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Comparison of simulation and analytical models for the distribution of a group of agents moving in random directions

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This study focused on simulation and analytical models of a multidimensional random walk of many agents. At each step, any agent can rotate by an arbitrary angle and continue moving in the direction selected. The analytical model used gives the probability of finding an agent in the circular area of the radius r in the given time moment. The model includes parameters that control the intensity of the walking process and the characteristics of the environment. Work on this topic mainly concerns the discrete case of a random walk. In this article, we consider the case of a random walk continuous in spatial coordinates. We obtain certain theoretical results for the analytical model with the help of the mathematical apparatus aimed at working with a generalized translation. With the help of a simulation, we reveal the meaning of the analytical model's parameters. Results can be used for modeling diffusion-like processes such as the spread of an epidemic, mechanical vibrations, forest fires, migration, and dissemination of information over a network.

KEYWORDS
Bessel function, generalized translation, migration model, multi-agent model, random walk, simulation model

MSC CLASSIFICATION
33C10, 60G50, 93A16

1 | INTRODUCTION

In this paper, we deal with planar and spatial random motion in cases when a moving particle can rotate at an arbitrary angle at each displacement. We studied the problem of theoretical description and simulation modeling of spatial random motion in an arbitrary direction and their comparison. The solution of the tasks set is of great importance since it confirms a theoretical model that gives instant results and may be used instead of slow simulation.

A brief historical overview will provide useful context. Since the founding of the "Educational Times," in 1847 under that heading, and then a separate edition entitled "Mathematical Questions," a large number of problems have been associated with probability and particularly with random walks. In 1865, M. W. Crofton[1] posed the problem of a traveler's movements along a river, which is formalized as a random walk in a straight line. In all likelihood, this was the first mathematical illustration of the random flight concept. Later, in a letter to *Nature* in 1905, Karl Pearson[2] suggested that the probability could be found that a man will be at a particular distance between r and $r + dr$ from origin O after n displacements if he started to move from a point O and walked one yard in a straight line, turned through any angle whatever and walked another yard in a second straight line, and so on. This random walk problem has attracted the interest of many researchers. In particular, Lord Rayleigh responded in 1880 that he had solved the random walk problem in the context of sound waves spreading.[3] Based on Lord Rayleigh's conceptions, in 1905, Jan Cornelis Kluyver[4] proposed a general solution to the random walk problem in terms of certain definite integrals, involving Bessel functions.

In 1906, Karl Pearson with his assistant wrote a detailed study of migration[5] using the Kluyver approach. In this article, we can see how difficult it was to study the proposed model containing integrals of Bessel functions without using computer systems and with access to only a limited mathematical apparatus of special functions. Here, the authors used graphical methods and calculated the characteristics of interest to them approximately and manually using power series. In all of the papers mentioned, the question regarded walks on a plane. Watson,[6, pp. 460–462] obtained a generalization of the random walk model in the case of an arbitrary number of space dimensions.

The first to study random motion the probability law of which forms a solution to the hyperbolic type equation was Sydney Goldstein in 1951.[7] He considered the simplest random walk on a real line, in which a particle placed at the origin moves changing its current speed in accordance with the simplest Poisson process with a constant parameter. This model was then examined in detail by Mark Katz[8] and Enzo Orsinger.[9] Subsequent generalizations of such a model are presented in articles.[10–14]

A more detailed historical overview is given in papers.[15,16]

The random walk introduced by Pearson (Pearson walks) has a large number of applications. The British theoretical physicist Lord Rayleigh studied the problem of the theory of sound, which is mathematically equivalent to the problem of a random walk.[3] He considered a set of oscillations, each with a unit amplitude, the same frequency, and an arbitrary phase, and posed the problem of finding the distribution of the resulting intensity. Nobel laureate Ronald Ross[17] presented a diffusion model of the random migration of mosquitoes when he studied the laws of the spread of malaria.

Pearson's paper[5] introduced the application of random walks to the description of migration models. Another interesting application where a special case of the random walk model appeared is a description of polymer conflagrations,[18, p. 142, formula 4.28b] The next application of Pearson random walks concerned the analysis of narrowband signals in noise.[19] Thereafter, a remarkable application was proposed for the motion of microorganisms on surfaces.[20,21] This type of model also appeared in the field of crystallography.[22]

It can be noted that interest in the problem of Pearson's random walks has not faded away in the 21st century. In paper,[23] uncorrelated and unbiased variants of the Pearson random walk in Euclidean spaces $\mathbb{R}^d$ without simulation were considered. Paper[24] deals with the representation of odd moments of the distribution of a four-step uniform Pearson random walk in even dimensions. Pearson random walks have also been applied to the modeling of transport and reaction of gasses in Vignoles.[25]

Historical and modern works on this topic mainly concern the discrete case of a random walk, in particular, for example, when a walk occurs along a lattice oriented parallel to the rectangular coordinate axes of a k-dimensional Euclidean space. Relatively little attention has been paid to the continuous case of a random walk in which the direction of a wandering object can change continuously from one step to the next. Why is this the case? The fundamental difference between the mathematical model of a random walk with an arbitrary, continuously changing angle of direction of a moving object from a walk on a grid is that a generalized translation is used instead of the usual one. The generalized translation is a singular integral operator, and the corresponding differential equations contain the Bessel operator instead of the usual derivative. Therefore, the reason that the model proposed by Rayleigh and Pearson did not receive substantial theoretical development was the lack of a suitable mathematical apparatus aimed at working with a generalized translation. In this paper, we present a mathematical description of a continuous random walk model, in which the direction of a wandering object can continuously change from one step to another, and show that it is supported by experimental data.

This study was motivated by applications in epidemiology, mathematical biology, vibration propagation, forest fire spread, crystallography, and other fields.

The results of this article were briefly discussed in the Russian language in the conference paper.[26]

2 | DESCRIPTION OF THE MODEL

In this section, we generate a quantitative model of random walks and some of its properties and compare it with established results.

