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Subalgebras of Heyting and De Morgan Heyting Algebras

Abstract. In this paper we obtain characterizations of subalgebras of Heyting algebras and De Morgan Heyting algebras. In both cases we obtain these characterizations by defining certain equivalence relations on the Priestley-type topological representations of the corresponding algebras. As a particular case we derive the characterization of maximal subalgebras of Heyting algebras given by M. Adams for the finite case.

Keywords: Lattices, Heyting algebras, De Morgan Heyting algebras, subalgebras, Priestley spaces, Heyting relations.

1. Introduction and Preliminaries

A *Heyting algebra* is an algebra $\langle H, \vee, \wedge, \rightarrow, 0, 1\rangle$ of type $(2, 2, 2, 0, 0)$ for which $\langle H, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $\rightarrow$ is the binary operation of relative pseudocomplementation (i.e., for $a, b, c \in H$, $a \wedge c \leq b$ iff $c \leq a \rightarrow b$).

A *De Morgan Heyting algebra* is an algebra $\langle L, \vee, \wedge, \rightarrow, ', 0, 1\rangle$ of type $\langle 2, 2, 2, 1, 0, 0\rangle$ such that $\langle L, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a *Heyting algebra* and $'$ is a unary operation satisfying the following identities:

(1) $(x \wedge y)' \approx x' \vee y'$

(2) $(x \vee y)' \approx x' \wedge y'$

(3) $x'' \approx x$.

These algebras were independently studied by H. Sankappanavar and A. Monteiro in [4] and [3], and they are the algebraic counterpart of the symmetric modal propositional calculus of Moisil.

The class of Boolean algebras is a familiar example of Heyting algebras, and it is known that there is a correspondence between the subalgebras of a Boolean algebra and certain equivalence relations defined on its Boolean

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space (see, for instance [8]). The objective of this article is to extend this correspondence to both the class of Heyting algebras and the class of De Morgan Heyting algebras.

In Section 2, we introduce the notion of a *Heyting relation* on a Heyting space, and we show that there is a one-to-one correspondence between the subalgebras of a Heyting algebra and the Heyting relations defined on its dual space. In addition, taking into account the duality between the category of Heyting algebras with homomorphisms and the category of Heyting spaces with h-morphisms, we obtain for the finite case, a characterization of maximal subalgebras of Heyting algebras. This result was first proved in a different way by M. Adams (see [1]).

In Section 3 we obtain a characterization of the subalgebras of a De Morgan Heyting algebra A via the notion of *De Morgan Heyting relation* defined on the topological space associated with A.

We give now a brief summary of the results about the duality between the category of Heyting algebras and the category of De Morgan algebras and certain topological spaces, based on the duality developed by H. A. Priestley. These results are essential to the development of this paper.

For a poset (partially ordered set) X and $U \subseteq X$, let $(U] = \{x \in X : x \leq y \text{ for some } y \in U\}$ and $[U) = \{x \in X : x \geq y \text{ for some } y \in U\}$. If $U = \{x\}$ we write $(x]$ and $[x)$ instead of $(\{x\}]$ and $[\{x\})$ respectively. U is *decreasing* if $U = (U]$ and U is *increasing* if $U = [U)$.

A triple $(X, \leq, \tau)$ is a *totally order disconnected topological space* if $(X, \leq)$ is a poset, τ is a topology on X, and for $x, y \in X$, if $x \not\leq y$ then there exists a clopen increasing $U \subseteq X$ such that $x \in U$ and $y \notin U$. A *compact* totally order disconnected space is called a *Priestley space*.

In [5] and [6] H.A.Priestley shows that the category of bounded distributive lattices and $(0,1)$-homomorphisms is dually equivalent to the category of Priestley spaces and order preserving continuous functions.

Since Heyting algebras are bounded distributive lattices, the category of Heyting algebras is isomorphic to a subcategory of bounded distributive lattices. A Heyting space is a Priestley space $(X, \leq, \tau)$ such that $(U]$ is clopen, for every clopen $U \subseteq X$. If X and X' are Heyting spaces, a (Heyting) morphism or h-morphism, is a continuous order-preserving map $f : X \to X'$ satisfying the following condition: if $f(x) \leq z$ then there exists x' such that $x \leq x'$ and $f(x') = z$. It was shown in [7] that the category of Heyting algebras with homomorphisms is dually equivalent to the category of Heyting spaces and h-morphisms.

Under the duality given above, if X is a Heyting space and $\mathbb{D}(X)$ denotes the set of clopen increasing subsets of X then $\langle \mathbb{D}(X), \cap, \cup, \to, \emptyset, X\rangle$ is a Heyting algebra, where $\to$ is given by $U_1 \to U_2 = (U_1 \cap U_2^c]^c$, for every $U_1, U_2 \in \mathbb{D}(X)$. In addition, if $f : X \to X'$ is an h-morphism then $\mathbb{D}(f) : \mathbb{D}(X') \to \mathbb{D}(X)$ defined by $\mathbb{D}(f)(U) = f^{-1}(U)$ is a Heyting homomorphism. Conversely, if L is a Heyting algebra then the set of prime filters of L, denoted by $\mathbb{X}(L)$, is a Heyting space, ordered by set inclusion with the topology having as a sub-basis the sets $\sigma_L(a) = \{P \in \mathbb{X}(L) : a \in P\}$ and $\mathbb{X}(L) \setminus \sigma_L(a)$ for $a \in L$. If $h : L \to L'$ is a Heyting homomorphism then $\mathbb{X}(h) : \mathbb{X}(L') \to \mathbb{X}(L)$ defined by $\mathbb{X}(h)(P) = h^{-1}(P)$ is an h-morphism. Moreover, the mapping $\sigma_L : L \to \mathbb{D}(\mathbb{X}(L))$ is a Heyting isomorphism and $\epsilon_X : X \to \mathbb{X}(\mathbb{D}(X))$ given by $\epsilon_X(x) = \{U \in \mathbb{D}(X) : x \in U\}$ is an isomorphism in the category of Heyting spaces.

If X is a Heyting space and $\phi : X \to X$ is an order reversing involutive ($\phi = \phi^{-1}$) homeomorphism then (X, ϕ) is called a *De Morgan Heyting space*. A dh-morphism is a map $f : X \to X'$ such that f is an h-morphism and $f \circ \phi_1 = \phi_2 \circ f$. The category whose objects are the De Morgan Heyting spaces with dh-morphisms is dually equivalent to the category of De Morgan Heyting algebras with (De Morgan Heyting) homomorphisms. Under this duality, if $\langle M, \wedge, \vee, \to, ', 0, 1\rangle$ is a De Morgan Heyting algebra and $\phi : \mathbb{X}(M) \to \mathbb{X}(M)$ is such that $\phi(P) = P'^c$, where $P' = \{a' \in M : a \in P\}$, then $(\mathbb{X}(M), \phi)$ is a De Morgan Heyting space. ϕ is called the Birula-Rasiowa transformation. If $\sigma(a) = U_a$ denotes the clopen increasing set that represents $a \in M$, then, under the duality given above, a' corresponds to the clopen increasing set $\phi(U_a)^c = \mathbb{X}(M) \setminus \phi(U_a)$. Conversely, if (X, ϕ) is a De Morgan Heyting space, $\langle \mathbb{D}(X), \cap, \cup, \to, ', \emptyset, X\rangle$ is a De Morgan Heyting algebra, where for $U \in \mathbb{D}(X)$, $U' = \phi(U)^c$. For further information see [2, 5, 6] and [7].

