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ON PROPERTIES OF RIEMANNIAN METRICS ASSOCIATED WITH B-ELLIPTIC OPERATORS

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Abstract. In this paper, we consider a Riemannian metric in which the Laplace–Beltrami operator coincides with a B-elliptic operator up to a factor.

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1. K-Homogeneous metrics. Let $\gamma = (\gamma_1, \dots, \gamma_n)$, where γ_i, $i = 1 \dots, n$, are fixed numbers such that

$$\sum_{i=1}^{n} \gamma_i^2 > 0.$$

We denote by $\mathbb{R}_+^n$ the set of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ such that $x_i \in \mathbb{R}$ if $\gamma_i = 0$ and $x_i \in (0, +\infty)$ if $\gamma_i \neq 0$. A variable x_i such that $\gamma_i \neq 0$ is said to be *exclusive*. As usually, we use the notation

$$(x)^\gamma = \prod_{i=1}^{n} x_i^{\gamma_i}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}_+^n.$$

Let a function $u(x)$ be twice continuously differentiable in $\mathbb{R}_+^n$.

We define the operator Δ_{B_γ} by the formula

$$\Delta_{B_\gamma} u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} \frac{\gamma_i}{x_i} \frac{\partial u}{\partial x_i}. \tag{1}$$

Operators of the form (1) were studied by I. A. Kipriyanov and his disciples (see [8–10]).

We state the following problem: Find a positive definite on $\mathbb{R}_+^n$, symmetric quadratic form (metric)

$$ds^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}\, dx_i\, dx_j,$$

such that the Laplace–Beltrami operator corresponding to this metric (see [3])

$$\Delta_\omega = \frac{1}{\sqrt{|g|}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sum_{j=1}^{n} g^{ik} \sqrt{|g|} \frac{\partial}{\partial x_k} \tag{2}$$

coincides with the operator Δ_{B_γ} up to a factor. Here functions g^{ij}, $i, j = 1, \dots, n$, are elements of the matrix $\|g^{ij}\|$, which is inverse to the matrix $\|g_{ij}\|$ (the covariant metric tensor), and

$$g = \det \|g_{ij}\|.$$

The study of elliptic partial differential operators using the Riemannian metric has a long history (see, e.g., [2, 6]).

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Theorem 1. *For $n \geq 3$, the elements of the matrix $\|g_{ij}\|$ are defined by the formula*

$$g_{ij} = \delta_{ij} \prod_{i=1}^{n} x_i^{K_i} = \delta_{ij} x^K, \quad i,j = 1,\ldots,n, \quad K = (K_1,\ldots,K_n), \tag{3}$$

where δ_{ij} is the Kronecker delta and

$$K_i = \frac{2}{n-2}\,\gamma_i. \tag{4}$$

Proof. Indeed, since $g_{ij} = 0$ for $i \neq j$, substituting (3) into (2), we obtain

$$\Delta_\omega u = \frac{1}{\sqrt{|g|}} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left(g^{kk} \sqrt{|g|} \frac{\partial u}{\partial x_k} \right), \tag{5}$$

where

$$|g| = g = x^{nK} = \prod_{i=1}^{n} \prod_{i=1}^{N} n\, x_i^{K_i} = \prod_{i=1}^{n} x_i^{2n\gamma_i/(n-2)}, \tag{6}$$

$$g^{kk} = x^{-K} = \prod_{i=1}^{n} x_i^{-2\gamma_i/(n-2)}. \tag{7}$$

Taking into account (6) and (7), we can rewrite (5) in the following form:

$$\begin{aligned}
\Delta_\omega u = \frac{1}{x^{nK/2}} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(x^{-K} x^{Kn/2} \frac{\partial u}{\partial x_j} \right)&\\
= x^{-K} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + x^{-Kn/2} \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \left(\prod_{l=1}^{n} x_l^{K_l(n-2)/2} \right) \frac{\partial u}{\partial x_j}&\\
= x^{-K} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + x^{-Kn/2} \sum_{j=1}^{n} \prod_{l=1}^{n} x_l^{K_l(n-2)/2} \frac{K_j(n-2)}{2} x_j^{-1} \frac{\partial u}{\partial x_j}&\\
= x^{-K} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + x^{-K} \sum_{j=1}^{n} \frac{K_j(n-2)}{2x_j} \frac{\partial u}{\partial x_j} = x^{-K} \Delta_{B_\gamma} u,&
\end{aligned}$$