2.1 | Quantitative model of random walks

Let agents $A = \{a_1, \dots, a_s\}$ be concentrated at the origin. So at time t_0 agent a_i, $i = \overline{1,s}$ has a position X_0. Then it starts to jump from the center, and at times $t_1, t_2, \dots, t_n$, it undergoes displacements $X_1, X_2, \dots, X_n$. The resultant at t_n is $S_n = X_0 + \sum_{m=1}^{n} X_m$. The displacements are assumed to be independent and the probability density of X_m is $p_m(X_m)$ and the probability density of S_n is required to be found.

Theorem 1. *Let l_m denotes the length of the mth jump, $m = 1, \dots, n$. Let $Pr(|S_n| \le r) = Pr(|S_n| \le r; l_1, l_2, \dots, l_n)$ be the probability that S_n lies inside or on a circle of radius r centered at origin O. Then the following formula for $Pr(|S_n| \le r)$ is valid*

$$Pr(|S_n| \le r) = \chi(r)\frac{2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^{\infty} J_{\frac{\nu}{2}}(rt) \prod_{m=1}^{n} j_{\frac{\nu}{2}-1}(l_m t) t^{\frac{\nu}{2}-1} dt, \tag{1}$$

where J_η is a Bessel function of the first kind[6] and $j_\eta(x) = \frac{2^\eta \Gamma(\eta+1)}{x^\eta} J_\eta(x)$ is a normalized Bessel function,[27]

$$\chi(r) = \begin{cases} 1 & \text{if } r \ne |S_n|; \\ \frac{1}{2} & \text{if } r = |S_n|. \end{cases}$$

Proof. In this problem, we suppose that all values of θ_m, $m = \overline{1,n}$ are not equally likely and depend on parameter $\nu > 1$. This expresses that the element of angle θ_m multiplied by the factor $\sin^{\nu-2}\theta_m$ and θ_m varies from 0 to π. Taking into account the formula

$$\int_0^{\pi} \sin^{\nu-2} t dt = \frac{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)},$$

we receive the probability that an agent will be in a circle of radius r after the nth jump for $n \ge 2$ is the $(n-1)$-tuple integral of the form

$$Pr(|S_n| \le r) = \left(\frac{\Gamma\left(\frac{\nu}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)}\right)^{n-1} \int_0^{\pi} \sin^{\nu-2}\theta_1 d\theta_1 \int_0^{\pi} \sin^{\nu-2}\theta_2 d\theta_2 \dots \int_0^{\pi} \sin^{\nu-2}\theta_{n-2} d\theta_{n-2} \int \sin^{\nu-2}\theta_{n-1} d\theta_{n-1}, \tag{2}$$

where the integration with respect to θ_{n-1} extends over the values of θ_{n-1}, which make $|S_n| \le r$. Formula (2) generalizes the formula from Watson,[6, p. 461] where $\nu = p \ge 2$ is only natural number.

When we use the Weber–Schafheitlin formula from Weinstein[28] (see also Watson,[6, p. 461]), for $\nu > 0$ of the form

$$\int_0^{\infty} J_{\nu-1}(st) J_\nu(rt) dt = \begin{cases} \frac{s^{\nu-1}}{r^\nu} & \text{for } 0 < s < r; \\ \frac{1}{2r} & \text{for } s = r; \\ 0 & \text{for } s > r. \end{cases} \tag{3}$$

So, if discontinuous factor

$$\chi(r)\frac{r^{\frac{\nu}{2}}}{|S_n|^{\frac{\nu}{2}-1}} \int_0^{\infty} J_{\frac{\nu}{2}-1}(|S_n| t) J_{\frac{\nu}{2}}(rt) dt = \begin{cases} 1 \text{ for } |S_n| \le r; \\ 0 \text{ for } |S_n| > r, \end{cases} \tag{4}$$

where

$$\chi(r) = \begin{cases} 1 & \text{if } r \ne |S_n|; \\ \frac{1}{2} & \text{if } r = |S_n| \end{cases}$$

is inserted in the $(n-1)$-tuple integral $Pr(|S_n| \le r)$, the range of values of θ_m may be taken to be from 0 to π

$$
\begin{aligned}
Pr(|S_n| \le r) = \chi(r) r^{\frac{\nu}{2}} & \left(\frac{\Gamma\left(\frac{\nu}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)} \right)^{n-1} \\
& \times \int_0^{\pi} \sin^{\nu-2}\theta_1 d\theta_1 \int_0^{\pi} \sin^{\nu-2}\theta_2 d\theta_2 \dots \int_0^{\pi} \sin^{\nu-2}\theta_{n-2} d\theta_{n-2} \\
& \times \int_0^{\pi} \sin^{\nu-2}\theta_{n-1} |S_n|^{1-\frac{\nu}{2}}\, d\theta_{n-1} \int_0^{\infty} J_{\frac{\nu}{2}-1}(|S_n|t) J_{\frac{\nu}{2}}(rt) dt \\
= \chi(r) r^{\frac{\nu}{2}} & \left(\frac{\Gamma\left(\frac{\nu}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)} \right)^{n-1} \int_0^{\pi} \sin^{\nu-2}\theta_1 d\theta_1 \int_0^{\pi} \sin^{\nu-2}\theta_2 d\theta_2 \dots \int_0^{\pi} \sin^{\nu-2}\theta_{n-2} d\theta_{n-2} \\
& \times \int_0^{\infty} J_{\frac{\nu}{2}}(rt) dt \int_0^{\pi} \frac{J_{\frac{\nu}{2}-1}(|S_n|t)}{|S_n|^{\frac{\nu}{2}-1}} \sin^{\nu-2}\theta_{n-1}\, d\theta_{n-1}.
\end{aligned}
$$

