

On the Stability of Stationary States in Diffusion Models in Biology and Humanities

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Abstract—We consider an initial-boundary value problem for the system of partial differential equations describing processes of growth and spread of substance in biology, sociology, economics and linguistics. We note from a general point of view that adding diffusion (migration) terms to ordinary differential equations, for example, to logistic ones, can in some cases improve sufficient conditions for the stability of a stationary solution. We give examples of models in which the addition of diffusion terms to ordinary differential equations changes the stability conditions of a stationary solution.

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1. INTRODUCTION

In this paper we study the stability of stationary solutions to differential equations and systems of partial differential equations that arise most often in mathematical biology when describing the quantitative growth and distribution of certain substances. Such substances include, in particular, biological populations. The first model of population growth, written in the form of a differential equation, was proposed by T. R. Malthus (Thomas Robert Malthus, 1766–1834) shortly after the discovery of differential and integral calculus (1798). This model considers a homogeneous population in conditions of unlimited food resources and habitat. At the same time, it is assumed that the growth rate of population is proportional to its size. The dynamics of such populations can be described by the ordinary differential equation (see, for example, [1–3]) $du/dt = Au$, where A is the innate rate of natural population growth. The solution of this equation is the function

$$u(t) = u(t_0) \exp(At), \quad (1)$$

that is, over time, the population size grows indefinitely according to the exponential law. In accordance with this law, an isolated population would develop under conditions of unlimited resources. In nature, such conditions are practically not found. A few examples of breeding species brought to places, where there is a lot of food and there are no competing populations and predators, are known to everyone (rabbits in Australia). The Malthus equation accurately describes the dynamics of a population of protozoa artificially created and maintained in conditions of an excess of food and space, for example, penicillin fungi grown in a cultivator, until the culture medium is depleted.

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Formula (1) adequately describes the process of population growth only for a limited period of time, since there comes a time when the growing population will exhaust the available resources. The population size may stabilize at some stable level, it may fluctuate regularly or irregularly, or it may contract. The behavior of a population whose size is stabilized at a certain equilibrium level is often described using the logistic equation proposed by P. F. Verhulst (Pierre François Verhulst, 1804–1849) in 1838:

$$\frac{du}{dt} = f(u) = ru \left(1 - \frac{u}{K}\right),$$

The solution of this equation has the form

$$u(t) = \frac{u(t_0) K \exp(rt)}{K - u(t_0) + \exp(At)}.$$

The direct study of this function shows that for small values of u , the Verhulst equation can be replaced with sufficient accuracy by the Malthus equation, and the growth is explosive exponential, with an increase in the value of t , the value of $u(t)$ approaches a constant value of K , called the capacity of the ecological niche of the population. Functions that satisfy such properties, differential equations that give such functions as solutions, as well as models that include such equations, are often called logistic.

It is well known that the partial stationary solution $u(t) \equiv 0$ of the Verhulst equation is unstable. This is easily verified by using the first linear approximation. In the general case, for checking the stability of a stationary solution $w \equiv \text{const}$ of equation $du/dt = f(u)$, that is, the solutions of equation $f(w) = 0$, we have to check the sign of the derivative $f'(u)$ at the point $u = w$. For the Verhulst equation, the function $f(u)$ has the form

$$f(u) = ru \left(1 - \frac{u}{K}\right),$$

its derivative has the form

$$\frac{df(u)}{du} = r - \frac{2ru}{K},$$

and at the point $u = 0$, it takes the value

$$\left. \frac{df(u)}{du} \right|_{u=0} = r > 0.$$

Since the value of the derivative $f'(u)$ at the point $u(t) \equiv 0$ is positive, the stationary solution of $u(t) \equiv 0$ is unstable.

Systems of differential equations simulate the growth of phenomena of various types, and for all such systems, studies of the stability of stationary solutions play an important role. These studies have a long history. In many cases, such models are based on ordinary differential equations. Although the theory of systems of ordinary differential equations has long been classical, interest in it does not fade away. In the last few decades, this is also due to the fact that such systems have found applications in modelling biological and social systems. From relatively recent works on mathematical biology, it is possible to indicate in this regard [4–10]. In [11], a discrete version of the Verhulst equation was used to study the spread of the COVID-19 coronavirus epidemic in Moscow. In the work [12], models of the origin and development of trends in painting are considered basing on the same principles as models of the growth of biological substances.

