International Scholarly Research Network
ISRN Applied Mathematics
Volume 2011, Article ID 679056, 19 pages
doi:10.5402/2011/679056

Research Article

Traffic Optimization at Junctions to Improve Vehicular Flows

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Received 14 April 2011; Accepted 22 May 2011

Academic Editor: E. J. Sellountos

The aim of this work is to improve urban traffic viability through an appropriate choice of yielding and stop signs or red and green phases for traffic lights in junctions with two entering and one exiting roads (junctions of 2×1 type). We consider a macroscopic fluid-dynamic model able to capture the traffic evolution. We analyze different functionals measuring networks performance in terms of average velocity, average traveling time, total flux, density, stop and go waves, average traveling time, weighted with the number of cars moving on roads, and kinetic energy. Right of way parameters which optimize the latter two functionals are obtained. Simulations of simple junctions of 2×1type have been used to test the correctness of the analytical results. Then, global performance of optimization procedures has been investigated on Re di Roma Square, in Italy. In particular, we discuss cases in which the functionals are optimized locally at each junction for different values of right of way parameters. We show that for the chosen initial data the only algorithm for the maximization of velocity assures globally the best performance for the network, also in terms of average traveling times and kinetic energy.

1. Introduction

The problem of traffic modeling is a very huge task, due to the complexity of the system to analyze. Many methods have been developed based on different approaches ranging from microscopic one, taking into account each single car, to kinetic and macroscopic ones, dealing with averaged quantities. Of course the understanding and control of traffic phenomena by means of simulation and optimization studies can be useful to eventually make decisions which may alleviate congestion, maximize flow traffic, reduce accidents, and other desirable ends.

Here rather than analyzing the movement of individual cars, we describe road networks behaviour using fluid-dynamic models; hence we treat traffic situations resulting from complex interaction of many vehicles. In particular, following [1] we determine the evolution of car traffic on each road with the first-order model of Lighthill, Whitham, and Richards (LWR model), which is on one side simple enough to permit a complete understanding of traffic flows and on the other side rich enough to detect important phenomena as queue formation. A key role in the networks modeling is played by the junctions. In order to capture the dynamics at a node, Cauchy problems with initial data constant on each road, called Riemann Problems at the node, have to be solved. The latter have a unique solution if we introduce some rules:

(A) the cars flow from an incoming road to outgoing ones according to the final destination;

(B) the number of cars passing the junction is maximized respecting rule (A).

For a junction of 1×2 type, that is, with one incoming road, labeled with 1, and two outgoing roads, labeled with 2, 3, the rule (A) is expressed by a coefficient α which describes the percentage of cars going from road 1 to road 2. Obviously $1 - \alpha$ is the percentage of cars moving towards the outgoing road 3.

If the junction has two incoming roads, labeled with 1 and 2, and one outgoing road, labeled with 3, (junction of 2×1 type), in order to solve the dynamics, we have to introduce the right of way parameter p. Supposing that C cars can enter into the junction, then pC cars come from road 1 and $(1 - p)C$ from road 2.

Assigning the initial density of all incoming and outgoing roads in a node, we compute the final equilibrium as function of the traffic distribution coefficient or/and of the right of way parameter depending on junction type. Such equilibrium, belonging to the admissible region for the final fluxes, is chosen according to a fixed strategy, and represents the solution of the Riemann Problem.

The aim of this work is the optimization of urban networks performance with junctions of 2×1 type through a suitable choice of the right of way parameter p. Observe that the choice of p corresponds to the use of yielding and stop signs or to the regulation of red and green phases for traffic lights.

Optimization problems for fluid-dynamic models have been already considered for car traffic: [2] is devoted to traffic light regulation, while [3] and [4] are more related to our analysis but focus on the case of smooth solutions (not developing shocks) and boundary control. A specific traffic regulation problem is addressed in [5]. Given a crossing with some expected traffic, is it preferable to construct a traffic circle or a light? The two solutions are studied in terms of flow control and the performances are compared. In [6–8], the traffic behaviour has been analyzed using four cost functionals, J_1, J_2, J_3, and J_5, measuring, respectively, cars average velocity, cars average traveling time, car velocity, weighted with cars quantity, traveling on the roads, that is, fluxes and stop and go waves (see also [9]). The optimization was done over right of way parameters and traffic distribution coefficients with the aim of maximizing J_1 and J_3 and minimizing J_2. In particular, two special cases of junctions have been considered for the optimization: the 2×1 and 1×2 cases.

In this work, we analyze the urban traffic behaviour introducing new functionals. From the solution of the Riemann Problem, we determine the average speeds at which drivers travel and we define additional functionals: J_4 which measures the density, J_6 the average traveling time, seen as the sum of the average traveling times on each road, weighted with the number of cars moving on it, and J_7 which gives information about the kinetic energy. Of course the aim is to optimize the choice of the parameters in order to minimize J_6 and to maximize J_7. Since in case of 1×2 junctions, the functionals J_6 and J_7 are optimized for the same values of the distribution coefficients which maximize and minimize J_1 and J_2, we focused our attention on the optimization of junctions of type 2×1, for which J_1 and J_6 have, in some cases, different optimal values. It is interesting to notice that in many cases (with the extreme case of functionals not depending on p) there is a set of optimal values of the right of way parameters.

The correctness of analytical optimization algorithms is tested through simulations. First we consider simple junctions of 2×1 type and compare different choices of parameters, statistical parameters, and optimal one. In particular, the initial density values on the roads are chosen in such way that optimization procedures, obtained evaluating the described cost functionals, give different optimal parameters.

Then, we studied the effects of the decentralized approach on the global performance of more complicated networks. According to this approach local optimal parameters at every junction of a complex network have been used. The discussed example regards Re di Roma Square, in Rome. We notice that the global effect of optimization is achieved using the optimal values for the velocity functional. Indeed, it is also shown that, considering the optimal values for the other cost functionals, the network becomes almost completely full, with consequent birth of congestion phenomena and reduction of cars fluxes. Finally we compare optimal algorithms with random ones. In the latter case, the right of way parameters are chosen randomly at every instant of time and for every junction independently. We see that the optimal algorithm based on the maximization of the velocity ensures better performance than random simulation.

The paper is organized as follows. In Section 2, we recall the basic definitions and the construction of solutions to Riemann Problems at junctions. The subsequent section is devoted to the introduction of the cost functionals and to the optimization of J_6 and J_7, compared with J_1 and J_2. Then, Section 4 reports simulation results first for simple junctions and then for Re di Roma Square.

2. Road Network Model

We consider a traffic network that is a finite collection of roads connected together by junctions. Formally, we introduce the following definition.