Using the definition of normalized Bessel function of the first kind j_ν (see Shishkina and Sitnik,[27, formula 1.19])

$$
j_\eta(x) = \frac{2^\eta \Gamma(\eta+1)}{x^\eta} J_\eta(x), \tag{5}
$$

we can write

$$
\begin{aligned}
Pr(|S_n| \le r) = \chi(r) 2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}} & \frac{\Gamma^{n-2}\left(\frac{\nu}{2}\right)}{\left(\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)\right)^{n-1}} \\
& \times \int_0^{\pi} \sin^{\nu-2}\theta_1 d\theta_1 \int_0^{\pi} \sin^{\nu-2}\theta_2 d\theta_2 \dots \int_0^{\pi} \sin^{\nu-2}\theta_{n-2} d\theta_{n-2} \\
& \times \int_0^{\infty} J_{\frac{\nu}{2}}(rt) t^{\frac{\nu}{2}-1} dt \int_0^{\pi} \frac{2^{\frac{\nu}{2}-1}\Gamma\left(\frac{\nu}{2}\right) J_{\frac{\nu}{2}-1}(|S_n|t)}{t^{\frac{\nu}{2}-1}|S_n|^{\frac{\nu}{2}-1}} \sin^{\nu-2}\theta_{n-1}\, d\theta_{n-1} \\
= \chi(r) 2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}} & \frac{\Gamma^{n-2}\left(\frac{\nu}{2}\right)}{\left(\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)\right)^{n-1}} \int_0^{\pi} \sin^{\nu-2}\theta_1 d\theta_1 \int_0^{\pi} \sin^{\nu-2}\theta_2 d\theta_2 \dots \int_0^{\pi} \sin^{\nu-2}\theta_{n-2} d\theta_{n-2} \\
& \times \int_0^{\infty} J_{\frac{\nu}{2}}(rt) t^{\frac{\nu}{2}-1} dt \int_0^{\pi} j_{\frac{\nu}{2}-1}(|S_n|t) \sin^{\nu-2}\theta_{n-1}\, d\theta_{n-1}.
\end{aligned}
$$

Since (see Figure 1)

$$
|S_n| = \sqrt{|S_{n-1}|^2 - 2|S_{n-1}| l_n \cos\theta_{n-1} + l_n^2}
$$

and for integral by θ_{n-1}, we obtain

$$\begin{aligned}
&\int_0^\pi j_{\frac{\nu}{2}-1}(|S_n|t)\sin^{\nu-2}\theta_{n-1}\,d\theta_{n-1}\\
&=\int_0^\pi j_{\frac{\nu}{2}-1}\left(\sqrt{|S_{n-1}|^2-2|S_{n-1}|l_n\cos\theta_{n-1}+l_n^2}\cdot t\right)\sin^{\nu-2}\theta_{n-1}\,d\theta_{n-1}\\
&=\frac{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\,{}^{\nu-1}T_{l_n}^{|S_{n-1}|}j_{\frac{\nu}{2}-1}(l_nt)
\end{aligned}$$

where $({}^{\gamma}T_x^y f)(x)$ is generalized translation[29] of the form

$$({}^{\gamma}T_x^y f)(x)=\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)}\int_0^\pi f\left(\sqrt{x^2+y^2-2xy\cos\theta}\right)\sin^{\gamma-1}\theta\,d\theta,\qquad \gamma>0.$$

The formula

$${}^{\gamma}T_x^y j_{\frac{\gamma-1}{2}}(x\xi)=j_{\frac{\gamma-1}{2}}(x\xi)\,j_{\frac{\gamma-1}{2}}(y\xi)$$

is known (see Shishkina and Sitnik,[27, formula 3.165]). Therefore, there follow

$$\begin{aligned}
&\int_0^\pi j_{\frac{\nu}{2}-1}\left(\sqrt{|S_{n-1}|^2-2|S_{n-1}|l_n\cos\theta_{n-1}+l_n^2}\cdot t\right)\sin^{\nu-2}\theta_{n-1}\,d\theta_{n-1}\\
&=\frac{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}j_{\frac{\nu}{2}-1}(l_nt)j_{\frac{\nu}{2}-1}(|S_{n-1}|t)
\end{aligned}$$

and

$$\begin{aligned}
&Pr(|S_n|\le r)\\
&=\chi(r)2^{1-\frac{\nu}{2}}r^{\frac{\nu}{2}}\frac{\Gamma^{n-2}\left(\frac{\nu}{2}\right)}{\left(\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)\right)^{n-1}}\frac{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\int_0^\pi\sin^{\nu-2}\theta_1d\theta_1\int_0^\pi\sin^{\nu-2}\theta_2d\theta_2\ldots\int_0^\pi\sin^{\nu-2}\theta_{n-2}d\theta_{n-2}\\
&\times\int_0^\infty J_{\frac{\nu}{2}}(rt)j_{\frac{\nu}{2}-1}(l_nt)j_{\frac{\nu}{2}-1}(|S_{n-1}|t)t^{\frac{\nu}{2}-1}dt\\
&=\chi(r)2^{1-\frac{\nu}{2}}r^{\frac{\nu}{2}}\frac{\Gamma^{n-3}\left(\frac{\nu}{2}\right)}{\left(\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)\right)^{n-2}}\int_0^\pi\sin^{\nu-2}\theta_1d\theta_1\int_0^\pi\sin^{\nu-2}\theta_2d\theta_2\ldots\int_0^\pi\sin^{\nu-2}\theta_{n-2}d\theta_{n-2}\\
&\times\int_0^\infty J_{\frac{\nu}{2}}(rt)j_{\frac{\nu}{2}-1}(l_nt)j_{\frac{\nu}{2}-1}(|S_{n-1}|t)t^{\frac{\nu}{2}-1}dt.
\end{aligned}$$

By repetitions of this process, we get (1). □

Corollary 1. *Since*

$$\frac{d}{dr}r^{\frac{\nu}{2}}J_{\frac{\nu}{2}}(rt)=tr^{\frac{\nu}{2}}J_{\frac{\nu}{2}-1}(rt)$$

