such that

$$
[T y, T y] \geq \alpha\|y\|^{2}, \quad \text { for every } y \in \mathcal{C}_{V}
$$

This implies that $\mathcal{C}_{T} \cap \mathcal{C}_{V}=\{0\}$ (which in turn implies the condition in Hypothesis 4.1). Later on, we modify Hypothesis 4.4 appropriately to express the results for the general case.

Hypothesis 4.4 is not sufficient to ensure the existence of solutions for every initial data point, as the next example shows.

Example 2. Assume that $\mathcal{H}=\mathcal{K}=\mathcal{E}=\ell_{2}(\mathbb{N})$, and consider the indefinite inner products

$$
[x, y]_{\mathcal{K}}=x_{1} \bar{y}_{1}-\sum_{k \geq 2} x_{k} \bar{y}_{k}, \quad x=\left(x_{k}\right)_{k \in \mathbb{N}}, y=\left(y_{k}\right)_{k \in \mathbb{N}} \in \ell_{2}(\mathbb{N})
$$

$$
[x, y]_{\mathcal{E}}=-x_{1} \bar{y}_{1}+\sum_{k \geq 2} x_{k} \bar{y}_{k}, \quad x=\left(x_{k}\right)_{k \in \mathbb{N}}, y=\left(y_{k}\right)_{k \in \mathbb{N}} \in \ell_{2}(\mathbb{N})
$$

Then, $\left(\mathcal{K},[\cdot, \cdot]_{\mathcal{K}}\right)$ and $\left(\mathcal{E},[\cdot, \cdot]_{\mathcal{E}}\right)$ are Krein spaces.
If $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ denotes the standard canonical basis of $\ell_{2}(\mathbb{N})$, and $\alpha>\beta>0$, consider the linear operators $T: \mathcal{H} \rightarrow \mathcal{K}$ and $V: \mathcal{H} \rightarrow \mathcal{E}$ given by

$$
\begin{array}{ll}
T e_{1}=\alpha e_{1}, & T e_{k}=e_{k} \quad \text { if } k \geq 2 \\
V e_{1}=\beta e_{1}, & V e_{k}=\left(1+\frac{1}{k}\right)^{1 / 2} e_{k} \quad \text { if } k \geq 2
\end{array}
$$

Both $T$ and $V$ are trivially surjective, $N(T)=N(V)=\{0\}$, and a few calculations show that, for $x=\left(x_{k}\right)_{k \in \mathbb{N}}=\sum_{k \geq 1} x_{k} e_{k} \in \ell_{2}(\mathbb{N})$,

$$
\begin{aligned}
T^{\#} T x & =\alpha^{2} x_{1} e_{1}-\sum_{k \geq 2} x_{k} e_{k}, \\
V^{\#} V x & =-\beta^{2} x_{1} e_{1}+\sum_{k \geq 2} x_{k}\left(1+\frac{1}{k}\right) e_{k}
\end{aligned}
$$

Hence, $T^{\#} T$ and $V^{\#} V$ are two indefinite operators acting on $\mathcal{H}$. Moreover, $T^{\#} T+\rho V^{\#} V$ is positive semidefinite if and only if $1 \leq \rho \leq \frac{\alpha^{2}}{\beta^{2}}$, i.e. $\rho_{-}=1$ and $\rho_{+}=\frac{\alpha^{2}}{\beta^{2}}$. Also,

$$
\gamma\left(T^{\#} T+\rho V^{\#} V\right)=\left\{\begin{array}{lll}
\rho-1 & , & 1 \leq \rho<\frac{\alpha^{2}+1}{\beta^{2}+1} \\
\alpha^{2}-\rho \beta^{2} & , & \frac{\alpha^{2}+1}{\beta^{2}+1} \leq \rho<\frac{\alpha^{2}}{\beta^{2}} \\
\frac{\alpha^{2}}{\beta^{2}}-1 & , & \rho=\frac{\alpha^{2}}{\beta^{2}}
\end{array}\right.
$$

Then, $R\left(T^{\#} T+\rho V^{\#} V\right)$ is closed for every $\rho \in\left(1, \frac{\alpha^{2}}{\beta^{2}}\right]$. Given $\rho \in\left(1, \frac{\alpha^{2}}{\beta^{2}}\right]$,

$$
N\left(T^{\#} T+\rho V^{\#} V\right)=\{0\}=N(T) \cap N(V),
$$

and $T^{\#} T+\rho V^{\#} V$ is a positive definite operator, or equivalently there exists $\gamma>0$ such that $[T y, T y] \geq \gamma\|y\|^{2}$ for every $y \in \mathcal{C}_{V}$.

However, Problem 1 does not admit solutions for every $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$. In fact, consider the vector $\left(0, V e_{1}\right)=\left(0, \beta e_{1}\right) \in \mathcal{K} \times \mathcal{E}$. By Theorem 3.5, Problem 11 admits a solution for $\left(0, \beta e_{1}\right)$ if and only if there exist $\lambda \in\left[1, \frac{\alpha^{2}}{\beta^{2}}\right]$ and $y \in \mathcal{C}_{V}$ such that

$$
\left(T^{\#} T+\lambda V^{\#} V\right)\left(e_{1}+y\right)=\lambda V^{\#} V e_{1}
$$

or equivalently,

$$
\begin{equation*}
\left(T^{\#} T+\lambda V^{\#} V\right) y=-T^{\#} T e_{1} \tag{4.3}
\end{equation*}
$$

On the one hand, note that $y=\left(y_{k}\right)_{k \in \mathbb{N}} \in \mathcal{C}_{V}$ if and only if

$$
\sum_{k \geq 2}\left(1+\frac{1}{k}\right)\left|y_{k}\right|^{2}=\beta^{2}\left|y_{1}\right|^{2}
$$

On the other hand, (4.3) is equivalent to

$$
\begin{aligned}
\left(\alpha^{2}-\lambda \beta^{2}\right) y_{1} & =-\alpha^{2} \\
{\left[\lambda\left(1+\frac{1}{k}\right)-1\right] y_{k} } & =0, \quad \text { for } k \geq 2
\end{aligned}
$$

In this case there is no $y \in \mathcal{C}_{V}$ satisfying (4.3) because the above equations imply

$$
0=\sum_{k \geq 2}\left(1+\frac{1}{k}\right)\left|y_{k}\right|^{2}=\beta^{2}\left|y_{1}\right|^{2}
$$

and thus $0=-\alpha^{2}$, leading to a contradiction.

Hypothesis 4.4 allows us to study a simpler equivalent problem, because in this case the operator pencil $P(\lambda)=T^{\#} T+\lambda V^{\#} V$ is regular: by Proposition 2.5] and 24, Cor. 4.14], $T^{\#} T+\rho V^{\#} V$ is positive definite for every $\rho \in\left(\rho_{-}, \rho_{+}\right)$. Let us fix $\rho=\frac{\rho_{-}+\rho_{+}}{2}$ for convenience, and define the following indefinite inner product on $\mathcal{K} \times \mathcal{E}$ :

$$
\begin{equation*}
\left[(w, z),\left(w^{\prime}, z^{\prime}\right)\right]_{\rho}=\left[w, w^{\prime}\right]_{\mathcal{K}}+\rho\left[z, z^{\prime}\right]_{\mathcal{E}}, \quad w, w^{\prime} \in \mathcal{K} \text { and } z, z^{\prime} \in \mathcal{E} \tag{4.4}
\end{equation*}
$$