that is,

$$\Delta_\omega u = x^{-K} \Delta_{B_\gamma} u. \qquad \square$$

We consider the set $\mathbb{R}_+^n$ equipped with a Riemannian metric

$$ds^2 = x^K \sum_{i=1}^{n} dx_i^2 \quad K \in \mathbb{R}, \tag{8}$$

as a Riemannian space. We denote this space by KI_n; its metric (8) is called the K-homogeneous metric.

Theorem 2. *For $n = 2$, the problem on the search for a metric satisfying Eq. ($\square$) has no solutions.*

Proof. Introduce the notation $E = g_{11}$, $F = g_{12} = g_{21}$, and $G = g_{22}$. Then

$$g = \det \|g_{ij}\| = EG - F^2, \quad g^{ij} = (-1)^{i+j} \frac{g_{ij}}{EG - F^2}.$$

Therefore,

$$\Delta_\omega u = \frac{G}{|g|} \frac{\partial^2 u}{\partial x_1^2} + \frac{E}{|g|} \frac{\partial^2 u}{\partial x_2^2} - 2\frac{F}{|g|} \frac{\partial^2 u}{\partial x_1 \partial x_2} + \Phi \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right), \tag{9}$$

where Φ is the term depending only on the first-order derivatives of the function u. The expression (9) is proportional to (1) if the condition $F \equiv 0$ holds; this implies

$$g = EG, \quad g^{11} = \frac{1}{E}, \quad g^{22} = \frac{1}{G}, \quad g_{12} = g_{21} = g^{12} = g^{21} = 0.$$

Therefore,

$$\Delta_\omega u = \frac{1}{\sqrt{|EG|}} \left(\frac{\partial}{\partial x_1} \left(\sqrt{\left|\frac{G}{E}\right|} \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\sqrt{\left|\frac{E}{G}\right|} \frac{\partial u}{\partial x_2} \right) \right)$$

$$= \frac{1}{E} \frac{\partial^2 u}{\partial x_1^2} + \frac{1}{G} \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial}{\partial x_1} \sqrt{\left|\frac{G}{E}\right|} \frac{\partial u}{\partial x_1} + \frac{\partial}{\partial x_2} \sqrt{\left|\frac{E}{G}\right|} \frac{\partial u}{\partial x_2}.$$

The first two terms must have the same coefficients, hence $E = G$. Then the last two terms vanish, which means the nonexistence of the desired metric for $n = 2$. □

2. Isometric transforms of K-homogeneous metrics. The necessary and sufficient condition under which the one-parameter group G with the infinitesimal operator

$$X = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}$$

is an isometry group is equivalent to the Killing conditions:

$$\sum_{s=1}^{n} \left(\xi_s \frac{\partial g_{ij}}{\partial x_s} + g_{is} \frac{\partial \xi_s}{\partial x_j} + g_{js} \frac{\partial \xi_s}{\partial x_i} \right) = 0, \quad i, j = 1, \dots, n.$$

Obviously,

$$\frac{\partial g_{ij}}{\partial x_s} = \delta_{ij} \frac{K_s x^K}{x_s}.$$

Therefore, the Killing equations take the form

$$\sum_{s=1}^{n} \left(\delta_{ij} \xi_s K_s x^{K-1} + x^K \left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} \right) \right) = 0, \quad i, j = 1, \dots, n.$$

Summing and dividing by x_K, we obtain

$$\delta_{ij} \sum_{s=1}^{N} \frac{\xi_s K_s}{x_s} + \frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} = 0, \quad i, j = 1, \dots, n. \tag{10}$$

For $i \neq j$, Eq. (10) can be written in the form

$$\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} = 0, \quad i, j = 1, \dots, n, \quad i \neq j. \tag{11}$$

For $i = j$ Eq. (10) can be written in the form

$$2 \frac{\partial \xi_j}{\partial x_j} + \sum_{s=1}^{n} \frac{K_s \xi_s}{x_s} = 0, \quad i = 1, \dots, n. \tag{12}$$