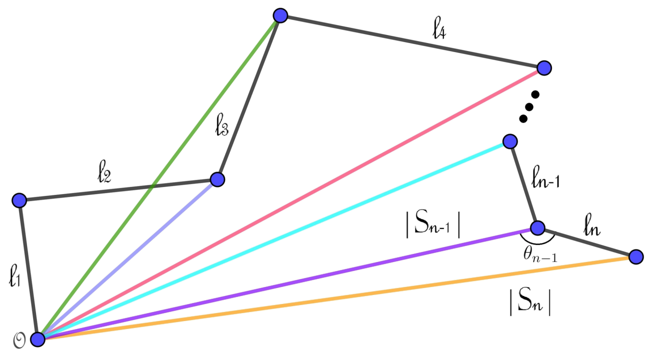


FIGURE 1 Migration scheme on the nth jump [Colour figure can be viewed at wileyonlinelibrary.com]

on differentiating $Pr(|S_n| \leq r)$ with respect to r, one gets the probability density

$$f_n(r) = \chi(r)\frac{2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty J_{\frac{\nu}{2}-1}(rt) \prod_{m=1}^{n} j_{\frac{\nu}{2}-1}(l_m t) t^{\frac{\nu}{2}} dt. \tag{6}$$

Remark 1. Since $\prod_{m=1}^{n} j_{\frac{\nu}{2}-1}(l_m t) = \left(\frac{2^{\frac{\nu}{2}-1}\Gamma\left(\frac{\nu}{2}\right)}{t^{\frac{\nu}{2}-1}}\right)^n (l_1 \cdot \ldots \cdot l_n)^{1-\frac{\nu}{2}} \prod_{m=1}^{n} J_{\frac{\nu}{2}-1}(l_m t)$, we can rewrite the integral in (6) in the form

$$\int_0^\infty J_{\frac{\nu}{2}-1}(rt) \prod_{m=1}^{n} j_{\frac{\nu}{2}-1}(l_m t) t^{\frac{\nu}{2}} dt = \frac{\left(2^{\frac{\nu}{2}-1}\Gamma\left(\frac{\nu}{2}\right)\right)^n}{(l_1 \cdot \ldots \cdot l_n)^{\frac{\nu}{2}-1}} \int_0^\infty t^{n\left(1-\frac{\nu}{2}\right)+\frac{\nu}{2}} \prod_{m=1}^{n} J_{\frac{\nu}{2}-1}(l_m t) J_{\frac{\nu}{2}-1}(rt) dt.$$

We can calculate the last integral using formula 2.12.44.5 from Prudnikov et al,[30, p. 207] of the form

$$\int_0^\infty x^{\alpha-1} \prod_{k=1}^{m} J_{\nu_k}(c_k x) dx = 2^{\alpha-1} c_m^{\nu_m-\mu} \frac{\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(\nu_m - \frac{\mu}{2} + 1\right)} \times \prod_{k=1}^{m} \frac{c_k^{\nu_k}}{\Gamma(\nu_k+1)} F_C^{(m-1)}\left(\frac{\mu}{2}, \frac{\mu}{2} - \nu_m; \nu_1 + 1, \ldots, \nu_{m-1}+1; \frac{c_1^2}{c_m^2}, \ldots, \frac{c_{m-1}^2}{c_m^2}\right),$$

where $c_k > 0$, $k = 1, 2, \ldots, n$, $c_m > c_1 + \ldots + c_{m-1}$, $-(\nu_1 + \ldots + \nu_m) < \alpha < \frac{m}{2} + 1$, $\mu = \alpha + \nu_1 + \ldots + \nu_m$,

$$F_C^{(n)}(a, b; c_1, \ldots, c_n; z_1, \ldots, z_n) = \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{(a)_{k_1+\ldots+k_n}(b)_{k_1+\ldots+k_n}}{(c_1)_{k_1}\ldots(c_n)_{k_n}} \frac{z_1^{k_1} \ldots z_n^{k_n}}{k_1! \ldots k_n!},$$

is the Lauricella function, and $(z)_n = z(z+1)\ldots(z+n-1)$, $n = 1, 2, \ldots$, $(z)_0 \equiv 1$ is the Pochhammer symbol. We obtain $\alpha = n\left(1 - \frac{\nu}{2}\right) + \frac{\nu}{2} + 1$, $m = n+1$, $\nu_k = \frac{\nu}{2} - 1$, $k = 1, \ldots, n+1$, $c_1 = l_1, \ldots, c_n = l_n$, $c_{n+1} = r$, $\mu = n\left(1 - \frac{\nu}{2}\right) + \frac{\nu}{2} + 1 + n\frac{\nu}{2} - n + \frac{\nu}{2} - 1 = \nu$. Therefore, if $l_1 + \ldots + l_n < r$,

$$f_n(r) = \chi(r)\frac{2r^{\frac{\nu}{2}-2}}{\Gamma\left(\frac{\nu}{2}\right)} F_C^{(n)}\left(\frac{\nu}{2}, 1; \underbrace{\frac{\nu}{2}, \ldots, \frac{\nu}{2}}_{n}; \frac{l_1^2}{r^2}, \ldots, \frac{l_n^2}{r^2}\right).$$

Remark 2. If $l_1 = l_2 = \dots = l_n = l$, then

$$Pr(|S_n| \le r) = \chi(r)\frac{2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty J_{\frac{\nu}{2}}(rt) j^n_{\frac{\nu}{2}-1}(lt) t^{\frac{\nu}{2}-1} dt. \tag{7}$$

Putting $l = \nu l_1$, we get

$$\begin{aligned} Pr(|S_n| \le r) &= \chi(r)\frac{2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty J_{\frac{\nu}{2}}(rt) j^n_{\frac{\nu}{2}-1}(\nu l_1 t) t^{\frac{\nu}{2}-1} dt = \{\nu t = \tau\} \\ &= \frac{1}{\nu}\chi(r)\frac{2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty J_{\frac{\nu}{2}}\left(r\frac{\tau}{\nu}\right) j^n_{\frac{\nu}{2}-1}(l_1 \tau) t^{\frac{\nu}{2}-1} d\tau. \end{aligned} \tag{8}$$

2.2 | Special cases of the quantitative model

Here, we compare the results of the model in the previous subsection with previously known models.

