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Effective heat conductivity of a composite with hexagonal lattice of perfectly conducting circular inclusions: An analytical solution

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Paper is devoted to the effective stationary heat conductivity for the fibre composite materials. We are aimed on getting on analytical expression for effective thermal conductivity coefficient. Asymptotic homogenization approach, based on the multiple scale perturbation method, is used. This allows to reduce the original boundary value problem in multiply connected domain to the sequence of boundary value problems in simply connected domains. These problems include: the local problem for the periodically repeated cell and homogenized problem with effective coefficient. It is shown that for densely packed and high contrast fibre composites, the cell problem can be solved analytically. For this aim, lubrication approach (asymptotics of thin layer) has been employed. We also generalise obtained solution to the case of medium-sized inclusions in the framework of the Padé approximants.

1 | INTRODUCTION

Periodically inhomogeneous composite materials, consisting of several components with different physical properties, are widely used in aircraft, rocket and shipbuilding, mechanical engineering, industrial and civil engineering and other areas of modern industry. The choice of various materials for the phases of composite and for inclusions forms allows obtaining materials with useful properties like high strength and rigidity, low thermal conductivity and so forth.

One of the most common approximations in the theory of composite materials is the homogenization theory [1]. Its essence is to replace the original inhomogeneous medium with a homogeneous one, in a certain sense equivalent to the original one. The properties of this homogeneous medium are called effective characteristics. Let us explain what we mean by effective properties in our case. A homogeneous medium with effective properties gives the same response to 'slow' perturbations as the original inhomogeneous medium ('slow' mean perturbations which characteristic period is much larger than the size of periodically repeating cells of a composite).

Various mathematical techniques are used to determine the effective characteristics. We single out the Rayleigh method [2–5], the Natanzon–Filshtinsky method [5–9], and the method of functional equations [5, 8]. The first two methods lead to an infinite system of linear algebraic equations, which is then reduced to a finite number of equations and solved numerically. The analysis of the reduced system makes it possible also to obtain approximate analytical dependences [2]. Such solutions are sufficiently accurate for finite values of the physical characteristics of inclusions or for dilute composites [3] or composite with moderate concentration [2]. The claims of some authors who present the reduction of the original problem to an infinite system of linear algebraic equations as an 'exact' solution or a 'closed-form solution' are groundless (see a detailed analysis of this problem in Refs. [4, 5]).

For dilute composites, there are many other approaches, for example, Mori–Tanaka approximation, approach based on Eshelby tensor, double-inclusion method and so forth. All of them lead to the same results, although authors often claim a significant expansion of the scope of their solutions applicability (see Andrianov and Mityushev [4]).

The method of functional equations allows to get a formally exact solution in which the Rayleigh and Natanzon–Filshtinsky coefficients are written out by formulas in the form of series [5, 8].

As a rule, the size of a repeated cell of a composite is significantly smaller than the size of the entire structure, which allows us to consider their ratio as a small parameter and use the asymptotic homogenization method (AHM) for PDEs with rapidly oscillating or periodically discontinuous coefficients [10–20]. This method can be briefly described as follows. First, periodically repeating boundary value problem ('cell problem') is isolated and its solution is sought under the boundary conditions of periodic continuation. Local coordinates are introduced ('fast variables') [21] and next the multiple scale perturbation method is applied [22]. Next, the averaging over local (fast) variables is performed. The efficiency of the homogenization method depends on possibility to solve cell problem. An exact analytical solution of the cell problem is available only in the simplest cases, for example, for layered composites [10, 11]. Often finite element method (FEM) or another numerical method is used for this purpose [12–19]. Numerical methods are of particular interest since they allow to determine the representative volume element (RVE) [16], to estimate boundary layers for composites of finite sizes, and to investigate the redistribution of temperature flows at the initial time instant [18, 19] and so forth. At the same time, the analytical expressions of the effective properties of composites is of fundamental interest. For obtaining the mentioned characteristics, the analytical solutions of the cell problem are highly required. First, for densely packed, high contrast composites, the accuracy of numerical methods decreases, and the computer time costs increase significantly. Second, analytical solutions can be used as benchmark for numerical algorithms. Third, the analytical solutions are convenient for preliminary engineering estimations and for optimal design.

An approximate analytical solving of the cell problem requires the use of additional small parameters [20, 21]. For two-phase media in which variations in the phase properties are small, weak-contrast expansions where developed [20, 23]. This approach is applicable at all volume fractions of inclusions. A drawback of weak-contrast expansions approach is that it is valid only when the two phase properties are nearly the same. Strong-contrast expansions based on powers of rational functions of the phase conductivities [24] yield a possibility for a slight extension of the area of applicability of the latter approach. On the other hand, though the methods that use a small volume fraction of inclusions (dilute composite materials) are well developed [25], but their limitation is obvious. It should be emphasized that the lattice approximation (replacing an inhomogeneous continuous medium with a discrete one) for the analytical homogenization of periodic composites has been proposed in Movchan et al. [3].

This paper stands as a continuation of the series of works devoted to the application of AHM to the fibre-reinforced composites of various structures (square [26–28], hexagonal [27, 28, 30–32]) and various type of inclusions (circular [26, 27, 30–32], square [29], rhombic [27] and curvilinear rhombic [27]). A distinctive feature of these works relates on the analytical solving of the cell problems. In particular, the following problems have been studied for the hexagonal lattice of circular inclusions. The dynamics of a composite membrane was considered in Andrianov et al. [31]. The problem of non-stationary heat transfer for large inclusions was solved in Andrianov et al. [32]. Stationary heat transfer in the cases of inclusions of large and close to limit possible sizes and high conductivity was studied in Kalamkarov et al. [30]. In this paper, we show that the transformation of the solution using Padé approximations fundamentally expands the scope of its application. Namely, we may compute the effective thermal conductivity coefficient for any sizes of absolutely conducting inclusions. In the limiting case, the leading term of the asymptotics is obtained, and it coincides with the well-known result [28].