It is easy to see that $\left(\mathcal{K} \times \mathcal{E},[\cdot, \cdot]_{\rho}\right)$ is a Krein space. Define the operator $L: \mathcal{H} \rightarrow \mathcal{K} \times \mathcal{E}$ by

$$
L x=(T x, V x), \quad x \in \mathcal{H} .
$$

The adjoint operator of $L$ with respect to the indefinite inner product $[\cdot, \cdot]_{\rho}$ in $\mathcal{K} \times \mathcal{E}$ is given by

$$
L^{\#}(w, z)=T^{\#} w+\rho V^{\#} z, \quad(y, z) \in \mathcal{K} \times \mathcal{E}
$$

and it is immediate that $L^{\#} L=T^{\#} T+\rho V^{\#} V$. Now consider the selfadjoint operator $G \in \mathcal{L}(\mathcal{H})$ given by

$$
\begin{equation*}
G:=\left(L^{\#} L\right)^{-1 / 2} V^{\#} V\left(L^{\#} L\right)^{-1 / 2} \tag{4.5}
\end{equation*}
$$

Then $P(\lambda)$ can be rewritten as

$$
T^{\#} T+\lambda V^{\#} V=\left(L^{\#} L\right)^{1 / 2}(I+(\lambda-\rho) G)\left(L^{\#} L\right)^{1 / 2}
$$

Hence, the operator pencil $T^{\#} T+\lambda V^{\#} V$ is congruent to the pencil $I+\gamma G$, where $\gamma=\lambda-\rho$. If $\kappa=\frac{\rho_{+}-\rho_{-}}{2}$, then $I+\gamma G$ is positive semidefinite if and only if $\gamma \in[-\kappa, \kappa]$, and positive definite if and only if $\gamma \in(-\kappa, \kappa)$, see [24, Prop. 3.11]. This reduction technique is very common in the operator pencils context, since the auxiliary pencil $P^{\prime}(\gamma)=I+\gamma G$ is easier to analyze, see e.g. 19]. A similar procedure is also applied in [26] for a constrained quadratic optimization problem in a finite dimensional setting.

Consider the neutral elements of the quadratic form $x \mapsto\langle G x, x\rangle$, i.e.

$$
Q(G):=\{x \in \mathcal{H}:\langle G x, x\rangle=0\} .
$$

Next, we determine sufficient conditions under which there exist $y \in Q(G)$ and $\gamma \in[-\kappa, \kappa]$ such that

$$
\begin{equation*}
(I+\gamma G) y=u_{0} \tag{4.6}
\end{equation*}
$$

for every vector $u_{0} \in \mathcal{H}$. Later on, we show that this implies that Problem 1 admits a solution for every initial data point. Solving (4.6) is equivalent to finding the vectors in $Q(G)$ which minimize the distance to the vector $u_{0} \in \mathcal{H}$ :

$$
\min \left\|y-u_{0}\right\|^{2} \quad \text { subject to } \quad y \in Q(G)
$$

In fact, the normal equation (4.6) is just the corresponding version of (3.8) for this minimal distance problem.

Consider the canonical decomposition of $G$ as the difference of two positive operators: there exist unique subspaces $\mathcal{H}_{ \pm} \subseteq \mathcal{H}$ and positive definite operators $G_{ \pm} \in \mathcal{L}\left(\mathcal{H}_{ \pm}\right)$such that

$$
\begin{equation*}
\mathcal{H}=\mathcal{H}_{+} \oplus \mathcal{H}_{-} \oplus N(G) \tag{4.7}
\end{equation*}
$$

and $G=\left(\begin{array}{rrr}G_{+} & 0 & 0 \\ 0 & -G_{-} & 0 \\ 0 & 0 & 0\end{array}\right)$ with respect to (4.7).
If $u_{0}=u_{0}^{+}+u_{0}^{-}+u_{0}^{0}$ with $u_{0}^{ \pm} \in \mathcal{H}_{ \pm}$and $u_{0}^{0} \in N(G)$, (4.6) translates into

$$
\left\{\begin{align*}
\left(I_{+}+\gamma G_{+}\right) y^{+} & =u_{0}^{+}  \tag{4.8}\\
\left(I_{-}-\gamma G_{-}\right) y^{-} & =u_{0}^{-} \\
y^{0} & =u_{0}^{0}
\end{align*}\right.
$$

where $y=y^{+}+y^{-}+y^{0}$ with $y^{ \pm} \in \mathcal{H}_{ \pm}$and $y^{0} \in N(G)$.
Consider the subspaces

$$
\begin{equation*}
\mathcal{N}_{ \pm}:=N(I \mp \kappa G) \tag{4.9}
\end{equation*}
$$

It is easy to check that $\mathcal{N}_{ \pm}=N\left(I_{ \pm}-\kappa G_{ \pm}\right)$. Since $\mathcal{N}_{ \pm}$is invariant for $G_{ \pm} \in$ $\mathcal{L}\left(\mathcal{H}_{ \pm}\right)$, its orthogonal complement in $\mathcal{H}_{ \pm}$,

$$
\mathcal{D}_{ \pm}:=\mathcal{H}_{ \pm} \ominus \mathcal{N}_{ \pm}
$$

is also an invariant subspace for $G_{ \pm}$. We call $\mathcal{D}_{ \pm}$the positive (negative) defect subspace of $\mathcal{N}_{ \pm}$.
Lemma 4.5. Given $u \in \mathcal{H}_{ \pm}$, decompose it as $u=v+w$ with $v \in \mathcal{N}_{ \pm}$and $w \in \mathcal{D}_{ \pm}$. Then, for every $\tau \in(-\kappa, \kappa)$,

$$
\left\|\left(I_{ \pm} \pm \tau G_{ \pm}\right)^{-1} u\right\|^{2}=\frac{\kappa^{2}}{(\kappa \pm \tau)^{2}}\|v\|^{2}+\left\|\left(I_{ \pm} \pm \tau G_{ \pm}\right)^{-1} w\right\|^{2}
$$

Proof. Given $\tau \in(-\kappa, \kappa)$, considering that $\left\|G_{ \pm}\right\|=\frac{1}{\kappa}$ we have that $I_{+}+\tau G_{+}$ is invertible, see [24, Prop. 3.11]. $\mathcal{N}_{+}$and $\mathcal{D}_{+}$are both invariant subspaces for $I_{+}+\tau G_{+}$. Also, since $\mathcal{N}_{+}=N\left(I_{+}-\kappa G_{+}\right)$, if $v \in \mathcal{N}_{+}$then

$$
\left(I_{+}+\tau G_{+}\right) v=\left(I_{+}-\kappa G_{+}\right) v+(\kappa+\tau) G_{+} v=\frac{\kappa+\tau}{\kappa} v
$$

Now, let $u=v+w \in \mathcal{H}_{+}$with $v \in \mathcal{N}_{+}$and $w \in \mathcal{D}_{+}$. Then,

$$
\left(I_{+}+\tau G_{+}\right)^{-1} u=\left(I_{+}+\tau G_{+}\right)^{-1} v+\left(I_{+}+\tau G_{+}\right)^{-1} w=\frac{\kappa}{\kappa+\tau} v+\left(I_{+}+\tau G_{+}\right)^{-1} w
$$

and it is immediate that

$$
\left\|\left(I_{+}+\tau G_{+}\right)^{-1} u\right\|^{2}=\frac{\kappa^{2}}{(\kappa+\tau)^{2}}\|v\|^{2}+\left\|\left(I_{+}+\tau G_{+}\right)^{-1} w\right\|^{2}
$$

The proof of the remaining norm equality is similar.
As a consequence of Lemma 4.5, if $u=v+w \in \mathcal{H}_{ \pm}$with $v \in \mathcal{N}_{ \pm}$and $w \in \mathcal{D}_{ \pm}$ is such that $v \neq 0$, then $\lim _{\tau \rightarrow \mp \kappa}\left\|\left(I_{ \pm} \pm \tau G_{ \pm}\right)^{-1} u\right\|=+\infty$.