1. For $\nu = 2$, we obtain Kluyver model[4] for the case when agents moving on a plain and the choice of the angle θ_m between $-\pi$ and π are equally probable. In this model, the probability that the distance from the starting point will be less or equal than r after n flights is

$$Pr(|S_n| \le r) = \chi(r) r \int_0^\infty J_1(rt) \prod_{m=1}^n J_0(l_m t) dt \tag{9}$$

and the probability density is

$$f_n(r) = \chi(r) r \int_0^\infty t J_0(rt) \prod_{m=1}^n J_0(l_m t) dt. \tag{10}$$

Here, we notice that $j_0(x) = J_0(x)$.

2. If $l_1 = l_2 = \dots = l_n = l$ in (9)–(10), then we receive the Pearson-Rayleigh model[2,3]

$$Pr(|S_n| \le r) = \chi(r) r \int_0^\infty J_1(rt) J_0^n(lt) dt,$$

$$f_n(r) = \chi(r) r \int_0^\infty t J_0(rt) J_0^n(lt) dt.$$

3. For $\nu = k \in \mathbb{N}$, $k \ge 3$ in (1), we obtain Watson model (see Watson,[6, pp. 460–462]). This case corresponds to the problem for the k-dimensional Euclidean space. If generalized polar coordinates (in which θ_m is regarded as a co-latitude) are used, the element of a generalized solid angle contains θ_m only in the factor $\sin^{p-2}\theta_m d\theta_m$ and θ_m varies from 0 to π. The symmetry with respect to the polar axis enables us to disregard the factor depending on the longitudes.

3 | MULTI-AGENT MODEL

Currently, many authors use multi-agent simulation models of epidemics, forest fire spread, animal migrations, and so forth. Models such as cellular automata in continuum mechanics are less well known (see, for example, review[31]). However, the relationship between the multi-agent models listed here and the corresponding models based on differential equations has not been studied sufficiently. Since, for the operation of a multi-agent model, we need to calculate the

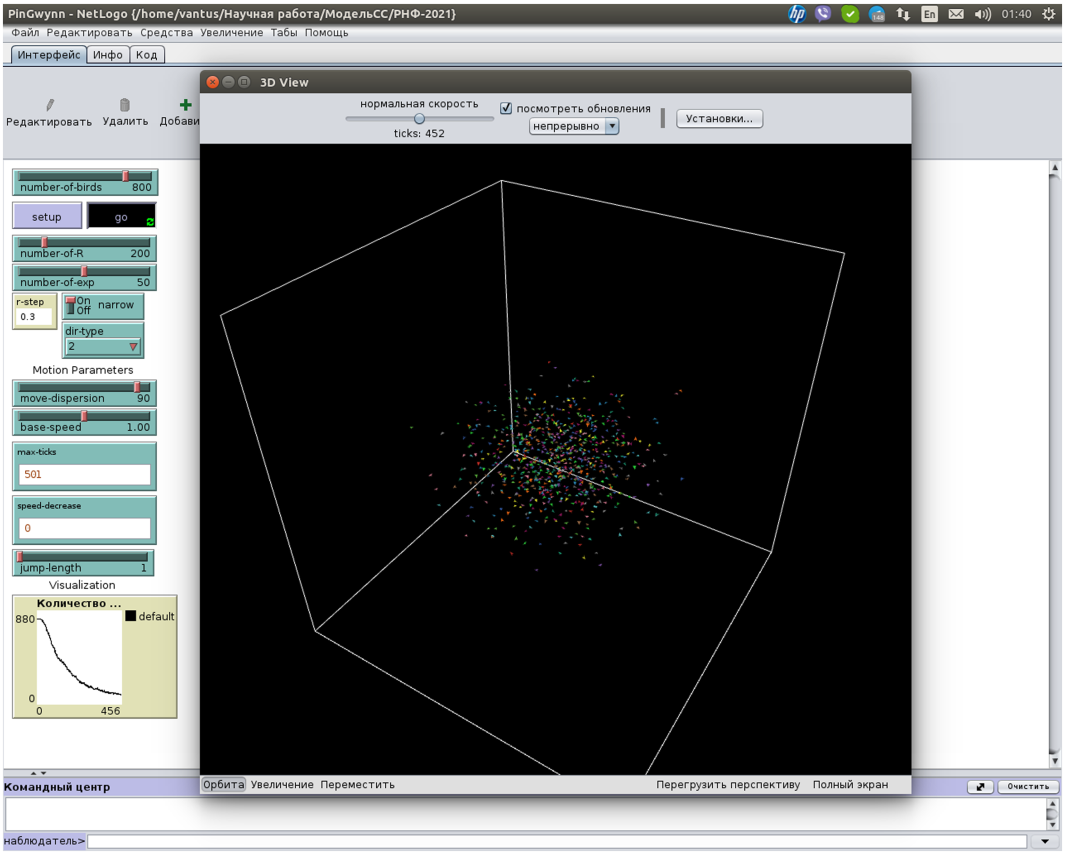


FIGURE 2 Agents movement model in NetLogo [Colour figure can be viewed at wileyonlinelibrary.com]

movements and interactions of each agent, then with a significant number of agents, the requirements for computing power for such a model can be very high. For example, to simulate a pandemic, it may be necessary to account for the interactions of hundreds of millions of agents. In this case, it seems appropriate to use the models mentioned based on differential equations.

Comparison of the multi-agent model and a model based on differential equations will determine how accurately the model based on differential equations describes the situation for a small number of agents and can help to justify the application of the model based on differential equations in the case of a very large number of agents.

Let $Ag = \{ag_1, \dots, ag_m\}$ be the set of agents. Each agent ag_i, $i = \overline{1, m}$ has the following parameters:

- coordinates (x_i, y_i),
- jump length l_i,
- movement speed v_i,
- direction of movement relative to the ordinate axis φ_i,
- direction of movement relative to the plane xOy (in the three-dimensional case) ψ_i.

All agents are initially located at the origin and then begin to move in a straight line in the direction determined by the angles φ_i and (in the three-dimensional case) ψ_i.

One agent ag_i, $i = \overline{1, m}$ or swarm-like group of agents ag_i, $i \in I \subset \overline{1, m}$ corresponds to one agent a_i, $i = \overline{1, s}$ according to Section 2.1. The discrete agent time does not necessarily correspond to real time. For example, when studying animal migrations or the spread of agricultural pests, it makes no sense to track the movement of animals every second. Instead, daily or weekly migration is tracked. Thus, one discrete time step can correspond to a day, a week, or a month.