2. Characterization of subalgebras of a Heyting algebra

In this section we characterize the subalgebras of a Heyting algebra by means of the space of its prime filters.

If $\mathcal{E}$ is an equivalence relation on a set A, $A/\mathcal{E}$ will denote the quotient set of A by $\mathcal{E}$ and $\overline{x}$ will stand for the equivalence class of an element $x \in A$.

Definition 2.1. *Let L be a Heyting algebra. For each $L_1 \subseteq L$, we define*

$$\mathbb{E}(L_1) = \{(P, Q) \in \mathbb{X}(L) \times \mathbb{X}(L) : P \cap L_1 = Q \cap L_1\}$$

Lemma 2.2. *$\mathbb{E}(L_1)$ is an equivalence relation on $\mathbb{X}(L)$.*

Proposition 2.3. *Let L be a Heyting algebra. If L_1 is a Heyting subalgebra of L and we consider the equivalence relation associated with L_1, $\mathbb{E}(L_1)$, then it satisfies the following condition*

$$(H) \qquad \text{if } P \subseteq Q \text{ then } \overline{P} \subseteq (\overline{Q}], \quad \text{for every } \ P, Q \in \mathbb{X}(L).$$

Proof. Let us consider the Heyting homomorphism $i : L_1 \to L$ such that $i(x) = x$. Then, under the Priestley duality we have that the map $\mathbb{X}(i) : \mathbb{X}(L) \to \mathbb{X}(L_1)$ defined by $\mathbb{X}(i)(P) = i^{-1}(P) = P \cap L_1$ is an h-morphism. Now suppose that $P \subseteq Q$, and consequently, $P \cap L_1 \subseteq Q \cap L_1$. $Let P_1 \in \overline{P}$. By definition of i, it follows that $i^{-1}(P_1) \subseteq i^{-1}(Q)$. Since $\mathbb{X}(i)$ is an h-morphism, we have that there exists $P' \in \mathbb{X}(L)$ such that $P_1 \subseteq P'$ and $\mathbb{X}(i)(P') = i^{-1}(Q)$, that is, $P' \cap L_1 = Q \cap L_1$. Therefore $P_1 \subseteq P'$ with $\overline{P'} = \overline{Q}$. This shows that $\overline{P} \subseteq (\overline{Q}]$. ■

Remark 2.4. If L_1 is a subalgebra of a Heyting algebra L and $\mathbb{E}(L_1)$ is the equivalence relation associated with L_1, the relation $\leq$ on $\mathbb{X}(L)/\mathbb{E}(L_1)$ given by

$$\overline{P} \leq \overline{Q} \Longleftrightarrow \overline{P} \subseteq (\overline{Q}]$$

is an order relation on $\mathbb{X}(L)/\mathbb{E}(L_1)$. In fact, it is easy to see that $\leq$ is reflexive and transitive. In order to prove that $\leq$ is antisymmetric, suppose that $\overline{P} \leq \overline{Q}$ and $\overline{Q} \leq \overline{P}$, that is, $\overline{P} \subseteq (\overline{Q}]$ and $\overline{Q} \subseteq (\overline{P}]$. From this, there exist $Q' \in \overline{Q}$ and $P' \in \overline{P}$ such that $P \subseteq Q'$ and $Q \subseteq P'$ and consequently $P \cap L_1 \subseteq Q' \cap L_1 = Q \cap L_1$ and $Q \cap L_1 \subseteq P' \cap L_1 = P \cap L_1$. Therefore, $\overline{P} = \overline{Q}$.

Lemma 2.5. *For each $P \in \mathbb{X}(L)$, the equivalence class $\overline{P}$ of P modulo $\mathbb{E}(L_1)$ is a closed and convex set.*

Proof. Let $P \in \mathbb{X}(L)$ and $P \cap L_1 = Q \in \mathbb{X}(L_1)$. Let $\mathbb{X}(i)$ be the h-morphism of Proposition 2.3. Since $\{Q\} \subseteq \mathbb{X}(L_1)$ is closed and $\mathbb{X}(i)$ is continuous, we have that $\mathbb{X}(i)^{-1}(Q) = \{R \in \mathbb{X}(L) : R \cap L_1 = Q\} = \overline{P}$ is closed. On the other hand, suppose that $P_1, P_2 \in \overline{P}$ and $P_1 \subseteq R \subseteq P_2$, so $P_1 \cap L_1 \subseteq R \cap L_1 \subseteq P_2 \cap L_1 = P_1 \cap L_1$. Then, we have that $P_1 \cap L_1 = R \cap L_1$, that is, $R \in \overline{P}$. ■

LEMMA 2.6. *The order $\leq$ given in Remark 2.4 is equivalent to*

$$\overline{P} \leq \overline{Q} \text{ if only if } P \cap L_1 \subseteq Q \cap L_1.$$

PROOF. Suppose that $\overline{P} \leq \overline{Q}$, that is, $\overline{P} \subseteq (\overline{Q}]$. Then, there exists $Q' \in \overline{Q}$ such that $P \subseteq Q'$. From this, we have that $P \cap L_1 \subseteq Q' \cap L_1 = Q \cap L_1$.

Conversely, suppose that $P \cap L_1 \subseteq Q \cap L_1$. In the proof of Proposition 2.3, we have already shown that this implies $\overline{P} \subseteq (\overline{Q}]$, that is, $\overline{P} \leq \overline{Q}$. ■

PROPOSITION 2.7. *Let L be a Heyting algebra and L_1 a subalgebra of L. Then $\mathbb{X}(L)/\mathbb{E}(L_1)$ is homeomorphic and order-isomorphic to $\mathbb{X}(L_1)$. In particular, $\mathbb{X}(L)/\mathbb{E}(L_1)$ is a Heyting space.*

PROOF. We first prove that $\mathbb{X}(L)/\mathbb{E}(L_1)$ and $\mathbb{X}(L_1)$ are homeomorphic as topological spaces. We know that $i : L_1 \to L$ is an injective Heyting homomorphism. Since i is 1-1, $\mathbb{X}(i)$ is surjective, what is easy to check. Therefore, the map $\mathbb{X}(i) : \mathbb{X}(L) \to \mathbb{X}(L_1)$ defined by $\mathbb{X}(i)(P) = i^{-1}(P) = P \cap L_1$ is a surjective h-morphism. Since $\mathbb{X}(i) : \mathbb{X}(L) \to \mathbb{X}(L_1)$ is a continuous function and $\mathbb{E}(L_1)$ is the equivalence relation on $\mathbb{X}(L)$ associated with $\mathbb{X}(i)$, there exists a continuous function $h : \mathbb{X}(L)/\mathbb{E}(L_1) \to \mathbb{X}(L_1)$ such that $\mathbb{X}(i) = h \circ \pi$, where π is the natural mapping. In addition, as $\mathbb{X}(L)$ and $\mathbb{X}(L_1)$ are compact Hausdorff spaces and $\mathbb{X}(i)$ is a surjective continuous function, it follows that $\mathbb{X}(L)/\mathbb{E}(L_1)$ and $\mathbb{X}(L_1)$ are homeomorphic as topological spaces.