Lemma 4.6. Given $u \in \mathcal{H}_{ \pm}$, if $\lim _{\tau \rightarrow \kappa}\left\|\left(I_{ \pm}-\tau G_{ \pm}\right)^{-1} u\right\|<+\infty$, then $u \in$ $R\left(I_{ \pm}-\kappa G_{ \pm}\right)$.

Proof. In the following we prove the statement for a vector in $\mathcal{H}_{+}$, the proof for vectors in $\mathcal{H}_{-}$is analogous. Let $u \in \mathcal{H}_{+}$be such that $\lim _{\tau \rightarrow \kappa}\left\|\left(I_{+}-\tau G_{+}\right)^{-1} u\right\|<$ $+\infty$. Then

$$
u \in \mathcal{D}_{+}=\overline{R\left(I_{+}-\kappa G_{+}\right)}
$$

Consider the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ in $\mathcal{D}_{+}$defined by $x_{n}=\left(I_{+}-\left(\kappa-\frac{1}{n}\right) G_{+}\right)^{-1} u$, with $n \in \mathbb{N}$. By hypothesis, $\left(x_{n}\right)_{n \in \mathbb{N}}$ is bounded. Assume that $M>0$ is such that $\left\|x_{n}\right\| \leq M$ for every $n \in \mathbb{N}$. Then,

$$
\begin{equation*}
\left\|\left(I_{+}-\kappa G_{+}\right) x_{n}-u\right\|=\frac{1}{n}\left\|G_{+}\left(I_{+}-\left(\kappa-\frac{1}{n}\right) G_{+}\right)^{-1} u\right\| \leq \frac{M}{n}\left\|G_{+}\right\| \rightarrow 0 \tag{4.10}
\end{equation*}
$$

We claim that $\left(x_{n}\right)_{n \in \mathbb{N}}$ is a Cauchy sequence. To prove it consider the sequence of positive definite operators $\left(\Delta_{n}\right)_{n \in \mathbb{N}}$ defined by

$$
\Delta_{n}=\left(I_{+}-\left(\kappa-\frac{1}{n}\right) G_{+}\right)^{-1}
$$

By the functional calculus for selfadjoint operators, given $m, n \in \mathcal{N}, m \geq n$ implies that $\Delta_{n} \leq \Delta_{m}$, and $\Delta_{n}$ commutes with $\Delta_{m}$, see e.g. [14]. Then,

$$
\begin{aligned}
0 \leq\left\|x_{n}\right\|^{2} & =\left\langle\Delta_{n} u, \Delta_{n} u\right\rangle=\left\langle\Delta_{n}\left(\Delta_{n}^{1 / 2} u\right), \Delta_{n}^{1 / 2} u\right\rangle \\
& \leq\left\langle\Delta_{m}\left(\Delta_{n}^{1 / 2} u\right), \Delta_{n}^{1 / 2} u\right\rangle=\left\langle\Delta_{m} u, \Delta_{n} u\right\rangle=\left\langle x_{m}, x_{n}\right\rangle
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left\|x_{n}-x_{m}\right\|^{2} & =\left\|x_{n}\right\|^{2}-2 \operatorname{Re}\left(\left\langle x_{n}, x_{m}\right\rangle\right)+\left\|x_{m}\right\|^{2} \\
& \leq\left\|x_{n}\right\|^{2}-2\left\|x_{n}\right\|^{2}+\left\|x_{m}\right\|^{2}=\left\|x_{m}\right\|^{2}-\left\|x_{n}\right\|^{2} \rightarrow 0
\end{aligned}
$$

as $m, n \rightarrow \infty$.
Since $\mathcal{D}_{+}$is closed, there exists $x \in \mathcal{D}_{+}$such that $\left\|x_{n}-x\right\| \rightarrow 0$ as $n \rightarrow \infty$. Thus, (4.10) says that $u=\left(I_{+}-\kappa G_{+}\right) x$, i.e. $u \in R\left(I_{+}-\kappa G_{+}\right)$.

Lemma 4.7. If there exist $y \in Q(G)$ and $\gamma \in[-\kappa, \kappa]$ such that $(I+\gamma G) y \in$ $\mathcal{H}_{ \pm} \backslash\{0\}$, then $\mathcal{N}_{\mp} \neq\{0\}$.

Proof. Suppose that $(I+\gamma G) y=u_{0}^{-}$, with $u_{0} \in \mathcal{H}_{-} \backslash\{0\}, y \in Q(G)$ and $\gamma \in[-\kappa, \kappa]$, and assume that $\mathcal{N}_{+}=\{0\}$. If $y=y^{+}+y^{-}+y^{0}$ with $y^{ \pm} \in \mathcal{H}_{ \pm}$ and $y^{0} \in N(G)$, by (4.8) we get $y^{+}=0$ and $y^{-} \neq 0$. But since $y \in Q(G)$,

$$
0=\langle G y, y\rangle=\left\langle G y^{-}, y^{-}\right\rangle=-\left\langle G_{-} y^{-}, y^{-}\right\rangle<0
$$

Then $\mathcal{N}_{+} \neq\{0\}$.
A similar argument for a vector $u_{0}^{+} \in \mathcal{H}_{+} \backslash\{0\}$ proves that $\mathcal{N}_{-} \neq\{0\}$.

Proposition 4.8. For every $u_{0} \in \mathcal{H}$ there exist $y_{0} \in Q(G)$ and $\gamma \in[-\kappa, \kappa]$ such that

$$
(I+\gamma G) y_{0}=u_{0}
$$

if and only if $\mathcal{N}_{+} \neq\{0\}$ and $\mathcal{N}_{-} \neq\{0\}$.
Proof. The necessity follows by Lemma 4.7. To prove the converse, assume that $\mathcal{N}_{+} \neq\{0\}$ and $\mathcal{N}_{-} \neq\{0\}$. Let $u_{0} \in \mathcal{H}$, and consider the decomposition $u_{0}=u_{0}^{+}+u_{0}^{-}+u_{0}^{0}$ with $u_{0}^{ \pm} \in \mathcal{H}_{ \pm}$and $u_{0}^{0} \in N(G)$, and the real valued functions $f_{ \pm}$defined by

$$
f_{ \pm}(\tau)=\left\|G_{ \pm}^{1 / 2}\left(I_{ \pm} \pm \tau G_{ \pm}\right)^{-1} u_{0}^{ \pm}\right\|, \quad \tau \in(-\kappa, \kappa)
$$

If there exists $\tau_{0} \in(-\kappa, \kappa)$ such that $f_{+}\left(\tau_{0}\right)=f_{-}\left(\tau_{0}\right)$, then setting $y_{0}=\left(I_{+}+\right.$ $\left.\tau_{0} G_{+}\right)^{-1} u_{0}^{+}+\left(I_{-}-\tau G_{-}\right)^{-1} u_{0}^{-}+u_{0}^{0}$ yields $y_{0} \in Q(G)$ and $\left(I+\tau_{0} G\right) y_{0}=u_{0}$.