Methods for introducing a small parameter characterizing the inhomogeneity of composite are based on the following steps. After introducing a small parameter, AHM [33] is used. The introduction of slow and fast variables gives the possibility to solve the original problem for a multiply connected domain in two steps. In the zero approximation, one obtains the boundary value problem for the periodically repeated cell in a simply connected domain. The homogenized equation is obtained after substituting the solution of the local problem into the equation of the first approximation and next averaging over fast variables. The found coefficients describe the effective properties of the homogeneous medium.

For the analytical solution of the cell problem, a small parameter is used, which represents the ratio of the distance between inclusions to the cell size. Further, the lubrication approach [34, 35] (thin layer asymptotics [36, 37]) is employed. The use of Padé approximants yields a solution that is also applicable to medium size inclusions.

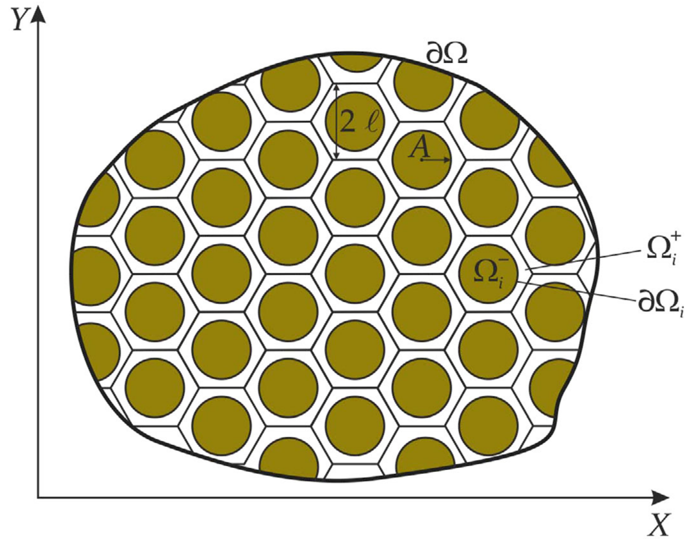


FIGURE 1 Composite under consideration

This paper is organized in the following way. In Section 2, we describe the main features of the physical problem under consideration. In Section 3, we state the main mathematical model. In Section 4, the approach for medium sized inclusions is described. In Section 5, two-sided estimates for the asymptotics of the effective parameter are presented and discussed. Last, in Section 6, the conclusions derived from this work are summarized.

2 | STATEMENT OF THE PROBLEM

The problem of stationary thermal conductivity for a composite with periodically arranged circular inclusions constituting a hexagonal lattice is considered (Figure 1) assuming an ideal contact between the matrix and the inclusions.

The heat conductivity problem is described by the following PDEs:

$$\lambda^{+}\left(\frac{\partial^2 u^+}{\partial x^2}+\frac{\partial^2 u^+}{\partial y^2}\right)=F(x,y)\quad \text{in}\quad \Omega_i^+, \tag{1}$$

$$\lambda^{-}\left(\frac{\partial^2 u^-}{\partial x^2}+\frac{\partial^2 u^-}{\partial y^2}\right)=F(x,y)\quad \text{in}\quad \Omega_i^-, \tag{2}$$

$$u^+=u^-,\quad \lambda^+\frac{\partial u^+}{\partial \mathbf{n}}=\lambda^-\frac{\partial u^-}{\partial \mathbf{n}}\quad \text{at}\quad \partial\Omega_i. \tag{3}$$

In Equations (1)–(9), the following notation is adopted: $x = X/l$; $y = Y/l$; $F(x, y) = l^2 f(x, y)$; u^+, u^- are the temperature distribution functions in the matrix and inclusions; λ_i^+, λ_i^- are the thermal conductivities of the phases of the composite, $\frac{\lambda^-}{\lambda^+} = \lambda$; $f(x, y)$ is the density of external heat sources; $\mathbf{n}$ stands for the outer normal to the inclusion contour.

Consider one of the terms in the expansion of the density of external heat sources in form of the Fourier series

$$F(x,y)=C\sin\frac{lx}{L_1}\sin\frac{ly}{L_2}. \tag{4}$$

We assume that the function $F(x, y)$ changes slowly in the sense that $l/L_i << 1$. In the latter case, it is possible to describe the behaviour of the composite in framework of effective medium theory [1, 33]. We introduce a small parameter $\varepsilon = \max\{l/L_i\} << 1$ and apply AHM. In accordance with AHM, based on the multiple scale asymptotic expansions [1, 33], the solution of the problem in the multiply connected domain (1)–(3) is represented as series in powers of the dimensionless small parameter ε, that is, we have

$$u^{\pm}=u_0(x,\,y)+\varepsilon\,u_1^{\pm}(x,\,y,\,\xi,\,\eta)+\varepsilon^2\,u_2^{\pm}(x,\,y,\,\xi,\,\eta)+\cdots \tag{5}$$

where ξ, η stand for ‘fast’ variables, $\xi = \frac{x}{\varepsilon}$, $\eta = \frac{y}{\varepsilon}$.