In order to prove that h is an order isomorphism, suppose that $\overline{P} \leq \overline{Q}$, hence, by Lemma 2.6, $P \cap L_1 \subseteq Q \cap L_1$ and consequently $h(\overline{P}) \subseteq h(\overline{Q})$. Conversely, suppose that $h(\overline{P}) \subseteq h(\overline{Q})$, that is, $P \cap L_1 \subseteq Q \cap L_1$. By Lemma 2.6 we have that $\overline{P} \leq \overline{Q}$. Taking into account that h and h^{-1} are order-isomorphisms then both, h and h^{-1}, are h-morphisms. Thus, h is an isomorphism between the Heyting spaces $\mathbb{X}(L)/\mathbb{E}(L_1)$ and $\mathbb{X}(L_1)$. ■

Motivated by condition (H) in the Proposition 2.3, we introduce the following notation and definition.

If $\mathcal{H}$ is a Heyting space and $\mathcal{E}$ is an equivalence relation on $\mathcal{H}$, let $\mathbb{S}(\mathcal{E})$ denote the family of subsets $U \in \mathbb{D}(\mathcal{H})$ satisfying the following condition:

$$(S) \qquad \text{if } a \in U \text{ then } \overline{a} \subseteq U.$$

DEFINITION 2.8. *Let $(\mathcal{H}, \leq, \tau)$ be a Heyting space. An equivalence relation $\mathcal{E}$ on $\mathcal{H}$ is called a* **Heyting relation** *if the following conditions are satisfied:*

(1) *$\overline{a}$ is a closed and convex set for each $a \in \mathcal{H}$.*

(2) *For all $a, b \in \mathcal{H}$ such that $a \leq b$, $\overline{a} \subseteq (\overline{b}]$.*

(3) *If* $\overline{a} \neq \overline{b}$, *there exists* $U \in \mathbb{S}(\mathcal{E})$ *that* **separates** *a and b, that is, there exists* $U \in \mathbb{S}(\mathcal{E})$ *such that either* $a \in U$ *and* $b \notin U$ *or* $b \in U$ *and* $a \notin U$.

The following result shows that for each subalgebra of a Heyting algebra, there exists a *Heyting relation* on its space of prime filters associated with it. Recall that $\sigma(a) = \{P \in \mathbb{X}(L) : a \in P\}$ is a clopen increasing set in $\mathbb{X}(L)$.

PROPOSITION 2.9. *If* L_1 *is a subalgebra of a Heyting algebra* L, *then* $\mathbb{E}(L_1)$ *is a Heyting relation on the Heyting space* $\mathbb{X}(L)$.

PROOF. It is immediate that $\mathbb{E}(L_1)$ is an equivalence relation on $\mathbb{X}(L)$. By Lemma 2.5, the elements of $\mathbb{X}(L)/\mathbb{E}(L_1)$ are convex and closed, that is, (1) holds. Besides, (2) follows from Proposition 2.3.

In order to prove (3), assume that $\overline{P}, \overline{Q} \in \mathbb{X}(L)/\mathbb{E}(L_1)$ and $\overline{P} \neq \overline{Q}$. Without loss of generality we can assume $\overline{P} \npreceq \overline{Q}$, that is, by Lemma 2.6, $P \cap L_1 \nsubseteq Q \cap L_1$. Then, there exists $a \in P \cap L_1$ and $a \notin Q \cap L_1$, so $P \in \sigma(a)$ and $Q \notin \sigma(a)$. Let us now prove that $\sigma(a) \in \mathbb{S}(\mathbb{E}(L_1))$.

Let $R \in \sigma(a)$ and $S \in \overline{R}$, then $R \cap L_1 = S \cap L_1$. Since $a \in L_1$, this shows that $a \in S$ and hence, $S \in \sigma(a)$. Therefore $\overline{R} \subseteq \sigma(a)$, which completes the proof. ∎

LEMMA 2.10. *Let* $\mathcal{H}$ *be a Heyting space and* $\mathcal{E}$ *a Heyting relation on* $\mathcal{H}$. *Define a binary relation* $\preceq$ *on* $\mathcal{H}/\mathcal{E}$ *as follows:*

$$\overline{a} \preceq \overline{b} \Longleftrightarrow \overline{a} \subseteq (\overline{b}].$$

Then $\preceq$ *is an ordering on* $\mathcal{H}/\mathcal{E}$.

PROOF. Clearly $\preceq$ is well defined, reflexive and transitive. Suppose that $\overline{a} \preceq \overline{b}$ and $\overline{b} \preceq \overline{a}$, then $a \leq b'$ and $b' \leq a'$, where $b' \in \overline{b}$ and $a' \in \overline{a}$. Since $\overline{a}$ is convex, we have that $b' \in \overline{a}$, that is, $\overline{a} = \overline{b}$. ∎

Observe that condition (3) in Definition 2.8 is not necessary for $\preceq$ to be an order.

Our next objective is to prove that if $\mathcal{H}$ is a *finite* Heyting space and $\mathcal{E}$ is an equivalence relation on $\mathcal{H}$, only condition (2) is necessary for $\mathcal{E}$ to be a Heyting relation. In order to show this we first prove the following result.

LEMMA 2.11. *If* $\mathcal{H}$ *is a finite Heyting space and* $\mathcal{E}$ *is an equivalence relation on* $\mathcal{H}$ *satisfying condition* (2) *then* $(\overline{a}]^c$ *is a clopen increasing set that satisfies* (S), *that is,* $(\overline{a}]^c \in \mathbb{S}(\mathcal{E})$.

Proof. Let $\overline{a} \in \mathcal{H}/\mathcal{E}$. Since the topology on $\mathcal{H}$ is the discrete topology, we have that $(\overline{a}]^c$ is a clopen increasing set.

In order to prove that $\mathcal{H}/\mathcal{E}$ satisfies condition (S), suppose that $b \in (\overline{a}]$ and $b' \in \overline{b}$, then $b \leq a'$, where $a' \in \overline{a}$. Hence, by condition (2), it follows that $\overline{b'} = \overline{b} \subseteq (\overline{a'}] = (\overline{a}]$. Thus $b' \in (\overline{a}]$.

This shows that (S) holds for $(\overline{a}]$, and consequently $(\overline{a}]^c$ also satisfies (S). ■

Proposition 2.12. *If $\mathcal{H}$ is a finite Heyting space, then, for each equivalence relation $\mathcal{E}$ defined on $\mathcal{H}$, condition* (2) *implies conditions* (1) *and* (3).