The vector

$$\xi = (\xi_1, \dots, \xi_n), \quad \xi_j = C\, x^p x_j, \tag{13}$$

where

$$p = (p_1, \dots, p_n), \quad p_1 = p_2 = \dots = p_n = \beta = -\sum_{l=1}^{n} \frac{K_l}{2} - 1, \tag{14}$$

is a solution of the system (12); this can be verified by a direct calculation. Substituting the representation (13) into Eqs. (11) and taking into account (14), we obtain

$$0 \equiv \frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} = C\beta x^p \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} \right), \quad i, j = 1, \ldots, n, \quad i \neq j.$$

Therefore,

$$p_1 = p_2 = \cdots = p_n = \beta = -\sum_{l=1}^{n} \frac{K_l}{2} - 1 = 0, \tag{15}$$

or, equivalently,

$$\sum_{l=1}^{n} K_l = -2. \tag{16}$$

Taking into account (4), we have

$$\sum_{i=1}^{n} \gamma_i = 2 - N. \tag{17}$$

3. Characteristics of K-homogeneous metrics in the case of one exclusive variable. One of the cases where the condition (15) (or, equivalently, (17)) is fulfilled is well known. The space KI_n is the Poincaré model of the n-dimensional Lobachevsky space. In what follows, we consider the case where $\gamma_1 = \gamma_2 = \cdots = \gamma_{n-1} = 0$, $\gamma_n \neq 0$. The metric (3) has the form

$$g_{ij} = \delta_{ij} x_n^K, \quad i, j = 1, \ldots, n, \tag{18}$$

where

$$K = \frac{2}{n-2}\gamma. \tag{19}$$

The following facts are proved by direct calculations.

Theorem 3. *The Christoffel symbols of the first kind corresponding to the metric* (8) *have the form*

$$\Gamma_{ij,k} = \frac{1}{2} K x_n^{K-1} \Big(\delta_{ik}\delta_{jn} + \delta_{jk}\delta_{in} - \delta_{ij}\delta_{kn} \Big).$$

Proof. From the definition of the Christoffel symbols of the first kind, taking into account (18)–(19), we obtain

$$\Gamma_{ij,k} = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k} \right) = \frac{1}{2} \Big(\delta_{ik}\delta_{jn} K x_n^{K-1} + \delta_{jk}\delta_{in} K x_n^{K-1} - \delta_{ij}\delta_{kn} K x_n^{K-1} \Big). \qquad \square$$

Theorem 4. *The Christoffel symbols of the second kind corresponding to the metric* (8) *have the form*

$$\Gamma_{ij}^k = \frac{K}{2x_n} \Big(\delta_{ik}\delta_{jn} + \delta_{jk}\delta_{in} - \delta_{ij}\delta_{kn} \Big).$$

Proof. From the definition of the Christoffel symbols of the second kind and Theorem 3 we have

$$\Gamma_{ij}^k = \sum_{h=1}^{n} g^{kh}\Gamma_{ij,h} = \frac{K}{2} \sum_{h=1}^{n} \delta_{kh} x_n^{-K} x_n^{K-1} \Big(\delta_{ih}\delta_{jn} + \delta_{jh}\delta_{in} - \delta_{ij}\delta_{hn} \Big)$$

$$= \frac{K}{2x_n} \sum_{h=1}^{n} \Big(\delta_{ki}\delta_{jn} + \delta_{kj}\delta_{in} - \delta_{ij}\delta_{kn} \Big). \qquad \square$$

Theorem 5. *The components of the Riemann tensor corresponding to the metric* (8) *have the form*

$$R_{ijk}^l = \left(\frac{K^2}{4x_n^2} - \frac{K}{2x_n^2} \right) \Big(\delta_{li}\delta_{in}\delta_{kn} + \delta_{ik}\delta_{jn}\delta_{ln} - \delta_{ij}\delta_{kn}\delta_{ln} - \delta_{lk}\delta_{in}\delta_{jn} \Big) + \frac{K^2}{4x_n^2} \Big(\delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj} \Big).$$

Proof. By definition, the components of the Riemann tensor are calculated by the formulas