In 1921, H. Hotelling (Harold Hotelling, 1895–1973) proposed to describe animal and human populations, taking into account migration patterns in addition to the logistic law (see, for example, [13]). In order to do this, he proposed an equation of the form

$$\frac{\partial p}{\partial t} = A(\xi - p)p + B\Delta p, \quad (2)$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator, A, B, ξ are given positive constants. This equation describes the growth and spread of a population. In this case, the values included in the equation have the following meaning: A is a population growth rate, B is a distribution rate, ξ is a population saturation threshold (or saturated density coefficient), p is a size (density) of the population, t is a time,

x, y are coordinates on the plane. The same equation is used to model the development of a malignant tumor [14]. The first term on the right side of the Hotelling equation is called a logistic term (growth component). The second term is usually called a diffusion term. It reflects the influence of migration (diffusion) processes on the change in population size. Equations and models reflecting changes in the size of biomass under the influence of logistic and diffusion components began to be called diffusion-logistic, and sometimes simply diffusion, implying, apparently, that the logistic component is present in the model by default. Such models can include not only one equation, but also systems of partial differential equations.

In this paper we consider a certain family of mathematical models with partial differential equations (models with distributed parameters), which are obtained from models with ordinary differential equations (models with concentrated parameters) by adding the so-called diffusion terms. The tendency of such sophistications of mathematical models can be traced in some works related to modelling the growth and distribution of populations, the growth and spread of infections, and the growth of tumors. In this regard, see first of all the monograph [2]. In the work [15], a diffusion model of a malignant tumor is presented. The mathematical model of glioma growth is based on the classical definition of cancer as uncontrolled proliferation of cells with the potential for invasion and metastasis, simplified for gliomas, which practically do not metastasize. This model is given in [14].

We are interested in the stability of stationary solutions of diffusion models. This issue is discussed in the book [2]. This book states that adding diffusion terms can change the stability of a stationary solution both for the worse and for the better. For models of a certain type, we try to concretize sufficient conditions for the stability of stationary solutions. Let's first point to a simple example. In the monograph by T. Puu [13], using the Lyapunov function (Alexander Mikhailovich Lyapunov, 1857–1918) of a special kind, a sufficient condition for the stability of a regular stationary solution w of the initial boundary value problem with a boundary condition of the first, second or third kind for the Hotelling equation (2) in a bounded domain was found in the form $w > \xi/2$. Later this condition was improved (weakened), namely, it was shown that the condition $w > \xi/2 - B/(2Ad^2)$ is sufficient for the stability of a stationary solution $w(x, y)$ of the initial boundary value problem with a boundary condition of condition of the first, second or third kind for the Hotelling equation (2) in a bounded domain with a diameter of d (see [16, 17]). This condition for $B > 0$ is satisfied for a small diameter of the domain Ω . It is necessary to pay attention to the following circumstance. The trivial solution is a stationary solution to the Hotelling equation both for $B = 0$ (in this case it coincides with the Verhulst equation) and for $B > 0$. However, in the case $B = 0$, as already noted, the trivial solution is unstable, and in the case $B > 0$ the trivial solution is stable. Below we extend the results of [16, 17] to one class of systems of partial differential equations.

2. INITIAL BOUNDARY VALUE PROBLEM UNDER STUDY

We consider the initial boundary value problem for a system of partial differential equations (see also [18, 19])

$$\frac{\partial u_s}{\partial t} = \vartheta_s \Delta u_s + F_s(u), \quad x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n, \quad t > 0, \tag{3}$$

$$\left(\mu_s u_s + \eta_s \frac{\partial u_s}{\partial \nu} \right) \Big|_{x \in \partial \Omega} = B_s(x), \quad \mu_s^2 + \eta_s^2 > 0, \quad \mu_s \geq 0, \quad \eta_s \geq 0, \\ \mu_s = const, \quad \eta_s = const, \tag{4}$$

$$u_s(x, 0) = u_s^0(x), \quad s = 1, \dots, m, \tag{5}$$

where Ω is a bounded domain with a piecewise smooth boundary $\partial \Omega$, $\vec{\nu} = \nu$ is a unit external normal vector to the boundary $\partial \Omega$ of the domain Ω , $u = (u_1(x, t), \dots, u_m(x, t))$, $\vartheta_s \geq 0$, $u = (u_1(x, t), \dots, u_m(x, t))$, $\vartheta_s \geq 0$, $s = 1, \dots, m$, Δ is the Laplace operator defined by the formula

$$\Delta v = \sum_{j=1}^n \partial^2 v / \partial x_j^2.$$

Certainly, we must require the fulfillment of the conditions for matching the initial and boundary data. However, in the framework of this work, we are moving away from this issue. We assume that all conditions of existence of the classical (regular) solutions to the problem under consideration are satisfied, and in addition, all the original functions have the necessary properties allowing us to perform all the operations that we perform below. If

$$\vartheta_s = 0, \quad s = 1, \dots, m \tag{6}$$

(in a model with concentrated parameters without diffusion terms), the variables x_1, \dots, x_n are included in equations (3) as parameters whose derivatives are not contained in these equations. If $\sum_{s=1}^m \vartheta_s^2 > 0$, then we deal with a system with distributed parameters.