Proof. We first prove that (2) implies (1). Let $\overline{a}$ be an equivalence class associated with $\mathcal{E}$ and suppose that $a \leq b \leq a_1$, where $a, a_1 \in \overline{a}$. Hence, by (2) we have that $\overline{a} \subseteq (\overline{b}]$ and $\overline{b} \subseteq (\overline{a}]$, that is, there exists $b_1 \in \overline{b}$ such that $a_1 \leq b_1$ and there exists $a_2 \in \overline{a}$ such that $b_1 \leq a_2$. By this procedure we can obtain the following sequence

$$a \leq b \leq a_1 \leq b_1 \leq a_2 \leq b_2 \dots$$

Since $\mathcal{H}$ is finite we have $a_n = b_n$ for some $n \in \mathbb{N}$, that is, $\overline{a} = \overline{b}$. Then, $\overline{a}$ is convex. Moreover, $\overline{a}$ is closed since the topology in $\mathcal{H}$ is the discrete topology.

To deduce (3) from (2), take $\overline{x} \neq \overline{y}$. First, we know that $\preceq$ is an order as it follows from (1) and (2). If $\overline{x} \prec \overline{y}$, by the previous lemma, $U = (\overline{x}]^c \in \mathbb{S}(\mathcal{E})$. If $y \notin (\overline{x}]^c$, then $y \in (\overline{x}]$, that is, there exists $x' \in \overline{x}$ such that $y \leq x'$, but then $\overline{y} \preceq \overline{x}$, which contradicts $\overline{x} \prec \overline{y}$. Therefore, $y \in U$ and $x \notin U$. If $\overline{x} \not\preceq \overline{y}$ and $\overline{y} \not\preceq \overline{x}$, we can proceed analogously to the above case to show that there exists $U = (\overline{x}]^c \in \mathbb{S}(\mathcal{E})$ such that $y \in U$ but $x \notin U$. This completes the proof. ■

The following results characterize the subalgebras of a Heyting algebra by means of the Heyting relations defined on its Heyting space.

Proposition 2.13. *Let $\mathcal{H}$ be a Heyting space. If $\mathcal{E}$ is a Heyting relation on $\mathcal{H}$, then $\mathbb{S}(\mathcal{E})$ is a Heyting subalgebra of $\mathbb{D}(\mathcal{H})$.*

Proof. It is clear that $\mathbb{S}(\mathcal{E})$ is a sublattice of $\mathbb{D}(\mathcal{H})$ and $\emptyset, \mathcal{H} \in \mathbb{S}(\mathcal{E})$. It only remains to show that $\mathbb{S}(\mathcal{E})$ is closed under $\rightarrow$, that is, if $U_1, U_2 \in \mathbb{S}(\mathcal{E})$, $(U_1 \cap U_2^c]^c \in \mathbb{S}(\mathcal{E})$. Since $U_1, U_2 \in \mathbb{D}(\mathcal{H})$ and $\mathbb{D}(\mathcal{H})$ is a Heyting algebra then $(U_1 \cap U_2^c]^c \in \mathbb{D}(\mathcal{H})$. Let us prove that $(U_1 \cap U_2^c]^c$ satisfies condition (S). Suppose that $a \in (U_1 \cap U_2^c]$, so $a \leq b$ where $b \in U_1 \cap U_2^c$. Thus, by (2), we have that $\overline{a} \subseteq (\overline{b}]$. Since U_1 and $U_2 \in \mathbb{S}(\mathcal{E})$, $\overline{b} \subseteq U_1 \cap U_2^c$ and consequently,

$\overline{a} \subseteq (\overline{b}] \subseteq (U_1 \cap U_2^c]$. This shows that if $a \notin (U_1 \cap U_2^c]$ then $(U_1 \cap U_2^c] \cap \overline{a} = \emptyset$, that is, $\overline{a} \subseteq (U_1 \cap U_2^c]^c$. Therefore $(U_1 \cap U_2^c]^c \in \mathbb{S}(\mathcal{E})$ and the proof is complete. ■

We have thus proved that there exists a correspondence between Heyting subalgebras and Heyting relations. The following results show that this correspondence is one-to-one. Recall that given a Heyting algebra L, the map $\sigma : L \to \mathbb{D}(\mathbb{X}(L))$ defined by $\sigma(a) = \{P \in \mathbb{X}(L) : a \in P\}$ is an isomorphism between Heyting algebras.

Proposition 2.14. *For each subalgebra L_1 of a Heyting algebra L, $\mathbb{S}(\mathbb{E}(L_1)) = \sigma_L(L_1)$.*

Proof. Suppose that $\sigma(a) \in \sigma(L_1)$ for $a \in L_1$. Then $\sigma(a) \in \mathbb{S}(\mathbb{E}(L_1))$. In fact, let $P \in \sigma(a)$ and $Q \in \overline{P}$, then $Q \cap L_1 = P \cap L_1$ and since $a \in L_1 \cap P$, it follows that $a \in Q$, that is, $Q \in \sigma(a)$.

In order to prove the opposite inclusion, suppose that $U \in \mathbb{S}(\mathbb{E}(L_1))$. Since U is a clopen increasing set, $U = \sigma(a)$ for some $a \in L$. Now suppose that $a \notin L_1$. Let F be the filter of L generated by $[a) \cap L_1$. Since $(a] \cap F = \emptyset$, by the Birkhoff-Stone theorem, there exists $P \in \mathbb{X}(L)$ such that $[a) \cap L_1 \subseteq P$ and $a \notin P$. Let I be the ideal of L generated by $(L \backslash P) \cap L_1$. Since $I \cap [a) = \emptyset$, again by the Birkhoff-Stone theorem, there exists $Q \in \mathbb{X}(L)$ such that $a \in Q$ and $Q \cap L_1 \cap (L \backslash P) = \emptyset$, that is, $Q \cap L_1 \subseteq P \cap L_1$ and therefore, $\overline{Q} \leq \overline{P}$. Since $\mathbb{E}(L_1)$ is a Heyting relation, it follows that there exists $P' \in \overline{P}$ such that $Q \subseteq P'$. Then, since $\sigma(a)$ is an increasing set, we have that $P' \in \sigma(a)$ and consequently $\overline{P} \subseteq \sigma(a)$. From this, $a \in P$, which is impossible. Therefore, $a \in L_1$ which completes the proof. ■

Recall that if L is a finite Heyting algebra, its prime filters are generated by the prime elements of L. This allows us to derive the following relation between the elements of $\mathbb{S}(\mathbb{E}(L_1))$ and L_1.

Proposition 2.15. *Let L be a finite Heyting algebra and L_1 a subalgebra of L. If $U = \{P_1, P_2, \ldots, P_n\} \in \mathbb{S}(\mathbb{E}(L_1))$, then $U = \bigcup_{i=1}^{n} \sigma(p_i)$, where P_i is generated by the prime element p_i for each $i = 1, 2, \ldots n$. Moreover, $\bigvee_{i=1}^{n} p_i \in L_1$.*

Proof. Suppose that $U = \{P_1, P_2, \ldots, P_n\} \in \mathbb{S}(\mathbb{E}(L_1))$. We denote by p_i the prime element which generates P_i, for each i. It is clear that

$U \subseteq \bigcup_{i=1}^{n} \sigma(p_i)$. In order to prove the other inclusion, suppose that $Q \in \bigcup_{i=1}^{n} \sigma(p_i)$, then $p_i \in Q$ for some i. From this, $P_i \subseteq Q$ and since U is an increasing set we have $Q \in U$. Therefore, $U = \bigcup_{i=1}^{n} \sigma(p_i) = \sigma(\bigvee_{i=1}^{n} p_i)$ and by Proposition 2.14, $\bigvee_{i=1}^{n} p_i \in L_1$. ■

Now recall that for every Heyting space $\mathcal{H}$, the map $\epsilon_X : X \to \mathbb{X}(\mathbb{D}(X))$ defined by $\epsilon_X(x) = \{U \in \mathbb{D}(X) : x \in U\}$ is a homeomorphism and an order-isomorphism.