Definition 2.1. A traffic network is given by a 4-tuple $(N, \mathcal{I}, \mathcal{F}, \mathcal{J})$ where

Cardinality

N is the cardinality of the network, that is, the number of roads in the network;

Lines

$\mathcal{I}$ is the collection of roads, modeled by intervals $I_k = [a_k, b_k] \subseteq \mathbb{R}, k = 1, \dots, N$;

Fluxes

$\mathcal{F}$ is the collection of flux functions $f_k : [0, \rho_k^{\max}] \mapsto \mathbb{R},\ k = 1, \dots, N$, with $\rho_k^{\max}$ the maximal density on road I_k;

Nodes

$\mathcal{J}$ is a collection of subsets of $\{\pm 1, \ldots, \pm N\}$ representing junctions. If $j \in J \in \mathcal{J}$, then the road $I_{|j|}$ is crossing at J as incoming road (i.e., at point b_i) if $j > 0$ and as outgoing road (i.e., at point a_i) if $j < 0$. For each junction $J \in \mathcal{J}$, we indicate by $\mathrm{Inc}(J)$ the set of incoming roads, that are I_i's such that $i \in J$, while by $\mathrm{Out}(J)$ the set of outgoing roads, that are I_i's such that $-i \in J$. We assume that each road is incoming for (at most) one node and outgoing for (at most) one node.

In what follows, we suppose that $f_k = f$ for $k = 1, \ldots, N$, but it is possible to generalize all definitions and results to the case of different fluxes f_k for each road I_k. In fact, all statements are in terms of fluxes values at junctions, thus it is sufficient that the ranges of fluxes intersect. On each road, we consider the LWR model, described by the equation (see [10, 11]),

$$\rho_t + f(\rho)_x = 0, \tag{2.1}$$

where $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, $\rho = \rho(t, x) \in [0, \rho_{\max}]$ is the density of cars, $\rho_{\max}$ is the maximal density, $f(\rho) = \rho v$ is the flux, and $v = v(\rho)$ the average velocity. For roads such that $i \notin \bigcup_{J \in \mathcal{J}} \mathrm{Inc}(J)$ and $b_i < +\infty$ or such that $i \notin \bigcup_{J \in \mathcal{J}} \mathrm{Out}(J)$ and $a_i > -\infty$, a boundary condition is needed. In throughout the paper, we use the following flux function:

$$f(\rho) = \rho(1 - \rho), \quad \rho \in [0, 1], \tag{2.2}$$

with a unique maximum $\sigma = 1/2$.

For a single conservation law (2.1) on a real line $\mathbb{R}$, a Riemann Problem (RP), the basic ingredient to construct approximate solutions to Cauchy Problems with wave front tracking algorithm, is a Cauchy problem for an initial data piecewise constant with only one discontinuity. In a similar way, an RP at a junction is a Cauchy Problem for an initial data constant on each incoming and outgoing road.

Fix a junction J with n incoming roads and m outgoing roads (junction of $n \times m$ type), where I_i, $i = 1, \ldots, n$, are the incoming roads and I_j , $j = n + 1, \ldots, n + m$, are the outgoing ones. Let $\rho = (\rho_1, \ldots, \rho_{n+m})$, $\rho_k \in [0, +\infty] \times I_k$ be the density vector for J.

Definition 2.2. A Riemann Solver (RS) for the junction J is a map $\mathrm{RS} : [0,1]^n \times [0,1]^m \mapsto [0,1]^n \times [0,1]^m$ that associates to a Riemann datum $\rho_0 = (\rho_{1,0}, \ldots, \rho_{n+m,0})$ at J a vector $\widehat{\rho} = (\widehat{\rho}_1, \ldots, \widehat{\rho}_{n+m})$ so that the solution on I_i, $i = 1, \ldots, n$, is given by the wave $(\rho_{i,0}, \widehat{\rho}_i)$ and on I_j, $j = n + 1, \ldots, n + m$ is given by the wave $(\widehat{\rho}_j, \rho_{j,0})$. We require the consistency condition:

$$(\mathbf{CC}) \quad \mathrm{RS}(\mathrm{RS}(\rho_0)) = \mathrm{RS}(\rho_0). \tag{2.3}$$

If $m \geq n$, it is possible to introduce an RS, based on the following rules ([1, 12]).

(A) At each junction J, we define a matrix $A = (\alpha_{j,i})$, that describes the traffic distribution from incoming to outgoing roads, where, for every $i \in \{1, \ldots, n\}$ and $j \in \{n + 1, \ldots, n + m\}$, $0 \leq \alpha_{j,i} \leq 1$ and $\sum_{j=n+1}^{n+m} \alpha_{j,i} = 1$. The ith column of A indicates the percentages of traffic that, from the incoming road I_i, are distributed to outgoing roads.

(B) respecting (A), drivers behave so as to maximize the flux through J.

In the case $m < n$, with $n \geq 2$ and $m = 1$, in order to define an RS, we fix also the right of way parameters vector $p = (p_1, \ldots, p_n)$ with $\sum_{i=1}^{n} p_i = 1$ and consider the following additional rule.

(P) Assume that not all cars can enter the outgoing roads, and let C be the amount that can do it. Then $p_i C$, $i = 1, \ldots, n$ cars come from the road I_i into the node.

For simplicity, we indicate by $\rho_i(t,x)$, $i = 1, \ldots, n$, the densities of the cars on the incoming roads and by $\rho_j(t,x)$, $j = n+1, \ldots, n+m$, those on outgoing roads. Let us introduce the notation

$$\gamma_k = f(\rho_k), \quad \widehat{\gamma}_k = f(\widehat{\rho}_k), \quad k = 1, \ldots, n+m. \tag{2.4}$$

Proposition 2.3. *Let $(\rho_{1,0}, \ldots, \rho_{n+m,0}) \in [0,1]$ be the initial densities of an RP at J. The maximum fluxes that can be obtained on the incoming roads and the outgoing ones, respectively, are given by (for a proof see [12]):*

$$\begin{aligned}
\gamma_i^{\max} &= \begin{cases} f(\rho_{i,0}), & \text{if } \rho_{i,0} \in \left[0, \dfrac{1}{2}\right] \\ f\left(\dfrac{1}{2}\right) = \dfrac{1}{4} & \text{if } \rho_{i,0} \in \left]\dfrac{1}{2}, 1\right], \end{cases} \quad i = 1, \ldots, n, \\
\gamma_j^{\max} &= \begin{cases} f\left(\dfrac{1}{2}\right) = \dfrac{1}{4}, & \text{if } \rho_{j,0} \in \left[0, \dfrac{1}{2}\right] \\ f(\rho_{j,0}) & \text{if } \rho_{j,0} \in \left]\dfrac{1}{2}, 1\right], \end{cases} \quad j = n+1, \ldots, n+m.
\end{aligned} \tag{2.5}$$

Now we focus on junctions of 2×1 type and indicate with 1 and 2 the entering roads and with 3 the exiting one. In this case, we need only one right of way parameter p. The solution to the RP with initial data $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0})$ is constructed in the following way. Since we want to maximize the through traffic (rule (B)), we set

$$\widehat{\gamma}_3 = \min\left\{\gamma_1^{\max} + \gamma_2^{\max}, \gamma_3^{\max}\right\}. \tag{2.6}$$

If $\widehat{\gamma}_3 = \gamma_1^{\max} + \gamma_2^{\max}$, then the solution of the RP is $\widehat{\gamma} = (\gamma_1^{\max}, \gamma_2^{\max}, \gamma_1^{\max} + \gamma_2^{\max})$. Consider now the case $\widehat{\gamma}_3 = \gamma_3^{\max}$ and the following conditions:

(A1) $p\gamma_3^{\max} < \gamma_1^{\max}$;

(A2) $(1-p)\gamma_3^{\max} < \gamma_2^{\max}$.