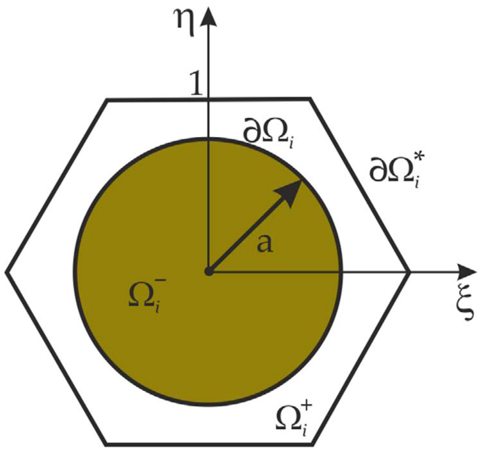


FIGURE 2 Periodically repeated composite cell: Ω_i^+ (Ω_i^-) is the matrix (inclusion) region ($a = A/l$)

Thus, the solution is divided into slowly changing and rapidly oscillating parts. Recall that the T-periodic function $f(\varepsilon,\theta)$ is called ε^{-1} oscillating, if [38] (Appendix 1, pp. 73, 74 in this paper)

$$0 < C_1 \le \int_0^T |f(\varepsilon,\theta)|^2 d\theta \le C_2 < \infty, \quad |\int_0^\alpha f(\varepsilon,\theta)d\theta| \le C\varepsilon, \qquad 0 \le \alpha \le T, \tag{6}$$

where C, C_1, C_2 are the positive constants.

We use the following normalization:

$$\iint_{\partial\Omega^+} u_i^+(x, y, \xi, \eta)d\xi d\eta + \lambda \iint_{\partial\Omega^-} u_i^-(x, y, \xi, \eta)d\xi d\eta = 0, \quad i = 1,2,3,\ldots \tag{7}$$

Functions $u_i^\pm$ satisfy the periodicity conditions

$$u_i^\pm(x, y, \xi+k, \eta+j) = u_i^\pm(x, y, \xi, \eta), \quad i = 1,2,3,\ldots, \tag{8}$$

where k, j are integer non-negative numbers.

Taking into account Ansatz (5), after an expansion in powers of a small parameter ε with regard to Equations (1)–(3), the solution of the original problem is divided into two stages. At the first stage, the solution of the local problem is determined, that is, the problem for the periodically repeating cell of the composite (Figure 2):

$$\frac{\partial^2 u_1^\pm}{\partial\xi^2} + \frac{\partial^2 u_1^\pm}{\partial\eta^2} = 0 \quad \text{in} \quad \Omega_i^\pm, \tag{9}$$

$$u_1^+ = u_1^-, \quad \frac{\partial u_1^+}{\partial\bar{\mathbf{n}}} - \lambda\frac{\partial u_1^-}{\partial\bar{\mathbf{n}}} = (\lambda-1)\frac{\partial u_0}{\partial\mathbf{n}} \quad \text{at} \quad \partial\Omega_i, \tag{10}$$

$$u_1^+ = 0 \quad \text{at} \quad \partial\Omega_i^*, \tag{11}$$

where $\frac{\partial}{\partial\bar{\mathbf{n}}}$, $\frac{\partial}{\partial\mathbf{n}}$ are the derivatives along the external normal to an inclusion contour regarding fast and slow variables, respectively, and they read

$$\frac{\partial}{\partial\bar{\mathbf{n}}} = \cos\alpha\frac{\partial}{\partial\xi} + \cos\beta\frac{\partial}{\partial\eta}, \qquad \frac{\partial}{\partial\mathbf{n}} = \cos\alpha\frac{\partial}{\partial x} + \cos\beta\frac{\partial}{\partial y}. \tag{12}$$

The second stage is to find the main ('slow') part of the solution $u_0(x, y)$ from the following homogenized PDE:

$$\bar{q}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}\right)+\frac{1}{\left|\Omega_i^*\right|}\left[\iint\limits_{\Omega_i^+}\left(\frac{\partial^2 u_1^+}{\partial x\partial \xi}+\frac{\partial^2 u_1^+}{\partial y\partial \eta}\right)d\xi\partial\eta+\lambda\iint\limits_{\Omega_i^-}\left(\frac{\partial^2 u_1^-}{\partial x\partial \xi}+\frac{\partial^2 u_1^-}{\partial y\partial \eta}\right)d\xi\partial\eta\right]=F, \tag{13}$$

where $\Omega_i^* = \Omega_i^+ \bigcup \Omega_i^-$, and $\bar{q} = \frac{|\Omega_i^+|+\lambda|\Omega_i^-|}{|\Omega_i^*|}$ stands for the Voigt-averaged parameter.

After substitution functions u_1^+ to Equation (13), u_1^-, the homogenized equation can be rewritten in the following way:

$$q_x\frac{\partial^2 u_0}{\partial x^2}+q_y\frac{\partial^2 u_0}{\partial y^2}=F \quad \text{in} \quad \Omega^*, \tag{14}$$

where Ω^* is a simply connected domain (in this case, a plane).

Effective characteristics q_x, q_y follow

$$q_x=\bar{q}+\frac{1}{\left|\Omega_i^*\right|}\left(\iint\limits_{\Omega_i^+}\frac{\partial u^+_{1(1)}}{\partial \xi}\,d\xi\partial\eta+\lambda\iint\limits_{\Omega_i^-}\frac{\partial u^-_{1(1)}}{\partial \xi}\,d\xi\partial\eta\right), \tag{15}$$

$$q_y=\bar{q}+\frac{1}{\left|\Omega_i^*\right|}\left(\iint\limits_{\Omega_i^+}\frac{\partial u^+_{1(2)}}{\partial \eta}\,d\xi\partial\eta+\lambda\iint\limits_{\Omega_i^-}\frac{\partial u^-_{1(2)}}{\partial \eta}\,d\xi\partial\eta\right), \tag{16}$$

where $u^\pm_{1(i)}$ $(i = 1, 2)$ are solutions of the local problem (9)–(11). We have

$$u_1^\pm = u^\pm_{1(1)}(\xi, \eta)\frac{\partial u_0}{\partial x}+u^\pm_{1(2)}(\xi, \eta)\frac{\partial u_0}{\partial y}. \tag{17}$$

Observe that the effective conductivity tensor of the regular hexagonal lattice is isotropic [8], and hence $q_x = q_y = q$.