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x_j} - \frac{\partial \Gamma^l_{ij}}{\partial x_k} + \sum_{m=1}^{n} \Big(\Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk}\Big).$$

We calculate the partial derivatives:

$$\frac{\partial \Gamma^k_{ij}}{\partial x_s} = -\frac{K}{2x_n^2}\delta_{sn}\Big(\delta_{ki}\delta_{jn} + \delta_{kj}\delta_{in} - \delta_{ij}\delta_{kn}\Big);$$

therefore,

$$\frac{\partial \Gamma^l_{ik}}{\partial x_j} = -\frac{K}{2x_n^2}\delta_{jn}\Big(\delta_{li}\delta_{kn} + \delta_{lk}\delta_{in} - \delta_{ik}\delta_{ln}\Big), \quad \frac{\partial \Gamma^l_{ij}}{\partial x_k} = -\frac{K}{2x_n^2}\delta_{kn}\Big(\delta_{li}\delta_{jn} + \delta_{lj}\delta_{in} - \delta_{ij}\delta_{ln}\Big).$$

We have

$$\frac{\partial \Gamma^l_{i}}{\partial x_j} - \frac{\partial \Gamma^l_{ij}}{\partial x_s} = -\frac{K}{2x_n^2}\Big(\delta_{jn}\delta_{lk}\delta_{in} - \delta_{jn}\delta_{ik}\delta_{ln} - \delta_{kn}\delta_{lj}\delta_{in} + \delta_{kn}\delta_{ij}\delta_{ln}\Big).$$

Now we calculate the last term in the definition. Taking into account Theorem 4, we obtain

$$\begin{aligned}
&\sum_{m=1}^{n} \Big(\Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk}\Big) \\
&\quad = \frac{K^2}{4x_n^2}\sum_{m=1}^{n}\Big(\delta_{mi}\delta_{kn}\delta_{lm}\delta_{jn} + \delta_{mi}\delta_{kn}\delta_{lj}\delta_{mn} - \delta_{mi}\delta_{kn}\delta_{mj}\delta_{ln} + \delta_{mk}\delta_{in}\delta_{lm}\delta_{jn} \\
&\quad + \delta_{mk}\delta_{in}\delta_{lj}\delta_{mn} - \delta_{mk}\delta_{in}\delta_{mj}\delta_{ln} - \delta_{ik}\delta_{mn}\delta_{lm}\delta_{jn} - \delta_{ik}\delta_{mn}\delta_{lj}\delta_{mn} + \delta_{ik}\delta_{mn}\delta_{mj}\delta_{ln} \\
&\quad - \delta_{mi}\delta_{jn}\delta_{lm}\delta_{kn} - \delta_{mi}\delta_{jn}\delta_{lk}\delta_{mn} + \delta_{mi}\delta_{jn}\delta_{mk}\delta_{ln} - \delta_{mj}\delta_{in}\delta_{lm}\delta_{kn} - \delta_{mj}\delta_{in}\delta_{lk}\delta_{mn} \\
&\quad + \delta_{mj}\delta_{in}\delta_{mk}\delta_{ln} + \delta_{ij}\delta_{mn}\delta_{lm}\delta_{kn} + \delta_{ij}\delta_{mn}\delta_{lk}\delta_{mn} - \delta_{ij}\delta_{mn}\delta_{mk}\delta_{ln}\Big).
\end{aligned}$$

Taking into account the properties of the Kronecker delta, in particular, the formulas $\delta_{il} = \delta_{li}$ and $\sum_{m=1}^{n} \delta_{mi}\delta_{lm} = \delta_{il}$, after identity transformations we arrive to the desired formulas. □

Theorem 6. *The components of the Ricci tensor corresponding to the metric* (8) *have the form*

$$R_{ij} = \frac{K}{4x_n^2}\Big((K-2)(2-n)\delta_{in}\delta_{jn} + \big(K(n-2)+2\big)\delta_{ij}\Big). \tag{20}$$

Proof. Substituting the formulas obtained in Theorem 5 into the definition of the components of the Ricci tensor

$$R_{ij} = \sum_{k=1}^{n} R^k_{ijk},$$

we obtain (20). □

Theorem 7. *The curvature of the space KI_n is calculated by the formula*

$$R = \frac{Kn(n-2)}{x_n^{K+2}} = \frac{2\gamma n}{x_n^{(2\gamma+2n-4)/(n-2)}}. \tag{21}$$