The solutions of the RP are the following:

(i) $(p\gamma_3^{\max}, (1-p)\gamma_3^{\max}, \gamma_3^{\max})$ if A_1 and A_2 are both satisfied;

(ii) $(\gamma_3^{\max} - \gamma_2^{\max}, \gamma_2^{\max}, \gamma_3^{\max})$ if A_1 is satisfied and A_2 is not satisfied;

(iii) $(\gamma_1^{\max}, \gamma_3^{\max} - \gamma_1^{\max}, \gamma_3^{\max})$ if A_2 is satisfied and A_1 is not satisfied.

The case of both A_1 and A_2 false is not possible, since it would be $\gamma_3^{\max} > \gamma_1^{\max} + \gamma_2^{\max}$.

From the flux function, we can express $\widehat{\rho}_k$ in terms of $\widehat{\gamma}_k$. In fact, solving the equation $\widehat{\rho}_k(1-\widehat{\rho}_k) = \widehat{\gamma}_k$, we get

$$\widehat{\rho}_k = \frac{1 + s_k\sqrt{1-4\widehat{\gamma}_k}}{2}, \quad k = 1,2,3, \tag{2.7}$$

with

$$s_i = \begin{cases} -1 & \text{if } \rho_{i,0} < \sigma,\ \gamma_1^{\max} + \gamma_2^{\max} \le \gamma_3^{\max}, \\ & \text{or } \rho_{i,0} < \sigma,\ \gamma_3^{\max} < \gamma_1^{\max} + \gamma_2^{\max},\ p_i\widehat{\gamma}_3 \ge \gamma_i^{\max}, \\ +1 & \text{if } \rho_{i,0} \ge \sigma, \\ & \text{or } \rho_{i,0} < \sigma,\ \gamma_3^{\max} < \gamma_1^{\max} + \gamma_2^{\max},\ p_i\widehat{\gamma}_3 < \gamma_i^{\max}, \end{cases} \quad i = 1,2,$$

$$s_3 = \begin{cases} -1 & \text{if } \rho_{3,0} \le \sigma, \\ & \text{or } \rho_{3,0} > \sigma,\ \gamma_1^{\max} + \gamma_2^{\max} < \gamma_3^{\max}, \\ +1 & \text{if } \rho_{3,0} > \sigma,\ \gamma_1^{\max} + \gamma_2^{\max} \ge \gamma_3^{\max}, \end{cases} \tag{2.8}$$

where

$$p_i = \begin{cases} p & \text{if } i = 1, \\ 1-p & \text{if } i = 2. \end{cases} \tag{2.9}$$

The velocity, in terms of $\widehat{\gamma}_k$, is given by $v(\widehat{\rho}_k) = (1 - s_k\sqrt{1-4\widehat{\gamma}_k})/2$, $k = 1,2,3$.

3. Cost Functionals

In this section, we introduce the functionals used to evaluate the network performance and report optimization results for the functionals J_6 and J_7, compared with J_1 and J_2. We focus again on junctions of 2×1 type and we define the following functionals:

J_1 measuring car average velocity:

$$J_1(t) = \sum_{k=1}^{3} \int_{I_k} v(\rho_k(t,x))dx, \tag{3.1}$$

J_2 measuring average traveling time:

$$J_2(t) = \sum_{k=1}^{3} \int_{I_k} \frac{1}{v(\rho_k(t,x))} dx, \tag{3.2}$$

J_3 measuring total flux of cars:

$$J_3(t) = \sum_{k=1}^{3} \int_{I_k} f(\rho_k(t,x)) dx, \tag{3.3}$$

J_4 measuring car density:

$$J_4(t) = \sum_{k=1}^{3} \int_0^t \int_{I_k} \rho_k(\tau,x) d\tau dx, \tag{3.4}$$

J_5 the Stop and Go Waves functional, measuring the velocity variation:

$$J_5(t) = \text{SGW} = \sum_{k=1}^{3} \int_0^t \int_{I_k} \left| Dv(\rho) \right| d\tau dx, \tag{3.5}$$

where $|Dv|$ is the total variation of the distributional derivative $D\rho$, which is a finite Radon measure,

J_6 measuring the kinetic energy:

$$J_6(t) = \sum_{k=1}^{3} \int_{I_k} f(\rho_k(t,x)) v(\rho_k(t,x)) dx, \tag{3.6}$$

J_7 measuring the average traveling time weighted with the number of cars moving on each road I_k:

$$J_7(t) = \sum_{k=1}^{3} \int_{I_k} \frac{\rho_k(t,x)}{v(\rho_k(t,x))} dx. \tag{3.7}$$

For a fixed time horizon $[0,T]$, our aim is to maximize $\int_0^T J_1(t)dt, \int_0^T J_3(t)dt, \int_0^T J_6(t)dt$ and to minimize $\int_0^T J_2(t)dt, \int_0^T J_7(t)dt$, choosing the right of way parameter $p_k(t)$. Since the solutions of such optimization control problems are too difficult, we reduce to the following problem.

(P_r) Consider a junction J of 2×1 type, the functionals J_k, $k = 1,2,3,6,7$, and the right of way parameter p_k as controls. We want to minimize $J_2(T)$, $J_7(T)$ and to maximize $J_1(T)$, $J_3(T)$, $J_6(T)$ for T sufficiently big.

Given the initial data, solving the RP, we determine the average velocity, the average traveling time, and the flux over the network as function of the right of way parameters p, then all the functionals depend on p. As was proved in [6–8], the functional $J_3(T)$ does not depend on the right of way parameter.

3.1. Optimization of J_6 and J_7

Let us consider the optimization of the functionals, measuring kinetic energy and weighted average traveling time. For T sufficiently big the functionals assume the form:

$$\begin{aligned} J_6(T) &= \sum_{k=1}^{3} f(\widehat{\rho}_k)v(\widehat{\rho}_k) = \sum_{k=1}^{3} \widehat{\gamma}_k \frac{\left(1 - s_k\sqrt{1-4\widehat{\gamma}_k}\right)}{2}, \\ J_7(T) &= \sum_{k=1}^{3} \frac{\widehat{\rho}_k}{v(\widehat{\rho}_k)} = \sum_{i=1}^{3} \frac{1 + s_k\sqrt{1-4\widehat{\gamma}_k}}{1 - s_k\sqrt{1-4\widehat{\gamma}_k}}, \end{aligned} \tag{3.8}$$

where s_k, $k = 1,2,3$ are defined in (2.8), $\widehat{\rho} = (\widehat{\rho}_1, \widehat{\rho}_2, \widehat{\rho}_3) = \mathrm{RS}(\rho_{1,0}, \rho_{2,0}, \rho_{3,0})$, and $\widehat{\gamma} = f(\widehat{\rho})$.

