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On the free implicative semilattice extension of a Hilbert algebra

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Hilbert algebras provide the equivalent algebraic semantics in the sense of Blok and Pigozzi to the implication fragment of intuitionistic logic. They are closely related to implicative semilattices. Porta proved that every Hilbert algebra has a free implicative semilattice extension. In this paper we introduce the notion of an optimal deductive filter of a Hilbert algebra and use it to provide a different proof of the existence of the free implicative semilattice extension of a Hilbert algebra as well as a simplified characterization of it. The optimal deductive filters turn out to be the traces in the Hilbert algebra of the prime filters of the distributive lattice free extension of the free implicative semilattice extension of the Hilbert algebra. To define the concept of optimal deductive filter we need to introduce the concept of a strong Frink ideal for Hilbert algebras which generalizes the concept of a Frink ideal for posets.

1 Introduction

In [14], Porta introduces the notion of the free Hertz algebra extension of a Hilbert algebra. Hertz algebras in the literature are also known as Brouwerian semilattices [11] and as implicative semilattices [13]. In the paper, we shall use the later terminology. A consequence of having these free extensions is that the category of Hilbert algebras is a reflective subcategory of the category of implicative semilattices.

In this paper we present a new definition of the free implicative semilattice extension of a Hilbert algebra, which, in our view, is simpler than that in [14]. We also give a different proof of the existence of this free extension. We also introduce several other categories of Hilbert algebras by considering several kinds of morphisms between Hilbert algebras and show how the categories we obtain are related to the categories of implicative semilattices.

Apart from the intrinsic interest of having free extensions, the fact that every Hilbert algebra has a free implicative semilattice extension is related to a Priestley style duality for Hilbert algebras, to which we shall devote a sequel to this paper. We describe briefly the main ideas relevant to the present paper needed to obtain the duality.

In [10], a Priestley style duality for bounded distributive join-semilattices is obtained and in [1, 2] a similar duality is obtained for bounded distributive meet-semilattices. Moreover in [1, 2] it is shown how to adapt the duality to distributive meet-semilattices with a top element. In both cases, in order to obtain the duality it is proved that the free distributive lattice extension of a distributive join(meet)-semilattice relative to the homomorphisms that preserve existing finite joins (meets) exists. In [3], the duality for distributive meet-semilattices worked out in [1, 2] is used to obtain a Priestley style duality for implicative semilattices, which as meet-semilattices are distributive. Let us give a hint at this Priestley style duality for implicative semilattices. The points of the dual space of a distributive meet-semilattice L with a top element can be seen as the intersections of the prime filters of the free distributive lattice extension of D of L relative to the homomorphisms that preserve existing finite joins. The Priestley like dual space of L is essentially the Priestley space of D plus a dense set that encodes L in the dual space. If L is an implicative semilattice, the free distributive lattice extension D of the meet-semilattice

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reduct of L may not be residuated, and therefore may not be a Heyting algebra, but the residual of two elements a, b of L exists in D and is the element $a \to b$ of L. The points of the dual space of L are again the restriction of the prime filters of D to L.

Now we can describe the points of the dual space of a Hilbert algebra in our duality as follows. Let $\boldsymbol{A}$ be a Hilbert algebra, L its free implicative semilattice extension and $D(L)$ the free distributive lattice extension of L. The points of the dual space of $\boldsymbol{A}$ will be the restriction to $\boldsymbol{A}$ of the prime filters of $D(L)$. These filters are characterized in an intrinsic way to the Hilbert algebra $\boldsymbol{A}$ in the present paper where are called optimal deductive filters. They will be used to provide an alternative proof of the existence of the free implicative semilattice extension of $\boldsymbol{A}$.

The paper is divided in seven sections. In the second section we introduce all the preliminary notions and results relevant to the paper. In the third section we discuss several notions of ideals for Hilbert algebras and introduce the concept of a strong Frink ideal. This concept is used in Section 4 to define the concept of an optimal deductive filter for Hilbert algebras. In Section 4 we also introduce the concept of separating family of deductive optimal filters. Section 5 is devoted to introducing and discussing several notions of morphisms between Hilbert algebras: homomorphisms, the semi-homomorphisms of [5], and the new concepts of homomorphisms and semi-homomorphisms with the sup-property. In Section 6 we present the simplified definition of the free implicative semilattice extension of a Hilbert algebra, give the proof of its uniqueness and existence using separating families of deductive optimal filters and discuss the relation of the categories of Hilbert algebras with homomorphisms and with semi-homomorphisms with corresponding categories of implicative semilattices. Finally, in Section 7 we discuss the relation between the deductive filters of a Hilbert algebra $\boldsymbol{A}$ and the filters of its free implicative semilattice extension L as well as the relation between the optimal deductive filters of $\boldsymbol{A}$ and the optimal deductive filters of L. Moreover, we also discuss the relation between the optimal deductive filters of $\boldsymbol{A}$ and the prime filters of the free distributive lattice extension D of L relative to the homomorphisms that preserve existing finite joins.

2 Preliminaries

In this section we fix the terminology adopted in the paper and introduce the main definitions of Hilbert algebra and implicative semilattice. We also recall the standard concepts and known results that we shall use subsequently.

We first recall some notions of filters and ideals for posets. Let $\mathbb{P} = \langle P, \leq \rangle$ be a poset. A set $Y \subseteq P$ is a *down-set* provided that for every $b \in P$ if $b \leq a$ for some $a \in Y$, then $b \in Y$. Dually, it is an *up-set* if for every $b \in P$, if $a \leq b$ for some $a \in Y$, then $b \in Y$. If $a \in P$,$\downarrow a$ denotes the down-set $\{b \in P : b \leq a\}$ and $\uparrow a$ the up-set $\{b \in P : a \leq b\}$. If $Y \subseteq P$, let Y^{u} denote the set of upper bounds of Y and Y^{l} the set of lower bounds. The two induced maps $(.)^{\mathrm{u}}$ and $(.)^{\mathrm{l}}$ on the powerset $\mathcal{P}(P)$ are the Galois connection of the relation $\leq$. Therefore the two compositions $(.)^{\mathrm{ul}}$ and $(.)^{\mathrm{lu}}$ are closure operators on P. Note that if $x \in P$, then $\downarrow x = (\{x\})^{\mathrm{ul}}$ and $\uparrow x = (\{x\})^{\mathrm{lu}}$.

A set $I \subseteq P$ is a *Frink ideal* if for every finite $X \subseteq I$, $(X)^{\mathrm{ul}} \subseteq I$ (cf. [9]). Dually, a set $F \subseteq P$ is a *Frink filter* of $\mathbb{P}$ if for every finite $X \subseteq F$, $(X)^{\mathrm{lu}} \subseteq F$. It immediately follows that Frink filters are up-sets and Frink ideals are down-sets. Note that P is both a Frink filter and a Frink ideal. A Frink ideal resp. a Frink filter, is proper if it is not P. Note also that if P is finite, then the Frink ideals are the closed sets of the closure operator $(.)^{\mathrm{ul}}$ and the Frink filters the closed sets of the closure operator $(.)^{\mathrm{lu}}$. In this paper we are interested in Frink ideals.

Note that a set $I \subseteq P$ is a Frink ideal if and only if for every $a_1, \ldots, a_n \in I$ and $c \in P$, $c \in I$ whenever $\bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow c$, and, moreover, if c is a minimum element of P then also $c \in I$.

A set $I \subseteq P$ is said to be an *ideal* of $\mathbb{P}$ if it is a nonempty updirected down-set, that is, a nonempty down-set of $\mathbb{P}$ such that for every $a, b \in I$ there is $c \in I$ with $a, b \leq c$. Similarly a set $F \subseteq P$ is said to be an *filter* of $\mathbb{P}$ if it is a nonempty downdirected up-set, that is, a nonempty up-set of $\mathbb{P}$ such that for every $a, b \in F$ there is $c \in F$ with $c \leq a, b$.

Proposition 2.1 *Every ideal of a poset is a Frink ideal and every filter is a Frink filter.*

Proof. Let I be an ideal of a poset $\mathbb{P}$. Let $X \subseteq I$ be finite. If $X = \varnothing$, then $X^{\mathrm{ul}} = \varnothing$ or $X^{\mathrm{ul}} = \{\bot\}$, if $\mathbb{P}$ has a minimum element $\bot$. In both cases $X^{\mathrm{ul}} \subseteq I$. So I is a Frink ideal. The case of filters is dealt in a similar way. □

A Frink filter is *prime* if it is a prime element in the lattice of Frink filters. Similarly, a Frink ideal is *prime* if it is a prime element in the lattice of Frink ideals.

The relation between Frink filters (ideals) and Frink ideals (filters) is established in [8]. It is as follows.

Proposition 2.2 *Let $F, I \subseteq P$ be a Frink filter and a Frink ideal respectively.*

(1) *F is a prime Frink filter if and only if $P - F$ is an ideal.*

(2) *I is a prime Frink ideal if and only if $P - I$ is a filter.*

Proof. (1) Let F be a prime Frink filter. Since F is an up-set, $P - F$ is a down-set. To prove that it is updirected, let $a, b \in P - F$. We need to show that $(P - F) \cap \{a, b\}^{\mathrm{u}} \neq \varnothing$. Note that $\{a, b\}^{\mathrm{u}} = \uparrow a \cap \uparrow b$. So if $(P - F) \cap \{a, b\}^{\mathrm{u}} = \varnothing$, then $\uparrow a \cap \uparrow b \subseteq F$, and since F is prime, $a \in F$ or $b \in F$, which is not possible. Suppose now that $P - F$ is an ideal. Let F_1, F_2 be Frink filters with $F_1 \cap F_2 \subseteq F$. Suppose that $F_1 \nsubseteq F$ and $F_2 \nsubseteq F$. Let $a \in F_1 - F$ and $b \in F_2 - F$. So, $a, b \in P - F$. Let $c \in P - F$ be such that $a, b \leq c$. Then $c \in \uparrow a \cap \uparrow b \subseteq F_1 \cap F_2 \subseteq F$, a contradiction. Hence $F_1 \subseteq F$ or $F_2 \subseteq F$. (2) can be proved by an analogous reasoning. □

2.1 Distributive meet-semilattices and implicative semilattices

A meet-semilattice $L = \langle L, \wedge \rangle$ is *distributive* provided that for every $a, b_1, b_2 \in L$ with $b_1 \wedge b_2 \leq a$, there exist $c_1, c_2 \in L$ such that $b_1 \leq c_1$, $b_2 \leq c_2$, and $a = c_1 \wedge c_2$, where the partial order $\leq$ is the meet-semilattice order (defined by $a \leq b$ iff $a \wedge b = a$). A *distributive meet-semilattice with top* is a distributive meet-semilattice $L = \langle L, \wedge, 1 \rangle$ such that L has a greatest element 1 w.r.t. $\leq$. A distributive meet-semilattice $L = \langle L, \wedge, 0, 1 \rangle$ is *bounded* if $\leq$ has a least element 0 and a greatest element 1 w.r.t. $\leq$. The class of distributive meet-semilattices is not a variety. For example the meet-semilattice given by the power set of $\{a, b, c\}$ is distributive but its sub-meet-semilattice $\{\{a\}, \{b\}, \{c\}, \varnothing\}$ is not.

An *implicative semilattice*, also known as Brouwerian semilattice [11] and as Hertz algebra [14], is a tuple $\langle L, \wedge, \rightarrow, 1 \rangle$ where $\langle L, \wedge, 1 \rangle$ is a meet-semilattice with top 1 and for every $a \in L$ the map $a \rightarrow (-) : L \rightarrow L$ is a right adjoint to the map $a \wedge (-) : L \rightarrow L$, that is for every $a, b, c \in L$,

$$a \wedge c \leq b \quad \text{iff} \quad c \leq a \rightarrow b.$$

For every implicative semilattice L, the meet-semilattice $\langle L, \wedge \rangle$ is distributive. Implicative semilattices form a variety; equational axiomatizations, as well as other informations, can be found in [6, 7, 11, 13].

Let L be a meet-semilattice with top. A nonempty set $F \subseteq L$ is a *filter* of L if it is an filter of the meet-semilattice order. This is equivalent to saying that (1) for every $a, b \in F$, $a \wedge b \in F$ and (2) for every $a \in F$ and $b \in L$, if $a \leq b$, then $b \in F$. We denote the set of filters of L by $\mathrm{Fi}L$. A filter $F \in \mathrm{Fi}L$ is *proper* if $F \neq L$. A proper filter F of L is *meet-prime* if it is a prime element of the lattice of filters, that is when for all filters F_1, F_2 of L, if $F_1 \cap F_2 \subseteq F$, then $F_1 \subseteq F$ or $F_2 \subseteq F$. Instead of meet-prime we shall simply say prime. Since the lattice of filters of a distributive meet-semilattice is distributive, the meet-prime filters are the meet-irreducible elements of that lattice.

In implicative semilattices the filters can be characterized as follows. Let L be an implicative semilattice. A set $F \subseteq L$ is a filter if and only if $1 \in F$ and for every $a, b \in L$, if $a, a \rightarrow b \in F$, then $b \in F$.

A set $I \subseteq L$ is a Frink ideal of L, F-*ideal* for short, if it is a Frink ideal of the meet-semilattice reduct. An F-ideal I is proper if $I \neq L$, and it is *prime* if it is proper and for every $a, b \in I$, $a \in I$ or $b \in I$ whenever $a \wedge b \in I$.

A (nonempty) set $F \subseteq L$ is said to be an *optimal filter* [1, 2] if it is a proper filter and $L - F$ is an F-ideal. Optimal filters separate filters from disjoint F-ideals; that is, if F is a filter and I is an F-ideal such that $F \cap I = \varnothing$, then there exists an optimal filter P such that $F \subseteq P$ and $P \cap I = \varnothing$ (cf. [1, 2] for a proof).

Let L, L' be distributive meet-semilattices. A homomorphism $h : L \rightarrow L'$ is a sup-homomorphism (cf. [1, 2]) if it preserves all existing finite joins, that is, if $X \subseteq L$ is finite and $\sup X$ exists in L, then $\sup(h[X])$ exists in L' and $h(\sup X) = \sup(h[X])$. Alternatively, $h : L \rightarrow L'$ is a sup-homomorphism if and only if

$$\bigcap_{i \leq n} \uparrow c_i \subseteq \uparrow b, \quad \text{then} \quad \bigcap_{i \leq n} \uparrow h(c_i) \subseteq \uparrow h(b)$$

for every $c_0, \ldots, c_n, b \in L$, and if L_1 has bottom element $\perp$, then $h(\perp)$ is a bottom element of L_2.