On the other hand, assume that $f_{+}(\tau)>f_{-}(\tau)$ for every $\tau \in(-\kappa, \kappa)$. By the functional calculus for selfadjoint operators, $f_{-}$is a monotone increasing function of $\tau$ on the interval $[0, \kappa)$. Since the extension of $f_{+}$is a continuous function of $\tau$ on the compact interval $[0, \kappa]$, it follows that

$$
\lim _{\tau \rightarrow \kappa}\left\|\left(I_{-}-\tau G_{-}\right)^{-1} u_{0}^{-}\right\| \leq \lim _{\tau \rightarrow \kappa}\left\|G_{-}^{-1 / 2}\right\| f_{-}(\tau)<+\infty
$$

Lemma 4.6 then assures that $u_{0}^{-} \in R\left(I_{-}-\kappa G_{-}\right)$. Now, since $\mathcal{N}_{-} \neq\{0\}$, let us choose $y \in \mathcal{N}_{-}$with $\|y\|=1$. Hence, considering that $G_{-} y=\frac{1}{\kappa} y$ and $\left(I_{-}-\kappa G_{-}\right)^{\dagger} u_{0}^{-} \perp y$, and setting

$$
y_{0}=\left(I_{+}+\kappa G_{+}\right)^{-1} u_{0}^{+}+\left(I_{-}-\kappa G_{-}\right)^{\dagger} u_{0}^{-}+\alpha_{-} y+u_{0}^{0}
$$

with

$$
\begin{equation*}
\alpha_{-}:=\left(\kappa\left(\left\|G_{+}^{1 / 2}\left(I_{+}+\kappa G_{+}\right)^{-1} u_{0}^{+}\right\|^{2}-\left\|G_{-}^{1 / 2}\left(I_{-}-\kappa G_{-}\right)^{\dagger} u_{0}^{-}\right\|^{2}\right)\right)^{1 / 2} \tag{4.11}
\end{equation*}
$$

yields $y_{0} \in Q(G)$ and $(I+\kappa G) y_{0}=u_{0}$.
A similar argument holds if we assume that $f_{+}(\tau)<f_{-}(\tau)$ for every $\tau \in$ $(-\kappa, \kappa)$, and thus the proof is complete.

Lemma 4.9. Under Hypothesis 4.4.

$$
\mathcal{N}_{ \pm} \subseteq\left(L^{\#} L\right)^{1 / 2}(N(T))^{\perp}
$$

Proof. We prove the statement for $\mathcal{N}_{+}$, a similar argument holds for $\mathcal{N}_{-}$. On the one hand, from

$$
T^{\#} T+\rho_{-} V^{\#} V=\left(L^{\#} L\right)^{1 / 2}(I-\kappa G)\left(L^{\#} L\right)^{1 / 2}
$$

it follows that $\mathcal{N}_{+}=\left(L^{\#} L\right)^{1 / 2}\left(N\left(T^{\#} T+\rho_{-} V^{\#} V\right)\right)$.
On the other hand, if $x \in N\left(T^{\#} T+\rho_{-} V^{\#} V\right)$ and $y \in N(T)$, then

$$
\rho_{-}\left\langle x, V^{\#} V y\right\rangle=\left\langle x,\left(T^{\#} T+\rho_{-} V^{\#} V\right) y\right\rangle=\left\langle\left(T^{\#} T+\rho_{-} V^{\#} V\right) x, y\right\rangle=0
$$

i.e., $x \in V^{\#} V(N(T))^{\perp}=L^{\#} L(N(T))^{\perp}$. Hence, we have that $N\left(T^{\#} T+\right.$ $\left.\rho_{-} V^{\#} V\right) \subseteq L^{\#} L(N(T))^{\perp}$. Applying $\left(L^{\#} L\right)^{1 / 2}$ to both sides of the inclusion,

$$
\mathcal{N}_{+} \subseteq\left(L^{\#} L\right)^{1 / 2}\left(\left(L^{\#} L\right)^{-1}\left(N(T)^{\perp}\right)\right)=\left(L^{\#} L\right)^{1 / 2}(N(T))^{\perp}
$$

We are now in conditions to state the main result of this section, establishing the necessary and sufficient conditions for Problem to admit a solution for every initial data point. We no longer assume that Hypothesis 4.4 hold.

Theorem 4.10. Assume that $N(T) \cap N(V)=\{0\}$. The following conditions are equivalent:
i) $\mathcal{Z}(w, z) \neq \varnothing$ for every $(w, z) \in \mathcal{K} \times \mathcal{E}$;
ii) there exists $\alpha>0$ such that $[T y, T y] \geq \alpha\|y\|^{2}$ for every $y \in \mathcal{C}_{V}$, and

$$
\begin{equation*}
\sup _{x \in \mathcal{P}^{-}(V)} \frac{[T x, T x]}{[V x, V x]} \quad \text { and } \quad \inf _{x \in \mathcal{P}^{+}(V)} \frac{[T x, T x]}{[V x, V x]} \tag{4.12}
\end{equation*}
$$

are attained.
Proof. ii) $\rightarrow i$ ): Suppose that item $i i$ holds and let $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$. Since the supremum and infimum in (4.12) being attained is equivalent to $\mathcal{N}_{+} \neq\{0\}$ and $\mathcal{N}_{-} \neq\{0\}$, by Proposition 4.8 for every $u_{0} \in \mathcal{H}$ there exist $\gamma \in[-\kappa, \kappa]$ and $\widetilde{y}_{0} \in Q(G)$ such that

$$
(I+\gamma G) \widetilde{y}_{0}=u_{0}
$$

Setting $u_{0}=\left(L^{\#} L\right)^{-1 / 2} T^{\#}\left(T V^{\dagger} z_{0}-w_{0}\right)$, applying $\left(L^{\#} L\right)^{1 / 2}$ to both sides of the equation, and taking $y_{0}=\left(L^{\#} L\right)^{-1 / 2} \widetilde{y}_{0}$ and $\lambda=\gamma+\rho$ the result follows.
i) $\rightarrow i i$ ): Assume that $\mathcal{Z}(w, z) \neq \varnothing$ for every $(w, z) \in \mathcal{K} \times \mathcal{E}$. By Proposition 4.3, it suffices to show that the infimum and supremum in (4.12) are attained, or equivalently, that $\mathcal{N}_{+} \neq\{0\}$ and $\mathcal{N}_{-} \neq\{0\}$. By Theorem 3.5, for every $w_{0} \in \mathcal{K}$ there exist $\lambda \in\left[\rho_{-}, \rho_{+}\right]$and $y \in \mathcal{C}_{V}$ such that $\left(T^{\#} T+\lambda V^{\#} V\right) y=T^{\#} w_{0}$.

Equivalently, $(I+\gamma G) \widetilde{y}=u_{0}$, where $\gamma=\lambda-\rho \in[-\kappa, \kappa], \widetilde{y}=\left(L^{\#} L\right)^{1 / 2} y \in Q(G)$ and $u_{0}=\left(L^{\#} L\right)^{-1 / 2} T^{\#} w_{0}$. Hence, for every

$$
u_{0} \in\left(L^{\#} L\right)^{1 / 2}(N(T))^{\perp}
$$

there exist $\gamma \in[-\kappa, \kappa]$ and $\widetilde{y} \in Q(G)$ such that $(I+\gamma G) \widetilde{y}=u_{0}$.
We now show that $\mathcal{H}_{+} \cap\left(L^{\#} L\right)^{1 / 2}(N(T))^{\perp}$ and $\mathcal{H}_{-} \cap\left(L^{\#} L\right)^{1 / 2}(N(T))^{\perp}$ are non trivial subspaces, which by Lemma 4.7 in turn implies that $\mathcal{N}_{-} \neq\{0\}$ and $\mathcal{N}_{+} \neq\{0\}$. Let us assume that $\rho>0$. It holds that $\left(L^{\#} L\right)^{1 / 2}\left(N\left(T^{\#} T\right)\right) \subseteq \mathcal{H}_{+}$. In fact, $T^{\#} T x=0$ if and only if $L^{\#} L x=\rho V^{\#} V x$, or equivalently, $\left(L^{\#} L\right)^{1 / 2} x=$ $\rho\left(L^{\#} L\right)^{-1 / 2} V^{\#} V x=\rho G\left(L^{\#} L\right)^{1 / 2} x$. If $\left(L^{\#} L\right)^{1 / 2} x=x_{+}+x_{-}+x_{0}$, with $x_{ \pm} \in$ $\mathcal{H}_{ \pm}$and $x_{0} \in N(G)$, then

$$
x_{+}=\rho G_{+} x_{+}, \quad x_{-}=-\rho G_{-} x_{-}, \quad \text { and } \quad x_{0}=\rho \cdot 0
$$