Proposition 2.16. *Let $\mathcal{H}$ be a Heyting space and $\mathcal{E}$ a Heyting relation on $\mathcal{H}$, then $\mathbb{E}(\mathbb{S}(\mathcal{E})) = \{(\epsilon(x), \epsilon(y)) : \overline{x} = \overline{y}\}$.*

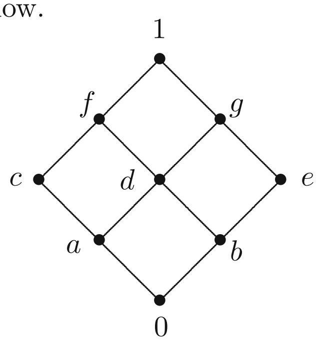
Proof. Suppose that $(\epsilon(x), \epsilon(y)) \notin \mathbb{E}(\mathbb{S}(\mathcal{E}))$ and $\overline{x} = \overline{y}$, then, $\epsilon(x) \cap \mathbb{S}(\mathcal{E}) \neq \epsilon(y) \cap \mathbb{S}(\mathcal{E})$. Without loss of generality we can assume there exists $U \in \mathbb{S}(\mathcal{E})$ such that $x \in U$ and $y \notin U$. Therefore, since $U \in \mathbb{S}(\mathcal{E})$, $\overline{x} = \overline{y}$ implies $x, y \in U$ which is a contradiction. This shows that $\{(\epsilon(x), \epsilon(y)) : \overline{x} = \overline{y}\} \subseteq \mathbb{E}(\mathbb{S}(\mathcal{E}))$.

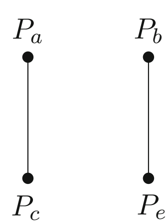
Now suppose that $\overline{x} \neq \overline{y}$, that is, $(\epsilon(x), \epsilon(y)) \notin \{(\epsilon(x), \epsilon(y)) : \overline{x} = \overline{y}\}$. Without loss of generality, by condition (3), we can assume that there exists $U \in \mathbb{S}(\mathcal{E})$ such that $x \in U$ and $y \notin U$. Hence, $\epsilon(x) \cap \mathbb{S}(\mathcal{E}) \neq \epsilon(y) \cap \mathbb{S}(\mathcal{E})$, that is, $(\epsilon(x), \epsilon(y)) \notin \mathbb{E}(\mathbb{S}(\mathcal{E}))$. This finishes the proof. ■

We have thus proved that there exists a one-to-one correspondence between subalgebras of a Heyting algebra and Heyting relations on its dual space.

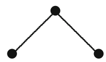
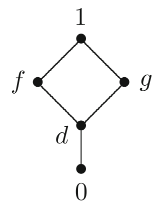
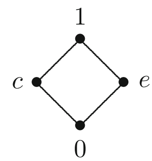
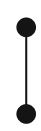
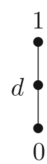
The next example shows how to obtain all the subalgebras of a finite Heyting algebra L. We have to find all the Heyting relations on its dual space, that is, all the possible partitions on $\mathbb{X}(L)$ that satisfy (2). Finally, we use Proposition 2.15 to obtain the subalgebra associated with each partition.

Consider the Heyting algebra L whose dual space is shown in the figure below.





The partitions associated with the Heyting relations on $\mathbb{X}(L)$ are shown in the following table.

$\mathbb{X}(L)/\mathcal{E}$	$(\mathbb{X}(L)/\mathcal{E}, \leq)$	Subalgebra
$\{\{P_a\}, \{P_c\}, \{P_b\}, \{P_e\}\}$	$\mathbb{X}(L)$	L
$\{\{P_c\}, \{P_e\}, \{P_a, P_b\}\}$		
$\{\{P_c, P_a\}, \{P_b, P_e\}\}$		
$\{\{P_a, P_b\}, \{P_c, P_e\}\}$		
$\{\{P_a, P_c, P_b\}, \{P_e\}\}$		
$\{\{P_a, P_e, P_b\}, \{P_c\}\}$		
$\{\{P_a, P_c, P_b, P_e\}\}$		

Condition (2) is necessary for $\mathbb{S}(\mathcal{E})$ to be a Heyting subalgebra as shown in the next example.

Let $\mathcal{E}$ be the equivalence relation whose associated partition is $\mathbb{X}(L)/\mathcal{E} = \{\{P_c, P_b, P_e\}, \{P_a\}\}$. It is clear that $\mathcal{E}$ does not satisfy (2) and $\mathbb{S}(\mathcal{E}) = \{\sigma(0), \sigma(a), \sigma(1)\}$ is not a Heyting subalgebra of $\mathbb{D}(\mathbb{X}(L))$.

In [1], M. Adams characterized maximal Heyting subalgebras. In order to prove this property for the finite case we will use the results we have already shown.

We first observe that if $\mathcal{H}$ is a Heyting space and $\mathcal{E}$ and $\mathcal{E}'$ are Heyting relations such that $\mathcal{E}' \subseteq \mathcal{E}$ then it is easily seen that $\mathbb{S}(\mathcal{E}) \subseteq \mathbb{S}(\mathcal{E}')$. In fact, there exists an antiisomorphism between the lattice of subalgebras of a Heyting algebra and the Heyting relations of its dual space.

Proposition 2.17. *Let $\mathcal{H}$ be a finite Heyting space and $\mathcal{E}$ a Heyting relation defined on $\mathcal{H}$. If $\mathbb{S}(\mathcal{E})$ is a maximal subalgebra then for each $\overline{x} \in \mathbb{X}(L)/\mathcal{E}$, $\overline{x}$ has at most two elements.*

Proof. Let $\mathcal{E}$ be a Heyting relation on $\mathcal{H}$ and $\mathcal{P} = \{\overline{x}_i\}_{i \in I}$ the partition associated with $\mathcal{E}$. Suppose that $\overline{x}_m$ has at least three elements. Let y be a minimal element in $\overline{x}_m$. We consider the following partition $\mathcal{P}'$ which is a refinement of $\mathcal{P}$.

(i) If $\overline{x}_i \prec \overline{x}_m$ we divide $\overline{x}_i$ in singleton classes, that is, for each $z \in \overline{x}_i$ we consider a new class $\overline{z}' = \{z\}$,

(ii) $\overline{x}'_{m_1} = \{y\}$; $\qquad \overline{x}'_{m_2} = \overline{x}_m \setminus \{y\}$.