If $\widehat{\gamma}_3 = \gamma_1^{\max} + \gamma_2^{\max}$, the solution of the RP does not depend on the parameter p and the same happens for the functionals J_6 and J_7 which do not depend on the parameter p. Hence we analyze the functionals in the case $\gamma_3^{\max} < \gamma_1^{\max} + \gamma_2^{\max}$, that is, $\widehat{\gamma}_3 = \gamma_3^{\max}$. Let us define

$$\begin{aligned} \beta^- &= \frac{\gamma_3^{\max} - \gamma_1^{\max}}{\gamma_1^{\max}}, \qquad p^+ = \frac{1}{1+\beta^-} = \frac{\gamma_1^{\max}}{\gamma_3^{\max}}, \\ \beta^+ &= \frac{\gamma_2^{\max}}{\gamma_3^{\max} - \gamma_2^{\max}}, \qquad p^- = \frac{1}{1+\beta^+} = \frac{\gamma_3^{\max} - \gamma_2^{\max}}{\gamma_3^{\max}}. \end{aligned} \tag{3.9}$$

It is easy to check that $\beta^- \le \beta^+$. Then, for $p \ge p^+$, A_1 is false and A_2 is true, for $p \le p^-$, A_1 is true and A_2 is false and, finally, for $p^- < p < p^+$, A_1 and A_2 are both true. From (2.7), neglecting the parts of the costs that do not depend on p, maximizing J_6 and minimizing J_7 is equivalent to maximize and minimize, respectively,

$$\widetilde{J}_6 = \begin{cases} \left(\gamma_3^{\max} - \gamma_2^{\max}\right)\left(1 - s_1\sqrt{1 - 4\left(\gamma_3^{\max} - \gamma_2^{\max}\right)}\right) + \gamma_2^{\max}\left(1 - s_2\sqrt{1 - 4\gamma_2^{\max}}\right), & 0 \le p \le p^-, \\ p\gamma_3^{\max}\left(1 - s_1\sqrt{1 - 4p\gamma_3^{\max}}\right) + (1-p)\gamma_3^{\max}\left(1 - s_2\sqrt{1 - 4(1-p)\gamma_3^{\max}}\right), & p^- < p < p^+, \\ \gamma_1^{\max}\left(1 - s_1\sqrt{1 - 4\gamma_1^{\max}}\right) + \left(\gamma_3^{\max} - \gamma_1^{\max}\right)\left(1 - s_2\sqrt{1 - 4\left(\gamma_3^{\max} - \gamma_1^{\max}\right)}\right), & p^+ \le p \le 1, \end{cases}$$

$$\widetilde{J}_7 = \begin{cases} \dfrac{1+s_1\sqrt{1-4\left(\gamma_3^{\max}-\gamma_2^{\max}\right)}}{1-s_1\sqrt{1-4\left(\gamma_3^{\max}-\gamma_2^{\max}\right)}} + \dfrac{1+s_2\sqrt{1-4\gamma_2^{\max}}}{1-s_2\sqrt{1-4\gamma_2^{\max}}}, & 0 \le p \le p^-, \\ \dfrac{1+s_1\sqrt{1-4p\gamma_3^{\max}}}{1-s_1\sqrt{1-4p\gamma_3^{\max}}} + \dfrac{1+s_2\sqrt{1-4(1-p)\gamma_3^{\max}}}{1-s_2\sqrt{1-4(1-p)\gamma_3^{\max}}}, & p^- < p < p^+, \\ \dfrac{1+s_1\sqrt{1-4\gamma_1^{\max}}}{1-s_1\sqrt{1-4\gamma_1^{\max}}} + \dfrac{1+s_2\sqrt{1-4\left(\gamma_3^{\max}-\gamma_1^{\max}\right)}}{1-s_2\sqrt{1-4\left(\gamma_3^{\max}-\gamma_1^{\max}\right)}}, & p^+ \le p \le 1. \end{cases} \tag{3.10}$$

Observe that when the condition A_1 is false and A_2 is true, or vice versa, the cost functionals J_6 and J_7 are constant with respect to p. To complete the analysis of the costs it is enough to take the derivatives with respect to p in the region where both A_1 and A_2 are true. The expressions are a bit long, so we do not report them, but it is straightforward to check that J_6 and J_7 are decreasing for $p < 1/2$ and increasing for $p > 1/2$, hence they have a minimum in $p = 1/2$.

We analyze the functionals J_6 and J_7 in the intervals: $[0, p^-]$, $]p^-, p^+[$ and $[p^+, 1]$. We search for absolute minimum in the case of J_6 and absolute maximum in the analysis of J_7, taking into account the possible values that s_k, $k = 1, 2, 3$, can assume in the above intervals:

(i) $s_1 = s_2 = 1, s_3 = \pm 1$ in $[0, p^-]$, $s_1 = s_2 = 1, s_3 = \pm 1$ in $]p^-, p^+[$, $s_1 = s_2 = 1, s_3 = \pm 1$ in $[p^+, 1]$;

(ii) $s_1 = s_2 = 1, s_3 = \pm 1$ in $[0, p^-]$, $s_1 = s_2 = 1, s_3 = \pm 1$ in $]p^-, p^+[$, $s_1 = -1, s_2 = 1, s_3 = \pm 1$ in $[p^+, 1]$;

(iii) $s_1 = 1, s_2 = -1, s_3 = \pm 1$ in $[0, p^-]$, $s_1 = s_2 = 1, s_3 = \pm 1$ in $]p^-, p^+[$, $s_1 = s_2 = 1, s_3 = \pm 1$ in $[p^+, 1]$;

(iv) $s_1 = 1, s_2 = -1, s_3 = \pm 1$ in $[0, p^-]$, $s_1 = s_2 = 1, s_3 = \pm 1$ in $]p^-, p^+[$, $s_1 = -1, s_2 = 1, s_3 = \pm 1$ in $[p^+, 1]$.

Remark 3.1. Since s_3 assumes the same values, 1 or -1, in all the intervals, (hence in the point of discontinuities) and the terms in which s_3 occurs do not depend on p, the optimal values are independent from s_3.

The functionals $J_2(T)$ and $J_7(T)$ are maximized for the same values of p. In fact we get the following theorem.

Theorem 3.2. *Consider a junction J of 2×1 type. For the flux function* (2.2)*, and T sufficiently big, the cost functionals $J_2(T)$ and $J_7(T)$ are optimized for the following values of p.*

(1) *Case $s_1 = s_2 = +1$, we have that*

(a) $p = 1/2$ *if* $\beta^- \le 1 \le \beta^+$ *or* $\gamma_2^{\max} = \gamma_3^{\max}$;

(b) $p \in [0, p^-]$ *if* $\beta^- \le \beta^+ \le 1$;

(c) $p \in [p^+, 1]$ *if* $1 \le \beta^- \le \beta^+$;

(2) *Case* $s_1 = -1 = -s_2$, *we have that*

(a) $p = 1/2$ *or* $p \in [p^+, 1]$ *if* $\beta^- \leq 1 \leq \beta^+$ *or* $\gamma_2^{\max} = \gamma_3^{\max}$;
(b) $p \in [0, p^-]$ *or* $p \in [p^+, 1]$ *if* $\beta^- \leq \beta^+ \leq 1$;
(c) $p \in [p^+, 1]$ *if* $1 \leq \beta^- \leq \beta^+$;