Let L be a distributive meet-semilattice. A *free distributive lattice extension* of L is a pair (D, j) where D is a distributive lattice and $j : L \to D$ is a one-to-one sup-homomorphism from L to the meet reduct of D that satisfies the following universal property: for every distributive lattice E and every sup-homomorphism $h : L \to E$ there exists a unique lattice homomorphism $\overline{h} : D \to E$ such that $h = \overline{h} \circ j$. In [1, 2], it is shown that:

Proposition 2.3

(1) *Every distributive meet-semilattice with top has a unique (up to isomorphism) free distributive lattice extension.*

(2) *The free distributive lattice extension of a distributive meet-semilattice L with top is (up to isomorphism) the unique distributive lattice $D(L)$ for which there is a one-to-one sup-homomorphism $j : L \to D(L)$ such that every element of $D(L)$ is the join of a finite set of elements of $j[L]$.*

Let L be a distributive meet-semilattice with top and let (D, j) be its distributive lattice free extension. There is an order isomorphism between the ordered set of the prime filters of D and the ordered set of the optimal filters of L, given by the map $j^{-1}[.]$ (cf. [1, 2]).

2.2 Hilbert algebras

Definition 2.4 A *Hilbert algebra* is a triple $\boldsymbol{A} = \langle A, \to, 1\rangle$, where A is a nonempty set, $\to$ a binary operation on A, $1 \in A$ and for every $a, b, c \in A$

(H1) $a \to (b \to a) = 1$,

(H2) $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$,

(H3) $a \to b = b \to a = 1$ implies $a = b$.

Hilbert algebras form a variety, equationally axiomatized in [7]. The $(\to, 1)$-reducts of implicative semilattices are Hilbert algebras.

Lemma 2.5 *In every Hilbert algebra $\boldsymbol{A}$,*

(1) $a \to a = 1$,

(2) $a \to 1 = 1$,

(3) $1 \to a = a$,

(4) $a \to (b \to c) = b \to (a \to c)$,

(5) $a \to (b \to c) = (a \to b) \to (a \to c))$,

(6) $a \to ((a \to b) \to b) = 1$,

(7) $a \to (a \to b) = a \to b$,

(8) $((a \to b) \to b) \to b = a \to b$,

(9) $(a \to b) \to ((b \to a) \to a) = (b \to a) \to ((a \to b) \to b)$.

We use the following notation. Let $\boldsymbol{A}$ be a Hilbert algebra and $b \in A$. We define inductively for $b \in A$ and for every sequence $a_0, \dots, a_n$ of elements of A the element $(a_n, \dots, a_0; b) \in A$ as follows:

$$(a_0; b) = a_0 \to b \quad \text{and} \quad (a_{n+1}, \dots, a_0; b) = a_{n+1} \to (a_n, \dots, a_0; b).$$

Thus, $(a_2, a_1, a_0; b) = a_2 \to (a_1 \to (a_0 \to b))$.

The following lemma states some facts that can be easily proved by induction.

Lemma 2.6 *Let $\boldsymbol{A}$ be a Hilbert algebra and let $a_0, \dots, a_n, b \in A$.*

(1) $(a_n, \dots, a_0; b) = (a_{\pi(n)}, \dots, a_{\pi(0)}; b)$, *for every permutation π of $\{0, \dots, n\}$,*

(2) $(a_n, \dots, a_0; 1) = 1$,

(3) $(a_n, \dots, a_0; b) = (a_n, \dots, a_1; a_0 \to b)$,

(4) $(a_n, \dots, a_0; b) = (a_n, \dots, a_m; (a_{m-1}, \dots, a_0; b))$.

In view of condition (1) of the lemma, if X is a nonempty finite subset of L and $a \in L$, we denote by $(X; a)$ the element $(a_n, \dots, a_0; a)$ where $a_0, \dots, a_n$ is an arbitrary enumeration of X.

In every Hilbert algebra $\boldsymbol{A}$ the relation $\leq$ defined by

$$a \leq b \quad \text{iff} \quad a \to b = 1$$

is a partial order with greatest element 1.

2.3 Deductive filters of Hilbert algebras

Definition 2.7 A subset F of a Hilbert algebra $\boldsymbol{A}$ is called a *deductive filter*[1] if the following two conditions are satisfied for every $a, b \in L$:

(1) $1 \in F$.

(2) if $a, a \to b \in F$, then $b \in F$.

The following facts are well-known.

Lemma 2.8 *Let F be a deductive filter, then*

(1) *F is an up-set,*

(2) *if $(a_n, \dots, a_m, a_{m-1}, \dots, a_0; b) \in F$ and $a_n, \dots, a_m \in F$, then $(a_{m-1}, \dots, a_0; b) \in F$,*

(3) *if $(a_n, \dots, a_0; b) \in F$ and $a_0, \dots, a_n \in F$, then $b \in F$.*

If $\boldsymbol{A}$ is a Hilbert algebra, then A is a deductive filter of $\boldsymbol{A}$, and the intersection of an arbitrary family of deductive filters of $\boldsymbol{A}$ is again a deductive filter of $\boldsymbol{A}$. Hence, for every $X \subseteq A$ there exists the least deductive filter containing X. This deductive filter is called the *deductive filter generated by* X. We denote it by $\langle X \rangle$. It can be characterized as follows:

$$a \in \langle X \rangle \quad \text{iff} \quad a = 1 \quad \text{or} \quad \exists\, a_0, \dots, a_n \in X : (a_n, \dots, a_0; b) = 1.$$

In particular, the deductive filter generated by $a \in A$ is the set $\{b \in A : a \to b = 1\}$, which is $\uparrow a = \{b \in A : a \leq b\}$, namely, the principal filter generated by a. A deductive filter F is *proper* if $F \neq A$.

We denote by $\mathrm{Dfi}\boldsymbol{A}$ the set of all deductive filters of $\boldsymbol{A}$. Then $\langle \mathrm{Dfi}\boldsymbol{A}, \cap, \vee \rangle$ is a bounded lattice, where $F_1 \vee F_2 = \langle F_1 \cup F_2 \rangle$, the infimum is $\{1\}$ and the supremum is A.

Note that if L is an implicative semilattice, the deductive filters of the Hilbert algebra reduct of L are exactly the nonempty filters of L.

Proposition 2.9 *If $\boldsymbol{A}$ is a Hilbert algebra, the lattice $\langle \mathrm{Dfi}\boldsymbol{A}, \cap, \vee \rangle$ is distributive.* [7]

A proper deductive filter F of a Hilbert algebra $\boldsymbol{A}$ is said to be *meet-prime* if it is a meet-prime element of the lattice $\mathrm{Dfi}\boldsymbol{A}$, that is, if it is proper and for any two deductive filters F_1, F_2 of $\boldsymbol{A}$ with $F_1 \cap F_2 \subseteq F$, we have $F_1 \subseteq F$ or $F_2 \subseteq F$. For simplicity, we shall call the meet-prime filters of a Hilbert algebra *prime*. Since the lattice $\mathrm{Dfi}\boldsymbol{A}$ is distributive, a deductive filter F is meet-prime if and only if it is a meet-irreducible element of $\mathrm{Dfi}\boldsymbol{A}$, that is, if and only if for any two deductive filters F_1, F_2 of $\boldsymbol{A}$ such that $F_1 \cap F_2 = F$, it holds that $F_1 = F$ or $F_2 = F$.

A proper deductive filter F of a Hilbert algebra $\boldsymbol{A}$ is said to be *completely meet-irreducible* if it is a completely meet-irreducible element of $\mathrm{Dfi}\boldsymbol{A}$, that is, if it is proper and for any family $\{F_i : i \in I\} \subseteq \mathrm{Dfi}\boldsymbol{A}$, such that $F = \bigcap_{i \in I} F_i$ there is $i \in I$ with $F = F_i$. Of course every completely meet-irreducible deductive filter is prime.

Proposition 2.10 *Let $\boldsymbol{A}$ be a Hilbert algebra.*

(1) *If F is a deductive filter of $\boldsymbol{A}$ and $a \in A - F$, then there exists a completely meet-irreducible deductive filter G of $\boldsymbol{A}$ such that $F \subseteq G$ and $a \notin G$,*

(2) *If $a, b \in A$ are such that $a \not\leq b$, then there exists a completely meet-irreducible deductive filter G such that $a \in G$ and $b \notin G$* [12].

[1] Deductive filters are usually called deductive systems, but since this last expression is used in other context to refer to logics, we prefer to call them deductive filters, as in [15].

3 Ideals of Hilbert algebras

In this section we discuss two notions of ideals for Hilbert algebras and the corresponding notions of being prime. The first notion we study is the poset notion of ideal applied to Hilbert algebras and the second a generalization of the notion of Frink ideal we call strong Frink ideal. This concept will allow to introduce the concept of optimal deductive filter in the next section.

Let $\boldsymbol{A}$ be a Hilbert algebra. An *ideal* of $\boldsymbol{A}$ is an ideal of the poset $\langle A, \leq \rangle$, where $\leq$ is the order defined by $\to$, that is, it is a nonempty updirected down-set. The set of ideals of a Hilbert algebra is not necessarily a closure system (i.e., a set closed under intersections of arbitrary families) even if we add the emptyset and, even with this addition, does not necessarily form a lattice. So we can not relay on a lattice to introduce a notion of prime ideal as we did in the case of deductive filters. Nevertheless there is a natural way to introduce such a notion.

Definition 3.1 An ideal I of a Hilbert algebra $\boldsymbol{A}$ is *prime* if it is proper and for every $a_0, \ldots, a_n \in A$, if $\langle a_0, \ldots, a_n \rangle \cap I \neq \varnothing$, then there is $i \leq n$ such that $a_i \in I$.

Note that this is the natural generalization of the notion of prime ideal for lattices. If L is a lattice, for every finite and nonempty $X \subseteq L$ let $\langle X \rangle$ be the filter generated by X; to say that I is an ideal with the property that for every finite and nonempty $X \subseteq L$ if $\langle X \rangle \cap I \neq \varnothing$ then $X \cap I \neq \varnothing$ is equivalent to saying that I is an ideal such that for every $a, b \in L$ if $a \wedge b \in I$, then $a \in I$ or $b \in I$.

The relation between prime deductive filters and prime ideals is as follows.

Proposition 3.2 *A subset F of a Hilbert algebra $\boldsymbol{A}$ is a prime deductive filter iff $I = A \setminus F$ is a prime ideal.*

Proof. Suppose F is a prime deductive filter of $\boldsymbol{A}$. It is obvious that $I = L \setminus F$ is a down-set. To show that it is updirected, let $a, b \in I$. If $(\uparrow a \cap \uparrow b) \cap I = \varnothing$, then $\uparrow a \cap \uparrow b \subseteq F$. Since F is prime, $\uparrow a \subseteq F$ or $\uparrow b \subseteq F$, and either $a \notin I$ or $b \notin I$. A contradiction. Hence $(\uparrow a \cap \uparrow b) \cap I \neq \varnothing$, and so I is an ideal. Finally, in order to show that I is prime suppose that $\langle a_0, \ldots, a_n \rangle \cap I \neq \varnothing$. If $a_0, \ldots a_n \notin I$, then $a_0, \ldots a_n \in F$. Therefore, $\langle a_0, \ldots, a_n \rangle \subseteq F$ and $F \cap I \neq \varnothing$. A contradiction. Hence, there is $i \leq n$ such that $a_i \in I$, and I is prime.

Conversely, suppose $I = L - F$ is a prime ideal. Since I is proper it is obvious that F satisfies condition (1) of Definition 2.7. To show that condition (2) is satisfied suppose $a, a \to b \in F$ and $b \notin F$. Then $b \in I \cap \langle a, a \to b \rangle$ since the later is the deductive filter generated by $\{a, a \to b\}$. Since I is prime, $a \in I$ or $a \to b \in I$, which is impossible. Therefore, $b \in F$. Finally, in order to show that F is prime suppose that $F_1 \cap F_2 \subseteq F$. If $F_1 \nsubseteq F$ and $F_2 \nsubseteq F$, then $F_1 \cap I \neq \varnothing$ and $F_2 \cap I \neq \varnothing$. So, there exist $a_1 \in F_1 \cap I$ and $a_2 \in F_2 \cap I$. Obviously $\uparrow a_1 \cap \uparrow a_2 \subseteq F_1 \cap F_2 \subseteq F$. On the other hand, since I is an ideal, $(\uparrow a_1 \cap \uparrow a_2) \cap I \neq \varnothing$. So, $F \cap I \neq \varnothing$. A contradiction. Hence either $F_1 \subseteq F$ or $F_2 \subseteq F$, and F is prime. □

Lemma 3.3 (Prime Deductive Filter Lemma, [5]) *Let $\boldsymbol{A}$ be a Hilbert algebra. If F is a deductive filter and I is a non-empty ideal of $\boldsymbol{A}$ such that $F \cap I = \varnothing$, then there exists a prime deductive filter G of $\boldsymbol{A}$ such that $F \subseteq G$ and $G \cap I = \varnothing$.*

In particular we have:

Corollary 3.4 *Let $\boldsymbol{A}$ be a Hilbert algebra, F a deductive filter of $\boldsymbol{A}$ and $a \in A - F$. Then there is a prime deductive filter G such that $a \notin G$ and $F \subseteq G$.*

Proof. If $a \in A - F$, then $\downarrow a$ is an ideal disjoint from F. By the Prime Deductive Filter Lemma we obtain the desired prime deductive filter. □

Corollary 3.5 *Every deductive filter F is the intersection of the prime deductive filters that contain F.*

The notion of Frink ideal for posets extends to the notion of Frink ideal for Hilbert algebras. Nevertheless we need a stronger notion here in order to define the optimal deductive filters of a Hilbert algebra.

Definition 3.6 Let $\boldsymbol{A}$ be a Hilbert algebra. A nonempty set $I \subseteq A$ is called a *strong Frink ideal* if it is a down-set and for every finite $X \subseteq I$ and every finite $Y \subseteq A$, if $X^{\mathrm{u}} \subseteq \langle Y \rangle$, then $\langle Y \rangle \cap I \neq \varnothing$.

Note that the last condition can equivalently be stated by saying that for every nonempty finite $X \subseteq I$ and every nonempty finite $Y \subseteq A$, if $X^{\mathrm{u}} \subseteq \langle Y \rangle$, then $\langle Y \rangle \cap I \neq \varnothing$, which is equivalent to saying that for every $a_0, \ldots, a_n \in I$ and every $b_0, \ldots, b_m \in A$,

$$\text{(1)} \qquad \text{if } \bigcap_{i \leq n} \uparrow a_i \subseteq \langle b_0, \ldots, b_m \rangle, \quad \text{then} \quad \langle b_0, \ldots, b_m \rangle \cap I \neq \varnothing.$$

Indeed, if $X = \varnothing$, then $X^{\mathrm{u}} = A$ and so if $X^{\mathrm{u}} \subseteq \langle Y \rangle$, obviously, $\langle Y \rangle \cap I \neq \varnothing$. Let us consider the case where $Y = \varnothing$ and assume that $X \subseteq I$ is nonempty, finite and $X^{\mathrm{u}} \subseteq \langle Y \rangle$. Since $\langle \varnothing \rangle = \{1\} = \langle \{1\} \rangle$, condition (1) above implies that $\langle \{1\} \rangle \cap I \neq \varnothing$, and hence we have $\langle Y \rangle \cap I \neq \varnothing$. In this case I must be A.