(iii) If $\overline{x}_i \not\preceq \overline{x}_m$, we consider $\overline{x}'_i = \overline{x}_i$.

It is clear that $\mathcal{P}'$ is a partition of $\mathcal{H}$. Let us prove that the associated equivalence relation $\mathcal{E}'$ is a Heyting relation.

First observe that $\mathcal{E}'$ satisfies (2). Indeed, suppose that $z \leq w$ and consider the following cases.

Suppose that $\overline{z} \not\preceq \overline{x}_m$. Then $\overline{w} \not\preceq \overline{x}_m$. In fact, if $\overline{w} \preceq \overline{x}_m$, there exists $t \in \overline{x}_m$ such that $w \leq t$, but then $z \leq w \leq t$ and so $\overline{z} \preceq \overline{x}_m$, which contradicts our assumption. Therefore, $\overline{z}' = \overline{z}$ and $\overline{w}' = \overline{w}$ and since $\mathcal{E}$ is a Heyting relation, we have that $\overline{z}' \subseteq (\overline{w}']$.

If $\overline{z} \preceq \overline{x}_m$ and $\overline{z} \neq \overline{x}_m$, then $\overline{z}' = \{z\}$ and so, $\overline{z}' \subseteq (\overline{w}']$.

Finally, suppose that $\overline{z} = \overline{x}_m$. If $\overline{z}' = \overline{x}_{m_1} = \{y\}$ then $\overline{z}' \subseteq (\overline{w}']$. If $\overline{z}' = \overline{x}_{m_2} = \overline{x}_m \setminus \{y\}$ then either $\overline{w} \not\preceq \overline{x}_m$ or $\overline{w} = \overline{x}_m$. In the first case, $\overline{w}' = \overline{w}$ and so $\overline{z}' = \overline{x}_{m_2} \subseteq (\overline{w}']$. And in the second one, $\overline{w}' = \overline{x}_{m_2}$ and so $\overline{z}' = \overline{x}_{m_2} \subseteq (\overline{w}']$.

This proves that $\mathcal{E}'$ satisfies (2). Therefore, since $\mathcal{H}$ is a finite Heyting space, by Proposition 2.12, $\mathcal{E}'$ also satisfies (1) and (3), which shows that $\mathcal{E}'$ is a Heyting relation.

Finally, taking into account that $\mathcal{P}'$ is a refinement of $\mathcal{P}$, $\mathcal{E}' \subseteq \mathcal{E}$, and then $\mathbb{S}(\mathcal{E}) \subsetneq \mathbb{S}(\mathcal{E}')$, so $\mathbb{S}(\mathcal{E})$ is not a maximal subalgebra. ■

The following proposition provides a characterization of maximal subalgebras and coincides with the result given by M. Adams in [1].

Proposition 2.18. *Let $\mathcal{H}$ be a finite Heyting space and $\mathcal{E}$ a Heyting relation on $\mathcal{H}$. Then $\mathbb{S}(\mathcal{E})$ is maximal if and only if each equivalence class associated with $\mathcal{E}$ is a singleton, except one which is of the form $\overline{x} = \{x_1, x_2\}$ where $[x_1) \cup \{x_2\} = [x_2) \cup \{x_1\}$.*

Proof. By Proposition 2.17 we know that if $\overline{x} \in \mathcal{H}/\mathcal{E}$ then $\overline{x}$ has at most two elements. We consider the following set

$$R = \{\overline{x} \in \mathcal{H}/\mathcal{E} : \overline{x} \text{ has exactly two elements } \}$$

and let $\overline{x}_m$ be a maximal element in R.

Let $\mathcal{P}'$ be the refinement of $\mathcal{H}/\mathcal{E}$ given by: $\overline{x}'_m = \overline{x}_m$ and $\overline{x}' = \{x\}$ for all $x \in \mathcal{H} \backslash \overline{x}_m$. We can now proceed analogously to the proof of Proposition 2.17 to show that the equivalence relation $\mathcal{E}'$ associated with $\mathcal{P}'$ is a Heyting relation. This proves that $\mathcal{H}/\mathcal{E}$ has a unique equivalence class $\overline{x} = \{x_1, x_2\}$ with two elements and the other equivalence classes are singletons. It remains to show that $[x_1) \cup \{x_2\} = [x_2) \cup \{x_1\}$.

Let $y \in [x_1) \cup \{x_2\}$, $y \neq x_1, x_2$, then $x_1 \leq y$ an so $\overline{x_1} \preceq \overline{y} = \{y\}$. By condition (2) in the definition of a Heyting partition we have that $x_2 \leq y$ and then $y \in [x_2) \cup \{x_1\}$.

In a similar way we can prove the other inclusion. Therefore, $[x_1) \cup \{x_2\} = [x_2) \cup \{x_1\}$.

Clearly, if $\mathcal{E}$ is an equivalence relation on $\mathcal{H}$ satisfying that each equivalence class has a unique element except one which satisfies $[x_1) \cup \{x_2\} = [x_2) \cup \{x_1\}$, then $\mathcal{E}$ is a Heyting relation and the associated subalgebra $\mathbb{S}(\mathcal{E})$ is maximal. ■

3. Characterization of De Morgan Heyting subalgebras

In this section we give necessary and sufficient conditions for the subalgebra associated with a Heyting relation on a De Morgan Heyting space to be a De Morgan Heyting subalgebra.

PROPOSITION 3.1. *Let L be a De Morgan Heyting algebra. If L_1 is a subalgebra of L and $\leq$ is the order defined on $\mathbb{X}(L)/\mathbb{E}(L_1)$ in the Remark* 2.4 *then the following conditions hold:*

(1) *$\overline{P}$ is closed and convex for each $P \in \mathbb{X}(L)$.*

(2) *If $\overline{P} \leq \overline{Q}$ then $\overline{P} \subseteq (\overline{Q}]$.*

(3) *If $\overline{P} \neq \overline{Q}$ then there exists $U \in \mathbb{S}(\mathbb{E}(L_1))$ that separates P and Q.*

(4) *for each $\overline{P} \in \mathbb{X}(L)/\mathbb{E}(L_1)$, $\phi(\overline{P}) \in \mathbb{X}(L)/\mathbb{E}(L_1)$, where $\phi(\overline{P}) = \{\phi(Q) : Q \in \overline{P}\}$.*

PROOF. Since L_1 is a Heyting subalgebra, by Proposition 2.9, $\mathbb{E}(L_1)$ satisfies conditions $(1) - (3)$. Let us prove (4).

If we consider the homomorphism between De Morgan Heyting algebras $i : L_1 \to L$ such that $i(x) = x$ then $\mathbb{X}(i) : \mathbb{X}(L) \to \mathbb{X}(L_1)$ given by $\mathbb{X}(i) = i^{-1}(P) = P \cap L_1$ is a dh-morphism and then

$$\phi_{L_1}(P \cap L_1) = \phi(P) \cap L_1, \qquad (1)$$

for all $P \in \mathbb{X}(L)$.

Taking into account this result, we will prove that $\phi(\overline{P}) = \overline{\phi(P)}$.