The last equation says that $x_{0}=0$, and $x_{-}=0$ because $\rho>0$ and $G_{-} \in$ $\mathcal{L}\left(\mathcal{H}_{-}\right)^{+}$. Therefore, $\left(L^{\#} L\right)^{1 / 2} x=x_{+}$. Then

$$
\left(L^{\#} L\right)^{1 / 2}(N(T)) \subseteq\left(L^{\#} L\right)^{1 / 2}\left(N\left(T^{\#} T\right)\right) \subseteq \mathcal{H}_{+}
$$

Hence,

$$
\mathcal{H}_{-} \subseteq\left(L^{\#} L\right)^{1 / 2}(N(T))^{\perp}
$$

and, by Lemma 4.7, $\mathcal{N}_{+} \neq\{0\}$. But by Lemma 4.9

$$
\mathcal{N}_{+} \subseteq \mathcal{H}_{+} \cap\left(L^{\#} L\right)^{1 / 2}(N(T))^{\perp}
$$

Then $\mathcal{H}_{+} \cap\left(L^{\#} L\right)^{1 / 2}(N(T))^{\perp} \neq 0$, which implies that $\mathcal{N}_{-} \neq\{0\}$. A similar argument holds for the case $\rho<0$, and thus the proof is complete.

## 5. Description of the set of solutions

In this section we consider a selfadjoint operator $G \in \mathcal{L}(\mathcal{H})$. Decomposing it as the sum of two positive operators with orthogonal ranges $G=G_{+}-$ $G_{-}$, by [24, Prop. 3.11] we have that $I+\gamma G$ is positive semidefinite if and only if $\gamma \in\left[-\left\|G_{+}\right\|^{-1},\left\|G_{-}\right\|^{-1}\right]$, and it is positive definite if and only if $\gamma \in$ $\left(-\left\|G_{+}\right\|^{-1},\left\|G_{-}\right\|^{-1}\right)$.

For simplicity, we assume that $\kappa:=\left\|G_{+}\right\|=\left\|G_{-}\right\|$. Hence, $I+\gamma G$ is positive semidefinite if and only if $\gamma \in[-\kappa, \kappa]$.

Also, we assume that the subspaces $\mathcal{N}_{+}$and $\mathcal{N}_{-}$given by (4.9) are non trivial, ensuring that for every $u \in \mathcal{H}$ there exist $y \in Q(G)$ and $\gamma \in[-\kappa, \kappa]$ such that

$$
\begin{equation*}
(I+\gamma G) y=u \tag{5.1}
\end{equation*}
$$

From now we consider a fixed vector $u_{0} \in \mathcal{H}$. If $(I+\gamma G) y=u_{0}$, for some $\gamma \in[-\kappa, \kappa]$ and $y \in Q(G)$, then from (4.8) it holds that $y^{0}=u_{0}^{0}$, and

$$
\left\{\begin{array}{l}
\left(I_{+}+\gamma G_{+}\right) y^{+}=u_{0}^{+} \\
\left(I_{-}-\gamma G_{-}\right) y^{-}=u_{0}^{-}
\end{array}\right.
$$

If $u_{0} \in N(G)$ then $y=u_{0}$ is the unique solution. On the other hand, if $u_{0} \notin N(G)$, then there is a unique $\gamma \in[-\kappa, \kappa]$ for any solution, as the next proposition shows.

Proposition 5.1. If $u_{0} \notin N(G)$ then there exists a unique $\gamma \in[-\kappa, \kappa]$ such that $(I+\gamma G) y=u_{0}$ admits a solution $y \in Q(G)$.

Proof. Let $u_{0} \notin N(G)$, and assume there exist $\gamma_{1}, \gamma_{2} \in[-\kappa, \kappa]$ and $y_{1}, y_{2} \in Q(G)$ such that

$$
\begin{align*}
& \left(I+\gamma_{1} G\right) y_{1}=u_{0} \\
& \left(I+\gamma_{2} G\right) y_{2}=u_{0} \tag{5.2}
\end{align*}
$$

On the one hand, since $\left\langle G y_{i}, y_{i}\right\rangle=0$ for $i=1,2,\left\langle u_{0}, y_{i}\right\rangle=\left\|y_{i}\right\|^{2}$. On the other hand,

$$
\begin{align*}
& \left\|y_{1}\right\|^{2}=\left\langle u_{0}, y_{1}\right\rangle=\left\langle\left(I+\gamma_{2} G\right) y_{2}, y_{1}\right\rangle=\left\langle y_{2}, y_{1}\right\rangle+\gamma_{2}\left\langle G y_{2}, y_{1}\right\rangle \\
& \left\|y_{2}\right\|^{2}=\left\langle u_{0}, y_{2}\right\rangle=\left\langle\left(I+\gamma_{1} G\right) y_{1}, y_{2}\right\rangle=\left\langle y_{1}, y_{2}\right\rangle+\gamma_{1}\left\langle G y_{1}, y_{2}\right\rangle \tag{5.3}
\end{align*}
$$

This implies that

$$
\begin{equation*}
\left(\gamma_{1}-\gamma_{2}\right)\left\langle G y_{1}, y_{2}\right\rangle=\left\|y_{1}\right\|^{2}-\left\|y_{2}\right\|^{2} \tag{5.4}
\end{equation*}
$$

By Cauchy-Schwarz inequality,

$$
\begin{aligned}
& \left\|y_{1}\right\|^{2}=\left|\left\langle u_{0}, y_{1}\right\rangle\right|=\left|\left\langle\left(I+\gamma_{2} G\right) y_{2}, y_{1}\right\rangle\right| \leq\left\|y_{2}\right\|\left\|y_{1}\right\|, \\
& \left\|y_{2}\right\|^{2}=\left|\left\langle u_{0}, y_{2}\right\rangle\right|=\left|\left\langle\left(I+\gamma_{1} G\right) y_{1}, y_{2}\right\rangle\right| \leq\left\|y_{1}\right\|\left\|y_{2}\right\| \text {, }
\end{aligned}
$$

and consequently $\left\|y_{1}\right\|=\left\|y_{2}\right\|$. By (5.4), this implies that $\gamma_{1}=\gamma_{2}$ or $\left\langle G y_{1}, y_{2}\right\rangle=$ 0 . However, if $\left\langle G y_{1}, y_{2}\right\rangle=0$, then from (5.3) it is easy to see that $y_{1}=y_{2}$, which in turn, by (5.2), implies that $\left(\gamma_{1}-\gamma_{2}\right) G y_{1}=0$. But $y_{1} \notin N(G)$ because $u_{0} \notin N(G)$, and hence $\gamma_{1}=\gamma_{2}$.