3. STATIONARY SOLUTION AND ITS STABILITY

Let $w = w(x) = (w_1(x_1, \dots, x_n), \dots, w_m(x_1, \dots, x_n))$ be a stationary solution of the initial boundary value problem (3)–(5), that is, the solution of the boundary value problem

$$\vartheta_s \Delta w_s + F_s(w) = 0, \quad x \in \Omega,$$

$$\left(\mu_s w_s + \eta_s \frac{\partial w_s}{\partial \nu} \right) \Big|_{x \in \partial \Omega} = B_s(x), \quad s = 1, \dots, m.$$

Suppose that the functions $F_s(u)$, $s = 1, \dots, m$, are differentiable at the point w . Then for sufficiently small deviations $z_s = z_s(x_1, \dots, x_n, t) = u_s - w_s$, $s = 1, \dots, m$, we have

$$F_s(u) = F_s(w + z) = F_s(w) + \sum_{k=1}^m b_{sk} z_k + \sum_{k=1}^m \epsilon_{sk}(z) z_k, \tag{7}$$

where

$$b_{sk} = \frac{\partial F_s(w)}{\partial z_k}, \quad \lim_{z \rightarrow 0} \epsilon_{sk}(z) = 0, \quad s, k = 1, \dots, m.$$

Substituting $u_s = w_s + z_s$ into equation (3) and taking into account (7), we get

$$\frac{\partial z_s}{\partial t} = \vartheta_s \Delta w_s + F_s(w) + \vartheta_s \Delta z_s + \sum_{k=1}^m b_{sk} z_k + \sum_{k=1}^m \epsilon_{sk}(z) z_k, \quad s = 1, \dots, m. \tag{8}$$

Considering that w is a stationary solution, we obtain from (8)

$$\frac{\partial z_s}{\partial t} = \vartheta_s \Delta z_s + \sum_{k=1}^m b_{sk} z_k + \sum_{k=1}^m \epsilon_{sk}(z) z_k, \quad s = 1, \dots, m.$$

We multiply each s th equation of system (8) by z_s and integrate the obtained equality over the domain Ω . Taking into account (7), we obtain

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} z_s^2 dx = \vartheta_s \int_{\Omega} z_s \Delta z_s dx + \int_{\Omega} \sum_{k=1}^m b_{sk} z_s z_k dx + \int_{\Omega} \sum_{k=1}^m \epsilon_{sk}(z) z_s z_k dx, \quad s = 1, \dots, m. \tag{9}$$

The last term on the right side of equality (9) with small deviations of z does not affect the sign of the entire sum and can be discarded. We apply the Green formula (see [20]) to the first term on the right side of this equation. As a result, we get

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} z_s^2 dx = -\vartheta_s \int_{\Omega} |\nabla z_s|^2 dx + \vartheta_s \int_{\partial \Omega} z_s \frac{\partial z_s}{\partial \nu} d\Gamma + \int_{\Omega} \sum_{k=1}^m b_{sk} z_s z_k dx, \quad s = 1, \dots, m, \tag{10}$$

where $d\Gamma$ is an arc element of the boundary $\partial \Omega$, i. e. the second term on the right side of equality (10) is a surface (for $n \geq 3$) or contour (for $n = 2$) integral of the first kind over the boundary of the domain Ω , or the sum of values at the ends of the interval Ω in the case of $n = 1$. In the integral over the boundary

for $\mu_s = 0$ or for $\eta_s = 0$ the integrand is equal to zero due to the boundary condition (4). From the same boundary condition, when $\mu_s \eta_s > 0$, we get

$$\left. \frac{\partial z_s}{\partial \nu} \right|_{\partial \Omega} = - \frac{\mu_s}{\eta_s} z_s \Big|_{\partial \Omega} .$$

Therefore, equality (10) can be rewritten as

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} z_s^2 dx = -\vartheta_s \int_{\Omega} |\nabla z_s|^2 dx - \vartheta_s \int_{\partial \Omega} \sigma_s z_s^2 d\Gamma + \int_{\Omega} \sum_{k=1}^m b_{sk} z_s z_k dx, \quad s = 1, \dots, m, \quad (11)$$

where $\sigma_s = \mu_s/\eta_s$ if $\mu_s \eta_s > 0$ or $\sigma = 0$ if $\mu_s \eta_s = 0$. Summing m equalities (11), we get

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} |z|^2 dx = \sum_{s=1}^m \left(-\vartheta_s \int_{\Omega} |\nabla z_s|^2 dx - \vartheta_s \int_{\partial \Omega} \sigma_s z_s^2 d\Gamma \right) + \int_{\Omega} \sum_{s=1}^m \sum_{k=1}^m \Theta_{sk} z_s z_k dx, \quad (12)$$

where $\Theta_{sk} = (b_{sk} + b_{ks})/2$. The sign of the left side of equality (12) is considered as an indicator of the stability of a trivial solution. Therefore, it is important to find the ratio of the terms in the right side, leading to the negativity of this expression. In parentheses on the right side, both the first term and the second term are not greater than zero. Next, we need to take into account the sign of the last term in the right part. Obviously, the negative definiteness of the quadratic form

$$\sum_{s=1}^m \sum_{k=1}^m \Theta_{sk} z_k z_s dx \quad (13)$$

will ensure the negativity of the left side of equality (12), and, therefore, the stability of the stationary solution. In the case of a model with concentrated parameters (a system of ordinary differential equations), that is, when conditions (6) are satisfied, negative definiteness of the quadratic form (13) is also a necessary condition for the stability of a trivial solution.