The set of all strong Frink ideals of $\boldsymbol{A}$ will be denoted by $\mathrm{sFId}(\boldsymbol{A})$. A strong Frink ideal I is *proper* if $I \neq A$.

In any lattice condition (1) is equivalent to

(2) $$\text{if } b_0 \wedge \cdots \wedge b_m \leq a_0 \vee \cdots \vee a_n, \quad \text{then} \quad b_0 \wedge \cdots \wedge b_m \in I,$$

if we take in (1) $\langle b_0, \ldots, b_m \rangle$ to refer to the filter generated by $b_0, \ldots, b_m$. Thus a strong Frink ideal in a lattice is just an ideal. Hence, condition (1) gives a natural generalization to Hilbert algebras of the concept of ideal for lattices.

Definition 3.7 A strong Frink ideal I of a Hilbert algebra $\boldsymbol{A}$ is *prime* if it is proper and for every finite set $X \subseteq A$ such that $\langle X \rangle \cap I \neq \varnothing$ it holds that $X \cap I \neq \varnothing$.

Lemma 3.8 *Let $\boldsymbol{A}$ be a Hilbert algebra.*

(1) *Every nonempty ideal is a strong Frink ideal.*

(2) *For every $a \in A, {\downarrow} a$ is a strong Frink ideal.*

Proof. (1) Suppose I is a nonempty ideal. Let $X \subseteq I$ be nonempty and finite and let $Y \subseteq A$ be finite and such that $X^{\mathrm{u}} \subseteq \langle Y \rangle$. Let $c \in I$ be such that for every $a \in X$ $a \leq c$. It exists because I is an ideal. Then, $c \in \langle Y \rangle$. Therefore $\langle Y \rangle \cap I \neq \varnothing$. Hence I is a strong Frink ideal.

(2) Because for every $a \in A$, ${\downarrow} a$ is a nonempty ideal. □

The relation between the Frink ideals of an implicative semilattice and the strong Frink ideals of their Hilbert algebra reduct is stated in the next proposition. This relation shows that the concept of strong Frink ideal is natural.

Proposition 3.9 *Let $L = \langle L, \to, \wedge, 1 \rangle$ be an implicative semilattice. A set $I \subseteq L$ is a prime Frink ideal of L if and only if it is a prime strong Frink ideal of the Hilbert algebra $\langle L, \to, 1 \rangle$.*

Proof. Let $I \subseteq L$ be a prime strong Frink ideal of $\langle L, \to, 1 \rangle$. It is immediate to see that I is a Frink ideal of L. Let us show that I is prime. Suppose that $a \wedge b \in I$. Then ${\uparrow}(a \wedge b) \subseteq \langle a, b \rangle$, because $a \to (b \to a \wedge b) = 1$. Therefore, since I is prime, $a \in I$ or $b \in I$. Conversely, if I is a prime Frink ideal of L, let $X \subseteq I$ and $Y \subseteq L$ be nonempty and finite with $X^{\mathrm{u}} \subseteq \langle Y \rangle$. Then $X^{\mathrm{u}} \subseteq {\uparrow} \bigwedge Y$. So, $\bigwedge Y \in I$. And because I is prime in L, $Y \cap I \neq \varnothing$. So I is prime in $\langle L, \to, 1 \rangle$. □

4 Optimal deductive filters

In [1, 2], the notion of optimal filter for distributive meet-semilattices is introduced and it is used to provide a topological duality for distributive meet-semilattices and for implicative semilattices. Recall that optimal filters are the filters which are complements of Frink ideals. In this section we introduce the analogous concept for Hilbert algebras using the concept of strong Frink ideal. We also discuss the notion of separating family of optimal deductive filters. This concept will be used in Section 6 to provide a proof of the existence of the free implicative semilattice extension of a Hilbert algebra.

Definition 4.1 A deductive filter F of a Hilbert algebra $\boldsymbol{A}$ is said to be *optimal* if $A - F$ is a strong Frink ideal. We denote the set of all optimal filters of $\boldsymbol{A}$ by $\mathrm{Opt}\boldsymbol{A}$.

Note that every optimal deductive filter is proper, because strong Frink ideals are nonempty.

Lemma 4.2 (Optimal Filter Lemma) *Let $\boldsymbol{A}$ be a Hilbert algebra. If F is a deductive filter and I is a strong Frink ideal of $\boldsymbol{A}$ with $F \cap I = \varnothing$, then there exists an optimal deductive filter G of $\boldsymbol{A}$ such that $F \subseteq G$ and $G \cap I = \varnothing$.*

Proof. Suppose that F is a deductive filter, I is a strong Frink ideal of $\boldsymbol{A}$ and $F \cap I = \varnothing$. Then F is proper. By Zorn's lemma we obtain a deductive filter G which is maximal in the set $\{F' \in \mathrm{Dfi}\boldsymbol{A} : F \subseteq F' \text{ and } F' \cap I = \varnothing\}$. We show that $A - G$ is a strong Frink ideal. First of all, since I is nonempty, G is proper, so $A - G$ is nonempty.

Also, from the fact that G is an up-set follows that $A - G$ is a down-set. Suppose now that $a_0, \dots, a_n \notin G$ and $\bigcap_{i\leq n} \uparrow a_i \subseteq \langle b_0, \dots, b_m\rangle$ for some $b_0, \dots, b_m \in A$. For every $i \leq m$ we have $\langle G \cup \{a_i\}\rangle \cap I \neq \varnothing$. Let for every $i \leq n$ $d_i \in \langle G \cup \{a_i\}\rangle \cap I$ and let $X_i \subseteq F$ be a finite set such that $d_i \in \langle X_i \cup \{a_i\}\rangle$. Consider the finite set $X = \bigcup_{i\leq n} X_i$. Then, for every $i \leq n$ $d_i \in \langle X \cup \{a_i\}\rangle = \langle X\rangle \vee \uparrow a_i$. Thus, $\uparrow d_i \subseteq \langle X\rangle \vee \uparrow a_i$ and $\bigcap_{i\leq n} \uparrow d_i \subseteq \bigcap_{i\leq n}(\langle X\rangle \vee \uparrow a_i)$. Using that (Prop. 2.9) the lattice of deductive filters is distributive we have $\bigcap_{i\leq n} \uparrow d_i \subseteq \langle X\rangle \vee \bigcap_{i\leq n} \uparrow a_i \subseteq \langle X\rangle \vee \langle b_0, \dots, b_m\rangle = \langle X \cup \{b_0, \dots, b_m\}\rangle$. Suppose now that $b_0, \dots, b_m \notin A - G$. Then $b_0, \dots, b_m \in G$. Since I is a strong Frink ideal, there is $c \in I \cap \langle X \cup \{b_0, \dots, b_m\}\rangle$. Therefore since $\langle X \cup \{b_0, \dots, b_m\}\rangle \subseteq G$, $G \cap I \neq \varnothing$, a contradiction. □

Corollary 4.3 *Every deductive filter F is the intersection of the optimal deductive filters that contain F.*

Proof. Let F be a deductive filter. If $a \notin F$, since $\downarrow a$ is a strong Frink ideal disjoint from F, the Optimal Filter Lemma provides an optimal deductive filter G such that $F \subseteq G$ and $a \notin G$. □

An immediate consequence is the following:

Corollary 4.4 *In every Hilbert algebra $\boldsymbol{A}$, for every $a, b \in A$,*

$$a \leq b \quad \textit{iff} \quad (\forall F \in \mathrm{Opt}\boldsymbol{A})(a \in F \to b \in F).$$

Lemma 4.5 *Let $\boldsymbol{A}$ be a Hilbert algebra. If F is a prime deductive filter* (*in particular, if it is a completely irreducible deductive filter*)*, then F is optimal.*

Proof. Suppose F is a prime deductive filter of $\boldsymbol{A}$. By Proposition 3.2, $I = A - F$ is a prime ideal, and therefore a strong Frink ideal. Hence F is optimal. □

For finite Hilbert algebras the converse of Lemma 4.5 holds.

Lemma 4.6 *Let $\boldsymbol{A}$ be a finite Hilbert algebra. Then a deductive filter of $\boldsymbol{A}$ is prime if and only if it is optimal.*

Proof. Suppose that $\boldsymbol{A}$ is a finite Hilbert algebra, and F is an optimal deductive filter of $\boldsymbol{A}$. Then $I = A - F$ is a prime strong Frink ideal of $\boldsymbol{A}$. Since A is finite, F is finite. Let us show that F is prime. Suppose F_1, F_2 are deductive filters such that $F_1 \cap F_2 \subseteq F$ but $F_1 \nsubseteq F$ and $F_2 \nsubseteq F$. Then let $a \in F_1 - F$ and $b \in F_2 - F$. It follows that $a, b \in I$. If $\uparrow a \cap \uparrow b \subseteq \langle F\rangle = F$, then $F \cap I \neq \varnothing$, which is not possible. Therefore, $\uparrow a \cap \uparrow b \nsubseteq \langle F\rangle$. Hence there is $c \in \uparrow a \cap \uparrow b$ such that $c \notin F$, which is impossible because $c \in F_1 \cap F_2 \subseteq F$. □

Proposition 4.7 *Let $\boldsymbol{A}$ be a Hilbert algebra. A set $I \subseteq A$ is a prime strong Frink ideal iff $A - I$ is an optimal deductive filter.*

Proof. Suppose that I is a prime strong Frink ideal. Since it is proper $A - I$ is nonempty. If we show that $A - I$ is a deductive filter we shall obtain that it is an optimal deductive filter. Since I is a down-set, $A - I$ is an up-set. So, being nonempty, $1 \in A - I$. Suppose now that $a, a \to b \in A - I$ and $b \in I$. Since $\uparrow b \subseteq \langle a, a \to b\rangle$ and I is prime $a \in I$ or $a \to b \in I$, a contradiction. Therefore, $b \in A - I$.

Suppose now that $A - I$ is an optimal deductive filter. Then I is a strong Frink ideal. We show that it is prime. Suppose $\bigcap_{i\leq n} \uparrow a_i \subseteq \langle b_0, \dots, b_m\rangle$ with $a_0, \dots, a_n \in I$ and $b_0, \dots, b_n \notin I$. It follows that $\langle b_0, \dots, b_n\rangle \subseteq A - I$. Since I is a strong Frink ideal, $\langle b_0, \dots, b_n\rangle \cap I \neq \varnothing$. Hence $(A - I) \cap I \neq \varnothing$, a contradiction. We conclude that I is prime. □

Corollary 4.8 *Let $\boldsymbol{A}$ be a Hilbert algebra and let $F \subseteq A$. Then F is an optimal deductive filter iff $A - F$ is a prime strong Frink ideal.*

4.1 Separating families of optimal deductive filters

Definition 4.9 Let $\boldsymbol{A}$ be a Hilbert algebra. A family $\mathcal{F}$ of optimal deductive filters of $\boldsymbol{A}$ is a *separating family* for $\boldsymbol{A}$ if for every deductive filter F of $\boldsymbol{A}$ and every $a \in A - F$ there is $G \in \mathcal{F}$ such that $F \subseteq G$ and $a \notin G$.

An example of a separating family different from the set of all optimal deductive filters is the set of all prime deductive filters. Another one, due to Proposition 2.10, is the set of all completely meet-irreducible filters.

It is well known that given an arbitrary poset $\langle X, \leq\rangle$ the operation $\Rightarrow$ on the set $\mathcal{P}^{\uparrow}(X)$ of the up-sets of X defined by

$$U \Rightarrow V := \{x \in X : \uparrow x \cap U \subseteq V\}$$

is the residual of $\cap$ in $\mathcal{P}^{\uparrow}(X)$, that is, for every $U, V, W \in \mathcal{P}^{\uparrow}(X)$,

$$U \cap V \subseteq W \iff U \subseteq V \Rightarrow W,$$

and that the algebra $\langle \mathcal{P}^{\uparrow}(X), \cap, \cup, \Rightarrow, X, \varnothing \rangle$ is a Heyting algebra. This implies that the algebra $\langle \mathcal{P}^{\uparrow}(X), \Rightarrow, X \rangle$ is a Hilbert algebra.

Let $\boldsymbol{A}$ be a Hilbert algebra and let $\mathcal{F}$ be a separating family for $\boldsymbol{A}$. Let $\varphi_{\mathcal{F}} : A \to \mathcal{P}(\mathcal{F})$ be the map defined by

$$\varphi_{\mathcal{F}}(a) = \{F \in \mathcal{F} : a \in F\},$$

for every $a \in A$. We extend this map to a map from $\mathcal{P}(A)$ to $\mathcal{P}(\mathcal{F})$ that we denote also by $\varphi_{\mathcal{F}}$ as follows:

$$\varphi_{\mathcal{F}}(X) = \{F \in \mathcal{F} : X \subseteq F\},$$

for every $X \subseteq A$. Note that $\varphi_{\mathcal{F}}(X) = \bigcap_{a \in X} \varphi_{\mathcal{F}}(a)$.

Let $\Rightarrow$ be the operation on $\mathcal{P}^{\uparrow}(\mathcal{F})$ associated with the poset $\langle \mathcal{F}, \subseteq \rangle$.