For $u_{0} \notin N(G)$, consider the set of solutions to (5.1),

$$
\Theta:=\left\{y \in Q(G):(I+\gamma G) y=u_{0}\right\}
$$

for the unique suitable $\gamma \in[-\kappa, \kappa]$. The following proposition describes the structure of the set $\Theta$, depending on whether $\gamma$ is an interior point of the interval or $\gamma= \pm \kappa$. Denote by $\mathcal{S}$ the unit sphere in $\mathcal{H}$, i.e.

$$
\mathcal{S}=\{x \in \mathcal{H}:\|x\|=1\} .
$$

Lemma 5.2. Let $u_{0} \notin N(G)$ and consider the unique $\gamma \in[-\kappa, \kappa]$ given by Proposition 5.1.
i) If $\gamma \in(-\kappa, \kappa)$, then

$$
\Theta=\left\{(I+\gamma G)^{-1} u_{0}\right\}
$$

ii) If $\gamma=\kappa$, then there exists $\alpha_{-} \geq 0$ such that

$$
\Theta=(I+\kappa G)^{\dagger} u_{0}+\alpha_{-} \cdot \mathcal{N}_{-} \cap \mathcal{S}
$$

iii) If $\gamma=-\kappa$, then there exists $\alpha_{+} \geq 0$ such that

$$
\Theta=(I-\kappa G)^{\dagger} u_{0}+\alpha_{+} \cdot \mathcal{N}_{+} \cap \mathcal{S}
$$

Proof. i) If $\gamma \in(-\kappa, \kappa)$ then $I+\gamma G$ is invertible. Hence, $y_{0}=(I+\gamma G)^{-1} u_{0}$.
ii) Suppose that $\gamma=\kappa$. Since
$Q(G)=\left\{y=y^{+}+y^{-}+y^{0}:\left\|G_{+}^{1 / 2} y^{+}\right\|=\left\|G_{-}^{1 / 2} y^{-}\right\|, y^{ \pm} \in \mathcal{H}_{ \pm}, y^{0} \in N(G)\right\}$,
writing $u_{0}=u_{0}^{+}+u_{0}^{-}+u_{0}^{0}$ the condition $(I+\kappa G) y_{0}=u_{0}$ leads to

$$
\left(I_{+}+\kappa G_{+}\right) y_{0}^{+}=u_{0}^{+} \quad, \quad\left(I_{-}-\kappa G_{-}\right) y_{0}^{-}=u_{0}^{-} \quad \text { and } \quad y_{0}^{0}=u_{0}^{0} .
$$

Then,

$$
y_{0}^{+}=\left(I_{+}+\kappa G_{+}\right)^{-1} u_{0}^{+} \quad \text { and } \quad y_{0}^{-}=\left(I_{-}-\kappa G_{-}\right)^{\dagger} u_{0}^{-}+v
$$

where $v \in \mathcal{N}_{-}$. If $v=0$, set $\alpha_{-}=0$. Otherwise, if $v \neq 0$, setting $\alpha_{-}:=\|v\|>0$ and $y_{-}:=\frac{v}{\|v\|} \in \mathcal{N}_{-} \cap \mathcal{S}$, we have that

$$
y_{0}=(I+\kappa G)^{\dagger} u_{0}+\alpha_{-} y_{-} .
$$

It only remains to show that $\alpha_{-}$is the same for every $y \in \Theta$. But, since $\left\|G_{+}^{1 / 2} y^{+}\right\|=\left\|G_{-}^{1 / 2} y^{-}\right\|, \alpha_{-}$is given by (4.11), and it does not depend on $y_{0}$ but only on $u_{0}$. Thus,

$$
\Theta=(I+\kappa G)^{\dagger} u_{0}+\alpha_{-} \cdot \mathcal{N}_{-} \cap \mathcal{S}
$$

An analogous procedure for the case $\gamma=-\kappa$ completes the proof.
As a consequence, as in Example 1 we can describe $\Theta$ by only analyzing which components of $u_{0}$ are null according to the decomposition $\mathcal{H}=\mathcal{N}_{+} \oplus$ $\mathcal{D}_{+} \oplus \mathcal{N}_{-} \oplus \mathcal{D}_{-} \oplus N(G)$.

Proposition 5.3. Consider $u_{0} \notin N(G)$ and write $u_{0}=v^{+}+w^{+}+v^{-}+w^{-}+u_{0}^{0}$, with $v^{ \pm} \in \mathcal{N}_{ \pm}, w^{ \pm} \in \mathcal{D}_{ \pm}$and $u_{0}^{0} \in N(G)$.
i) If $u_{0} \in \mathcal{H}_{ \pm}$, then there exists $\alpha_{ \pm}>0$ such that

$$
\Theta=(I \pm \kappa G)^{\dagger} u_{0}+\alpha_{\mp} \cdot \mathcal{N}_{\mp} \cap \mathcal{S}
$$

ii) If $v^{+} \neq 0$ and $v^{-} \neq 0$, then $\gamma \in(-\kappa, \kappa)$ and

$$
\Theta=\left\{(I+\gamma G)^{-1} u_{0}\right\}
$$

Proof. i) Assume that $u_{0} \in \mathcal{H}_{+}$and consider $y_{0}=y_{0}^{+}+y_{0}^{-}+y_{0}^{0} \in \Theta$ with $y_{0}^{ \pm} \in \mathcal{H}_{ \pm}$and $y_{0}^{0} \in N(G)$. If $\gamma \in[-\kappa, \kappa]$ is such that $(I+\gamma G) y_{0}=u_{0}$, then

$$
\left(I_{+}+\gamma G_{+}\right) y_{0}^{+}+\left(I_{-}-\gamma G_{-}\right) y_{0}^{-}+y_{0}^{0}=u_{0}=v^{+}+w^{+}+u_{0}^{0}
$$

Since $\left(I_{-}-\gamma G_{-}\right) y_{0}^{-}=0$ and $y_{0}^{-} \neq 0$, it holds that $\gamma=\kappa$. The result then follows from Lemma 5.2 The proof is analogous when $u_{0} \in \mathcal{H}_{-}$.
ii) Assuming that $v^{+} \neq 0$ and $v^{-} \neq 0$, following the same ideas of Proposition 4.8, we show that there exists $\gamma \in(-\kappa, \kappa)$ such that

$$
\left\|G_{+}^{1 / 2}\left(I_{+}+\gamma G_{+}\right)^{-1}\left(v^{+}+w^{+}\right)\right\|=\left\|G_{-}^{1 / 2}\left(I_{-}-\gamma G_{-}\right)^{-1}\left(v^{-}+w^{-}\right)\right\|
$$

which implies that the vector $y_{0}:=(I+\gamma G)^{-1} u_{0}$ belongs to $\Theta$ (because $y_{0} \in$ $Q(G)$ and $\left.(I+\gamma G) y_{0}=u_{0}\right)$.

Consider the real valued functions $g_{ \pm}$defined by

$$
g_{ \pm}(\tau)=\left\|G_{ \pm}^{1 / 2}\left(I_{ \pm} \pm \tau G_{ \pm}\right)^{-1}\left(v^{ \pm}+w^{ \pm}\right)\right\|^{2}, \quad \tau \in(-\kappa, \kappa)
$$

Since $G_{ \pm}^{1 / 2}$ and $\left(I_{ \pm} \pm \tau G_{ \pm}\right)^{-1}$ commute, and $G_{ \pm}^{1 / 2} v^{ \pm}=\kappa^{-1 / 2} v^{ \pm}$, Lemma 4.5 implies that

$$
\begin{aligned}
g_{ \pm}(\tau) & =\frac{\kappa^{2}}{(\kappa \pm \tau)^{2}}\left\|G_{ \pm}^{1 / 2} v^{ \pm}\right\|^{2}+\left\|\left(I_{ \pm} \pm \tau G_{ \pm}\right)^{-1} G_{ \pm}^{1 / 2} w^{ \pm}\right\|^{2} \\
& =\frac{\kappa}{(\kappa \pm \tau)^{2}}\left\|v^{ \pm}\right\|^{2}+\left\|G_{ \pm}^{1 / 2}\left(I_{ \pm} \pm \tau G_{ \pm}\right)^{-1} w^{ \pm}\right\|^{2}, \quad \text { for every } \tau \in(-\kappa, \kappa)
\end{aligned}
$$

Since the operator $I_{-}+\kappa G_{-}$is invertible, it follows that $g_{-}$is bounded on $(-\kappa, 0)$. Analogously, $g_{+}$is bounded on $(0, \kappa)$. On the other hand, since $v^{ \pm} \neq 0$, it is immediate that

$$
\lim _{\tau \rightarrow-\kappa} g_{+}(\tau)=+\infty \quad \text { and } \quad \lim _{\tau \rightarrow \kappa} g_{-}(\tau)=+\infty
$$

Hence, it is readily seen that there exists $\gamma \in(-\kappa, \kappa)$ such that $g_{-}(\gamma)=g_{+}(\gamma)$, or equivalently,

$$
\left\|G_{+}^{1 / 2}\left(I_{+}+\gamma G_{+}\right)^{-1}\left(v^{+}+w^{+}\right)\right\|=\left\|G_{-}^{1 / 2}\left(I_{-}-\gamma G_{-}\right)^{-1}\left(v^{-}+w^{-}\right)\right\|
$$

Thus, $\Theta=\left\{(I+\gamma G)^{-1} u_{0}\right\}$.
As it is illustrated by Case 3 in Example 1 if $u_{0}$ does not belong to $\mathcal{H}_{+}$nor to $\mathcal{H}_{-}$and also $v^{-}=0$ or $v^{+}=0$ (which is the only situation not covered by Proposition 5.3), it is not possible to assert whether $\Theta$ is a singleton.