We used the open source NetLogo[32] system (Figure 2) as a multi-agent modeling environment. Wolfram Mathematica 12 was used to process the obtained data.[*]

[*]See the code repository in https://bitbucket.org/Vantus/pingwynn.

(A) (B)

FIGURE 3 Two models of agents movement, $t = 375$. (A) Movement of agents with equal probability of choosing any direction of motion. (B) Movement of agents preferring motion along the Oy axis [Colour figure can be viewed at wileyonlinelibrary.com]

In order to understand the meaning of parameter ν of the theoretical model, we built two simulation models of the movement of agents. In the first one (Figure 3A), corresponding to even ν, the agent's direction angle $\varphi \in [0, 2\pi)$ with respect to the Oy axis and the agent's inclination angle $\psi \in [0, 2\pi)$ with respect to the plane xOy in the three-dimensional case are uniformly distributed random variables that were calculated every discrete moment of time. We call a model of this type isotropic.

In the second model (Figure 3B), corresponding to odd ν, the agent chose in the one-dimensional case the angle $\varphi_0 = 0$ or $\varphi_0 = \pi$ with equal probabilities and then chose the uniform random $\varphi_1 \in [-\alpha, \alpha)$, $\alpha \leq \pi$, eventually making a movement at an angle $\varphi = \varphi_0 + \varphi_1$. In the two-dimensional case, the agent chose the agent's direction angle $\varphi \in [0, 2\pi)$ relative to the Oy axis as a uniformly distributed random variable and the agent's inclination angle relative to the xOy plane as a uniformly distributed random variable $\psi \in [-\alpha, \alpha)$. This model will be called anisotropic.

We performed the experiments to calculate the value $\rho(t, r) = N(t, r)/m$, where $N(t, r)$ is the number of agents in a circle of radius r centered at the origin at the time moment t. Then we compare the value of $\rho(t, r)$ with the cumulative distribution function of agents $Pr(|S_t| \leq r)$. The cumulative distribution function of agents $Pr(|S_t| \leq r)$ in the continuous model was described in Section 2.

We carried out several series of computational experiments. These included 51 experiments on the movement of agents from the origin in each series for the dimensions $n = 2$ and $n = 3$ and various types of agents' movement. From the values of the function ρ at the time moments n and for the radii r in these experiments, we calculated the mean value $\bar{\rho}(n, r)$, and compare it with $Pr(|S_n| \leq r)$, for $\nu = 2$ and $\nu = 3$, respectively.

The coefficient of determination R^2 was used as a measure for estimating the quality of the approximation. The value R^2 for the sample $\{y_1, \ldots, y_l\}$ with the mean value $\bar{y}$ and for the predicted values $\{f_1, \ldots, f_l\}$ is defined as the following:

$$R^2 = 1 - \frac{SS_{\text{res}}}{SS_{\text{tot}}},$$

where

$$SS_{\text{tot}} = \sum_{i=1}^{l} (y_i - \bar{y})^2, \; SS_{\text{res}} = \sum_{i=1}^{l} (y_i - f_i)^2.$$

Let $m = 1000$, $l_i = 1$, $\nu_i = 1$, $i = \overline{1, m}$, agents move along the plane according to the isotropic model. An example of the dependencies is shown in Figures 4 and 5. For the function $\bar{\rho}(t, 4.5)$, the coefficient of determination is $R^2 = 0.993696$ (first 150 time moments), and for $\bar{\rho}(15, r)$, the coefficient of determination is $R^2 = 0.9967$ (r from 0.3 to 9 with a step of

FIGURE 4 Comparison of theoretical and multi-agent models with fixed radius $r = 4.5$ for two-dimensional (2D) and three-dimensional (3D) cases [Colour figure can be viewed at wileyonlinelibrary.com]

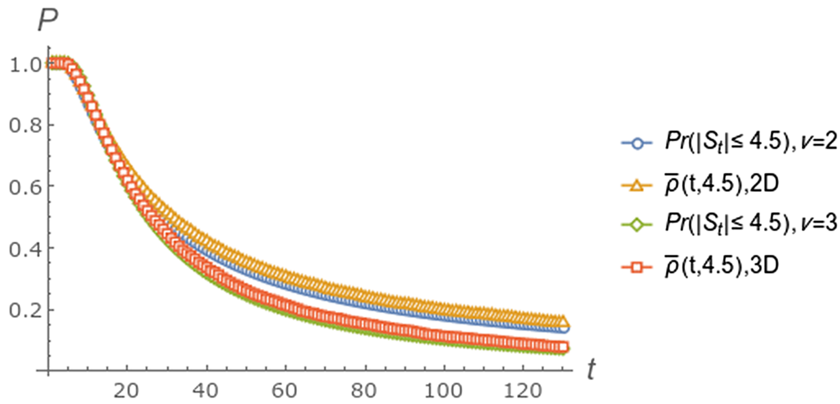


FIGURE 5 Comparison of theoretical and multi-agent models at fixed time $t = 15$ for two-dimensional (2D) and three-dimensional (3D) cases [Colour figure can be viewed at wileyonlinelibrary.com]