Theorem 4.10 *The map $\varphi_{\mathcal{F}} : A \to \mathcal{P}^{\uparrow}(\mathcal{F})$ is an embedding from the Hilbert algebra $\boldsymbol{A}$ to $\langle \mathcal{P}^{\uparrow}(\mathcal{F}), \Rightarrow, \mathcal{F} \rangle$.*

Proof. Since the family $\mathcal{F}$ is separating, it follows that if $a, b \in A$ are different, then $\varphi_{\mathcal{F}}(a) \neq \varphi_{\mathcal{F}}(b)$. We now show that $\varphi_{\mathcal{F}}(a \to b) = \varphi_{\mathcal{F}}(a) \Rightarrow \varphi_{\mathcal{F}}(b)$. Suppose that $P \in \varphi_{\mathcal{F}}(a \to b)$, i.e., $a \to b \in P$. If $P \subseteq Q$ and $Q \in \varphi_{\mathcal{F}}(a)$, then since $a \to b \in Q$, it follows that $b \in Q$, and hence $Q \in \varphi_{\mathcal{F}}(b)$. This shows that $\varphi_{\mathcal{F}}(a \to b) \subseteq \varphi_{\mathcal{F}}(a) \Rightarrow \varphi_{\mathcal{F}}(b)$. To prove the other inclusion, assume that $P \in \varphi_{\mathcal{F}}(a) \Rightarrow \varphi_{\mathcal{F}}(b)$ and $a \to b \notin P$. Then consider the filter $F = \langle (P \cup \{a\} \rangle$. It follows that $b \notin F$, for if $b \in F$ there are $c_0, \dots, c_n \in P$ such that $(c_n, \dots, c_0, a; b) = 1$. Thus, $(c_n, \dots, c_0, a; b) = 1 \in P$. Hence, $a \to b \in P$, which contradicts the assumption. Since $\mathcal{F}$ is a separating family, there exists $Q \in \mathcal{F}$ such that $F \subseteq Q$ and $b \notin Q$. Then, since $a \in P$ and $P \subseteq Q$, $b \in Q$ follows from the assumption that $P \in \varphi_{\mathcal{F}}(a) \Rightarrow \varphi_{\mathcal{F}}(b)$, and we obtain a contradiction. Therefore $a \to b \in P$. Finally it is clear that $\varphi_{\mathcal{F}}(1) = \mathcal{F}$. □

Corollary 4.11 *If $\boldsymbol{A}$ is a Hilbert algebra and $\mathcal{F}$ is a separating family for $\boldsymbol{A}$, then $\varphi_{\mathcal{F}}$ is an isomorphism between $\boldsymbol{A}$ and $\langle \varphi_{\mathcal{F}}[A], \Rightarrow, \mathcal{F} \rangle$.*

We proceed to show that the closure of $\varphi_{\mathcal{F}}[A]$ under finite intersections is closed under the operation $\Rightarrow$.

Lemma 4.12 *Let $\boldsymbol{A}$ be a Hilbert algebra and let $\mathcal{F}$ be a separating family for $\boldsymbol{A}$. For all $a_0, \dots, a_n, b \in A$,*

$$\varphi_{\mathcal{F}}(a_0) \cap \dots \cap \varphi_{\mathcal{F}}(a_n) \Rightarrow \varphi_{\mathcal{F}}(b) = \varphi_{\mathcal{F}}((a_n, \dots, a_0; b)).$$

Proof. Let $P \in \varphi_{\mathcal{F}}(a_0) \cap \dots \cap \varphi_{\mathcal{F}}(a_n) \Rightarrow \varphi_{\mathcal{F}}(b)$. Suppose that $(a_n, \dots, a_0; b) \notin P$. So $b \neq 1$. Consider the filter $G = \langle P \cup \{a_0, \dots, a_n\} \rangle$. Then $b \notin G$, for if $b \in G$ there are $c_0, \dots, c_n \in x$ such that $(c_n, \dots, c_0, a_n, \dots, a_0; b) = 1$. Thus, $(a_n, \dots, a_0; b) \in P$, which is absurd. Since $\mathcal{F}$ is a separating family, let $Q \in \mathcal{F}$ be such that $G \subseteq Q$ and $b \notin Q$. Then, since $a \in Q$ and $P \subseteq Q$, $b \in Q$ and we obtain a contradiction. Therefore $(a_n, \dots, a_0; b) \in P$. Suppose now that $P \in \varphi_{\mathcal{F}}((a_n, \dots, a_0; b))$ and $P \subseteq Q \in \mathcal{F}$ is such that $Q \in \varphi_{\mathcal{F}}(a_0) \cap \dots \cap \varphi_{\mathcal{F}}(a_n)$. It follows that $b \in Q$, so $Q \in \varphi_{\mathcal{F}}(b)$. Therefore, $P \in \varphi_{\mathcal{F}}(a_0) \cap \dots \cap \varphi_{\mathcal{F}}(a_n) \Rightarrow \varphi_{\mathcal{F}}(b)$. □

The lemma implies:

Proposition 4.13 *Let $\boldsymbol{A}$ be a Hilbert algebra and let $\mathcal{F}$ be a separating family for $\boldsymbol{A}$. The closure of $\varphi_{\mathcal{F}}[A]$ under finite intersections is closed under $\Rightarrow$.*

Proof. Let $a_0, \dots, a_n, b_0, \dots, b_m \in A$. Then

$$\bigcap_{i \leq n} \varphi_{\mathcal{F}}(a_i) \Rightarrow \bigcap_{j \leq m} \varphi_{\mathcal{F}}(b_j) = \bigcap_{j \leq m} \left[\bigcap_{i \leq n} \varphi_{\mathcal{F}}(a_i) \Rightarrow \varphi_{\mathcal{F}}(b_j) \right] = \bigcap_{j \leq m} \varphi_{\mathcal{F}}((a_n, \dots, a_0; b_j)).$$

Thus we obtain the desired conclusion. □

Definition 4.14 Let $\boldsymbol{A}$ be a Hilbert algebra and let $\mathcal{F}$ be a separating family for $\boldsymbol{A}$. We denote by $L_{\mathcal{F}}(\boldsymbol{A})$ the algebra with domain the closure of $\varphi_{\mathcal{F}}[A]$ under finite intersections and operations $\cap$ and $\Rightarrow$, restricted to this set.

Corollary 4.15 *Let $\boldsymbol{A}$ be a Hilbert algebra and let $\mathcal{F}$ be a separating family for $\boldsymbol{A}$. Then $L_{\mathcal{F}}(\boldsymbol{A})$ is an implicative semilattice, and therefore its meet-semilattice reduct is a distributive meet-semilattice.*

Lemma 4.16 *Let $\boldsymbol{A}$ be a Hilbert algebra and $\mathcal{F}$ a separating family for $\boldsymbol{A}$. If $X, X_0, \ldots, X_n$ are nonempty finite subsets of A, then*

$$\bigcap_{i\leq n}\langle X_i\rangle \subseteq \langle X\rangle \quad \textit{iff} \quad \varphi_{\mathcal{F}}(X) \subseteq \bigcup_{i\leq n}\varphi_{\mathcal{F}}(X_i).$$

Proof. Suppose that $\bigcap_{i\leq n}\langle X_i\rangle \subseteq \langle X\rangle$. Let $P\in\varphi_{\mathcal{F}}(X)$ and assume that $P\notin\bigcup_{i\leq n}\sigma(X_i)$. Then $\langle X\rangle\subseteq P$ and for every $i\leq n$ there is $a_i\in X_i - P$. Thus $\bigcap_{i\leq n}\uparrow a_i \subseteq \bigcap_{i\leq n}\langle X_i\rangle \subseteq \langle X\rangle$. Since, P is optimal, $A-P$ is a strong Frink ideal, and so $\langle X\rangle\cap(A-P)\neq\varnothing$, a contradiction because $\langle X\rangle\subseteq P$. Suppose now that $\varphi_{\mathcal{F}}(X)\subseteq\bigcup_{i\leq n}\varphi_{\mathcal{F}}(X_i)$. Assume that $a\in\bigcap_{i\leq n}\langle X_i\rangle$. Then for every $i\leq n$, $(X_i;a)=1$. Suppose that $a\notin\langle X\rangle$. Let $Q\in\mathcal{F}$ be such that $\langle X\rangle\subseteq Q$ and $a\notin Q$. Then $Q\in\varphi_{\mathcal{F}}(X)$ and therefore $Q\in\bigcup_{i\leq n}\varphi_{\mathcal{F}}(X_i)$. It follows that there is $i\leq n$ such that $X_i\subseteq Q$. Then, since $1\in Q$, $(X_i;a)\in Q$. It follows that $a\in Q$, a contradiction. Therefore $a\in\langle X\rangle$. □

Corollary 4.17 *Let $\boldsymbol{A}$ be a Hilbert algebra. For any separating families $\mathcal{F}$ and $\mathcal{F}'$ for $\boldsymbol{A}$ and all nonempty finite subsets $X, X_0, \ldots, X_n$ of A,*

$$\varphi_{\mathcal{F}}(X) \subseteq \bigcup_{i\leq n}\varphi_{\mathcal{F}}(X_i) \quad \textit{iff} \quad \varphi_{\mathcal{F}'}(X) \subseteq \bigcup_{i\leq n}\varphi_{\mathcal{F}'}(X_i).$$

5 Morphisms between Hilbert algebras

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Hilbert algebras. We shall consider different concepts of structure preserving maps from $\boldsymbol{A}$ to $\boldsymbol{B}$. One is monotone map. Another is semi-homomorphism. A map $h: A\to B$ is a *semi-homomorphism* [5] from $\boldsymbol{A}$ to $\boldsymbol{B}$ if $h(1)=1$ and for every $a,b\in A$, $h(a\to b)\leq h(a)\to h(b)$. A third one is (algebraic) homomorphism. A map $h: A\to B$ is a *homomorphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$ if $h(1)=1$ and for every $a,b\in A$, $h(a\to b)=h(a)\to h(b)$. We shall consider also the maps that in addition have the sup-property we introduce below in Definition 5.3.

Next proposition explains the interest of the notion of semi-homomorphism.

Proposition 5.1 *Let L, L' be two implicative semilattices and let $h: L\to L'$. Then h is a $(\wedge,1)$-homomorphism if and only if it is a semi-homomorphism from the Hilbert algebra reduct of L to the Hilbert algebra reduct of L'.*

Proof. Suppose h is a $(\wedge,1)$-homomorphism from L to L'. Then $h(1)=1$ and h is monotone. Moreover for $a,b\in L$, $h(a\to b)\wedge h(a) = h((a\to b)\wedge a)\leq h(b)$, because $(a\to b)\wedge a\leq b$. Therefore, $h(a\to b)\leq h(a)\to h(b)$. So, h is a semi-homomorphism. Suppose now that h is a semi-homomorphism from the Hilbert algebra reduct of L to the Hilbert algebra reduct of L'. Thus $h(1)=1$. Moreover, h is monotone. Let $a,b\in L$. If $a\leq b$ then $a\to b=1$, so $1\leq h(a)\to h(b)$. Hence $h(a)\leq h(b)$. This implies that if $a,b\in L$, then $h(a\wedge b)\leq h(a)\wedge h(b)$. To prove the other inequality note that $a\to(b\to(a\wedge b))=1$. So $1\leq h(a)\to(h(b\to(a\wedge b))$. Hence $h(a)\leq h(b\to(a\wedge b))\leq h(b)\to h(a\wedge b)$. It follows that $h(a)\wedge h(b)\leq h(a\wedge b)$. □

Proposition 5.2 *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Hilbert algebras and $h: A\to B$.*

(1) *h is monotone iff $h^{-1}[X]$ is an up-set of $\boldsymbol{A}$ for every up-set $X\subseteq B$,*

(2) *h is a semi-homomorphism iff $h^{-1}[G]$ is a deductive filter of $\boldsymbol{A}$ for every deductive filter G of $\boldsymbol{B}$.*

Proof. The proof of (1) is straightforward. A proof of (2) can be found in [5]. □

Definition 5.3 Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Hilbert algebras. A map $h: A\to B$ has the *sup-property* if for every $c_1,\ldots,c_n,b_0,\ldots,b_m\in A$,

$$\text{(sup)} \qquad \text{if } \bigcap_{i\leq n}\uparrow c_i \subseteq \langle b_0,\ldots,b_m\rangle, \quad \text{then} \quad \bigcap_{i\leq n}\uparrow h(c_i) \subseteq \langle h(b_0),\ldots,h(b_m)\rangle.$$

A *sup-semi-homomorphism* is a semi-homomorphism with the sup-property. Similarly a *sup-homomorphism* is a homomorphism with the sup-property, and a *sup-embedding* is an embedding that is also a sup-homomorphism.

Next proposition provides a characterization of sup-semi-homomorphisms and sup-homomorphisms.

Proposition 5.4 *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Hilbert algebras. Let $h : \boldsymbol{A} \to \boldsymbol{B}$ be a monotone map. The following statements are equivalent*:

(1) *h has the sup-property,*

(2) *for every prime strong Frink ideal I of $\boldsymbol{B}$, $h^{-1}[I] = \varnothing$ or $h^{-1}[I]$ is a prime strong Frink ideal of $\boldsymbol{A}$.*

(3) *for every optimal deductive filter P of $\boldsymbol{B}$, $h^{-1}[P] = A$ or $h^{-1}[P]$ is an optimal deductive filter of $\boldsymbol{A}$.*

Proof. Let $h : \boldsymbol{A} \to \boldsymbol{B}$ be a monotone map.

(1) implies (2). Suppose that h has the sup-property. Let I be a prime strong Frink ideal of $\boldsymbol{B}$. Suppose that $h^{-1}[I] \neq \varnothing$. Let us show that $h^{-1}[I]$ is a prime strong Frink ideal of $\boldsymbol{A}$. Obviously, $h^{-1}[I]$ is a nonempty down-set of $\boldsymbol{A}$, because h is monotone and I is a down-set. Suppose $c_1, \ldots, c_n \in h^{-1}[I]$, $b_0, \ldots, b_m \in A$ and $\bigcap_{i \le n} \uparrow c_i \subseteq \langle b_0, \ldots, b_m \rangle$. We have to show that $h^{-1}[I] \cap \{b_0, \ldots, b_m\} \neq \varnothing$. Since h has the sup-property,

$$\bigcap_{i \le n} \uparrow h(c_i) \subseteq \langle h(b_0), \ldots, h(b_m) \rangle.$$

Moreover, $h(c_1), \ldots, h(c_n) \in I$, because $c_1, \ldots, c_n \in h^{-1}[I]$. Thus, $I \cap \{h(b_0), \ldots, h(b_m)\} \neq \varnothing$, because I is a prime strong Frink ideal. It follows that $h^{-1}[I] \cap \{b_0, \ldots, b_m\} \neq \varnothing$. Therefore, $h^{-1}[I]$ is a prime strong Frink ideal.

(2) implies (3). Assume (2). Let P be an optimal filter of $\boldsymbol{B}$. Then $B - P$ is a prime strong Frink ideal. So, by (2) $h^{-1}[B - P] = \varnothing$ or $h^{-1}[B - P]$ is a prime strong Frink ideal of $\boldsymbol{A}$. Therefore, $A - h^{-1}[B - P]$ is an optimal deductive filter or $A - h^{-1}[B - P] = A$. Since $h^{-1}[P] = A - h^{-1}[B - P]$, $h^{-1}[P]$ is an optimal deductive filter or $h^{-1}[P] = A$.