The main difficulty of the approach concerns on finding a solution to the cell problem. To solve the problem, the so-called lubrication approach [34, 35] is employed. It should be noticed that from a mathematical point of view, the latter approach can be treated as thin layer asymptotics [36, 37].

3 | SOLVING A CELL PROBLEM BY THIN LAYER ASYMPTOTICS

We deal with densely packed high contrast composite ($a \to 1, \lambda \to \infty$). The essence of thin layer asymptotics (lubrication approach) is to replace the boundary value problem defined in the original region by a counterpart problem in the region with a simpler geometry [34–37]. In the next step, the solution of simplified problem is extended to the original region.

Let us determine the effective parameter of composite q_y in the direction of the axis y (or η with regard to 'fast' coordinates). The following steps are employed while achieving a solution.

1. First, we replace the outer hexagonal contour of the cell with a circle of radius b (Figure 3).
   In polar coordinates ($r=\sqrt{\xi^2+\eta^2}, \theta=\arctan(\eta/\xi)$), the cell problem is described by the following equations:

$$\frac{\partial^2 u_1^+}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u_1^+}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 u_1^+}{\partial \theta^2}=0 \quad \text{in} \quad \Omega_i^+, \tag{18}$$

$$\frac{\partial^2 u_1^-}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u_1^-}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 u_1^-}{\partial \theta^2}=0 \quad \text{in} \quad \Omega_i^-, \tag{19}$$

$u_1^+ = u_1^-$ at $r = a$,

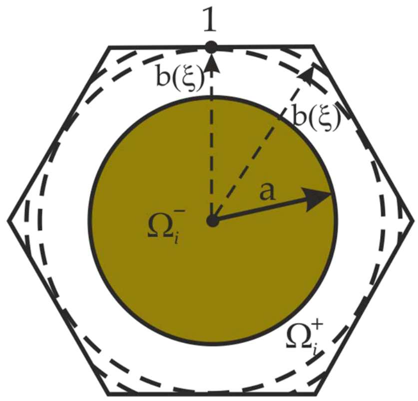


FIGURE 3 Lubrication approach model for a hexagonal lattice

$$\frac{\partial u_1^+}{\partial r} - \lambda \frac{\partial u_1^-}{\partial r} = (\lambda - 1)\left(\frac{\partial u_0}{\partial x}\cos\theta + \frac{\partial u_0}{\partial y}\sin\theta\right) \quad \text{at} \quad r = a, \tag{20}$$

$$u_1^+ = 0 \quad \text{at} \quad r = b. \tag{21}$$

Solution to the boundary value problem (16)–(21) has the following form:

$$u_1^- = A_1 r\cos\theta + A_2 r\sin\theta, \tag{22}$$

$$u_1^+ = \left(B_1 r + \frac{C_1}{r}\right)\cos\theta + \left(B_2 r + \frac{C_2}{r}\right)\sin\theta, \tag{23}$$

where A_i, B_i, C_i $(i = 1, 2)$ are the functions of slow variables determined by the conditions (20) and (21). They can be written as follows:

$$A_1 = -\frac{(\lambda - 1)\left(b^2 - a^2\right)}{(b^2 + a^2) + \lambda(b^2 - a^2)}\frac{\partial u_0}{\partial x}, \qquad B_1 = \frac{(\lambda - 1)a^2}{(b^2 + a^2) + \lambda(b^2 - a^2)}\frac{\partial u_0}{\partial x}, \tag{24}$$

$$C_1 = -\frac{(\lambda - 1)a^2 b^2}{(b^2 + a^2) + \lambda(b^2 - a^2)}\frac{\partial u_0}{\partial x}, \quad A_2 = A_1, \; B_2 = B_1, \; C_2 = C_1\left(\frac{\partial u_0}{\partial x} \to \frac{\partial u_0}{\partial y}\right). \tag{25}$$

2. We suppose that the outer contour of cell b is described by the formulas

$$b = \begin{cases} b(\xi) = \sqrt{1 + \xi^2} & \text{for} \quad 0 \le \xi \le \frac{1}{\sqrt{3}} \\ b(\xi) = 2\sqrt{\xi^2 - \sqrt{3}\,\xi + 1} & \text{for} \quad \frac{1}{\sqrt{3}} \le \xi \le \frac{2}{\sqrt{3}} \end{cases} \tag{26}$$

Figure 4 shows the fourth part of the cell.

3. For obtaining the effective parameter q_y, integration in expression (13) is performed over the original region of the hexagonal cell, taking into account relation (26), that is, A_2, B_2, C_2 are treated as functions of ξ.