(3) *Case* $s_1 = +1 = -s_2$, *we have that*

(a) $p = 1/2$ *or* $p \in [0, p^-]$ *if* $\beta^- \leq 1 \leq \beta^+$;
(b) $p \in [0, p^-]$ *if* $\beta^- \leq \beta^+ \leq 1$;
(c) $p \in [0, p^-]$ *or* $p \in [p^+, 1]$ *if* $1 \leq \beta^- \leq \beta^+$;
(d) $p = 1/2$ *or* $p \in [p^+, 1]$ *if* $\gamma_2^{\max} = \gamma_3^{\max}$;

(4) *Case* $s_1 = s_2 = -1$, *we have that*

(a) $p = 1/2$ *or* $p \in [p^+, 1]$ *if* $\beta^- \leq 1 \leq \beta^+$, *with* $\beta^-\beta^+ > 1$ *or* $\gamma_2^{\max} = \gamma_3^{\max}$;
(b) $p = 1/2$ *or* $p \in [0, p^-]$ *if* $\beta^- \leq 1 \leq \beta^+$, *with* $\beta^-\beta^+ < 1$;
(c) $p = 1/2$ *or* $p \in [0, p^-] \cup [p^+, 1]$ *if* $\beta^- \leq 1 \leq \beta^+$, *with* $\beta^-\beta^+ = 1$;
(d) $p \in [0, p^-]$ *if* $\beta^- \leq \beta^+ \leq 1$,
(e) $p \in [p^+, 1]$, *if* $1 \leq \beta^- \leq \beta^+$;

where $\beta^- = (\gamma_3^{\max} - \gamma_1^{\max})/\gamma_1^{\max}$, $\beta^+ = \gamma_2^{\max}/(\gamma_3^{\max} - \gamma_2^{\max})$, $p^+ = \gamma_1^{\max}/\gamma_3^{\max}$, $p^- = (\gamma_3^{\max} - \gamma_2^{\max})/\gamma_3^{\max}$.

In the particular case $\gamma_1^{\max} = \gamma_2^{\max} = \gamma_3^{\max}$, the functionals J_2 and J_7 are optimized for $p = 1/2$.

The maximization of the functionals $J_1(T)$ and $J_6(T)$ is reached, in some cases, for different values of the right of way parameter, as reported in the following theorem, in which the optimization analysis of the new functional $J_6(T)$ is compared with the results obtained in [6] for $J_1(T)$.

Theorem 3.3. *Consider a junction* J *of* 2×1 *type. For the flux function* (2.2), *and* T *sufficiently big, the cost functionals* $J_1(T)$ *and* $J_6(T)$ *are optimized for the following values of* p.

(1) *Case* $s_1 = s_2 = +1$, *we have that*

(a) $p \in [0, p^-]$ *if* $\beta^- \leq 1 \leq \beta^+$, *with* $\beta^-\beta^+ > 1$, *or* $1 \leq \beta^- \leq \beta^+$, *or* $\gamma_2^{\max} = \gamma_3^{\max}$;
(b) $p \in [0, p^-] \cup [p^+, 1]$ *if* $\beta^- \leq 1 \leq \beta^+$, *with* $\beta^-\beta^+ = 1$;
(c) $p \in [p^+, 1]$ *if* $\beta^- \leq 1 \leq \beta^+$, *with* $\beta^-\beta^+ < 1$ *or* $\beta^- \leq \beta^+ \leq 1$;

(2) *Case* $s_1 = -1 = -s_2$, *we have that*

for $J_1(T), p \in [p^+, 1]$;

for $J_6(T)$,

(a) $p \in [p^+, 1]$ *if* $\beta^- \leq 1 \leq \beta^+$, *or* $\beta^- \leq \beta^+ \leq 1$;
(b) $p \in [0, p^-]$ *or* $p \in [p^+, 1]$ *if* $1 \leq \beta^- \leq \beta^+$, *or* $\gamma_2^{\max} = \gamma_3^{\max}$;

(3) *Case* $s_1 = +1 = -s_2$, *we have that*

for $J_1(T), p \in [0, p^-]$;

for $J_6(T)$,

(a) $p \in [0, p^-]$ *if* $\beta^- \leq 1 \leq \beta^+$, *or* $1 \leq \beta^- \leq \beta^+$, *or* $\gamma_2^{\max} = \gamma_3^{\max}$;

(b) $p \in [0, p^-]$ *or* $p \in [p^+, 1]$ *if* $\beta^- \leq \beta^+ \leq 1$;

(4) *Case* $s_1 = s_2 = -1$, *we have that*

for $J_1(T)$,

(a) $p \in [0, p^-]$ *if* $\beta^- \leq 1 \leq \beta^+$, *with* $\beta^-\beta^+ < 1$, *or* $\beta^- \leq \beta^+ \leq 1$;

(b) $p \in [0, p^-] \cup [p^+, 1]$ *if* $\beta^- \leq 1 \leq \beta^+$, *with* $\beta^-\beta^+ = 1$;

(c) $p \in [p^+, 1]$ *if* $\beta^- \leq 1 \leq \beta^+$, *with* $\beta^-\beta^+ > 1$, *or* $1 \leq \beta^- \leq \beta^+$, *or* $\gamma_2^{\max} = \gamma_3^{\max}$;

for $J_6(T)$,

(a) $p \in [0, p^-]$ *or* $p \in [p^+, 1]$ *if* $\beta^- \leq 1 \leq \beta^+$, *with* $\beta^-\beta^+ > 1$, *or* $\beta^-\beta^+ < 1$, *or* $\gamma_2^{\max} = \gamma_3^{\max}$;

(b) $p \in [0, p^-] \cup [p^+, 1]$ *if* $\beta^- \leq 1 \leq \beta^+$, *with* $\beta^-\beta^+ = 1$;

(c) $p \in [p^+, 1]$ *if* $\beta^- \leq \beta^+ \leq 1$;

(d) $p \in [0, p^-]$ *if* $1 \leq \beta^- \leq \beta^+$;

where $\beta^- = (\gamma_3^{\max} - \gamma_1^{\max})/\gamma_1^{\max}$, $\beta^+ = \gamma_2^{\max}/(\gamma_3^{\max} - \gamma_2^{\max})$, $p^+ = \gamma_1^{\max}/\gamma_3^{\max}$, $p^- = (\gamma_3^{\max} - \gamma_2^{\max})/\gamma_3^{\max}$.

The functionals J_1 and J_6 are maximized for $p = 0$ or $p = 1$ if $\gamma_1^{\max} = \gamma_2^{\max} = \gamma_3^{\max}$.

Remark 3.4. In the cases in which two possible optimal candidates exist, we evaluate the corresponding values of the cost functionals. The optimal p is given by the value that minimizes or maximizes, respectively, the functionals $J_1(T)$, $J_6(T)$ and $J_2(T)$, $J_7(T)$. If we have a set of optimal values $[0, p^-]$, we can choose $p = p^-$ or $p = p^- - \varepsilon$, while in the case in which the optimal right of way parameter belongs to $[p^+, 1]$ we can take as optimal value $p = p^+$ or $p = p^+ + \varepsilon$, with ε small and positive.