(3) implies (1). Assume (3). We show that h has the sup-property. Let $c_1, \ldots, c_n, b_0, \ldots, b_m \in A$ be such that $\bigcap_{i \le n} \uparrow c_i \subseteq \langle b_0, \ldots, b_m \rangle$. Suppose that $\bigcap_{i \le n} \uparrow h(c_i) \nsubseteq \langle h(b_0), \ldots, h(b_m) \rangle$. Let $c \in \bigcap_{i \le n} \uparrow h(c_i)$ be such that

$$c \notin \langle h(b_0), \ldots, h(b_m) \rangle.$$

Then there is an optimal filter P with $\langle h(b_0), \ldots, h(b_m) \rangle \subseteq P$ and $c \notin P$. So, for every $i \le n$ $h(c_i) \notin P$. Therefore, $c_i \in A - h^{-1}[P]$, for every $i \le n$; hence $h^{-1}\boldsymbol{A} \neq A$, and $b_j \in h^{-1}[P]$, for every $j \le m$. The assumption implies that $h^{-1}[P]$ is optimal. Therefore, $A - h^{-1}[P]$ is a prime strong Frink ideal. Hence $\{b_0, \ldots, b_m\} \cap (A - h^{-1}[P]) \neq \varnothing$, a contradiction. Thus, h has the sup-property. □

Proposition 5.5 *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Hilbert algebras both with a bottom element. Let $h : \boldsymbol{A} \to \boldsymbol{B}$ be a monotone map. The following statements are equivalent*:

(1) *h has the sup-property and $h(\bot^{\boldsymbol{A}}) = \bot^{\boldsymbol{B}}$,*

(2) *for every prime strong Frink ideal I of $\boldsymbol{B}$, $h^{-1}[I]$ is a prime strong Frink ideal of $\boldsymbol{A}$.*

(3) *for every optimal deductive filter P of $\boldsymbol{B}$, $h^{-1}[P]$ is an optimal deductive filter of $\boldsymbol{A}$.*

Proof. Let $h : \boldsymbol{A} \to \boldsymbol{B}$ be a monotone map.

(1) implies (2). Suppose that h has the sup-property and $h(\bot^{\boldsymbol{A}}) = \bot^{\boldsymbol{B}}$. Let I be a prime strong Frink ideal of $\boldsymbol{B}$. We know from the previous proposition that $h^{-1}[I]$ is a prime strong Frink ideal of $\boldsymbol{A}$ or $h^{-1}[I] = \varnothing$. But since $h(\bot^{\boldsymbol{A}}) = \bot^{\boldsymbol{B}} \in I$, we have $\bot^{\boldsymbol{A}} \in h^{-1}[I]$. So, $h^{-1}[I] \neq \varnothing$ and therefore $h^{-1}[I]$ is a prime strong Frink ideal.

(2) implies (3). Assume (2). Let P be an optimal filter of $\boldsymbol{B}$. Then $B - P$ is a prime strong Frink ideal. So, by (2) $h^{-1}[B - P]$ is a prime strong Frink ideal of $\boldsymbol{A}$. Therefore, $h^{-1}[P] = A - h^{-1}[B - P]$ is an optimal deductive filter.

(3) implies (1). From the previous proposition follows that h has the sup-property. Now we show that $h(\bot^{\boldsymbol{A}}) = \bot^{\boldsymbol{B}}$. Suppose that $\bot^{B} < h(\bot^{\boldsymbol{A}})$. Let P be an optimal deductive filter of $\boldsymbol{B}$ such that $h(\bot^{\boldsymbol{A}}) \in P$ and $\bot^{\boldsymbol{B}} \notin P$. Then since $\bot^{\boldsymbol{A}} \in h^{-1}[P]$, $h^{-1}[P] = A$ and it is not proper, a contradiction. Thus $h(\bot^{\boldsymbol{A}}) = \bot^{\boldsymbol{B}}$. □

For one-to-one homomorphisms satisfying condition (sup) on the Definition 5.3 the reverse of the implication in the condition (sup) also holds.

Lemma 5.6 *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Hilbert algebras. If $h : \boldsymbol{A} \to \boldsymbol{B}$ is a one-to-one sup-homomorphism, then for every $c_1, \dots, c_n, b_0, \dots, b_m \in A$,*

$$\text{if} \quad \bigcap_{i \leq n} \uparrow h(c_i) \subseteq \langle h(b_0), \dots, h(b_m) \rangle, \quad \text{then} \quad \bigcap_{i \leq n} \uparrow c_i \subseteq \langle b_0, \dots, b_m \rangle$$

and also if $a \in A$ is such that $h(a)$ is a bottom element of $\boldsymbol{B}$, then a is a bottom element of $\boldsymbol{A}$.

Proof. Suppose that $\bigcap_{i \leq n} \uparrow h(c_i) \subseteq \langle h(b_0), \dots, h(b_m) \rangle$. Let $d \in \bigcap_{i \leq n} \uparrow c_i$. Then $h(d) \in \bigcap_{i \leq n} \uparrow h(c_i)$ and therefore $h(d) \in \langle h(b_0), \dots, h(b_m) \rangle$). Hence,

$$(h(b_0), \dots, h(b_m); h(d)) = 1$$

and so $h((b_0, \dots, b_m; d)) = 1$. Since h is one-to-one, $(b_0, \dots, b_m; d) = 1$. Thus, $d \in \langle b_0, \dots, b_m \rangle$.

Suppose now that $a \in A$ is such that $h(a)$ is a bottom element of $\boldsymbol{B}$. Let $b \in A$. Then $\uparrow h(b) \subseteq \uparrow h(a)$. Thus from what we already proved follows that $\uparrow b \subseteq \uparrow a$ and hence $a \leq b$. □

Examples of sup-homomorphisms are provided by the maps $\varphi_{\mathcal{F}}$ associated with separating families $\mathcal{F}$ of optimal deductive filters. Let $\mathcal{F}$ be a separating family of optimal deductive filters for a Hilbert algebra $\boldsymbol{A}$. Recall that $L_{\mathcal{F}}(\boldsymbol{A})$ is the implicative semilattice with domain the closure of $\varphi_{\mathcal{F}}[A]$ under finite intersections together with the operations $\cap$ and $\Rightarrow$.

Proposition 5.7 *The map $\varphi_{\mathcal{F}} : \boldsymbol{A} \to L_{\mathcal{F}}(\boldsymbol{A})$ is a one-to-one sup-homomorphism and if $\boldsymbol{A}$ has a bottom element $\perp$ then $\varphi_{\mathcal{F}}(\perp)$ is a bottom element of $L_{\mathcal{F}}(\boldsymbol{A})$.*

Proof. We know that $\varphi_{\mathcal{F}}$ is a homomorphism. Let us prove that it is a sup-homomorphism. Suppose $\bigcap_{i \leq} \uparrow a_i \subseteq \langle b_0, \dots, b_m \rangle$ with $a_1, \dots, a_n, b_0, \dots, b_m \in A$. If $d \in \bigcap_{i \leq n} \uparrow \varphi_{\mathcal{F}}(a_i)$, let $c_0, \dots, c_k \in A$ be such that $d = \varphi_{\mathcal{F}}(c_0) \cap \dots \cap \varphi_{\mathcal{F}}(c_k)$; then for every $j \leq k$, $\varphi_{\mathcal{F}}(c_j) \in \bigcap_{i \leq n} \uparrow \varphi_{\mathcal{F}}(a_i)$. Since $\varphi_{\mathcal{F}}$ is one-to-one this implies that $c_j \in \bigcap_{i \leq} \uparrow a_i$, and therefore $c_j \in \langle b_0, \dots, b_m \rangle$, for every $j \leq k$. Now, if $c_j \in \langle b_0, \dots, b_m \rangle$, then $(b_0, \dots, b_m; c_j) = 1$ and it follows that

$$(\varphi_{\mathcal{F}}(b_0), \dots, \varphi_{\mathcal{F}}(b_m); \varphi_{\mathcal{F}}(c_j)) = 1.$$

Hence, $\varphi_{\mathcal{F}}(c_j) \in \langle \varphi_{\mathcal{F}}(b_0), \dots, \varphi_{\mathcal{F}}(b_m) \rangle$ for every $j \leq k$. So, $d \in \langle \varphi_{\mathcal{F}}(b_0), \dots, \varphi_{\mathcal{F}}(b_m) \rangle$. □

Lemma 5.8 *The composition of maps with the sup-property has the sup-property.*

Proof. Let $h : \boldsymbol{A} \to \boldsymbol{B}$, $f : \boldsymbol{B} \to \boldsymbol{C}$ be maps of Hilbert algebras with the sup-property. We prove that $f \circ h : \boldsymbol{A} \to \boldsymbol{C}$ has the sup-property. Suppose that $c_1, \dots, c_n, b_0, \dots, b_m \in A$ are such that $\bigcap_{i \leq} \uparrow c_i \subseteq \langle b_0, \dots, b_m \rangle$. Then

$$\bigcap_{i \leq n} \uparrow h(c_i) \subseteq \langle h(b_0), \dots, h(b_m) \rangle,$$

because h has the sup-property. Now since f has the sup-property

$$\bigcap_{i \leq n} \uparrow f(h(c_i)) \subseteq \langle f(h(b_0)), \dots, f(h(b_m)) \rangle.$$

This shows that $f \circ h$ has the sup-property. □

The sup-property for a morphism says that it preserves what we might call virtual finite suprema. The property that corresponds to preserving virtual arbitrary suprema is the following. Let $\boldsymbol{A}, \boldsymbol{B}$ be Hilbert algebras. A map $h : A \to B$ has the *infinite sup-property* if for every set $\{c_k : k \in K\} \subseteq A$ and every $X \subseteq A$,

$$\text{(inf-sup)} \qquad \text{if} \ \bigcap_{k \in K} \uparrow c_k \subseteq \langle X \rangle, \ \text{then} \ \bigcap_{k \in K} \uparrow h(c_k) \subseteq \langle h[X] \rangle.$$

Let $\boldsymbol{A}$ be a Hilbert algebra. A deductive filter F of $\boldsymbol{A}$ is *completely prime* if it is a completely prime element of the lattice $\mathrm{Dfi}\boldsymbol{A}$.

Proposition 5.9 *Let $\boldsymbol{A}, \boldsymbol{B}$ be Hilbert algebras and let $h : A \to B$ a map with the infinite sup-property. Then for every completely prime deductive filter F of $\boldsymbol{B}$, $h^{-1}[F]$ is a completely prime deductive filter of $\boldsymbol{A}$.*

Proof. Suppose that F is a completely prime deductive filter of $\boldsymbol{B}$. Let $\{G_k : k \in K\}$ be a family of deductive filters of $\boldsymbol{A}$ such that $\bigcap_{k\in K} G_k \subseteq h^{-1}[F]$. Suppose that for every $k \in K$, $G_k \nsubseteq h^{-1}[F]$. So let for every $k \in K$, $a_k \in G_k - h^{-1}[F]$. Then $\bigcap_{k\in K} \uparrow a_k \subseteq \bigcap_{k\in K} G_k \subseteq h^{-1}[F] = \langle h^{-1}[F]\rangle$. Therefore by the infinite sup-property, $\bigcap_{k\in K} \uparrow h(a_k) \subseteq \langle h[h^{-1}[F]]\rangle$. Since $\langle h[h^{-1}[F]]\rangle \subseteq F$ we obtain that $\bigcap_{k\in K} \uparrow h(a_k) \subseteq F$. So, since F is completely prime, let $k \in K$ be such that $\uparrow h(a_k) \subseteq F$. This implies that $c_k \in h^{-1}[F]$, a contradiction. □

6 The free implicative semilattice extension of a Hilbert algebra

As we mentioned in the introduction, Porta [14] shows that every Hilbert algebra has a free implicative semilattice extension. In this section we provide a useful simple characterization of the free implicative semilattice extension of a Hilbert algebra as well as a different proof of its existence.

The free implicative semilattice extension of a Hilbert algebra $\boldsymbol{A}$ can be obtained in the following way. We consider the poset $\mathrm{Opt}\boldsymbol{A}$ of the optimal deductive filters of $\boldsymbol{A}$ ordered by inclusion and the algebra $\mathcal{P}^{\uparrow}(\mathrm{Opt}\boldsymbol{A})$ of its up-sets, alternatively any separating family can do the job of $\mathrm{Opt}\boldsymbol{A}$. The map that sends any element a of $\boldsymbol{A}$ to the set of optimal filters that contain a is a Hilbert algebra embedding when we consider the residual operation $\Rightarrow$ of the intersection operation $\cap$ in $\mathcal{P}^{\uparrow}(\mathrm{Opt}\boldsymbol{A})$. The closure under finite intersections of the image of $\boldsymbol{A}$ by the embedding turns out to be closed under the operation $\Rightarrow$ in $\mathcal{P}^{\uparrow}(\mathrm{Opt}\boldsymbol{A})$. The subalgebra that results is the free implicative semilattice extension of $\boldsymbol{A}$.

Let $\boldsymbol{A}$ be a Hilbert algebra. A pair $\langle L, e\rangle$, where L is an implicative semilattice and e a one-to-one homomorphism from $\boldsymbol{A}$ to $\langle L, \to, 1\rangle$ is an *implicative semilattice envelope* of $\boldsymbol{A}$ if for every $a \in L$ there is a finite $X \subseteq A$ such that $a = \bigwedge e[X]$. Note that since $\bigwedge e[\varnothing] = 1 = e(1) = \bigwedge e[\{1\}]$, we can say in the definition that for every $a \in L$ there is a finite and nonempty $X \subseteq A$ such that $a = \bigwedge e[X]$.

Proposition 6.1 *Let $\boldsymbol{A}$ be a Hilbert algebra. If $\langle L, e\rangle$ is an implicative semilattice envelope, then e is a sup-homomorphism from $\boldsymbol{A}$ to $\langle L, \to, 1\rangle$. Moreover, for every finite set $X \subseteq A$, $\langle X\rangle = A$ if and only if $\bigwedge e[X]$ is a bottom element of L. In particular, if $\boldsymbol{A}$ has a bottom element $\bot$, then $e(\bot)$ is a bottom element of L.*

Proof. Suppose that c_i with $i \le n$ and $b_0, \dots, b_m$ are elements of A such that $\bigcap_{i\le n} \uparrow c_i \subseteq \langle b_0, \dots, b_m\rangle$. Assume that $x \in \bigcap_{i\le n} \uparrow e(c_i)$. Let $a_0, \dots, a_k \in A$ be such that $x = e(a_0) \wedge \dots \wedge e(a_k)$. Then for every $i \le n$ and every $j \le k$, $e(c_i) \le e(a_j)$. So, $e(c_i) \to e(a_j) = 1$. Since e is a one-to-one homomorphism, it follows that $c_i \to a_j = 1$, and so $c_i \le a_j$, for every $i \le n$ and every $j \le k$. Thus, for every $j \le k$, $a_j \in \langle b_0, \dots, b_m\rangle$. This means that $(b_0, \dots, b_m; a_j) = 1$. So, $e((b_0, \dots, b_m; a_j) = e(1)$. Since e is a homomorphism, $(e(b_0), \dots, e(b_m); e(a_j)) = e(1)$. This implies that in L, $e(b_0) \wedge \dots \wedge e(b_m) \le e(a_0) \wedge \dots \wedge e(a_k) = x$. So, it follows that $x \in \langle e(a_0), \dots, e(a_k)\rangle$.