Let $\phi(R) \in \phi(\overline{P})$ then $R \in \overline{P}$, an so $R \cap L_1 = P \cap L_1$. Thus, $\phi_{L_1}(R \cap L_1) = \phi_{L_1}(P \cap L_1)$. From (1), we have that $\phi(R) \cap L_1 = \phi(P) \cap L_1$ and consequently $\phi(R) \in \overline{\phi(P)}$. Therefore, $\phi(\overline{P}) \subseteq \overline{\phi(P)}$.

Suppose now that $R \in \overline{\phi(P)}$. Then $R \cap L_1 = \phi(P) \cap L_1$. Thus, $\phi_{L_1}(R \cap L_1) = \phi_{L_1}(\phi(P) \cap L_1))$, and so $\phi(R) \cap L_1 = \phi(\phi(P)) \cap L_1$. It follows that $\phi(R) \in \overline{P}$ and so $R \in \phi(\overline{P})$.

Consequently we have that for all $\overline{P} \in \mathbb{X}(L)/\mathbb{E}(L_1), \phi(\overline{P}) \in \mathbb{X}(L)/\mathbb{E}(L_1)$. This completes the proof. ■

REMARK 3.2. In section 1 we defined the order $\leq$ on $\mathbb{X}(L)/\mathbb{E}(L_1)$ by means of (H). If L is a De Morgan Heyting algebra, $\leq$ also satisfies

$$(DH) \qquad \text{if } \overline{P} \leq \overline{Q} \text{ then } \overline{Q} \subseteq [\overline{P}).$$

In fact, suppose that $\overline{P} \leq \overline{Q}$, then, by Lemma 2.6, $P \cap L_1 \subseteq Q \cap L_1$. From this $\phi_{L_1}(Q \cap L_1) \subseteq \phi_{L_1}(P \cap L_1)$ and so $\overline{\phi(Q)} \leq \overline{\phi(P)}$. Therefore, $\overline{\phi(Q)} \subseteq (\overline{\phi(P)}]$, then $\phi(\overline{\phi(Q)}) \subseteq \phi((\overline{\phi(P)}])$, that is, $\phi(\overline{\phi(Q)}) \subseteq [\phi(\overline{\phi(P)})) = [\phi(\phi(\overline{P})))$. Finally we have that $\overline{Q} \subseteq [\overline{P})$.

Condition (4) in Proposition 3.1 allows us to define the map Φ from $\mathbb{X}(L)/\mathbb{E}(L_1)$ into $\mathbb{X}(L)/\mathbb{E}(L_1)$ in the following way $\Phi(\overline{P}) = \{\phi(Q) : Q \in \overline{P}\}$.

Proposition 3.3. *Let L be a De Morgan Heyting algebra and L_1 a subalgebra of L, the pair $(\mathbb{X}(L)/\mathbb{E}(L_1), \Phi)$ with the quotient topology and ordered by $\leq$ defined in Lemma 2.4 is isomorphic to the De Morgan Heyting space $(\mathbb{X}(L_1), \phi_{L_1})$.*

Proof. In order to see that $(\mathbb{X}(L)/\mathbb{E}(L_1), \Phi)$ is a De Morgan Heyting space, we will show that Φ is an order reversing involutive homeomorphism.

Let us prove that Φ is an anti-isomorphism of period 2, that is $\Phi(\Phi(\overline{P})) = \overline{P}$ and $\overline{P} \leq \overline{Q}$ if and only if $\Phi(\overline{Q}) \leq \Phi(\overline{P})$.

In the proof of Proposition 3.1 we proved that $\Phi(\overline{P}) = \overline{\Phi(P)}$. Then we have $\Phi(\Phi(\overline{P})) = \Phi(\overline{\Phi(P)}) = \overline{\Phi(\Phi(P))} = \overline{P}$.

Let us show now that if $\overline{P} \leq \overline{Q}$ then $\Phi(\overline{Q}) \leq \Phi(\overline{P})$.

Since $\overline{P} \leq \overline{Q}$, by Lemma 2.6, $P \cap L_1 \subseteq Q \cap L_1$ and then $\phi_{L_1}(Q \cap L_1) \subseteq \phi_{L_1}(P \cap L_1)$. From (1), we have that $\phi(Q) \cap L_1 \subseteq \phi(P) \cap L_1$, that is, $\overline{\phi(Q)} \leq \overline{\phi(P)}$ and consequently $\Phi(\overline{Q}) \leq \Phi(\overline{P})$.

Since $\Phi = \Phi^{-1}$ then it follows that if $\Phi(\overline{P}) \leq \Phi(\overline{Q})$ then $\overline{Q} \leq \overline{P}$.

Let us show that Φ is continuous.

Consider an open set $U = \{\overline{P_i}, i \in I\} \subseteq \mathbb{X}(L)/\mathbb{E}(L_1)$. Then $\bigcup_{i\in I} \overline{P}_i$ is an open set in $\mathbb{X}(L)$. Since ϕ is a homeomorphism, $\phi[\bigcup_{\overline{P}_i \in U} \overline{P}_i]$ is an open set. Consequently, we have that $\Phi(U) = \phi[\bigcup_{i\in I} \overline{P}_i]$ is open. On the other hand, $\Phi^{-1}(U) = \Phi(U)$ and thus we have that $\Phi^{-1}(U)$ is an open set.

From this Φ is an order reversing involutive homeomorphism.

In Proposition 2.7 we showed that $h : \mathbb{X}(L)/\mathbb{E}(L_1) \to \mathbb{X}(L_1)$ given by $\mathbb{X}(i) = h \circ \pi$ is a morphism between Heyting spaces. We now prove that h is a morphism in the category of De Morgan Heyting spaces, that is, $h \circ \Phi = \phi_{L_1} \circ h$.

In order to prove this, observe that $\mathbb{X}(i)$ is a morphism in the category of De Morgan spaces and the natural application π satisfies $\Phi \circ \pi = \pi \circ \phi$. In fact, $(\Phi \circ \pi)(P) = \Phi(\overline{P}) = \overline{\phi(P)} = (\pi \circ \phi)(P)$.

Thus, we have, $(h\circ\Phi)(\overline{P}) = (h\circ\Phi\circ\pi)(P) = (h\circ\pi\circ\phi)(P) = (\mathbb{X}(i)\circ\phi)(P) = (\phi_{L_1} \circ \mathbb{X}(i))(P) = (\phi_{L_1} \circ h \circ \pi)(P)) = (\phi_{L_1} \circ h)(\overline{P})$.

This shows that h is an isomorphism between the De Morgan Heyting spaces $\mathbb{X}(L)/\mathbb{E}(L_1)$ and $\mathbb{X}(L_1)$. ■

Definition 3.4. *Let $(\mathcal{H}, \leq, \tau, \phi)$ be a De Morgan Heyting space. An equivalence relation $\mathcal{E}$ is called a De Morgan Heyting relation, if the following conditions hold:*

(1) *$\overline{a}$ is convex and closed for each $a \in \mathcal{H}$.*

(2) *$\overline{a} \subseteq (\overline{b}]$, for all $a, b \in \mathcal{H}$ such that $a \leq b$.*

(3) *If* $\overline{a} \neq \overline{b}$ *then there exists* $U \in \mathbb{S}(\mathcal{E})$ *that separates* a *and* b.