4. Simulations

In this section, we present some simulation results in order to test the optimization algorithms both for single junctions and complex networks. The aim is to verify the correctness of the analytical results and then to analyze the effects of different control procedures, applied locally at each junction, on the global performances of networks. The approximation of the conservation laws, that describe the density evolution for each road of the network (see [13]), is made by the numerical scheme of Godunov ([14]), with space step $\Delta x = 0.01$. The time step is determined by the CFL condition ([15]), equal to 0.5.

4.1. Single Junctions

We consider single junctions of 2×1 type, namely, junctions consisting of two incoming roads, 1 and 2, and one outgoing road, 3, in order to verify the goodness of optimization procedures. Then we compare cost functionals behaviour using right of way parameters, that optimize the

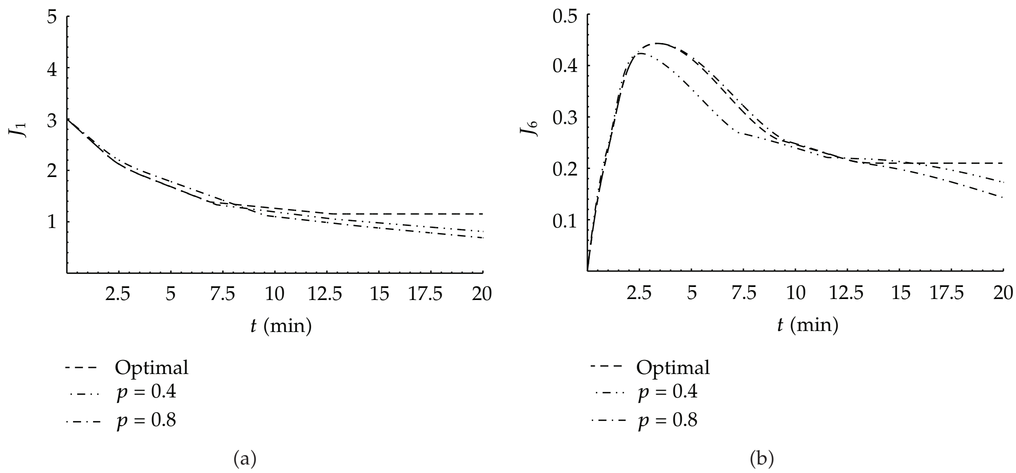


Figure 1: Comparison among J_1 computed by optJ_1 (a), J_6 computed by optJ_6 (b), and fixed cases for $p = 0.4$ and $p = 0.8$.

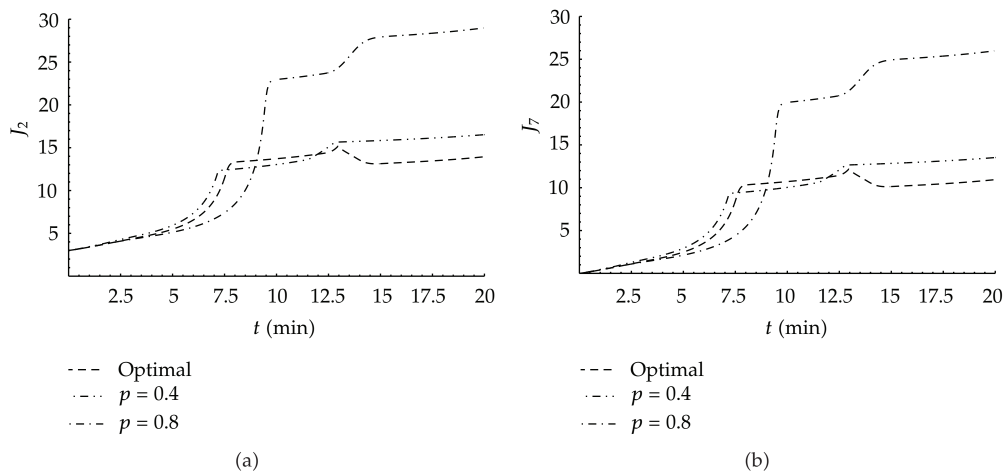
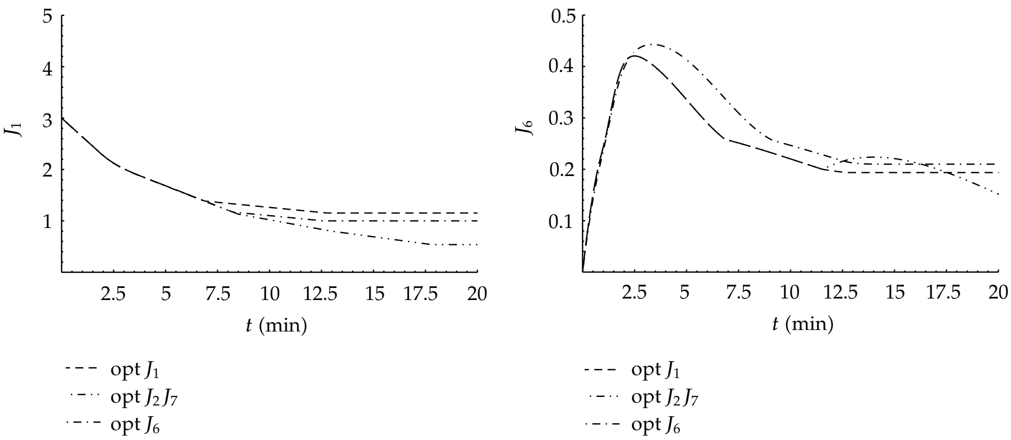


Figure 2: Comparison among J_2 (a) and J_7 (b) computed by optJ_2J_7, and fixed cases for $p = 0.4$ and $p = 0.8$.

functionals (*optimal case*) and fixed right of way parameters (*fixed case*), according to which p is chosen by the user. The evolution of the traffic is simulated in a time interval $[0, T]$, where $T = 20$ min for the flux function (2.2). As for the initial conditions on the roads, we assume that, at the starting instant of simulation ($t = 0$), all roads are empty. Moreover, for roads 1, 2, and 3, also boundary data, $\rho_{i,\,b}$, $i \in \{1, 2, 3\}$, have to be considered. Precisely, we choose $\rho_{i,\,b} = 0.3$, $\rho_{2,\,b} = 0.4$, and $\rho_{3,\,b} = 0.1$.

The motivation for the choice of such boundary data is that it allows to capture the case in which different optimization procedures, one for J_1, indicated as "optJ_1", one for J_6, "optJ_6", and one for both J_2 and J_7, "optJ_2J_7", give origin to different optimal values.

As you can see from Figures 1 and 2, the optimal algorithms for J_1, J_2, J_6, and J_7 improve traffic conditions with respect to the fixed cases. Notice that, when we are not in the steady state, it could occur that some fixed simulations can behave better than the optimal



(a) (b)

Figure 3: Comparison among J_1 (a) and J_6 evaluated by optJ_1, opt$J_2 J_7$ and optJ_6 (b).

Table 1: Boundary data for the four simulation cases.