After carrying out the transformations, we pass to the limit $\lambda \to \infty$. As a result, we find the asymptotic expression for the effective parameter $q_y^{(\infty)}$ for large inclusion sizes, which reads

$$q_y^{(\infty)} = \frac{2\sqrt{3}\,a^2}{\sqrt{1 - a^2}}\arctan\frac{\sqrt{3}}{3\sqrt{1 - a^2}} + 1 + \frac{\sqrt{3}\,a^2}{3}\left(\frac{\pi}{4} - \frac{3}{2}\arcsin\frac{\sqrt{3}}{3a}\right)$$
$$+ \frac{4\sqrt{3}\,a^2}{3\sqrt{1 - a^2}}\left[\arctan\left(\left(\sqrt{3}\,a - \sqrt{3a^2 - 1}\right)\sqrt{\frac{1 - a}{1 + a}}\right)\right. \tag{27}$$

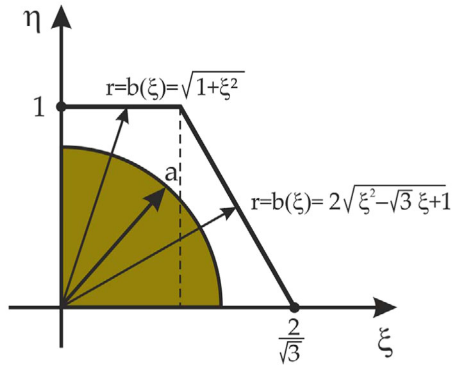


FIGURE 4 Approximation of the outer contour of a cell

$$-\frac{1}{4}\arctan\frac{2\left(\sqrt{3}\,a-1-\sqrt{3a^2-1}\right)\sqrt{1-a^2}}{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\,a\right)a+\sqrt{3a^2-1}\left(a-2+\sqrt{3}\,a\right)+2}$$

$$\left.-\frac{1}{8}\arctan\frac{2\sqrt{1-a^2}}{a}\right]-\frac{a^2}{4}\ln\frac{\left(2+3a^2+2\sqrt{3\left(3a^2-1\right)}\right)}{\left(4-3\,a^2\right)}.$$

In the limiting case $a\to 1$ relation (27), taking into account the leading term of the asymptotics and the first correction to it of order $(1-a^2)^0$, yields the following formula:

$$q^{(\infty)}_{y\ sympt}=\frac{\sqrt{3}\,\pi}{\sqrt{1-a^2}}+1+\frac{\sqrt{3}\,\pi}{12}-\frac{\sqrt{3}}{2}\arcsin\frac{\sqrt{3}}{3}-\frac{1}{4}\ln\left(5+2\sqrt{6}\right)-\sqrt{3}\left(\sqrt{3}+\sqrt{2}\right). \tag{28}$$

Keller in Ref. [39] considered the problem of densely packed high contrast square lattice of circular inclusions. He assumed that the effective conductivity of this medium mainly depends upon the fluxes between 'neighbours' (closely spaced inclusions) and he obtained the asymptotic formula of effective conductivity. Essentially, he used the same ideas as the lubrication approach [34, 35] (thin layer asymptotics [36, 37]). Berlyand and Novikov generalized this approach for the case of densely packed high contrast hexagonal lattice of circular inclusions [40] (formula 3.10 in their paper). In the notation adopted in our paper, this formula is written as follows:

$$q_{sympt}=\frac{\sqrt{3}\pi}{\sqrt{1-a^2}}. \tag{29}$$

4 | SOLUTION FOR MEDIUM SIZE INCLUSIONS

The asymptotic expression for the effective parameter (27) was obtained for large inclusion sizes and can be used for $a\geq\frac{1}{\sqrt{3}}\approx 0.5774$. In this regard, it is of interest to generalize the proposed approach to the case of medium-sized inclusions, that is, for $a\leq\frac{1}{\sqrt{3}}$ (Figure 5).

The use of the Equations (9)–(11), taking into account conditions (26) and after the averaging procedure, yields to the following formula:

$$q_{med.\,\mathrm{incl.}}=1+\frac{\sqrt{3}\,a^2}{\sqrt{1-a^2}}\arctan\frac{\sqrt{3}}{3\sqrt{1-a^2}}+\frac{\sqrt{3}\,a^2}{3}\left[\frac{2}{\sqrt{1-a^2}}\left(2\arctan\sqrt{\frac{1-a}{1+a}}+\arctan\frac{a}{\sqrt{1-a^2}}\right)-\frac{\pi}{2}\right]. \tag{30}$$

Effective parameter $q_{med.\,\mathrm{incl.}}$ corresponds to the composite with $\lambda\to\infty$ and inclusions of medium size ($a\leq\frac{1}{\sqrt{3}}$).

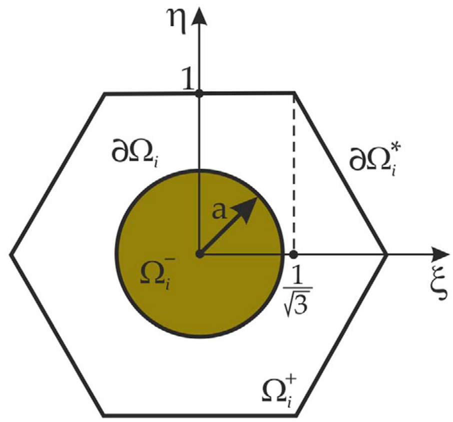


FIGURE 5 Periodically repeated cell of a composite with an inclusion of medium size

5 | TWO-SIDED ESTIMATES OF THE ASYMPTOTICS OF THE EFFECTIVE THERMAL CONDUCTIVITY PARAMETER

Solution (30) for medium-sized inclusions is not applicable in the case of their large sizes and has wrong asymptotics (29) at $\lambda \to \infty$, $a \to 1$. For overcoming this drawback, we transform relation (30) with a help of the Padé approximants (PA) [41].