Let $X \subseteq A$ be finite. Suppose that $\langle X\rangle = A$. If X is empty, then $\langle X\rangle = \{1\}$ and so $\boldsymbol{A}$ is the one element Heyting algebra. In this case L is isomorphic to $\boldsymbol{A}$. Suppose that X is nonempty and let $X = \{a_0, \dots, a_n\}$. Then for every $a \in A$, $a \in \langle a_0, \dots, a_n\rangle$ and therefore $(a_0, \dots, a_n; a) = 1$. Hence $(e(a_0), \dots, e(a_n); e(a)) = 1$. Thus, $e(a_0) \wedge \dots \wedge e(a_n) \le e(a)$. Since every element of L is of the form $\bigwedge e[Y]$ for some finite and nonempty $Y \subseteq A$, it follows that $\bigwedge e[X] = e(a_0) \wedge \dots \wedge e(a_n)$ is a bottom element of L. Suppose now that $\bigwedge e[X]$ is a bottom element of L. If X is empty, then $\bigwedge e[X] = 1$ and so L is a one element Hilbert algebra. Then $\boldsymbol{A}$ is also a one element Hilbert algebra and so L is isomorphic to $\boldsymbol{A}$ and $\langle X\rangle = A$. Suppose now that X is nonempty and let $X = \{a_0, \dots, a_n\}$. We show that $\langle a_0, \dots, a_n\rangle = A$. This is equivalent to showing that for every $a \in A$, $(a_0, \dots, a_n; a) = 1$. Let $a \in A$. Then since $e(a_0) \wedge \dots \wedge e(a_n) = \bigwedge e[X] \le e(a)$, $(e(a_0), \dots, e(a_n); e(a)) = e(1)$. So, $(a_0, \dots, a_n; a) = 1$ as desired. □

Corollary 6.2 *Let $\boldsymbol{A}$ be a Hilbert algebra. If $\langle L, e\rangle$ is an implicative semilattice envelope and has a bottom element, then there is a finite set $X \subseteq A$ such that $\langle X\rangle = A$.*

A pair $\langle L, e\rangle$, where L is an implicative semilattice and e a one-to-one homomorphism from $\boldsymbol{A}$ to $\langle L, \to, 1\rangle$ is a *free implicative semilattice extension* of $\boldsymbol{A}$ if the following universal property holds: for every implicative semilattice L' and every homomorphism $g : \boldsymbol{A} \to \langle L', \to'\rangle$, there is a unique homomorphism $\overline{g} : L \to L'$ such that $g = \overline{g} \circ e$, moreover, if g is one-to-one, then $\overline{g}$ is one-to-one. By a standard categorical argument, if a Hilbert algebra has a free implicative semilattice extension, it is unique up to isomorphism. That is:

Proposition 6.3 *Let $\boldsymbol{A}$ be a Hilbert algebra and let $\langle L, e\rangle$ and $\langle L', e'\rangle$ be free implicative semilattice extensions of $\boldsymbol{A}$. Then there is an isomorphism $h : L \to L'$ such that $e' = h \circ e$.*

We shall show that for every Hilbert algebra $\boldsymbol{A}$, $\langle L, e\rangle$ is an implicative semilattice envelope of $\boldsymbol{A}$ if and only if it is an implicative semilattice free extension of $\boldsymbol{A}$.

Lemma 6.4 *Let $\boldsymbol{A}$ be a Hilbert algebra and $L = \langle L, \to, \wedge, 1\rangle$ an implicative semilattice. Let $e : A \to L$ be an embedding of $\boldsymbol{A}$ into $\langle L, \to, 1\rangle$. Then the subalgebra of the implicative semilattice generated by $e[A]$ is the closure of $e[A]$ under finite meets.*

Proof. Let S be the closure of $e[A]$ under finite meets. We show that S is already closed under $\to$. Let $a_0, \ldots, a_n, b_0, \ldots b_m \in A$. Note that

$$\begin{aligned} e(a_0) \wedge \cdots \wedge e(a_n) \to e(b_0) \wedge \cdots \wedge e(b_n) &= (e(a_0), \ldots, e(a_n); e(b_0) \wedge \cdots \wedge e(b_n)) \\ &= (e(a_0), \ldots, e(a_n); e(b_0)) \wedge \cdots \wedge (e(a_0), \ldots, e(a_n); e(b_m)) \\ &= e((a_0, \ldots, a_n; b_0)) \wedge \cdots \wedge e((a_0, \ldots, a_n; b_0)). \end{aligned}$$

Thus, $e(a_0) \wedge \cdots \wedge e(a_n) \to e(b_0) \wedge \cdots \wedge e(b_n) \in S$. □

Proposition 6.5 *Let $\boldsymbol{A}$ be a Hilbert algebra. If $\langle L, e\rangle$ is a free implicative semilattice extension of $\boldsymbol{A}$, then L is the closure of $e[A]$ under finite meets (in L), so $\langle L, e\rangle$ is an implicative semilattice envelope of $\boldsymbol{A}$.*

Proof. Let $S(e[A])$ be the implicative semilattice obtained as the subalgebra of L generated by $e[A]$. The map $e : A \to S(e[A])$ is a Hilbert algebra embedding. By the universal property defining the implicative semilattice free extension, let $\overline{e} : L \to S(e[A])$ be the unique homomorphism such that $e = \overline{e} \circ e$. We show that $L = S(e[A])$. Let $a \in L$. Then $\overline{e}(a) \in S(e[A])$, so let $b_0, \ldots, b_m \in A$ be such that $\overline{e}(a) = e(b_0) \wedge \cdots \wedge e(b_m)$. Now we prove that $\overline{e}(e(b_0) \wedge \cdots \wedge e(b_m)) = e(b_0) \wedge \cdots \wedge e(b_m)$. This is obvious because $\overline{e}(e(b_0) \wedge \cdots \wedge e(b_m)) = \overline{e}(e(b_0)) \wedge \cdots \wedge \overline{e}(e(b_m))) = e(b_0) \wedge \cdots \wedge e(b_m)$. Thus, $\overline{e}(a) = \overline{e}(e(b_0) \wedge \cdots \wedge e(b_m))$. Since $\overline{e}$ is one-to-one, $a = e(b_0) \wedge \cdots \wedge e(b_m)$. Thus $a \in S(e[A])$. □

Proposition 6.6 *If $\langle L, e\rangle$ is an implicative semilattice envelope of a Hilbert algebra $\boldsymbol{A}$, then for every implicative semilattice L' and every semi-homomorphism $g : \boldsymbol{A} \to \langle L', \to'\rangle$, there is a unique $(\wedge, 1)$-homomorphism $\overline{g} : L \to L'$ such that $g = \overline{g} \circ e$. Moreover,*

(1) *if g is a homomorphism, then $\overline{g}$ is a homomorphism,*

(2) *if g is a one-to-one homomorphism, then $\overline{g}$ is a one-to-one homomorphism,*

(3) *if g is onto, then $\overline{g}$ is onto.*

Proof. Let $L' = \langle L', \to', \wedge', 1\rangle$ be an implicative semilattice and let $g : \boldsymbol{A} \to \langle L', \to'\rangle$ a semi-homomorphism. In order to define $\overline{g}$ we first show that for all $a_0, \ldots, a_n, b_0, \ldots, b_m \in A$, if $e(a_0) \wedge \cdots \wedge e(a_n) = e(b_0) \wedge \cdots \wedge e(b_m)$, then $g(a_0) \wedge \cdots \wedge g(a_n) = g(b_0) \wedge \cdots \wedge g(b_m)$. Suppose that $a_0, \ldots, a_n, b_0, \ldots, b_m \in A$ are such that $e(a_0) \wedge \cdots \wedge e(a_n) = e(b_0) \wedge \cdots \wedge e(b_m)$. We prove that $g(a_0) \wedge \cdots \wedge g(a_n) \leq g(b_0) \wedge \cdots \wedge g(b_m)$, and a similar argument gives the other inequality. By assumption, $e(a_0) \wedge \cdots \wedge e(a_n) \leq e(b_i)$, for every $i \leq m$. Hence $(e(a_0), \ldots, e(a_n); e(b_i)) = 1 = e(1)$. So, $e((a_0, \ldots, a_n; b_i)) = e(1)$, and since e is one-to-one, $(a_0, \ldots, a_n; b_i) = 1$. Therefore, $g((a_0, \ldots, a_n; b_i)) = g(1)$, so since g is a semi-homomorphism, $1 = g(1) = g((a_0, \ldots, a_n; b_i)) \leq (g(a_0), \ldots, g(a_n); g(b_i))$. Therefore, $(g(a_0), \ldots, g(a_n); g(b_i)) = 1$. It follows that $g(a_0) \wedge \cdots \wedge g(a_n) \leq g(b_i)$, for every $i \leq m$. Hence, $g(a_0) \wedge \cdots \wedge g(a_n) \leq g(b_0) \wedge \cdots \wedge g(b_m)$.

We define $\overline{g} : L \to L'$ by

$$\overline{g}(a) = g(a_0) \wedge \cdots \wedge g(a_n)$$

for every $a \in L$, where $a_0, \ldots, a_n \in A$ are such that $a = e(a_0) \wedge \cdots \wedge e(a_n)$. The considerations above show that this definition is sound. Moreover it is obvious from the definition that $g = \overline{g} \circ e$.

Now we prove that $\overline{g}$ is a $(\wedge, 1)$-homomorphism. Let $a, b \in L$. Suppose that $a = e(a_0) \wedge \cdots \wedge e(a_n)$ with $a_0, \ldots, a_n \in A$ and $b = e(b_0) \wedge \cdots \wedge e(b_m)$ with $b_0, \ldots, b_m \in A$. So $a \wedge b = e(a_0) \wedge \cdots \wedge e(a_n) \wedge e(b_0) \wedge \cdots \wedge e(b_m)$. Hence, by definition of $\overline{g}$, $\overline{g}(a \wedge b) = g(a_0) \wedge \cdots \wedge g(a_n) \wedge g(b_0) \wedge \cdots \wedge g(b_m)$. Therefore, $\overline{g}(a \wedge b) = \overline{g}(a) \wedge \overline{g}(b)$. Now, since $1 = e(1)$, $\overline{g}(1) = g(1) = 1$. If g is a homomorphism we also have that $\overline{g}(a \to b) = \overline{g}(a) \to' \overline{g}(b)$. Indeed, an easy computation shows that

$$a \to b = e((a_0, \ldots, a_n; b_0)) \wedge \cdots \wedge e((a_0, \ldots, a_n; b_m)).$$

Then

$$\overline{g}(a \to b) = g((a_0, \ldots, a_n; b_0)) \wedge \cdots \wedge g((a_0, \ldots, a_n; b_m)).$$

Again an easy computation shows that

$$\begin{aligned} &g((a_0, \ldots, a_n; b_0)) \wedge \cdots \wedge g((a_0, \ldots, a_n; b_m)) \\ &\quad = (g(a_0), \ldots, g(a_n); g(b_0)) \wedge \cdots \wedge (g(a_0), \ldots, g(a_n); g(b_m)) \\ &\quad = g(a_0) \wedge \cdots \wedge g(a_n) \to' g(b_0) \wedge \cdots \wedge g(b_m). \end{aligned}$$

Since

$$g(a_0) \wedge \cdots \wedge g(a_n) \to' g(b_0) \wedge \cdots \wedge g(b_m) = \overline{g}(a) \to' \overline{g}(b)$$

we obtain $\overline{g}(a \to b) = \overline{g}(a) \to' \overline{g}(b)$.

To conclude we prove uniqueness. Suppose that $f : L \to L'$ is a $(\wedge, 1)$-homomorphism such that $g = f \circ e$. We prove that for every $a \in L$, $\overline{g}(a) = f(a)$. Suppose that $a = e(b_0) \wedge \cdots \wedge e(b_n)$ with $b_0, \ldots, b_n \in A$. Then $\overline{g}(a) = g(b_0) \wedge \cdots \wedge g(b_n) = f(e(b_0)) \wedge \cdots \wedge f(e(b_n)) = f(e(b_0) \wedge \cdots \wedge e(b_n)) = f(a)$.

Suppose now that g is a one-to-one homomorphism and $a, b \in L$ are such that $\overline{g}(a) = \overline{g}(b)$. Assume that $a = e(a_0) \wedge \cdots \wedge e(a_n)$ and $b = e(b_0) \wedge \cdots \wedge e(b_m)$, with $a_0, \ldots, a_n, b_0, \ldots, b_m \in A$. Then $g(a_0) \wedge \cdots \wedge g(a_n) = \overline{g}(a) = \overline{g}(b) = g(b_0) \wedge \cdots \wedge g(b_m)$. It follows that $(g(a_0), \ldots, g(a_n); g(b_i)) = 1$ for every $i \leq m$ and $(g(b_0), \ldots, g(b_n); g(a_j)) = 1$ for every $j \leq n$. So, $g((a_0, \ldots, a_n; b_i)) = g(1)$ and $g((b_0, \ldots, b_n; a_j)) = g(1)$. Since g is one-to-one,

$$(a_0, \ldots, a_n; b_i) = 1 \quad \text{and} \quad (b_0, \ldots, b_n; a_j) = 1.$$

This implies that $(e(a_0), \ldots, e(a_n); e(b_i)) = e(1)$ and $(e(b_0), \ldots, e(b_n); e(a_j)) = e(1)$, from which follows that $e(a_0) \wedge \cdots \wedge e(a_n) \leq e(b_i)$ and $e(b_0) \wedge \cdots \wedge e(b_n) \leq e(a_j)$. Consequently, $a = e(a_0) \wedge \cdots \wedge e(a_n) = e(b_0) \wedge \cdots \wedge e(b_m) = b$. So $\overline{g}$ is one-to-one.

The last part of the proposition is obvious. □

Theorem 6.7 *For every Hilbert algebra $\boldsymbol{A}$, if $\langle L, e\rangle$ and $\langle L', e'\rangle$ are implicative semilattice envelopes of $\boldsymbol{A}$, then L and L' are isomorphic by an isomorphism $h : L \to L'$ such that $e' = h \circ e$.*

In [14], Porta defines the Hertz free extension of a Hilbert algebra $\boldsymbol{A}$ as a pair $\langle L, e\rangle$ which using our terminology is an implicative semilattice envelope of $\boldsymbol{A}$ with the universal property above. From what we have already seen it follows that Porta's definition has some redundant conditions.