Case	$\rho_{1,b}$	$\rho_{2,b}$	$\rho_{3,b}$
A	0.112	0.139	0.846
B	0.183	0.139	0.782
C	0.112	0.183	0.673
D	0.301	0.412	0.101

ones. This is due to the fact that the analytical results are always obtained for big times. In Figure 3, we present the differences between the optimization algorithms for cost functionals J_1 and J_6. As expected, J_1 and J_6 are higher, respectively, applying optJ_1 and optJ_6. Moreover, simulating the junction with the parameter p obtained using the control procedures optJ_1, optJ_6, and opt$J_2 J_7$, the behaviours of J_1 and J_6 are different, confirming the theoretical results that, depending on the initial data, the functionals can be optimized for different values of the right of way parameter.

As for a single junction of 2×1 type, various optimization algorithms can be applied to set the priority parameters, we tried to understand which is the best policy to adopt for the improvements of traffic conditions. In order to discriminate among the different optimization approaches, we analyze the behaviour of the Stop and Go Waves functional (SGW), which measures the velocity variations on roads. We simulate the traffic evolution for a junction in four different situations, denoted by A, B, C, and D, in a time interval $[0, T]$, where T is 30 min for cases A and B and 50 min for cases C and D. We assume that at the beginning of the simulation all the roads are empty. Boundary data are reported in Table 1, and chosen in such way to test various cases of Theorems 3.2 and 3.3.

For such case studies, optimal values for the right of way parameter p are as shown in Table 2.

In Figures 4 and 5, the behaviour of SGW, applying optJ_1, opt$J_2 J_7$, and optJ_6, in the different simulation cases, is depicted. In the steady state, SGW is lower when the algorithm optJ_1 is used, and this implies a lower probability of car accidents. We can conclude that it is suitable to choose the parameter p which maximizes the average velocities. Notice that in

Table 2: Optimal right of way parameters for simulation cases A, B, C, and D.

Case	optJ_1	optJ_2J_7	optJ_6
A	$p \in [0.769,\ 1]$	$p = 0.5$	$p \in [0,\ 0.769]$
B	$p \in [0,\ 0.294]$	$p \in [0,\ 0.294]$	$p \in [0.882,\ 1]$
C	$p \in [0.454,\ 1]$	$p \in [0.454,\ 1]$	$p \in [0,\ 0.318]$
D	$p \in [0,\ 0.040]$	$p = 0.5$	$p \in [0.840,\ 1]$

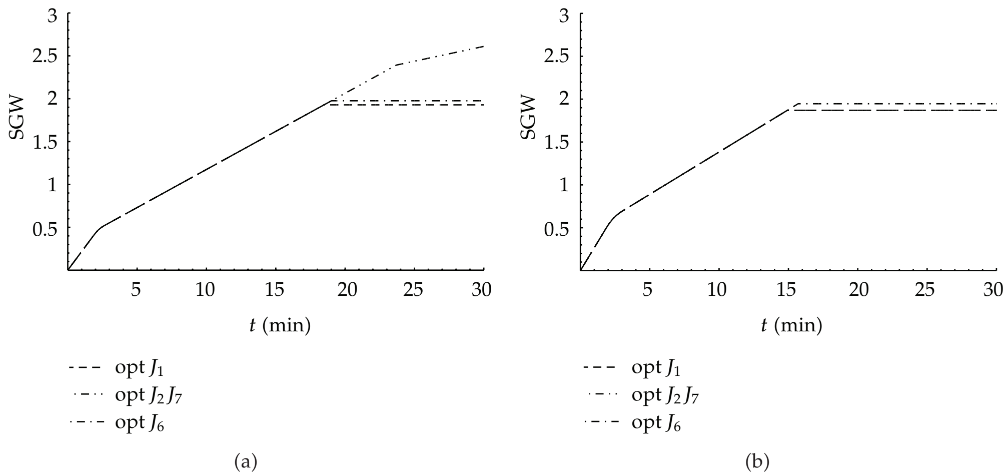


Figure 4: Comparison among SGW, evaluated by optJ_1, optJ_2J_7, and optJ_6, in case A (a) and case B (b).

cases B and C, the optimal algorithms optJ_1 and optJ_2J_7 give the same set of p, hence curves overlap.

4.2. Simulation of Traffic for Re Di Roma Square

This subsection is devoted to the description of simulative results for a real urban network, Re di Roma Square, a big traffic circle inside the urban network of Rome, in Italy. The choice of this case study is justified by the presence of congestion phenomena, which could be avoided through an opportune choice of network parameters.

The topology of the square, represented in Figure 6, is described by 12 roads, which form the circle: $1R$, $2R$, $3R$, $4R$, $5R$, $6R$, $7R$, $8R$, $9R$, $10R$, $11R$, and $12R$; 12 roads, connecting the inner roads with outside: aosta_ent, aosta_exi, vercelli, pinerolo, appia_sud_ent, appia_sud_exi, albalonga_ent, albalonga_exi, cerveteri_ent, cerveteri_exi, appia_nord_ent, appia_nord_exi.

As it is shown, Re di Roma square is formed by junctions of 2×1 type $(1, 3, 5, 7, 9, 11)$, in white, and junctions of 1×2 type $(2, 4, 6, 8, 10, 12)$, in black. The traffic distribution coefficients at 1×2 junctions are completely determined by road capacities (and the characteristics of the nearby portion of the Rome urban network); hence only right of way parameters for 2×1 junctions can be chosen as control parameters. Table 3 reports the distribution coefficients used for simulations.

The evolution of the traffic flows is simulated in a time interval $[0, T]$, where $T = 30\,\text{min}$. We assume that, at the starting instant of simulation, all roads are empty. We use

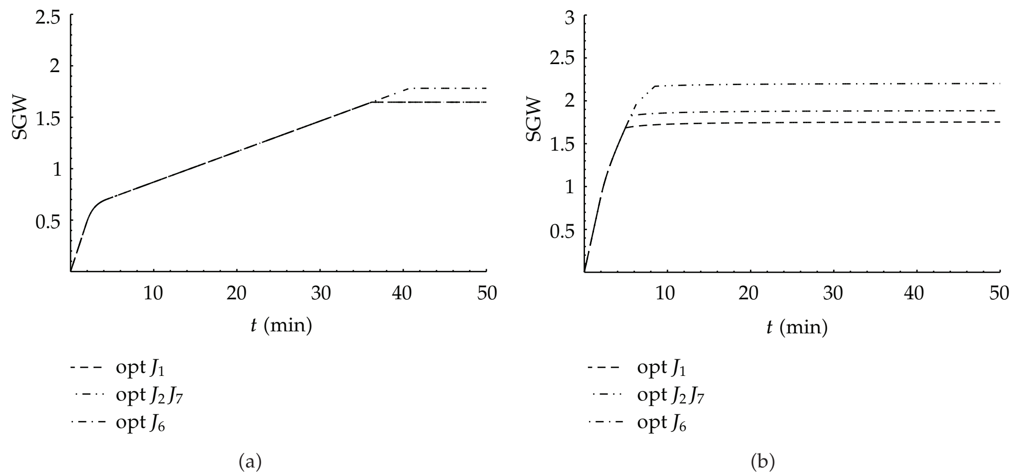


Figure 5: Comparison among SGW, evaluated by optJ_1, opt$J_2 J_7$, and optJ_6, in case C (a) and case D (b).