We rewrite relation (30) as follows:

$$q_{med.\,\mathrm{incl.}} = 1 + \frac{2\sqrt{3}\,a^2}{\sqrt{1-a^2}}\left(\arctan\frac{\sqrt{3}}{3\sqrt{1-a^2}} + \frac{2}{3}\arctan\sqrt{\frac{1-a}{1+a}}\right) - \frac{\sqrt{3}\,\pi a^2}{6} + \frac{\sqrt{3}\,a^2}{\sqrt{1-a^2}}\left(\frac{2}{3}\arctan\frac{a}{\sqrt{1-a^2}} - \arctan\frac{\sqrt{3}}{3\sqrt{1-a^2}}\right) \tag{31}$$

and construct a sequence of PA $q^*_{[2/2]}, q^*_{[2/4]}, \ldots$ for expression

$$q^* = \frac{\sqrt{3}\,a^2}{\sqrt{1-a^2}}\left(\frac{2}{3}\arctan\frac{a}{\sqrt{1-a^2}} - \arctan\frac{\sqrt{3}}{3\sqrt{1-a^2}}\right). \tag{32}$$

Then, we get

$$q^*_{[2/2]} = -\frac{1}{2}\,\frac{\pi^3 a^2}{\sqrt{3}(\pi+4)\pi a - \frac{1}{4}\left(2\sqrt{3}\pi^2 + 9\pi - 64\sqrt{3}\right)a^2},$$

$$q^*_{[2/4]} = -\frac{\sqrt{3}}{2}\pi^5 a^2 \Big/ \Big(3(\pi+4a)\pi^3 - \frac{3}{4}\left(2\pi^2 + 3\sqrt{3}\pi - 64\right)\pi^2 a^2 - 2\left(2\pi^2 + 9\sqrt{3}\pi - 96\right)\pi a^3 - \frac{1}{32}\left(12\pi^4 + 9\sqrt{3}\pi^3 + 94\pi^2 + 3456\sqrt{3}\pi - 24576\right)a^4\Big),$$

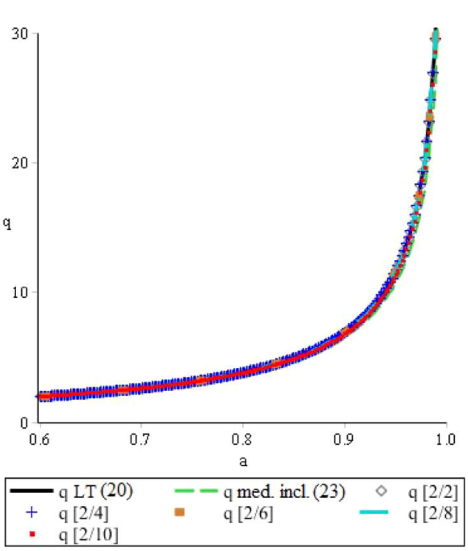
$$q^*_{[2/6]} = -\frac{3}{2}\pi^7 a^2 \Big/ \Big(3\sqrt{3}(\pi+4a)\pi^5 - \frac{3}{4}\left(2\sqrt{3}\pi^2 + 9\pi - 64\sqrt{3}\right)\pi^4 a^2 - 2\left(2\sqrt{3}\pi^2 + 27\pi - 96\sqrt{3}\right)\pi^3 a^3 - \frac{1}{32}\left(12\sqrt{3}\pi^4 + 27\pi^3 + 94\sqrt{3}\pi^2 + 10368\pi - 24576\sqrt{3}\right)\pi^2 a^4 - \frac{1}{20}\left(32\sqrt{3}\pi^4 + 315\pi^3 + 1215\sqrt{3}\pi^2 + 34560\pi - 61440\sqrt{3}\right)\pi a^5 - \frac{1}{960}\left(180\sqrt{3}\pi^6 + 1603\sqrt{3}\pi^4 + 153495\pi^3 - 589440\sqrt{3}\pi^2 + 8294400\pi - 11796480\sqrt{3}\right)a^6\Big),$$

$$
\begin{aligned}
q^*_{[2/8]} = &-\frac{3\sqrt{3}}{2}\pi^9 a^2 \Big/ \Big(9(\pi+4a)\pi^7 - \frac{9}{4}\left(2\pi^2+3\sqrt{3}\pi-64\right)\pi^6 a^2 - 6\left(2\pi^2+9\sqrt{3}\pi-96\right)\pi^5 a^3 \\
&- \frac{1}{32}\left(36\pi^4+27\sqrt{3}\pi^3+282\pi^2-10368\sqrt{3}\pi-73728\right)\pi^4 a^4 - \frac{1}{20}(96\pi^4+315\sqrt{3}\pi^3-3645\pi^2 \\
&+34560\sqrt{3}\pi-184320\Big)\pi^3 a^5 - \frac{1}{320}(180\pi^6+1603\pi^4+51165\sqrt{3}\pi^3-589440\pi^2 \\
&+2764800\sqrt{3}\pi-11796480\sqrt{3}\Big)\pi^2 a^6 - \frac{1}{2800}\Big(7680\pi^6+14490\sqrt{3}\pi^5+360255\pi^4+3534300\pi^3 \\
&-35817600\pi^2+116121600\sqrt{3}\pi-412876800\Big)\pi a^7 \\
&- \frac{1}{35840}(12600\pi^8-8505\sqrt{3}\pi^7+125863\pi^6+3109554\sqrt{3}\pi^5 \\
&- 58215948\pi^4+310464000\sqrt{3}\pi^3-2750791680\pi^2+6936330240\sqrt{3}\pi-21139292160\Big)a^8\Big),
\end{aligned}
\tag{33}
$$