Now we provide a proof of existence of the implicative semilattice envelope (or the free implicative semilattice extension) for Hilbert algebras.

Theorem 6.8 *For every Hilbert algebra $\boldsymbol{A}$ there exists a unique* (*up to isomorphism*) *implicative semilattice envelope.*

Proof. Let $\mathcal{F}$ be any separating family for a Hilbert algebra $\boldsymbol{A}$. By Corollary 4.15 we know that $L_{\mathcal{F}}(\boldsymbol{A})$ is an implicative semilattice. Also we know that the map $\varphi_{\mathcal{F}}$ is a one-to-one sup-homomorphism. So, $\langle L_{\mathcal{F}}(\boldsymbol{A}), \varphi_{\mathcal{F}}\rangle$ is an implicative semilattice envelope of $\boldsymbol{A}$. By Theorem 6.7 this implicative semilattice envelope is unique up to isomorphism. □

The free implicative semilattice extension construction can be turned into a functor between the appropriate categories. We consider the four categories described in the next table:

Category	Objects	Morphisms
$\mathbf{Hil^s}$	Hilbert algebras	semi-homomorphisms
$\mathbf{Hil}$	Hilbert algebras	homomorphisms
$\mathbf{IMSL^s}$	implicative semilattices	$(\wedge, 1)$-homomorphisms
$\mathbf{IMSL}$	implicative semilattices	homomorphisms

Proposition 6.9 *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Hilbert algebras. Let $(L(\boldsymbol{A}), e_{\boldsymbol{A}})$ and $(L(\boldsymbol{B}), e_{\boldsymbol{B}})$ be respectively their free implicative semilattice extensions. If $h : \boldsymbol{A} \to \boldsymbol{B}$ is a (semi-) homomorphism, then there is a unique $((\wedge, 1)$-) homomorphism $\overline{h} : L(\boldsymbol{A}) \to L(\boldsymbol{B})$ such that $e_{\boldsymbol{B}} \circ h = \overline{h} \circ e_{\boldsymbol{A}}$.*

Proof. Let $h : \boldsymbol{A} \to \boldsymbol{B}$ be a semi-homomorphism. Note that $e_{\boldsymbol{B}} \circ h : \boldsymbol{A} \to L(\boldsymbol{B})$ is a semi-homomorphism of Hilbert algebras. So there is a unique $(\wedge, 1)$-homomorphism $\overline{e_{\boldsymbol{B}} \circ h} : L(\boldsymbol{A}) \to L(\boldsymbol{B})$ such that $e_{\boldsymbol{B}} \circ h = \overline{e_{\boldsymbol{B}} \circ h} \circ e_{\boldsymbol{A}}$. We let $\overline{h} = \overline{e_{\boldsymbol{B}} \circ h}$. If $h : \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then $e_{\boldsymbol{B}} \circ h : \boldsymbol{A} \to L(\boldsymbol{B})$ is a homomorphism, and so $\overline{e_{\boldsymbol{B}} \circ h} : L(\boldsymbol{A}) \to L(\boldsymbol{B})$ is also a homomorphism. □

Hence we have a functor from the category $\mathbf{Hil}$ to the category $\mathbf{IMSL}$ that maps every Hilbert algebra to its free implicative semilattice extension and every homomorphism h between Hilbert algebras to the homomorphism $\overline{h}$ between their free implicative semilattice extensions. This functor is left adjoint to the forgetful functor from $\mathbf{IMSL}$ to $\mathbf{Hil}$ that forgets the meet operation. This implies that $\mathbf{Hil}$ can be taken as a reflective subcategory of $\mathbf{IMSL}$.

Similarly, we have a functor from the category $\mathbf{Hil^s}$ to the category $\mathbf{IMSL^s}$ that maps every Hilbert algebra to its free implicative semilattice extension and every semi-homomorphism h between Hilbert algebras to the $(\wedge, 1)$-homomorphism $\overline{h}$ between their free implicative semilattice extensions. This functor is left adjoint to the forgetful functor from $\mathbf{IMSL^s}$ to $\mathbf{Hil^s}$ that forgets the meet operation and so $\mathbf{Hil^s}$ can be taken as a reflective subcategory of $\mathbf{IMSL^s}$.

A useful property of the implicative semilattice envelope of a Hilbert algebra is the following.

Lemma 6.10 *Let $\boldsymbol{A}$ be a Hilbert algebra and let $\langle L, e\rangle$ be its implicative semilattice envelope. Then for every finite $X \subseteq A$ and every $a \in A$,*

$$\bigwedge e[X] \leq e(a) \quad \textit{iff} \quad a \in \langle X\rangle.$$

Proof. Suppose that $\bigwedge e[X] \leq e(a)$. Then for every $c \in A$ such that $a \leq c$ we have that $e(c) \in \langle e[X]\rangle$. Hence $\uparrow e(a) \subseteq \langle e[X]\rangle$. Since e is one-to-one, it follows that $a \in \langle X\rangle$. Suppose now that $a \in \langle X\rangle$. So $\uparrow a \subseteq \langle X\rangle$. Since e is a sup-homomorphism, $\uparrow e(a) \subseteq \langle e[X]\rangle$. It follows that $\bigwedge e[X] \leq e(a)$. □

From the lemma it follows that if $\boldsymbol{A}$ is a Hilbert algebra and $\langle L, e\rangle$ is its implicative semilattice envelope, then for every finite $X, Y \subseteq A$, $\bigwedge e[X] = \bigwedge e[Y]$ if and only if $\langle X\rangle = \langle Y\rangle$. This shows that the implicative semilattice envelope of a Hilbert algebra $\boldsymbol{A}$ can be constructed from the set of finite subsets of A modulo the equivalence relation defined on them by: $X \equiv Y$ iff $\langle X\rangle = \langle Y\rangle$. The construction in [14] to prove the existence of free implicative semilattice extension of a Hilbert algebra follows this path.

Lemma 6.11 *Let $\boldsymbol{A}$ be a Hilbert algebra and let $\langle L, e\rangle$ be its implicative semilattice envelope. Then for every finite and non-empty $X \subseteq A$ and every finite and non-empty $Y \subseteq A$,*

$$\bigcap_{a\in X} \uparrow a \subseteq \bigvee_{b\in Y} \uparrow b \quad \textit{iff} \quad \bigcap_{a\in X} \uparrow_L e(a) \subseteq \uparrow_L \bigwedge_{b\in Y} e(b).$$

In other words,

$$\bigcap_{a\in X} \uparrow a \subseteq \langle Y\rangle \quad \textit{iff} \quad \bigcap_{a\in X} \uparrow_L e(a) \subseteq \uparrow_L \bigwedge e[Y].$$

Proof. Assume that $\bigcap_{a\in X} \uparrow a \subseteq \bigvee_{b\in Y} \uparrow b$. Suppose that Z is a finite subset of A and $\bigwedge e[Z] \in \bigcap_{a\in X} \uparrow_L e(a)$. Then $e(a) \leq \bigwedge e[Z]$ for every $a \in X$, and therefore, for every $a \in X$ and every $c \in Z$, $e(a) \leq e(c)$. This implies that $a \leq c$, for every $c \in Z$. Thus, $Z \subseteq \bigcap_{a \in X} \uparrow a$. Hence, $Z \subseteq \langle Y \rangle$ and therefore $e[Z] \subseteq \langle e[Y] \rangle$. This implies that $\bigwedge e[Z] \in \bigwedge_{b \in Y} e(b)$.

Suppose now that $\bigcap_{a \in X} \uparrow_L e(a) \subseteq \uparrow_L \bigcap_{b \in Y} e(b)$. Let $d \in \bigcap_{a \in X} \uparrow a$. Then $e(d) \in \bigcap_{a \in X} \uparrow_L e(a)$. Hence, $e(d) \in \uparrow_L \bigwedge_{b \in Y} e(b)$. Thus, $\bigwedge e[Y] \leq e(b)$. This implies that $b \in \langle Y \rangle$. □

7 The relation between the deductive filters of a Hilbert algebra and the filters of its free implicative semilattice extension

In [14], Porta establishes the relation between the deductive filters of a Hilbert algebra $\boldsymbol{A}$ and the filters of its free implicative semilattice extension $\langle L, e \rangle$: the map $e^{-1}[.]$ is an order isomorphism between the filters of L and the deductive filters of $\boldsymbol{A}$. We provide first a proof of this fact for the interested reader and then we proceed to show that the isomorphism restricts to an order isomorphism between the ordered set of the optimal filters of L and the ordered set of the optimal deductive filters of $\boldsymbol{A}$. In the way of doing this we need to explore the relation between prime strong Frink ideals of $\boldsymbol{A}$ and prime Frink ideals of L. We also discuss the relation between the ordered set of the optimal deductive filters of $\boldsymbol{A}$ and the ordered set of the prime filters of the free distributive lattice extension of L relative to the homomorphisms that preserve existing finite joins.

To conclude the paper, and for the sake of completeness, we present an interesting characterization of the implicative semilattice free extension of a Hilbert algebra obtained by Porta in [14]. And we also show that the lattice of congruences of a Hilbert algebra $\boldsymbol{A}$ is isomorphic to the lattice of congruences of its free implicative semilattice extension, and we describe the isomorphism.

Lemma 7.1 *Let $\boldsymbol{A}$ be a Hilbert algebra and $\langle L, e \rangle$ its implicative semilattice envelope.*

(1) *If F a deductive filter of $\boldsymbol{A}$, then the filter $[e[F])$ of L generated by $e[F]$ is such that $e^{-1}[[e[F])] = F$*

(2) *If G is a filter of L, then $e^{-1}[G]$ is a deductive filter of $\boldsymbol{A}$.*

(3) *If G is a filter of L, then $G = [e[e^{-1}[G]])$.*

Proof. (1) Assume that F is a deductive filter of $\boldsymbol{A}$. Consider the filter $G = [e[F])$ of L generated by $e[F]$. It is clear that $F \subseteq e^{-1}[G]$. Now, if $a \in e^{-1}[G]$, then $e(a) \in G$, hence there is a finite $X \subseteq F$ such that $\bigwedge e[X] \leq e(a)$. Then by Lemma 6.10 $a \in \langle X \rangle$. Therefore $a \in F$.

(2) Suppose $a, a \to b \in e^{-1}[G]$. Then $e(a), e(a \to b) \in G$. Thus, $e(a), e(a) \to e(b) \in G$, and since $e(a) \to e(b) = e(b) \in G$, it follows that $e(b) \in G$. Therefore, $b \in e^{-1}[G]$. Moreover, $e(1) \in G$ and so $1 \in e^{-1}[G]$.

(3) Let G be a filter of L. Suppose that $\bigwedge e[X] \in G$ with X a nonempty finite subset of A. Let $a \in X$. Then $\bigwedge e[X] \leq e(a)$ and hence $e(a) \in G$. Therefore, $a \in e^{-1}[G]$. Thus, $e(a) \in e[e^{-1}[G]]$, for every $a \in X$. Therefore, $\bigwedge e[X] \in [e[e^{-1}[G]])$. To prove the other inclusion, suppose $\bigwedge e[X] \in [e[e^{-1}[G]])$. Let $Z \subseteq A$ be a finite set such that $Z \subseteq e^{-1}[G]$ and $\bigwedge e[Z] \leq \bigwedge e[X]$. By Lemma 6.10 it follows that $\langle X \rangle \subseteq \langle Z \rangle$. Therefore, $X \subseteq e^{-1}[G]$. Hence for every $a \in X$, $e(a) \in G$. We obtain that $\bigwedge e[X] \in G$. □

Proposition 7.2 *Let $\boldsymbol{A}$ be a Hilbert algebra and let $\langle L, e \rangle$ be its implicative semilattice envelope. The map $[e[.]) : \mathrm{Dfi}\boldsymbol{A} \to \mathrm{Fi}L$ establishes an order isomorphism between the deductive filters of $\boldsymbol{A}$ and the filters of L, whose inverse is the map $e^{-1}[.]$.*

Proof. From (1) in Lemma 7.1 it follows that $[e[.])$ is one-to-one. Moreover, from (3) of that lemma it follows that it is onto. From (1) it also follows that $e^{-1}[.]$ is the inverse of $[e[.])$. Now, if $F, F' \in \mathrm{Dfi}\boldsymbol{A}$ and $F \subseteq F'$ it is immediate that $[e[F]) \subseteq [e[F'])$. Moreover, using Lemma 7.1, if $[e[F]) \subseteq [e[F'])$, follows that $F = e^{-1}[[e[F])] \subseteq e^{-1}[[e[F'])] = F'$. □

In particular, since any order isomorphism sends meet-prime elements to meet-prime elements, it follows that if $\boldsymbol{A}$ is a Hilbert algebra and $\langle L, e \rangle$ is its implicative semilattice envelope, then $F \subseteq A$ is a prime deductive filter of $\boldsymbol{A}$ if and only if there exists a prime filter G of L such that $F = e^{-1}[G]$.

The isomorphism described in Proposition 7.2 maps optimal deductive filters to optimal filters and conversely. We proceed to show this.

Proposition 7.3 *Let $\boldsymbol{A}$ be a Hilbert algebra and $\langle L, e\rangle$ its implicative semilattice envelope. If I is a prime Frink ideal of L, then $e^{-1}[I]$ is a prime strong Frink ideal of $\boldsymbol{A}$.*

Proof. Suppose that I is a prime Frink ideal of L. Then it is a prime strong Frink ideal of $\langle L, \rightarrow, 1\rangle$. Since e is a sup-homomorphism, $e^{-1}[I]$ is a prime strong Frink ideal of $\boldsymbol{A}$ or $e^{-1}[I] = \varnothing$. We show the last case does not hold. Let $b \in I$. Then $b = \bigwedge e[X]$ for some finite and nonempty set $X \subseteq A$. Since I is prime, there is $a \in X$ such that $e(a) \in I$, and therefore $a \in e^{-1}[I]$. □

Proposition 7.4 *Let $\boldsymbol{A}$ be a Hilbert algebra and $\langle L, e\rangle$ its implicative semilattice envelope. Let $I \subseteq A$, then I is a strong Frink ideal of $\boldsymbol{A}$ iff $\downarrow_L e[I]$ is an Frink ideal of L. Moreover, I is a prime strong Frink ideal iff $\downarrow_L e[I]$ is a prime Frink ideal.*

Proof. Suppose that I is a strong Frink ideal of $\boldsymbol{A}$. In order to show that $\downarrow_L e[I]$ is a prime Frink ideal, let $X_0, \dots, X_n, Y$ be nonempty finite subsets of A such that $\bigwedge e[X_0], \dots, \bigwedge[X_n] \in \downarrow_L e[I]$ and $\bigcap_{i\leq n} \uparrow_L \bigwedge[X_i] \subseteq \uparrow_L \bigwedge e[Y]$. There are $a_0, \dots, a_n \in I$ such that $\bigwedge e[X_i] \leq e(a_i)$, for every $i \leq n$. Then, $\bigcap_{i\leq n} \uparrow_L e(a_i) \subseteq \uparrow_L \bigwedge e[Y]$. By Lemma 6.11, we have $\bigcap_{i\leq n} \uparrow a_i \subseteq \langle Y\rangle$. Since I is a strong Frink ideal it follows that $\langle Y\rangle \cap I \neq \varnothing$. Let $c \in I \cap \langle Y\rangle$. Then $\bigwedge e[Y] \leq e(c) \in e[I]$. Therefore, $\bigwedge e[Y] \in \downarrow_L e[I]$.