Proof. The formula (21) follows from the definition of the curvature

$$R = \sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}R_{ij}$$

and the formulas obtained above. □

Remark 1. There are no twice continuously differentiable changes of variables that reduce an equation of the form

$$\Delta_{B_\gamma} u \equiv \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} + \frac{\gamma}{x_n}\frac{\partial u}{\partial x_n} = f, \quad \gamma \neq 0,$$

to an equation of the form

$$\Delta u \equiv \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = f.$$

Indeed, if such a substitution exists, then there exists a coordinate transform turning a metric with nonzero curvature to a metric with zero curvature, which is impossible.

4. Geodesic lines for K-homogeneous metrics.

Theorem 8. *The system of equations of geodesic lines of the space KI_n is reduced to the first-order system*

$$\frac{dx_k}{ds} = \frac{C_k}{x_n^K}, \quad k = 1, \dots, n-1, \qquad \left(\frac{dx_n}{ds}\right)^2 = \frac{C_n}{x_n^K} - \frac{B^2}{x_n^{2K}}, \tag{22}$$

where $B = \sqrt{\sum\limits_{k=1}^{n-1} C_k^2}$.

Proof. The system of equations for geodesic lines of a given metric $\|g_{ij}\|$ has the form

$$\frac{d^2 x_k}{ds^2} + \sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma_{ij}^k \frac{dx_i}{ds}\frac{dx_j}{ds} = 0, \quad k = 1, 2, \dots, n,$$

where s is the natural parameter (arclength). In our case, using the Christoffel symbols calculated above, we rewrite this system in the form

$$\frac{d^2 x_k}{ds^2} + \frac{K}{x_n}\frac{dx_n}{ds}\frac{dx_k}{ds} = 0, \qquad k = 1, \dots, n-1, \tag{23}$$

$$\frac{d^2 x_n}{ds^2} - \frac{K}{2x_n}\sum_{i=1}^{n}\left(\frac{dx_i}{ds}\right)^2 + \frac{K}{2x_n}\left(\frac{dx_n}{ds}\right)^2 = 0. \tag{24}$$

Equations (23) can be written in the form

$$x_n^{-K}\frac{d}{ds}\left(x_n^K \frac{dx_k}{ds}\right) = 0, \quad k = 1, \dots, n-1. \tag{25}$$

Multiplying (25) by x_n^K, integrating, and dividing by x_n^K, we obtain

$$\frac{dx_k}{ds} = \frac{C_k}{x_n^K}, \quad k = 1, \dots, n-1. \tag{26}$$

Substituting (26) into (24), we have

$$\frac{d^2 x_n}{ds^2} - \frac{KB^2}{2x_n^{2K+1}} + \frac{K}{2x_n}\left(\frac{dx_n}{ds}\right)^2 = 0, \tag{27}$$

where B was defined above. Equations (27) admits reducing of the order. We set $p = p(x_n) = dx_n/ds$ and $v = p^2$; then $d^2x_n/ds^2 = p'p = v'/2$. Equation (27) takes the form

$$v' + \frac{K}{x_n}v = \frac{B^2 K}{x_n^{2K+1}} \quad \Longleftrightarrow \quad \frac{d}{dx_n}(x_n^K v) = \frac{B^2 K}{x_n^{K+1}}.$$

Integrating and dividing by x_n^K, we obtain

$$v = p^2 = \left(\frac{dx_n}{ds}\right)^2 = \frac{C_n}{x_n^K} - \frac{B^2}{x_n^{2K}}. \qquad \square$$

It is well known (see [1]) that geodesic lines possess the property

$$\sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}\frac{dx_i}{ds}\frac{dx_j}{ds} = \text{const}\,.$$

In the case considered, this leads to the equality

$$\sum_{i=1}^{n} x_n^K \left(\frac{dx_i}{ds}\right)^2 = \text{const}\,. \tag{28}$$

From (22) we can easily obtain that the constant in Eq. (28) coincides with C_n.

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