Let us consider the diffusion model with distributed parameters. In this case, it is possible to weaken the sufficient condition for the stability of a stationary solution. For this purpose, we use the Steklov–Poincare–Friedrichs inequality (see [21, p. 62; 22, p. 150; 23, 24])

$$\int_{\Omega} |\nabla z_s|^2 dx \geq \frac{1}{d^2} \int_{\Omega} z_s^2 dx,$$

where $d = \text{diam } \Omega$ is a diameter of the domain Ω . Hence

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} |z|^2 dx \leq - \sum_{s=1}^m \frac{\vartheta_s}{d^2} \int_{\Omega} z_s^2 dx - \sum_{s=1}^m \vartheta_s \int_{\partial \Omega} \sigma_s z_s^2 d\Gamma + \int_{\Omega} \sum_{s=1}^m \sum_{k=1}^m \Theta_{sk} z_k z_s dx.$$

Now we can assert that a sufficient condition for the stability of a stationary solution is the negative definiteness of the quadratic form

$$\sum_{s=1}^m \sum_{k=1}^m A_{sk} z_k z_s, \quad (14)$$

where

$$A_{sk} = \Theta_{sk} - \delta_{ks} \vartheta_s / d^2. \quad (15)$$

4. EXAMPLES OF MATHEMATICAL MODELS IN BIOLOGY

4.1. The Lotka–Volterra Model

As an example we will consider the Lotka–Volterra system (Alfred James Lotka, 1880–1949; Vito Volterra, 1860–1940), which in a non-diffusion version has long been one of the main tools

in mathematical ecology, genetics, and the mathematical theory of selection and evolution (see, for example, [1, 10]):

$$\partial u_s / \partial t = \left(\phi_s - \sum_{j=1}^m \phi_j u_j \right) u_s + \vartheta_s \Delta u_s, \quad s = 1, \dots, m.$$

Coefficients (15) of the quadratic form (14) have the form

$$A_{ks} = \delta_{ks} \left(\phi_s - \sum_{j=1}^m \phi_j w_j - \vartheta_s / d^2 \right) - (\phi_k w_s + \phi_s w_k) / 2. \quad (16)$$

It is clear that due to the subtraction of positive terms ϑ_s / d^2 from diagonal elements of a matrix of quadratic form with coefficients (16), a sufficient condition for negative definiteness of a quadratic form in the diffusion model is less rigid than in the non-diffusion case. This can be demonstrated quite clearly by the example of a trivial solution. Condition $\Phi_s < \vartheta_s / d^2$, $s = 1, \dots, m$, is sufficient for the stability of a trivial solution of this system. For $\vartheta_s > 0$, $s = 1, \dots, m$, that is, in the case of a diffusion model with distributed parameters, this condition is satisfied for domains with a small diameter. In the case of two equations, the model considered in this example turns into a predator-prey model. Without taking into account migrations (diffusion), the system of equations in the simplest version of this model has the following form (see, for example, [2])

$$\frac{\partial u_1}{\partial t} = \alpha u_1 - \beta u_1 u_2, \quad \frac{\partial u_2}{\partial t} = \kappa \beta u_1 u_2 - m u_2.$$

The initial conditions are given in the form $u_s(x, 0) = u_s^0$, $s = 1, 2$. By adding the diffusion terms, we obtain the following system of partial differential equations

$$\frac{\partial u_1}{\partial t} = \alpha u_1 - \beta u_1 u_2 + \vartheta_1 \Delta u_1, \quad \frac{\partial u_2}{\partial t} = \kappa \beta u_1 u_2 - m u_2 + \vartheta_2 \Delta u_2. \quad (17)$$

We consider the system of equations (17) in a bounded domain Ω with a diameter d , with a piecewise smooth boundary and we require the solution to fulfill the boundary conditions

$$\left(\mu_s u_s + \eta_s \frac{\partial u_s}{\partial \nu} \right) \Big|_{x \in \partial \Omega} = B_s(x), \quad \mu_s^2 + \eta_s^2 > 0, \quad \mu_s \eta_s \geq 0, \quad \mu_s = \text{const}, \quad \eta_s = \text{const}, \quad (18)$$

and initial conditions

$$u_s(x, 0) = u_s^0(x), \quad s = 1, 2. \quad (19)$$

Let $w = (w_1, w_2)$ be a stationary solution of the problem (17)–(19). Coefficients (15) of a quadratic form (14) for system (17) have the form

$$A_{11} = \alpha - \beta w_2 - \vartheta_1 / d^2, \quad A_{22} = \kappa \beta w_1 - m - \vartheta_2 / d^2, \quad A_{12} = A_{21} = \beta (\kappa w_2 - w_1) / 2.$$