To end this section, we show how these previous results can be applied to describe the set of solutions to Problem 1. We assume that $N(T) \cap N(V)=\{0\}$ and $\mathcal{Z}(w, z) \neq \varnothing$ for every $(w, z) \in \mathcal{K} \times \mathcal{E}$.

Consider an initial data point $\left(w_{0}, z_{0}\right) \in \mathcal{K} \times \mathcal{E}$ and a fixed vector $x_{0} \in \mathcal{H}$ such that $V x_{0}=z_{0}$. By Theorem 3.5, the set of solutions to Problem 1 is $\mathcal{Z}\left(w_{0}, z_{0}\right)=x_{0}+\Omega$ with the set $\Omega$ given by
$\Omega:=\left\{y \in \mathcal{C}_{V}:\left(T^{\#} T+\lambda V^{\#} V\right) y=-T^{\#}\left(T x_{0}-w_{0}\right)\right.$ for some $\left.\lambda \in\left[\rho_{-}, \rho_{+}\right]\right\}$.
Considering the operator $G$ given by (4.5) and setting $u_{0}:=-\left(L^{\#} L\right)^{-1 / 2} T^{\#}\left(T x_{0}-\right.$ $\left.w_{0}\right), \Omega$ can be alternatively described as

$$
\Omega=\left\{y \in \mathcal{C}_{V}:(I+\gamma G)\left(L^{\#} L\right)^{1 / 2} y=u_{0} \quad \text { for some } \gamma \in[-\kappa, \kappa]\right\}
$$

Since $\left(L^{\#} L\right)^{1 / 2}\left(\mathcal{C}_{V}\right)=Q(G)$, it follows that $\Omega=\left(L^{\#} L\right)^{-1 / 2}(\Theta)$ and thus

$$
\mathcal{Z}\left(w_{0}, z_{0}\right)=x_{0}+\left(L^{\#} L\right)^{-1 / 2}(\Theta)
$$

We establish now the main result of this section.
Theorem 5.4. There exists an open and dense subset $\mathcal{M}$ of $\mathcal{K} \times \mathcal{E}$ such that $\mathcal{Z}(w, z)$ is a singleton for every $(w, z) \in \mathcal{M}$.
Proof. The set
$\widetilde{\mathcal{M}}=\left\{u=v^{+}+w^{+}+v^{-}+w^{-}+u^{0} \in \mathcal{H}: v^{ \pm} \in \mathcal{N}_{ \pm} \backslash\{0\}, w^{ \pm} \in \mathcal{D}_{ \pm}, u^{0} \in N(G)\right\}$
is non empty, open and dense in $\mathcal{H}$. In fact, $\widetilde{\mathcal{M}}$ is non empty as a consequence of the assumption that $\mathcal{N}_{ \pm} \neq \varnothing$ and Lemma 4.9, while the remaining conditions follow immediately. By Proposition 5.3, $\Theta$ is a singleton for every $u \in \widetilde{\mathcal{M}}$.

Finally, considering the operator $A: \mathcal{K} \times \mathcal{E} \rightarrow \mathcal{H}$ given by

$$
A(w, z)=-\left(L^{\#} L\right)^{-1 / 2} T^{\#}\left(T V^{\dagger} z-w\right), \quad(w, z) \in \mathcal{K} \times \mathcal{E}
$$

yields $R(A)=\left(L^{\#} L\right)^{1 / 2}(N(T))^{\perp}$ is a closed subspace, and consequently $\mathcal{M}:=$ $A^{-1}(\widetilde{\mathcal{M}})$ is an open and dense subset of $\mathcal{K} \times \mathcal{E}$. Hence, $\mathcal{Z}(w, z)$ is a singleton for every $(w, z) \in \mathcal{M}$.

Remark 5.5. An immediate consequence of Hypothesis 4.4 is that $N(T) \cap$ $N(V)=\{0\}$. However, the condition in this hypothesis can be slightly modified in order to address the case in which this intersection is non trivial. Indeed, the following conditions are equivalent:
i) $\mathcal{Z}(w, z) \neq \varnothing$ for every $(w, z) \in \mathcal{K} \times \mathcal{E}$;
ii) there exists $\alpha>0$ such that

$$
[T y, T y] \geq \alpha\|y\|^{2}, \quad \text { for every } y \in \mathcal{C}_{V} \cap(N(T) \cap N(V))^{\perp}
$$

and

$$
\sup _{x \in \mathcal{P}^{-}(V)} \frac{[T x, T x]}{[V x, V x]} \quad \text { and } \quad \inf _{x \in \mathcal{P}^{+}(V)} \frac{[T x, T x]}{[V x, V x]}
$$

are attained.

As a result, there exists an open and dense subset $\mathcal{M}$ of $\mathcal{K} \times \mathcal{E}$ such that, instead of a singleton, the set of solutions to Problem 1 is an affine manifold parallel to the subspace $N(T) \cap N(V)$ i.e. for every $(w, z) \in \mathcal{M}$,

$$
\mathcal{Z}(w, z)=\widetilde{x}_{(w, z)}+N(T) \cap N(V)
$$

where $\widetilde{x}_{(w, z)}$ is a particular solution to Problem 1 with initial data $(w, z)$.

## 6. Application: Indefinite abstract mixed splines

The abstract mixed problem in Hilbert spaces was originally proposed by A. I. Rozhenko and V. A. Vasilenko in [40], and it can be stated as follows. Let $\left(\mathcal{H},\langle\cdot, \cdot\rangle_{\mathcal{H}}\right),\left(\mathcal{K}_{1},\langle\cdot, \cdot\rangle_{\mathcal{K}_{1}}\right),\left(\mathcal{K}_{2},\langle\cdot, \cdot\rangle_{\mathcal{K}_{2}}\right)$ and $\left(\mathcal{E},\langle\cdot, \cdot\rangle_{\mathcal{E}}\right)$ be Hilbert spaces, and consider (bounded) surjective operators $U: \mathcal{H} \rightarrow \mathcal{K}_{1}, W: \mathcal{H} \rightarrow \mathcal{K}_{2}$ and $V: \mathcal{H} \rightarrow \mathcal{E}$. Given $\left(w_{0}, z_{0}\right) \in \mathcal{K}_{2} \times \mathcal{E}$ and $\mu \in \mathbb{R}$, analize the existence of

$$
\min _{x \in \mathcal{H}}\left(\|U x\|_{\mathcal{K}_{1}}^{2}+\mu\left\|W x-w_{0}\right\|_{\mathcal{K}_{2}}^{2}\right), \quad \text { subject to } V x=z_{0}
$$

and if the minimum exists, find the set of arguments at which it is attained.
The abstract mixed splines problem is a generalization of the abstract interpolating and smoothing splines problems proposed by Atteia in [3]. For a complete exposition on these subjects see [4, 7, 13].