Assume now that $\downarrow_L e[I]$ is a Frink ideal of L. To show that I is a strong Frink ideal of $\boldsymbol{A}$ suppose that $a_0, \dots, a_n \in I$ and $\bigcap_{i\leq n} \uparrow a_i \subseteq \langle X\rangle$ with X a finite subset of A. Then, by Lemma 6.11, $\bigcap_{i\leq n} \uparrow_L e(a_i) \subseteq \uparrow_L \bigwedge e[X]$. Since $e(a_i) \in \downarrow_L e[I]$ for every $i \leq n$, $\bigwedge e[X] \in \downarrow_L e[I]$. Let $a \in I$ be such that $\bigwedge e[X] \leq e(a)$. This implies that $a \in \langle X\rangle$, hence $I \cap \langle X\rangle \neq \varnothing$.

Assume now that I is a prime strong Frink ideal. Suppose that $\bigwedge e[X] \wedge \bigwedge e[Y] \in \downarrow_L e[I]$ with X, Y finite subsets of A. Then $\bigwedge e[X \cup Y] \in \downarrow_L e[I]$. Let $c \in I$ be such that $\bigwedge e[X \cup Y] \leq e(c)$. It follows that $c \in \langle X \cup Y\rangle$. Since I is prime, $I \cap (X \cup Y) \neq \varnothing$. Therefore there is $a \in I \cap X$ or $b \in I \cap Y$. In the first case $\bigwedge e[X] \in \downarrow_L e[I]$ and in the second case $\bigwedge e[Y] \in \downarrow_L e[I]$. Finally, if $\downarrow_L e[I]$ is a prime Frink ideal of L, from Proposition 7.3 it follows that $e^{-1}[\downarrow_L e[I]]$ is a prime strong Frink ideal of $\boldsymbol{A}$ or $e^{-1}[\downarrow_L e[I]] = \varnothing$. This last possibility does not hold since $I \neq \varnothing$. We show that $I = e^{-1}[\downarrow_L e[I]]$. The inclusion $I \subseteq e^{-1}[\downarrow_L e[I]]$ is clear. Suppose that $a \in e^{-1}[\downarrow_L e[I]]$. Then $e(a) \in \downarrow_L e[I]$. Let $c \in I$ be such that $e(a) \leq e(c)$. This implies that $a \leq c$. Therefore, $a \in I$. We conclude that $e^{-1}[\downarrow_{Le} e[I]] \subseteq I$. □

Corollary 7.5 *Let $\boldsymbol{A}$ be a Hilbert algebra and $\langle L, e\rangle$ its implicative semilattice envelope. The map $\downarrow_L e[.]$ establishes an order isomorphism between the set of prime strong Frink ideals of $\boldsymbol{A}$ and the set of prime Frink ideals of L, ordered both by inclusion, whose inverse is the map $e^{-1}[.]$.*

Proof. Let I be a prime strong Frink ideal of $\boldsymbol{A}$. Then $\downarrow_L e[I]$ is a prime Frink ideal of L. We show that $e^{-1}[\downarrow_L e[I]] = I$. The inclusion $I \subseteq e^{-1}[\downarrow_L e[I]]$ is obvious. Suppose $a \in e^{-1}[\downarrow_L e[I]]$. Then $e(a) \in \downarrow_L e[I]$. Let $b \in I$ be such that $e(a) \leq e(b)$. Then $a \leq b$. So $a \in I$. Moreover, since I is nonempty, $e^{-1}[\downarrow_L e[I]]$ is nonempty.

Let now H be a prime Frink ideal of L. Then by Proposition 7.3 this set is a strong Frink ideal of $\boldsymbol{A}$ and $\downarrow_L e[e^{-1}[H]]$ is a Frink ideal of L. Let us show that $H = \downarrow_L e[e^{-1}[H]]]$. The inclusion $\downarrow_L e[e^{-1}[H]] \subseteq H$ is clear. Suppose that $a \in H$. Then $a = e(a_0) \wedge \dots \wedge e(a_n)$ for some $a_0, \dots, a_n \in A$. So, since H is prime, $e(a_i) \in H$ for some $i \leq n$. Thus $e(a_i) \in e[e^{-1}[H]]$ and therefore, $a = e(a_0) \wedge \dots \wedge e(a_n) \in \downarrow_L e[e^{-1}[H]]$. □

Corollary 7.6 *Let $\boldsymbol{A}$ be a Hilbert algebra and $\langle L, e\rangle$ be its implicative semilattice envelope. A set $F \subseteq A$ is an optimal deductive filter of $\boldsymbol{A}$ iff there is an optimal filter G of L such that $e^{-1}[G] \neq A$ and $F = e^{-1}[G]$.*

Proof. Suppose that G is an optimal filter of L such that $e^{-1}[G] \neq A$. Then $J = L - G$ is a prime strong Frink ideal of L. Hence, by Proposition 7.4 $e^{-1}[J]$ is a prime strong Frink ideal of $\boldsymbol{A}$ or $e^{-1}[J] = \varnothing$. Since $e^{-1}[J] = A - e^{-1}[G]$, the second of the two alternatives is impossible, because $e^{-1}[G] \neq A$. It follows that $e^{-1}[G]$ is an optimal deductive filter.

Suppose now that F is an optimal deductive filter of $\boldsymbol{A}$. Then $I = A - F$ is a strong Frink ideal of $\boldsymbol{A}$, and F as well as I are nonempty. By Proposition 7.4, $\downarrow_L e[I]$ is a prime Frink ideal of L such that $e^{-1}[\downarrow_L e[I] = I$. So $L - \downarrow_L e[I]$ is an optimal deductive filter of L and $e^{-1}[L - \downarrow_L e[I]] = A - e^{-1}[\downarrow_L e[I] = A - I = F$. □

From Corollary 7.6 and Lemma 7.1 we obtain:

Proposition 7.7 *Let $\boldsymbol{A}$ be a Hilbert algebra and $\langle L, e\rangle$ its implicative semilattice envelope. The map $[e[.])$ establishes an order isomorphism between the set of optimal deductive filters of $\boldsymbol{A}$ and the set of optimal filters G of L such that $e^{-1}[G] \neq A$ ordered both by inclusion, whose inverse is the map $e^{-1}[.]$.*

In Corollary 7.6 and Proposition 7.7 we need the restriction to the set of optimal filters G of L such that $e^{-1}[G] \neq A$ because in a Hilbert algebra $\boldsymbol{A}$ without a bottom element there might exist a nonempty finite set X such that the deductive filter generated by X is A. In this case the implicative semilattice envelope $\langle L, e\rangle$ has a bottom element, namely $\bigwedge e[X]$, and $L - \{\bigwedge e[X]\}$ is an optimal filter and $e^{-1}[L - \{\bigwedge e[X]\}] = A$, which is not optimal.

Let DLat, IMSL and Hil be respectively the varieties of distributive lattices with a top element, implicative semilattices and Hilbert algebras. Let $D : \mathsf{IMSL} \to \mathsf{DLat}$ be the map that assigns to every implicative semilattice $L = \langle L, \to, \wedge, 1\rangle$ the free distributive lattice extension $\langle D(L), j\rangle$ of $\langle L, \wedge, 1\rangle$, as introduced in [1,2]. Recall from the preliminaries section that for every $L \in \mathsf{DLat}$, the map $j^{-1}[.]$ obtained from the embedding $j : L \to D(L)$ establishes an order isomorphism between the ordered set of the prime filters of $D(L)$ and the ordered set of the optimal filters of L. Let now $L : \mathsf{Hil} \to \mathsf{IMSL}$ be the map that assigns to every Hilbert algebra $\boldsymbol{A}$ its free implicative semilattice extension $\langle L, e\rangle$.

Proposition 7.8 *Let $\boldsymbol{A}$ be a Hilbert algebra, $\langle L, e\rangle$ its implicative semilattice envelop and $\langle D(L), j\rangle$ the free distributive lattice extension of L. Then the ordered set of the prime filters F of $D(L)$ such that $(j \circ e)^{-1}[F] \neq A$ is isomorphic by the map $(j \circ e)^{-1}[.]$ to the ordered set of the optimal deductive filters of $\boldsymbol{A}$.*

P r o o f. $e^{-1}[.]$ establishes an order isomorphism between the optimal filters G of L such that $e^{-1}[G] \neq A$ and the optimal filters of $\boldsymbol{A}$, and $j^{-1}[.]$ establishes an order isomorphism between the prime filters of $D(L)$ and the optimal filters of L. So, $(j \circ e)^{-1}[.]$ establishes an isomorphism between the ordered set of the prime filters F of $D(L)$ such that $(j \circ e)^{-1}[F] \neq A$ and the ordered set of the optimal deductive filters of $\boldsymbol{A}$. □

Porta also gives the following interesting characterization of the implicative semilattice free extension of a Hilbert algebra which for completeness we reproduce here.

Proposition 7.9 *Let $\boldsymbol{A}$ be a Hilbert algebra, L an implicative semilattice and $e : A \to L$ an embedding of $\boldsymbol{A}$ to $\langle L, \to, 1\rangle$. The following properties are equivalent*:

(1) *(L, e) is the implicative semilattice free extension of $\boldsymbol{A}$.*

(2) *The map $e^{-1}[.] : \mathrm{Fi}(L) \to \mathrm{Dfi}\boldsymbol{A}$ is a one-to-one lattice homomorphism*

(3) *The map $[e[.]) : \mathrm{Dfi}\boldsymbol{A} \to \mathrm{Fi}(L)$ is onto $\mathrm{Fi}(L)$.*

P r o o f. By Proposition 7.2, (1) implies (2) and (3). Suppose now (2). Let us consider the subalgebra $L' = S(e[A])$ of L. Let $x \in L$ and let $G = \{y \in L : x \leq y\}$. Let $G' = G \cap L'$, which is a filter of L'. Let H be the up-set generated by G' in L. We show that H is indeed a filter of L. Let $a, b \in H$, then let $a', b' \in G'$ be such that $a' \leq a$ and $b' \leq b$. Since $a' \wedge b' \in G'$ and $a' \wedge b' \leq a \wedge b$, $a \wedge b \in H$. Now we show that $G \cap e[A] = H \cap e[A]$. On the one hand, $G \cap e[A] \subseteq G \cap L' \subseteq H$. On the other hand, if $a \in H \cap e[A]$, then let $b \in G'$ be such that $b \leq a$. Then $x \leq b$, so $x \leq a$. Therefore, $a \in G$. It follows that $e^{-1}[H] = e^{-1}[G]$. So from the hypothesis we obtain that $G = H$. Therefore, $x \in H$. Thus there is $b \in G'$ such that $b \leq x$. But since $b \in G$, $x \leq b$, and so $x = b$. Since $b \in L'$, $x \in L'$. Thus $L = L'$ and we conclude that (2) implies (1). To finish the proof we show that (3) implies (1). Suppose (3). Let $x \in L$, and let, as before, $G = \{y \in L : x \leq y\}$. Since $G \in \mathrm{Fi}(L)$, by the assumption there is $F \in \mathrm{Dfi}\boldsymbol{A}$ such that $[e[F]) = G$. So, since $x \in G$, there are $b_0, \dots, b_n \in F$ such that $e(b_0) \wedge \cdots \wedge e(b_n) \leq x$. Moreover, since $e[F] \subseteq G$, we have $x \leq e(b_i)$ for every $i \leq n$. So, $e(b_0) \wedge \cdots \wedge e(b_n) = x$, and therefore, $x \in S(e[A])$. □

It is well known that on every Hilbert algebra $\boldsymbol{A}$ the lattice of deductive filters of $\boldsymbol{A}$ is isomorphic to the lattice of congruences of $\boldsymbol{A}$ by the isomorphism

$$F \mapsto \theta_F = \{\langle a, b\rangle : a \to b, b \to a \in F\},$$

whose inverse is given by

$$\theta \mapsto F_\theta = \{a \in A : a\theta 1\}.$$

Also it is well known that the lattice of congruences of an implicative semilattice L is isomorphic to the lattice of filters of L. The isomorphism and its inverse are defined as above. Therefore, since the lattice of filters of the implicative semilattice envelope of $\boldsymbol{A}$ is isomorphic to the lattice of deductive filters of $\boldsymbol{A}$, the lattice of congruences of $\boldsymbol{A}$ is isomorphic to the lattice of congruences of the implicative semilattice envelope of $\boldsymbol{A}$. We end the paper by a detailed description of the isomorphism.

Proposition 7.10 *Let $\boldsymbol{A}$ be a Hilbert algebra and let $\langle L, e\rangle$ be its implicative semilattice envelope. The map form* $\mathrm{Con}L$ *to* $\mathrm{Con}\boldsymbol{A}$ *given by*

$$\theta \mapsto \{\langle a, b\rangle \in A \times A : e(a)\theta e(b)\}$$

is an isomorphism.

Proof. Let $\theta \in \mathrm{Con}L$. Then let $F_\theta = \{a \in L : a\theta 1\}$. This set is a filter of L. So $e^{-1}[F_\theta]$ is a deductive filter of $\boldsymbol{A}$. Consider the congruence $\theta_{F_\theta} = \{\langle a, b\rangle : a \to b, b \to a \in e^{-1}[F_\theta]\}$ of $\boldsymbol{A}$. Then $\langle a, b\rangle \in \theta_{F_\theta}$ iff $e(a \to b), e(b, \to a) \in F_\theta$ iff $e(a) \to e(b), e(b) \to e(a) \in F_\theta$ iff $e(a) \to e(b)\theta 1$, $e(b) \to e(a)\theta 1$ iff $e(a)\theta e(b)$. Thus, $\theta_{F_\theta} = \{\langle a, b\rangle \in A \times A : e(a)\theta e(b)\}$. □

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