This quadratic form is negatively defined if and only if the following equalities are satisfied

$$A_{11} = \alpha - \beta w_2 - \vartheta_1 / d^2 < 0, \quad (20)$$

$$A_{11} A_{22} - A_{12}^2 = (\alpha - \beta w_2 - \vartheta_1 / d^2) (\kappa \beta w_1 - m - \vartheta_2 / d^2) - \beta^2 (\kappa w_2 - w_1)^2 / 4 > 0. \quad (21)$$

Conditions (20), (21) are sufficient for the asymptotic stability of the stationary solution w . Obviously, if the stationary solution is constant, these inequalities are certainly fulfilled in domains with small diameters. Therefore, stationary solutions of the diffusionless model $w_1 = w_2 = 0$ and $w_1 = m / (\kappa \beta)$, $w_2 = \alpha / \beta$, while remaining stationary solutions in the diffusion model in a domain with a small diameter (of course, with an appropriate set of initial and boundary conditions), change the nature of stability, namely, they become asymptotically stable. On the other hand, as shown in [2], a large value of a domain diameter can result in the so-called diffusion instability.

4.2. Diffusion Models of Oncological Processes

Let us consider the immune response model described in [15]. Let u_1 be a linear density of dividing cells, $q = q(x, t)$ be a linear density of lymphocytes. Then the mathematical model describing the interaction of dividing cells and lymphocytes, assuming the absence of their interaction with normal and dead cells, has the form

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= D_1 \frac{\partial^2 u_1}{\partial x^2} + \mu_1 u_1 - \gamma_{12} u_1 q, \\ \frac{\partial q}{\partial t} &= D_4 \frac{\partial^2 q}{\partial x^2} - v \frac{\partial q}{\partial x} - \gamma_{21} u_1 q. \end{aligned} \tag{22}$$

The model described in [15] is assumed to have the pointwise appearance of dividing cells. We consider a model with smooth distribution of dividing cells at the initial moment of time. Then we will have initial conditions of the form

$$u_1(x, 0) = u_1^0(x) \in C^\infty([0, l]), \quad q(x, 0) = q^0. \tag{23}$$

When choosing boundary conditions, it is assumed that

$$\left. \frac{\partial u_1}{\partial x} \right|_{x=0} = 0, \quad q \Big|_{x=0} = q^0; \quad \left. \frac{\partial u_1}{\partial x} \right|_{x=l} = 0, \quad q \Big|_{x=l} = q^0.$$

A refined sufficient condition for the asymptotic stability of a stationary solution consists in the fulfillment of two inequalities [25]:

$$\mu_1 - \gamma_{12} w_2 - 2D_1/l^2 < 0, \tag{24}$$

$$\begin{aligned} 4(\mu_1 - \gamma_{12} w_2 - 2D_1/l^2) \left(-\exp(-\sigma x) \gamma_{21} w_1 - \frac{\sigma^2 D_4}{\exp(\sigma l) - 1 - l\sigma} \right) \\ - (\gamma_{12} w_1 + \exp(\sigma l) \gamma_{21} w_2)^2 > 0. \end{aligned} \tag{25}$$

If we formally pass to a point model from model (22), discarding diffusion and convective terms with partial derivatives with respect to the spatial variable, we obtain the system of ordinary differential equations

$$\frac{du_1}{dt} = \mu_1 u_1 - \gamma_{12} u_1 q, \quad \frac{dq}{dt} = -\gamma_{21} u_1 q. \tag{26}$$

It is obvious that a vector function with coordinates $u_1 = 0, q = 0$ is an unstable stationary solution of system (26). However, for system (22), the same vector function may also turn out to be a stable stationary solution if the following condition is satisfied:

$$\mu_1 - 2D_1/l^2 < 0. \tag{27}$$

Condition (25) for the zero solution turns out to be superfluous, since it becomes a consequence of condition (27).

In the work [25], in order to prove the sufficiency of conditions (24), (25), the authors had to use a weighted version of the Steklov–Poincaré–Friedrichs inequality because of the convective term $-v\partial q/\partial x$ in the second equation. Another weighted variant is used in the work [26], where it is shown that the condition

$$M \frac{(\gamma + 1)^2}{4h^2} - A(s - 2p_0) > 0$$

is sufficient for the asymptotic stability of the stationary solution p_0 of the initial boundary value problem

$$\frac{\partial p}{\partial t} = Ap(s - p) + MB_\gamma p, \quad \frac{\partial p(0, t)}{\partial r} = 0, \quad p(h, t) = \beta, \quad t \geq 0.$$

Here B_γ is the Bessel operator defined by the formula

$$B_\gamma p = \frac{\partial^2 p}{\partial r^2} + \frac{\gamma}{r} \frac{\partial p}{\partial r} = r^{-\gamma} \frac{\partial}{\partial r} \left(r^\gamma \frac{\partial p}{\partial r} \right).$$

Very extensive information, including impressive literature reviews, about boundary value problems, functional spaces related to singular differential equations with the Bessel operator, see in [27–30].