Generalizations to Krein spaces of the abstract interpolating and smoothing splines problems have been studied before 20, 23]. in particular, a generalization of the abstract mixed splines problem was also proposed in [20].

The following indefinite abstract mixed splines problem follows as a natural generalization of this family of problems. Given a Hilbert space $\left(\mathcal{H},\langle\cdot, \cdot\rangle_{\mathcal{H}}\right)$, and Krein spaces $\left(\mathcal{K}_{1},[\cdot, \cdot]_{\mathcal{K}_{1}}\right),\left(\mathcal{K}_{2},[\cdot, \cdot]_{\mathcal{K}_{2}}\right)$ and $\left(\mathcal{E},[\cdot, \cdot]_{\mathcal{E}}\right)$, let $U \in \mathcal{L}\left(\mathcal{H}, \mathcal{K}_{1}\right)$, $W \in \mathcal{L}\left(\mathcal{H}, \mathcal{K}_{2}\right)$ and $V \in \mathcal{L}(\mathcal{H}, \mathcal{E})$ be (bounded) surjective operators.

Problem 2. Given $\mu \neq 0$, and $\left(w_{0}, z_{0}\right) \in \mathcal{K}_{2} \times \mathcal{E}$, analyze the existence of

$$
\begin{gathered}
\min _{x \in \mathcal{H}}\left([U x, U x]_{\mathcal{K}_{1}}+\mu\left[W x-w_{0}, W x-w_{0}\right]_{\mathcal{K}_{2}}\right) \\
\quad \text { subject to }\left[V x-z_{0}, V x-z_{0}\right]_{\mathcal{E}}=0
\end{gathered}
$$

and if the minimum exists, find the set of arguments at which it is attained.
If $V^{\#} V$ is semidefinite then Problem 2 becomes the abstract mixed splines problem analyzed in 20]. We proceed now to describe how this problem can be studied in the context of the ILSP analyzed in this paper, in the case when $V^{\#} V$ is indefinite.

Given $\mu \neq 0$, define the inner product on $\mathcal{K}_{1} \times \mathcal{K}_{2}$ as in (4.4) and assume that $U^{\#} U+\mu W^{\#} W$ is indefinite. Also, defining the operator $T: \mathcal{H} \rightarrow \mathcal{K}_{1} \times \mathcal{K}_{2}$ by

$$
\begin{equation*}
T x:=(U x, W x), \quad x \in \mathcal{H} \tag{6.1}
\end{equation*}
$$

it is immediate that Problem 2 is equivalent to the following: given $\left(w_{0}, z_{0}\right) \in$ $\mathcal{K}_{2} \times \mathcal{E}$, analyze the existence of

$$
\begin{equation*}
\min _{x \in \mathcal{H}}\left[T x-\left(0, w_{0}\right), T x-\left(0, w_{0}\right)\right]_{\mu}, \text { subject to }\left[V x-z_{0}, V x-z_{0}\right]_{\mathcal{E}}=0 \tag{6.2}
\end{equation*}
$$

and if the minimum exists, find the set of arguments at which it is attained. Hence, it is clear that this is a particular case of Problem 1. Moreover, if $w_{0}=0$ and $T$ is surjective, then (6.2) reduces to the indefinite abstract splines problem considered in [23] with initial data $z_{0} \in \mathcal{E}$. The following proposition provides a necessary and sufficient condition for this particular case.

Proposition 6.1. The operator $T$ defined in (6.1) is surjective if and only if

$$
\mathcal{H}=N(U)+N(W)
$$

Proof. Assume $R(T)=\mathcal{K}_{1} \times \mathcal{K}_{2}$, and let $(u, 0) \in \mathcal{K}_{1} \times \mathcal{K}_{2}$. Then there exists $y \in \mathcal{H}$ such that $(U y, 0)=T y=(u, 0)$. Consequently, $y \in N(W)$ and since $u \in \mathcal{K}_{1}$ is arbitrary $\mathcal{K}_{1}=U(N(W))$ follows. Thus, $\mathcal{H}=U^{-1}(U(N(W)))=$ $N(U)+N(W)$.

Conversely, assume that $\mathcal{H}=N(U)+N(W)$. Then $N(U)^{\perp} \cap N(W)^{\perp}=\{0\}$. Given $(u, w) \in N\left(T^{\#}\right)$, we have that

$$
U^{\#} u=-\mu W^{\#} w \in R\left(U^{\#}\right) \cap R\left(W^{\#}\right)=N(U)^{\perp} \cap N(W)^{\perp}=\{0\}
$$

and $\underline{(u, w)}=(0,0)$ because $U^{\#}$ and $W^{\#}$ are injective. Therefore, $N\left(T^{\#}\right)=\{0\}$ and $\overline{R(T)}=\mathcal{K}_{1} \times \mathcal{K}_{2}$.

Since $U(N(W))=U(\mathcal{H})=\mathcal{K}_{1}$, it follows that $U(N(W))$ is closed. Now, consider a sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ in $\mathcal{H}$ such that $T x_{n} \rightarrow(y, z)$ for some $(y, z) \in$ $\mathcal{K}_{1} \times \mathcal{K}_{2}$. Then, for each $n \in \mathbb{N}$ consider $u_{n}=P_{N(W)} \perp x_{n}$ and $v_{n}=P_{N(W)} x_{n}$. Then, $W u_{n}=W x_{n}=z_{n} \rightarrow z$ and, since $u_{n} \in N(W)^{\perp}, u_{n}=W^{\dagger} W u_{n} \rightarrow W^{\dagger} z$. Therefore, $U u_{n} \rightarrow U W^{\dagger} z$ and

$$
U v_{n}=U x_{n}-U u_{n} \rightarrow y-U W^{\dagger} z
$$

The closedness of $U(N(W))$ implies that there exists $u \in N(W)$ such that $y-U W^{\dagger} z=U u$. Hence, $U\left(W^{\dagger} z+u\right)=y$ and $W\left(W^{\dagger} z+u\right)=z+W u=z$. Thus, $T\left(W^{\dagger} z+u\right)=(y, z)$ and the range of $T$ is closed, thus completing the proof.

Now for a fixed $\rho \neq 0$ we define a new indefinite inner product on $\mathcal{K}_{1} \times \mathcal{K}_{2} \times \mathcal{E}$. If $u, u^{\prime} \in \mathcal{K}_{1}, w, w^{\prime} \in \mathcal{K}_{2}$ and $z, z^{\prime} \in \mathcal{E}$,

$$
\left[(u, w, z),\left(u^{\prime}, w^{\prime}, z^{\prime}\right)\right]_{\rho}:=\left[u, u^{\prime}\right]_{\mathcal{K}_{1}}+\mu\left[w, w^{\prime}\right]_{\mathcal{K}_{2}}+\rho\left[z, z^{\prime}\right]_{\mathcal{E}}
$$

It is easy to see that the space $\mathcal{K}_{1} \times \mathcal{K}_{2} \times \mathcal{E}$ is a Krein space with this indefinite inner product. Also, defining the operator $L: \mathcal{H} \rightarrow \mathcal{K}_{1} \times \mathcal{K}_{2} \times \mathcal{E}$ by

$$
\begin{equation*}
L x:=(T x, V x)=(U x, W x, V x), \quad x \in \mathcal{H} \tag{6.3}
\end{equation*}
$$

it is immediate that

$$
L^{\#} L=U^{\#} U+\mu W^{\#} W+\rho V^{\#} V
$$

By means of the operators $T$ and $L$ defined in (6.1) and (6.3) respectively, the results concerning the ILSP analyzed in this paper can be directly applied.

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