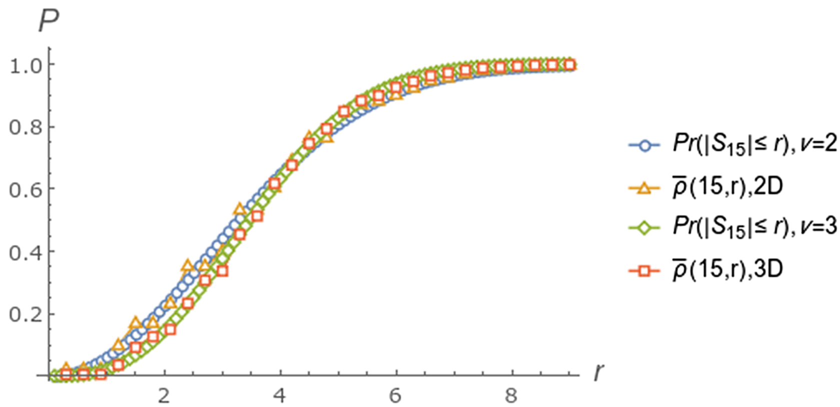
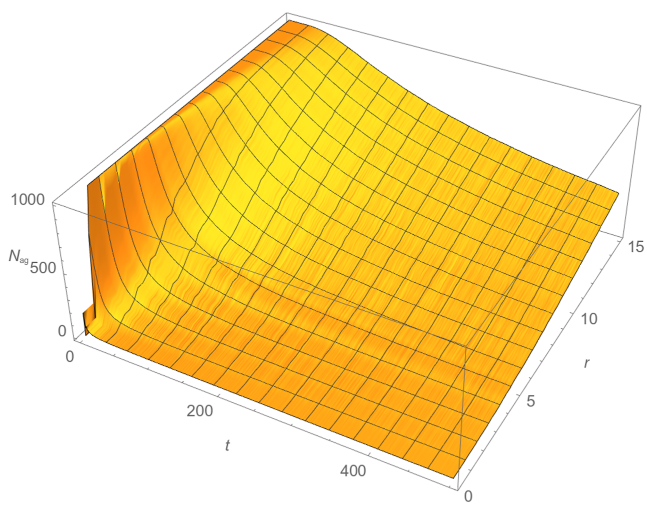


FIGURE 6 General view of the function $N_{ag} = m\bar{\rho}$ of the average number of agents in a circle of radius r at time t for multi-agent models [Colour figure can be viewed at wileyonlinelibrary.com]



0.3), and similar values were obtained for other values of the parameters. Thus, the continuous model with $\nu = 2$ predicts well the behavior of the multi-agent model on the plane. The general view of $m\bar{\rho}$ is shown in Figure 6.

The parameter ν from Equation (8) signifies the agents' speed, which is confirmed by experiments with $\nu = \nu_i = 0.5$ and $\nu_i = 0.25$, $i = \overline{1, m}$ (Figures 7 and 8). For the function $\bar{\rho}(t, 4.5)$, the coefficient of determination $R^2 = 0.982718$ for $\nu = 0.25$, for $\nu = 0.5$ $R^2 = 0.99165$ (first 400 time moments).

Now, let the agents move in three-dimensional space according to an isotropic model: In this case, the continuous model with $\nu = 3$ predicts the behavior of the multi-agent well. For example, for $\rho(t, 4.5)$, $R^2 = 0.998016$ (first 130 times), and for $\rho(15, r)$, $R^2 = 0.999029$ (r from 0.3 to 9 with a step of 0.3).

Non-integer values of the parameter $\nu > 1$ permit the application of the theoretical model to the movement of agents according to the anisotropic model. The movement of agents on the plane along the chosen direction with a deviation of

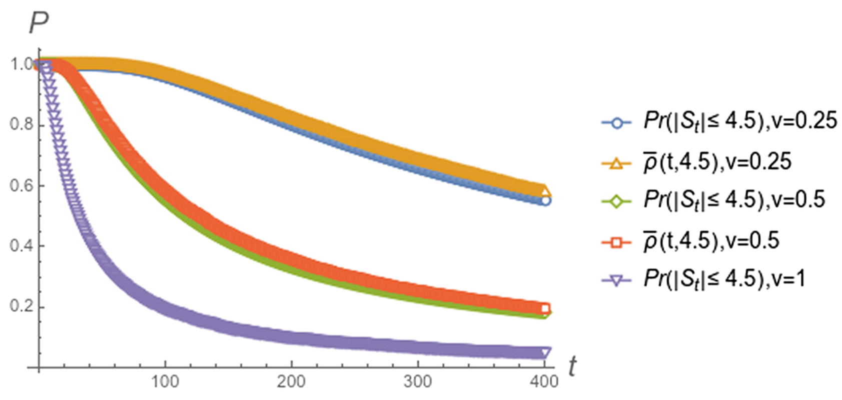


FIGURE 7 Comparison of the values of $\bar{\rho}(t, 4.5)$ and $Pr(|S_t| \leq 4.5)$ for the continuous model at different speeds ν of movement of agents [Colour figure can be viewed at wileyonlinelibrary.com]

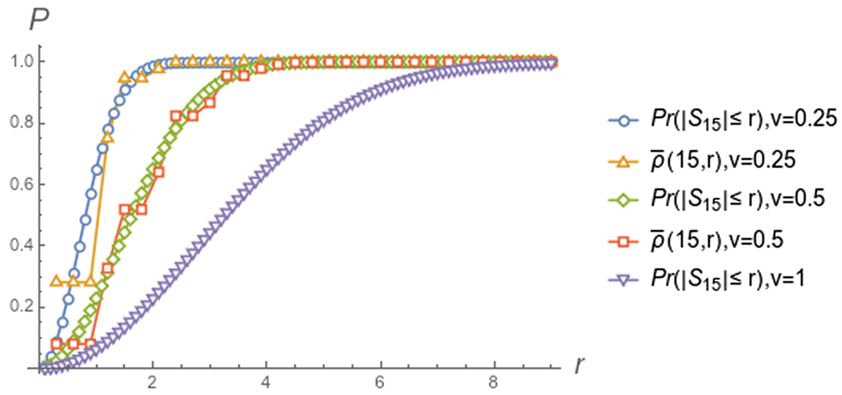


FIGURE 8 Comparison of the values of $\bar{\rho}(15, r)$ and $Pr(|S_{15}| \leq r)$ for the continuous model at different speeds ν of movement of agents [Colour figure can be viewed at wileyonlinelibrary.com]