4.3. Kermak–McKendrick Epidemic Model

As an example we consider the classical epidemic model of W. Kermak and A. McKendrick

$$\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI - \gamma I, \quad \frac{dR}{dt} = \gamma I,$$

where β and γ are positive constants. This model, proposed in 1927 (see [31]), characterizes the changes in the number of susceptible (S), infected (I), and recovered (R) individuals in the population in question. Subsequently, based on this model, an impressive number of models have been created, clarifying it and adapting to various situations. We change the model by adding a diffusion term to the right side of each equation, after that we obtain a system of partial differential equations

$$\frac{dS}{dt} = -\beta SI + \vartheta_1 \Delta S, \quad (28)$$

$$\frac{dI}{dt} = \beta SI - \gamma I + \vartheta_2 \Delta I, \quad (29)$$

$$\frac{dR}{dt} = \gamma I + \vartheta_3 \Delta R. \quad (30)$$

We consider system (28)–(30) in a domain $\Omega \subset \mathbb{R}^2$ with a piecewise smooth boundary $\Gamma = \partial\Omega$. We introduce uniform designations, which are consistent with the original model: $u_1 = u_1(x, t) = S$, $u_2 = u_2(x, t) = I$, $u_3 = u_3(x, t) = R$. Let us impose additional conditions on the solution (u_1, u_2, u_3) –boundary:

$$\left(\mu_j u_j + \eta_j \frac{\partial u_j}{\partial \nu} \right) \Big|_{x \in \partial\Omega} = B_j(x), \quad \mu_j^2 + \eta_j^2 > 0, \quad \mu_j \geq 0, \quad \eta_j \geq 0,$$

–initial:

$$u_j(x, 0) = w_j(x), \quad j = 1, 2, 3.$$

Here $B_j(x) \in C(\partial\Omega)$, $w_j(x) \in C^2(\Omega) \cap C(\bar{\Omega})$, $j = 1, 2, 3$, $\bar{\Omega} = \Omega \cup \partial\Omega$. Let the vector $w = (w_1(x), w_2(x), w_3(x))$ be a stationary solution of system (28)–(30), that is, the solution of the system

$$-\beta w_1 w_2 + \vartheta_1 \Delta w_1 = 0,$$

$$\beta w_1 w_2 - \gamma w_2 + \vartheta_2 \Delta w_2 = 0,$$

$$\gamma w_2 + \vartheta_3 \Delta w_3 = 0,$$

satisfying the boundary conditions

$$\left(\mu_j w_j + \eta_j \frac{\partial w_j}{\partial \nu} \right) \Big|_{x \in \partial\Omega} = B_j(x), \quad \mu_j^2 + \eta_j^2 > 0, \quad \mu_j \geq 0, \quad \eta_j \geq 0.$$

This time, we obtain the inequality for the derivative of the Lyapunov function

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} z^2 dx \leq \int_{\Omega} \sum_{k,j=1}^3 A_{kj} z_k z_j dx - \int_{\partial\Omega} \sum_{j=1}^3 \vartheta_j g_j d\Gamma + \int_{\Omega} (\beta z_1 z_2^2 - \beta z_1^2 z_2) dx,$$

where

$$A_{11} = -\beta w_2 - \frac{\vartheta_1}{d^2}, \quad A_{22} = \beta w_1 - \gamma - \frac{\vartheta_2}{d^2}, \quad A_{33} = -\frac{\vartheta_3}{d^2},$$

$$A_{12} = (\beta w_2 - \beta w_1)/2, \quad A_{23} = \gamma/2, \quad A_{13} = 0.$$

Using the Sylvester criterion, we obtain the following sufficient conditions for negative definiteness of a quadratic form

$$A_{11} = -\beta w_2 - \frac{\vartheta_1}{d^2} < 0, \quad (31)$$

$$A_{11}A_{22} - A_{12}^2 = \left(-\beta w_2 - \frac{\vartheta_1}{d^2}\right) \left(-\beta w_1 - \gamma - \frac{\vartheta_2}{d^2}\right) - \frac{1}{4}\beta(w_2 - w_1) > 0, \tag{32}$$

$$(A_{11}A_{22} - A_{12}^2) A_{33} - A_{11}A_{23}^2 \tag{33}$$

$$= \left(\left(-\beta w_2 - \frac{\vartheta_1}{d^2}\right) \left(-\beta w_1 - \gamma - \frac{\vartheta_2}{d^2}\right) - \frac{1}{4}\beta(w_2 - w_1)\right) \left(-\frac{\vartheta_3}{d^2}\right) - \left(-\beta w_2 - \frac{\vartheta_1}{d^2}\right) \frac{\gamma^2}{4} < 0. \tag{34}$$