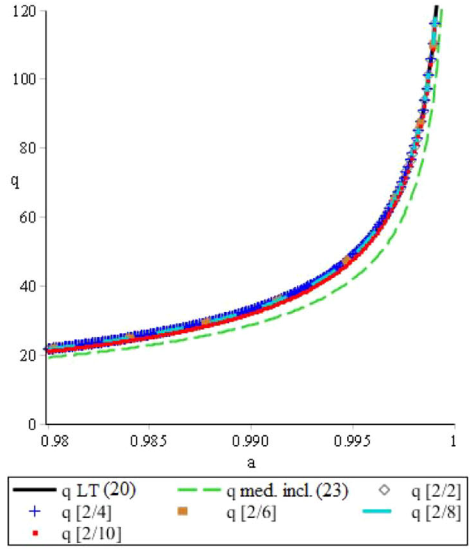
$$
\begin{aligned}
q^*_{[2/10]} = &-\frac{9}{2}\pi^{11} a^2 \Big/ \Big(9\sqrt{3}(\pi+4a)\pi^9 - \frac{9}{4}\left(2\sqrt{3}\pi^2+9\pi-64\sqrt{3}\right)\pi^8 a^2 \\
&- 6\left(2\sqrt{3}\pi^2+27\pi-96\sqrt{3}\right)\pi^7 a^3 - \frac{1}{32}\Big(36\sqrt{3}\pi^4+81\pi^3+282\sqrt{3}\pi^2 \\
&+ 31104\pi-73728)\pi^6 a^4 - \frac{1}{20}(96\sqrt{3}\pi^4+945\pi^3-3645\sqrt{3}\pi^2+103680\pi-184320\sqrt{3}\Big)\pi^5 a^5 \\
&- \frac{1}{320}\left(180\sqrt{3}\pi^6+1603\sqrt{3}\pi^4+153495\pi^3-589440\sqrt{3}\pi^2+8294400\pi-11796480\sqrt{3}\right)\pi^4 a^6 \\
&- \frac{1}{2800}\Big(7680\sqrt{3}\pi^6+43470\pi^5-360255\sqrt{3}\pi^4+10602900\pi^3 \\
&-35817600\sqrt{3}\pi^2+348364800\pi-412876800\sqrt{3}\Big)\pi^3 a^7 \\
&- \frac{1}{35840}\Big(12600\sqrt{3}\pi^8-25515\pi^7+125863\sqrt{3}\pi^6+9328662\pi^5 \\
&- 58215948\sqrt{3}\pi^4+931392000\pi^3-2740039680\sqrt{3}\pi^2+2079006720\pi-21139292160\sqrt{3}\Big)\pi^2 a^8 \\
&- \frac{1}{26880}\Big(49152\sqrt{3}\pi^8+82134\pi^7-2489953\sqrt{3}\pi^6+74331432\pi^5 \\
&- 367895220\sqrt{3}\pi^4+4380687360\pi^3-11421204480\sqrt{3}\pi^2+71345111040\pi-63417876480\sqrt{3}\Big)\pi a^9 \\
&- \frac{1}{2150400}\Big(529200\sqrt{3}\pi^{10}-1990170\pi^9+6730776\sqrt{3}\pi^8+316328085\pi^7-2909134880\sqrt{3}\pi^6 \\
&+ 50865513300\pi^5-210050507520\sqrt{3}\pi^4+2063016345600\pi^3-4802582937600\sqrt{3}\pi^2 \\
&+ 25684239974400\pi-20293720473600\sqrt{3}\Big)a^{10}\Big).
\end{aligned}
$$

A further increase in the order of PA is inexpedient, because starting from $q^*_{[2/12]}$, the PA sequence diverges. Formula for the effective parameter

$$
q_{Pade} = 1 + \frac{2\sqrt{3}a^2}{\sqrt{1-a^2}}\left(\arctan\frac{\sqrt{3}}{3\sqrt{1-a^2}} + \frac{2}{3}\arctan\sqrt{\frac{1-a}{1+a}}\right) - \frac{\sqrt{3}\pi a^2}{6} + q^*_{[2/10]}
\tag{34}
$$



a) $0.6 \le a < 1$



b) $0.98 \le a < 1$

FIGURE 6 The effective parameter described by the PA sequence

gives accurate results to any size $0 \le a < 1$ for absolutely conducting inclusions. At $a \to 1$ one obtains the following leading term of the asymptotics (29):

$$q_{Pade} = \frac{\sqrt{3}\pi}{\sqrt{1-a^2}} - 5 + \frac{2\sqrt{3}}{3} - \frac{\pi\sqrt{3}}{6} - \frac{9\pi^{11}}{2}\Bigg[\frac{567\sqrt{3}}{256}\pi^{10} - \left(\frac{216513}{2048} - \frac{512\sqrt{3}}{7}\right)\frac{\pi^9}{5} - \left(1020681 - \frac{22137707\sqrt{3}}{40}\right)\frac{\pi^8}{4480} - \left(\frac{266514867}{16} - \frac{26330161\sqrt{3}}{3}\right)\frac{\pi^7}{8960} - \left(1877769 - \frac{191470627\sqrt{3}}{168}\right)\frac{\pi^6}{160} - \left(\frac{77374953}{16} - 2284453\sqrt{3}\right)\frac{\pi^5}{64} - \left(287388 - \frac{16903659\sqrt{3}}{80}\right)\pi^4 - 3\left(513324 - 190784\sqrt{3}\right)\pi^3 - 3072\left(864 - 919\sqrt{3}\right)\pi^2 - 147456\left(81 - 16\sqrt{3}\right)\pi + 9437184\sqrt{3}\Bigg]^{-1} + O\left(\sqrt{1-a}\right), \tag{35}$$