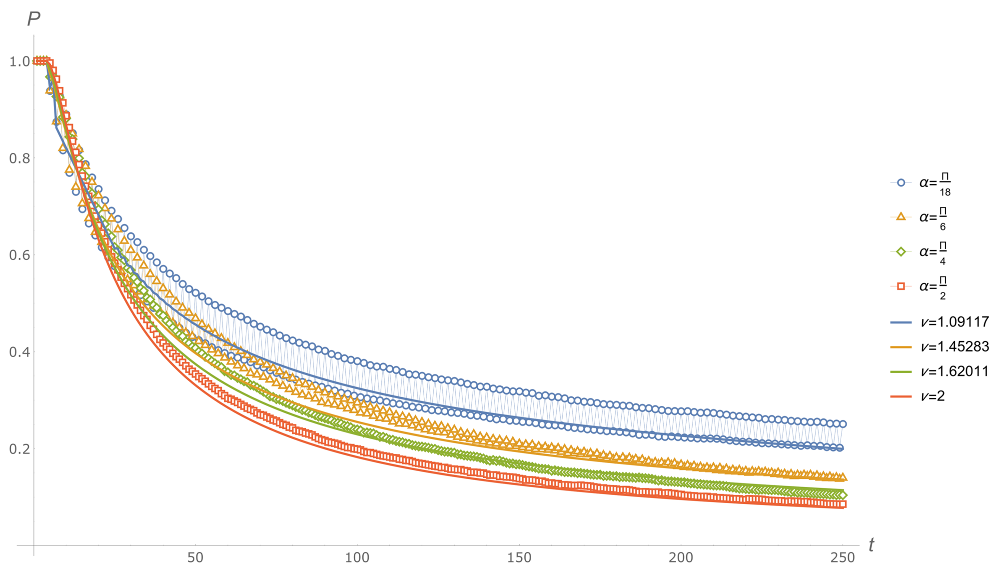


FIGURE 9 Values $\bar{\rho}(t, 4.5)$ (markers) and $Pr(|S_t| \leq 4.5)$ (line) for anisotropic model [Colour figure can be viewed at wileyonlinelibrary.com]

$0 < \alpha < \pi$ from it will correspond to $1 < \nu < 2$ (Figures 9 and 10). The movement of agents in three-dimensional space along the chosen plane with deviation $-\pi < \alpha < \pi$ will correspond to $2 < \nu < 3$.

The exact correspondence between ν and α has not been completely studied, but we can say that, in the two-dimensional case, a good approximation of this dependence will be

$$\nu = \ln\left(\frac{2\alpha}{\pi}(e^2 - e) + e\right).$$

Examples of the fitness of $Pr(|S_t| \leq 4.5)$ for $\bar{\rho}(t, 4.5)$ for different values α and ν for the first 250 time moments can be found in Table 1 and in Figures 9 and 10.

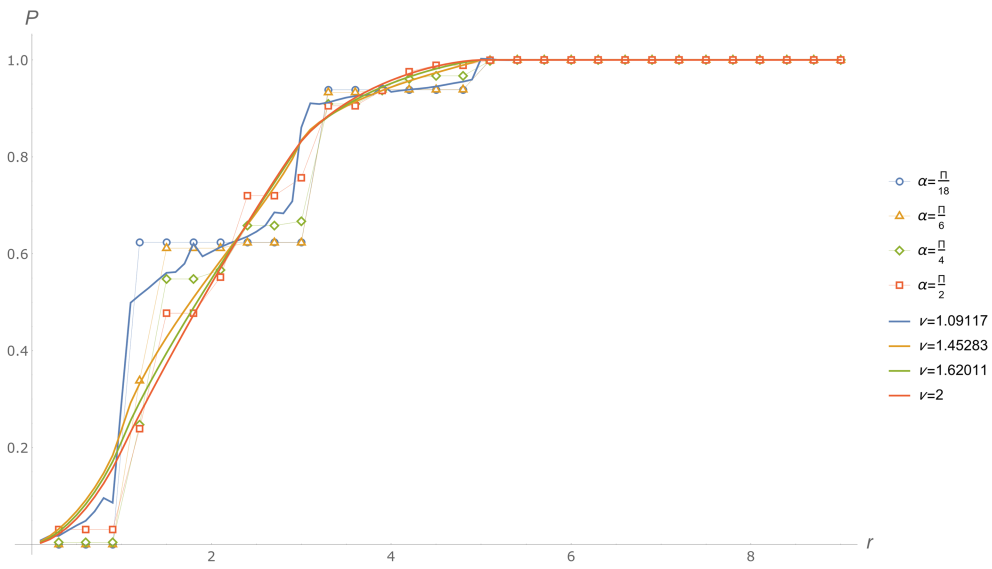


FIGURE 10 Values $\bar{\rho}(5, r)$ (markers) and $Pr(|S_5| \leq r)$ (line) for anisotropic model [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 The comparison of $Pr(|S_t| \leq 4.5)$ and $\bar{\rho}(t, 4.5)$ for different values α and ν

α	ν	R^2	Comments
$\pi/18$	1.09117	0.927494	Point $t = 24$ was dropped
$\pi/6$	1.45283	0.975205	Points $t \in [8,24]$ were dropped
$\pi/4$	1.62011	0.990793	

Some points could not be calculated due to difficulties in calculating the integral (9) by standard means. Such points have been excluded. In the future, we can recalculate these points using Remark 1 with the Lauricella function. We may note that in this case, the function $\bar{\rho}$ oscillates, and the function $Pr(|S_t| \leq r)$ corresponds to some smoothing of $\bar{\rho}$ but not $\bar{\rho}$ itself, unlike previous cases.

4 | CONCLUSION

We investigated the relationship between a random walk model based on a generalized translation and multi-agent models of migration. The mathematical apparatus of our studies includes the method of transmutation operators,[27] random walk theory, special functions, and multi-agent simulation. The relevance of the model based on the generalized translation is based on the fact that we can apply the model in cases where there are too many agents and the use of the simulation model is not rational. With the help of simulation, we clarified the meaning of the parameters of differential equations that model random movements of agents. We found how those parameters influence the agent's speed and preferable direction of the movement.

The practical application of the results obtained lies in the field of modeling processes such as diffusion, for example, in the spread of an epidemic, mechanical vibrations, forest fires, migration, dissemination of information over a social network, as well as over computer networks, and others.

Further research may be related to obtaining an equation corresponding to the studied distribution density function for an arbitrary ν and obtaining a limit central theorem.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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