These conditions are verifiable in practice, in computer simulations. Note that if we consider a trivial stationary solution $w_1 = w_2 = w_3 = 0$ in the framework of this model, conditions (31)–(34) will take the form

$$A_{11} = \frac{\vartheta_1}{d^2} > 0, \tag{35}$$

$$A_{11}A_{22} - A_{12}^2 = \frac{\vartheta_1}{d^2} \left(\gamma + \frac{\vartheta_2}{d^2}\right) > 0, \tag{36}$$

$$(A_{11}A_{22} - A_{12}^2) A_{33} - A_{11}A_{23}^2 = \frac{\vartheta_1}{d^2} \left(\frac{\gamma^2}{4} - \left(\gamma + \frac{\vartheta_2}{d^2}\right) \frac{\vartheta_3}{d^2}\right) < 0. \tag{37}$$

In a model with concentrated parameters, that is, when $\vartheta_1 = \vartheta_2 = \vartheta_3 = 0$, conditions (35)–(37) are not fulfilled, the trivial solution is unstable. In a model with distributed parameters, that is, when $\vartheta_1\vartheta_2\vartheta_3 > 0$, conditions (35), (36) are fulfilled in any case, therefore, condition (37) becomes a substantial sufficient condition for the stability of a trivial solution. This condition, obviously, can be rewritten as

$$\frac{\gamma^2}{4} - \left(\gamma + \frac{\vartheta_2}{d^2}\right) \frac{\vartheta_3}{d^2} < 0.$$

The fulfillment of this condition is possible for a domain with a small diameter. For domains with a large diameter, this condition is not satisfied.

5. MODELS IN HUMANITIES

5.1. The Hotelling Equation

We have already given one example above. The Verhulst model has an unstable trivial solution, which becomes stable if we add a diffusion term and proceed to the Hotelling model in a domain with a small diameter. Let’s note that the Hotelling equation describes the growth of a natural language vocabulary [32]).

5.2. On Modelling the Interaction of Language Groups

The model considered below in a non-diffusion form is borrowed from the work [12]. We supplement it to a diffusive form and give the quantities involved in it a different meaning. We still consider a new interpretation to be possible, basing on the already-mentioned principle of analogies in models constructing (see [33]). We realize that obtaining reliable data about the input values of the model, as well as about the desired values for its experimental verification, is a separate time-consuming task, which is not considered here.

Let $u_1 = u_1(x_1, x_2, t)$ and $u_2 = u_2(x_1, x_2, t)$ be the number (in arbitrary units) of groups of speakers of two languages (adverbs, dialects) living in the common territory. We will assume that the second group has “aggressiveness” in the following sense: due to the influence of the participants of the second group, a transition from the first group to the second group is possible. The system of ordinary differential equations, describing this process, has the form

$$\frac{\partial u_1}{\partial t} = \mu u_1(u_1 - \alpha)(1 - u_1) - u_1 u_2, \tag{38}$$

$$\frac{\partial u_2}{\partial t} = -\beta(b - u_1)u_2, \tag{39}$$

where $\alpha \in [0, 1)$, μ , β , and b are some parameters. We add to the system (38)–(39) the initial conditions

$$u_1|_{t=0} = u_1^0 \in [0, 1], \quad u_2|_{t=0} = u_2^0 \in [0, 1).$$

In the diffusion version, the system takes the form

$$\frac{\partial u_1}{\partial t} = \mu u_1(u_1 - \alpha)(1 - u_1) - u_1 u_2 + \vartheta_1 \Delta u_1, \quad (40)$$

$$\frac{\partial u_2}{\partial t} = -\beta(b - u_1)u_2 + \vartheta_2 \Delta u_2, \quad (41)$$

where $\vartheta_1 \geq 0$, $\vartheta_2 \geq 0$ (for $\vartheta_1 = 0$, $\vartheta_2 = 0$ we obtain the system (38)–(39) from [12]). We assume that the system is considered in a bounded domain Ω with a piecewise smooth boundary Γ . We add boundary conditions

$$\left(\eta_{11} u_1 + \eta_{12} \frac{\partial u_1}{\partial \nu} \right) \Big|_{\Gamma} = \zeta_1(x_1, x_2),$$

$$\left(\eta_{21} u_2 + \eta_{22} \frac{\partial u_2}{\partial \nu} \right) \Big|_{\Gamma} = \zeta_2(x_1, x_2),$$

where ν is the unit vector of the external normal to the boundary Γ , $\eta_{s1}^2 + \eta_{s2}^2 > 0$, $\eta_{s1}\eta_{s2} > 0$, $s = 1, 2$. Everywhere further we will consider regular solutions with sufficient smoothness, what, in particular, entails the fulfillment of all the necessary conditions for matching the initial and boundary data.