(4) *For each* $\overline{a} \in \mathcal{H}/\mathcal{E}$, $\phi(\overline{a}) \in \mathcal{H}/\mathcal{E}$.

REMARK 3.5. Note that by Proposition 3.1, $\mathbb{E}(L_1)$ is a De Morgan Heyting relation.

PROPOSITION 3.6. *Let* $\mathcal{H}$ *be a De Morgan Heyting space. If* $\mathcal{E}$ *is a De Morgan Heyting relation then the set* $\mathbb{S}(\mathcal{E})$ *is a De Morgan Heyting subalgebra of* $\mathbb{D}(\mathcal{H})$.

PROOF. We have already proved that $\mathbb{S}(\mathcal{E})$ is a Heyting subalgebra. It remains to prove that if $U \in \mathbb{S}(\mathcal{E})$ then $U' = \phi(U)^c \in \mathbb{S}(\mathcal{E})$.

We know that $\phi(U)^c$ is a clopen increasing set. Furthermore, if $P \in \phi(U)$ then $P = \phi(Q)$ where $Q \in U$ and by condition (4), $\phi(\overline{Q}) = \overline{P}$. On the other hand, $\overline{Q} \subseteq U$, because $U \in \mathbb{S}(\mathcal{E})$ and consequently $\overline{P} \subseteq \phi(U)$. This shows that $\phi(U)$ satisfies condition (S), and consequently $U' = \phi(U)^c$ also satisfies condition (S). ■

Analogously to the case of Heyting algebras, if L is a De Morgan Heyting algebra and L_1 is a subalgebra of L then $\mathbb{S}(\mathbb{E}(L_1)) = \sigma_L(L_1)$. In addition, if $\mathcal{H}$ is a De Morgan Heyting space and $\mathcal{E}$ is a De Morgan Heyting relation we have that $\mathbb{E}(\mathbb{S}(\mathcal{E})) = \{(\epsilon(x), \epsilon(y)) : \overline{x} = \overline{y}\}$.

We have thus proved that there exists a one-to-one correspondence between De Morgan Heyting subalgebras and De Morgan Heyting relations.

By Proposition 3.3 we know that there exists the same number of prime filters in a De Morgan Heyting algebra as equivalence classes on its associated relation. Now, we will show, in the finite case, how we can obtain each prime element of a subalgebra from the elements of each corresponding equivalence classes. This is not always possible in the case of Heyting algebras.

LEMMA 3.7. *If* L_1 *is a De Morgan Heyting subalgebra of* L *and* L *is finite, for each* $\overline{P} = \{P_1, P_2, \ldots P_n\}$, $\overline{Q} = \{Q_1, Q_2, \ldots, Q_m\}$ *in* $\mathbb{X}(L)/\mathbb{E}(L_1)$ *such that* $\overline{P} \leq \overline{Q}$, *we have that* $\bigvee_{j=1}^{m} q_j \leq \bigvee_{i=1}^{n} p_i$, *where* P_i *is generated by the prime element* p_i *and* Q_j *is generated by the prime element* q_j.

PROOF. Suppose that $Q_j \in \overline{Q}$. Since $\overline{P} \leq \overline{Q}$, by (DH) it follows that there exists $P_i \in \overline{P}$ such that $P_i \subseteq Q_j$. From this $q_j \leq p_i$ for some i and consequently $\bigvee_{j=1}^{m} q_j \leq \bigvee_{i=1}^{n} p_i$. ■

PROPOSITION 3.8. *Let* L *be a finite De Morgan Heyting algebra and* L_1 *a subalgebra of* L. *For each* $\overline{P} = \{P_1, P_2, \ldots, P_n\} \in \mathbb{X}(L)/\mathbb{E}(L_1)$ *we have* *that* $\bigvee_{i=1}^{n} p_i \in L_1$*, where* p_i *is the prime element generating* P_i*, for each* $P_i \in \overline{P}$*. Moreover, every prime element of* L_1 *has this form.*

Proof. Let $\overline{P} = \{P_1, P_2, \ldots, P_n\} \in \mathbb{X}(L)/\mathbb{E}(L_1)$ and consider the set $[\overline{P})$. Since $\mathbb{X}(L)$ is finite then $[\overline{P})$ is a clopen increasing set. Let us prove that $[\overline{P}) \in \mathbb{S}(\mathbb{E}(L_1))$. In order to prove that, it remains to prove that (S) holds.

Suppose that $Q \in [\overline{P})$, then there exists $R \in \overline{P}$ such that $R \subseteq Q$ and so $\overline{R} = \overline{P} \leq \overline{Q}$. By condition (DH), it follows that $\overline{Q} \subseteq [\overline{P})$, which proves that $[\overline{P}) \in \mathbb{S}(\mathbb{E}(L_1))$.

On the other hand, by Proposition 2.15,

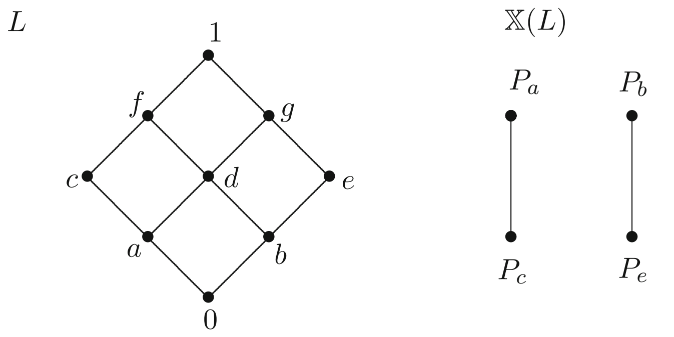
$$[\overline{P}) = \bigcup_{Q_i \in [\overline{P})} \sigma(q_i) = \sigma(\bigvee_{Q_i \in [\overline{P})} q_i).$$

But by Lemma 3.7 we have that $\sigma(\bigvee_{Q_i \in [\overline{P})} q_i) = \sigma(\bigvee_{i=1}^{n} p_i)$. Hence, again by Proposition 2.15 we obtain $\bigvee_{i=1}^{n} p_i \in L_1$. ■

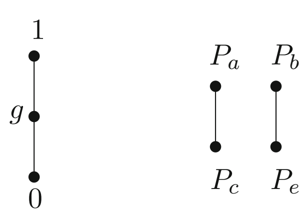
We have thus proved that if p_i is the prime element generating P_i for each $P_i \in \overline{P}$ then $\bigvee_{i=1}^{n} p_i$ is a prime elements of L_1. Moreover, every prime element of L_1 is of this form. Then we can compute all the elements of L_1, since they are joins of the prime elements.

In the next example we show that this does not happen in general.

Consider the Heyting algebra



and the Heyting subalgebra L_1 with its associated partition $\mathbb{X}(L)/\mathbb{E}(L_1)$:



If we consider $\overline{P_c} = \{P_c\}$, $\bigvee_{P_i \in \overline{P_c}} p_i = c$ but $c \notin L_1$.

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