or

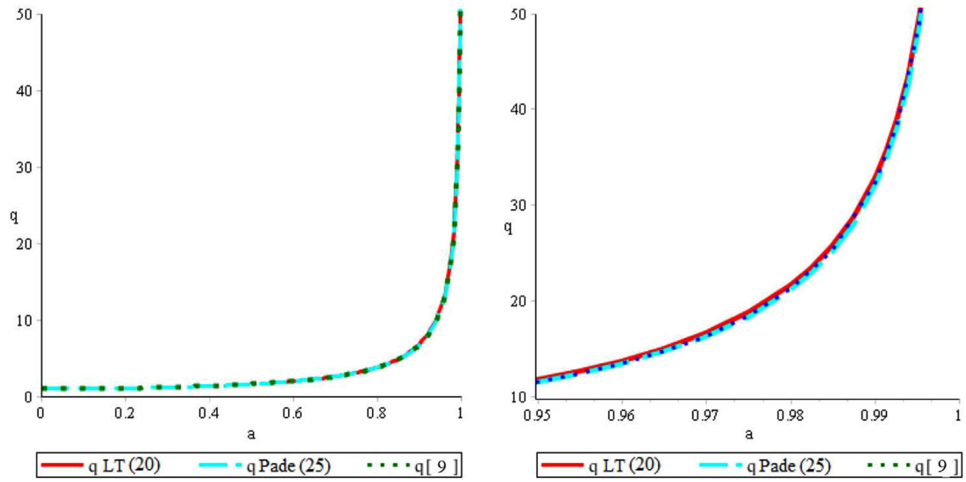
$$q_{Pade} = \frac{\sqrt{3}\pi}{\sqrt{1-a^2}} - 6.9345. \tag{36}$$

For comparison, the result obtained in Gluzman et al. [9] has the form

$$q = \frac{5.18766}{\sqrt{0.9069 - c}} - 6.2371 = \frac{\sqrt{3}\pi}{\sqrt{1-a^2}} - 6.2371, \tag{37}$$

where $c = \frac{\pi a^2}{2\sqrt{3}}$ is the volume fraction of inclusions.

Figure 6a,b presents the solutions described by PA $q^*_{[2/2]}, q^*_{[2/4]}, \dots, q^*_{[2/10]}$. On the other hand, in Figure 7a,b graphs of the effective parameter for absolutely conducting inclusions are shown in comparison with the formula (7.5.44) obtained in reference [9].



a) $0 \leq a < 1$ b) $0.95 \leq a < 1$

FIGURE 7 Effective parameter for absolutely conducting inclusions

Thus, solutions based on thin layer asymptotics (27) and PA (33), (34) give the upper and lower bounds for the asymptotics of the effective parameter in the case of absolutely conducting inclusions. It should be mentioned that the distance between upper and lower bounds is small for any values of the size of the inclusions.

6 | CONCLUDING REMARKS

Our results are based on analytical homogenization theory. First, we have used a small parameter characterizing the inhomogeneity of the medium and the multiple scale asymptotic method. The original problem has been separated to local (on a periodically repeated cell) and global (for an equation with effective characteristics in a simply connected domain) problem. Next, we have employed a small parameter characterizing a high volume fraction of inclusions and we have used thin layer asymptotics (lubrication approach). Application of the PA yielded the effective heat conductivity parameter for large, close to the limit possible size inclusions. Upper and low bounds for solution were obtained. An important feature of the proposed solution is the fact that it correctly describes the asymptotics of the effective conductivity at $\lambda \to \infty$, $\lambda = 0$ and $a \to 1$.

The exact formula for any regular array (4.2.28) in the series form, the polynomial approximation equations (7.2.6) and the corresponding transformed formula (7.5.44) are obtained in reference [9]. Formula (4.2.28) is exact, but written in the form of a series slowly convergent in the critical regime. Therefore, one needs asymptotic analysis in order to extract the singularity. Renormalization approach is applied in Gluzman et al. [9] directly to (4.2.28) to resolve this task. We have employed an alternate approach to the same problem of the critical percolation regime based on the Pade approximations. Comparison of relation (34) with formula (7.5.44) reported in Gluzman et al. [9] showed excellent accuracy of the obtained by us results.

In the era of artificial intelligence and big data, it is imperative to emphasize the importance of analytical solutions. As noted in Choy [42] 'In spite of huge progress in computational techniques and technology, we are still barely able to simulate, let alone accurately compute, experimentally measurable quantities for systems of no more than a few thousand atoms at the microscopic level. Hence one can say that some form of effective medium theory is indispensable for our understanding in the end. Nor is it really fruitful to do otherwise'.

Some scientists assert that the area of applicability of the asymptotic solutions is very restricted. But asymptotic methods give (as remarked by Laplace) the better results in proportion to its being more necessary ([43], p.1). They allow to obtain results for extreme values of the parameters, that is, just in those situations where numerical methods have fundamental problems. Analytical solutions serve as a good benchmark for numerical solutions, allowing one to estimate the applicability limits of the latter. And, finally, they can be easily combined with the numerical algorithms.

We considered the regular arrangement of inclusions. Since regular case possesses some extreme properties [25, 44–47], the obtained effective parameters can serve as estimation of the effective parameters of random composites. In addition, it is possible to obtain analytical estimates of the effective properties for shaking geometry composites by employing the security-spheres approach [25].

In addition, the found analytical expressions for the effective properties can be used in the optimal design of composites [48]. For a future work, it is of interest to study the problems of bending of composite plates with hexagonal lattice of circular inclusions and compare the results with known refined theories [49–51].

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CONFLICT OF INTEREST

The authors declare that there is no potential conflict of interest.

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