Applying the above reasoning in this case, we get the inequality

$$\frac{1}{2} \frac{\partial}{\partial t} \int_0^l |z|^2 dx \leq \int_0^l (a_{11} z_1^2 + 2a_{12} z_1 z_2 + a_{22} z_2^2) dx,$$

where d is the diameter of the domain Ω ,

$$a_{11} = 2\mu v_1 + 2\mu\alpha v_1 - \mu\alpha - 3\mu v_1^2 - v_2 - \frac{\vartheta_1}{d^2},$$

$$a_{12} = a_{21} = \frac{1}{2}(\beta v_2 - v_1), \quad a_{22} = \beta v_1 - b\beta - \frac{\vartheta_2}{d^2}.$$

A refined sufficient condition for the asymptotic stability of a stationary solution is the negative definiteness of the quadratic form

$$\sum_{k=1}^2 \sum_{j=1}^2 a_{kj} \xi_k \xi_j. \quad (42)$$

Certainly, checking this condition, although somewhat cumbersome, is still quite feasible, especially with the help of a computer.

It should be noted that taking into account diffusion phenomena can also introduce new knowledge about the object of research or about the model. Let's turn to the primary source of the model under consideration. In the work [12] it is indicated that the system of ordinary differential equations (38)–(39) has four stationary points. Note that each of them is also a stationary solution of the system (40), (41) of partial differential equations with diffusion terms. In order to illustrate our thesis about new information about the object, we consider two of these stationary points.

1. $v_1 = \alpha$, $v_2 = 0$.

At this stationary point, the eigenvalues of the matrix of the right side of the equations (38), (39) will have the form $\lambda_1 = \mu\alpha(1 - \alpha) > 0$, $\lambda_2 = -\beta(b - \alpha)$. Since one of the eigenvalues is positive, this stationary point will be unstable. We will study the influence of nonzero diffusion terms. It turns out that

in this case the quadratic form (42) may be negatively defined if certain conditions are satisfied. In this case we have

$$a_{11} = \mu\alpha(1 - \alpha) - \vartheta_1/d^2, \quad a_{12} = -\alpha/2, \quad a_{22} = \beta(\alpha - b) - \vartheta_2/d^2.$$

According to the Sylvester criterion, for the negative definiteness of the form (42), it is necessary and sufficient to fulfill the conditions

$$a_{11} = \mu\alpha(1 - \alpha) - \vartheta_1/d^2 < 0, \\ a_{11}a_{22} - a_{12}^2 = \Phi_1(d; \alpha, \beta, \mu, b) = (\mu\alpha(1 - \alpha) - \vartheta_1/d^2) (\beta(\alpha - b) - \vartheta_2/d^2) - \alpha^2/4 > 0.$$

It is quite obvious that each of these two conditions will be satisfied for sufficiently small d .

2. $v_1 = 1, v_2 = 0$.

At this point $\lambda_1 = -\mu(1 - \alpha) < 0, \lambda_2 = -\beta(b - 1)$. The inequality $b > 1$ is a necessary and sufficient condition for the stability of the point $(1, 0)$, considered as a stationary point of a system of ordinary differential equations (38), (39). Considering (40), (41), we have

$$a_{11} = -\mu(1 - \alpha) - \vartheta_1/d^2, \quad a_{12} = -1/2, \quad a_{22} = \beta(1 - b) - \vartheta_2/d^2.$$

Sylvester's criterion for the negative definiteness of a quadratic form in this case will lead to the conditions

$$a_{11} = -\mu(1 - \alpha) - \vartheta_1/d^2 < 0, \\ a_{11}a_{22} - a_{12}^2 = \Phi_2(d; \alpha, \beta, \mu, b) = (-\mu(1 - \alpha) - \vartheta_1/d^2) (\beta(1 - b) - \vartheta_2/d^2) - 1/4 > 0.$$

This time, the first condition is satisfied. The second condition can be satisfied for small values of d , not only for $b > 1$, but also for $b \leq 1$. Also worthy of note is the point $v_1 = 0, v_2 = 0$. At this stationary point, the eigenvalues of the matrix at the right side of equations (38)–(39) will be negative, and therefore this point will be stable. In this case, adding diffusion terms does not give anything new.

CONCLUSION

In this paper, we obtained a sufficient condition for the asymptotic stability of a stationary solution in a system of partial differential equations obtained from autonomous systems of ordinary differential equations by adding diffusion terms. It is noted that the nature of stability of constant stationary solutions changes when models with concentrated parameters are replaced by models with distributed parameters (in domains with small diameters for the better). It should be added that the obtained results also indicate the absence of the phenomenon known as Turing instability in many cases of domains with small diameters. In terms of population models, this fact means that migration processes contribute to the stability of stationary states in small domains.

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