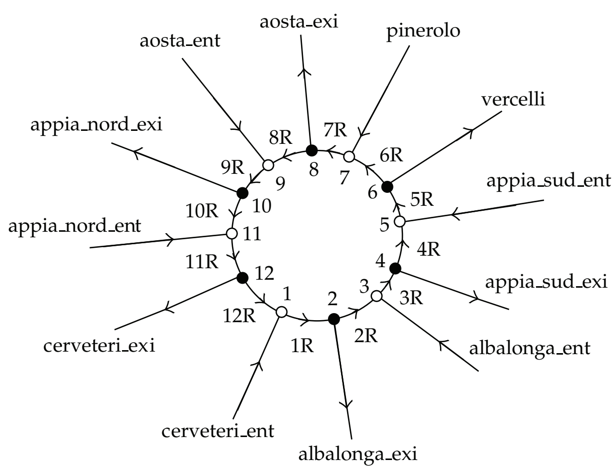


Figure 6: Topology of Re di Roma Square.

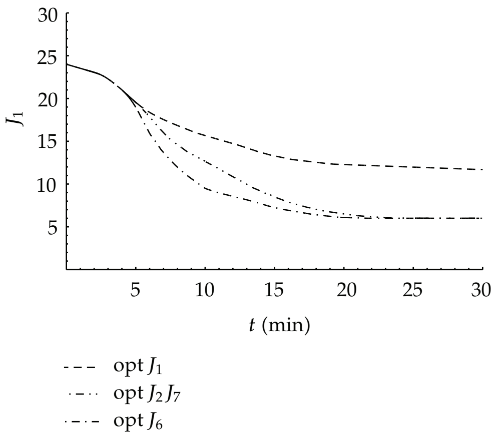
boundary conditions for roads with not infinite endpoints. In particular, roads cerveteri_exi, albalonga_exi, appia_sud_exi, vercelli, aosta_exi, and appia_nord_exi have a boundary data equal to 0.4, while the other ones equal to 0.35.

We analyze two simulation cases: right of way parameters, that optimize the cost functionals (*optimal case*); dynamic random parameters (*dynamic random case*), which means that right of way parameters change randomly at every step of the simulation process.

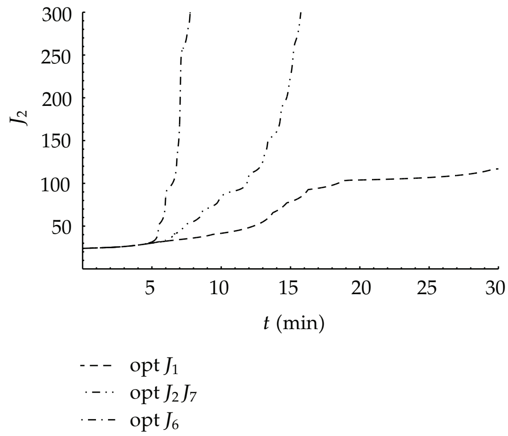
The aim is to investigate the effects of optJ_1, optJ_6, and opt$J_2 J_7$ on the global performances of the network in order to understand which is the best optimization algorithm (the algorithm realizing the better viability conditions on the whole Re di Roma square). In the following pictures, we report the time behaviour evolutions of the cost functionals J_1, J_2, J_6, and J_7, fixing at every junction the values of the right of way parameters obtained by the optimization procedures.

Table 3: Traffic distribution parameters for junctions of 1 × 2 type.

Junction i	$\alpha_{iR,(i-1)R}$	$\alpha_{i,(i-1)R}$
$i = 2$	0.866071	0.133929
$i = 4$	0.459854	0.540146
$i = 6$	0.800971	0.199029
$i = 8$	0.730612	0.269388
$i = 10$	0.536050	0.463950
$i = 12$	0.753927	0.246073



(a)



(b)

Figure 7: Behaviour of the optimal cost functionals J_1 (a) and J_2 (b).

The optimizations of local type, like the ones that we are considering here, could not necessarily imply global optimization for big networks, as we can see in Figures 7, 8, 10, 11. This is due to various factors, mainly depending on the network topology and on traffic loads.

First of all, notice that in the case study the algorithm optJ_1 allows a global optimization for the whole network. It is evident in Figure 7, where J_1 and J_2 are, respectively, the highest and the lowest. Hence, the use of opt$J_2 J_7$ and optJ_6, as control procedures, does not guarantee better performance of traffic flows. The goodness of optJ_1 for global performances is confirmed by the behaviour of J_2. In fact, opt$J_2 J_7$ and optJ_6 can let J_2 explode, that is, the traffic circle is stuck and the time to run inside goes to infinity. This situation is more evident in Figure 8, where we can capture another important aspect: the total kinetic energy J_6, on the whole network, tends to zero when optJ_1 is not used. This means that the cars flux is going to zero, as evident from Figure 9, hence roads inside the circle are becoming full. A consequence of this phenomenon is also visible in J_7 evolution, that tends to infinity.

In Figure 10, J_4 and J_5 (Stop and Go Waves functional, SGW) behaviour are depicted. There are no optimization algorithms for these functionals, and they are computed directly using optJ_1, opt$J_2 J_7$, and optJ_6. It is evident that the amount of traffic load, visible in J_4, tends to decrease using optJ_1. Moreover, the behaviour of J_5, that measures the velocity variation, indicates that the use of optJ_1 leads to more regular densities on roads, giving advantages in terms of security.

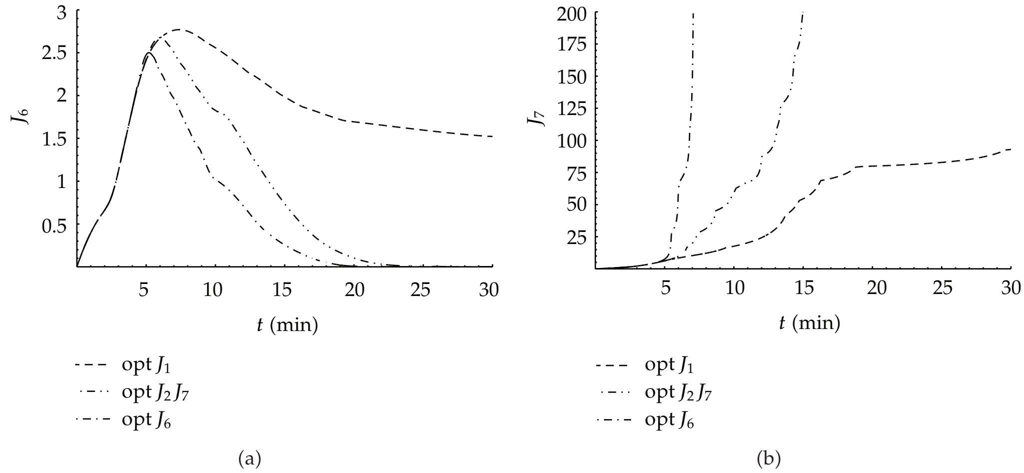


Figure 8: Behaviour of the optimal cost functionals J_6 (a) and J_7 (b).

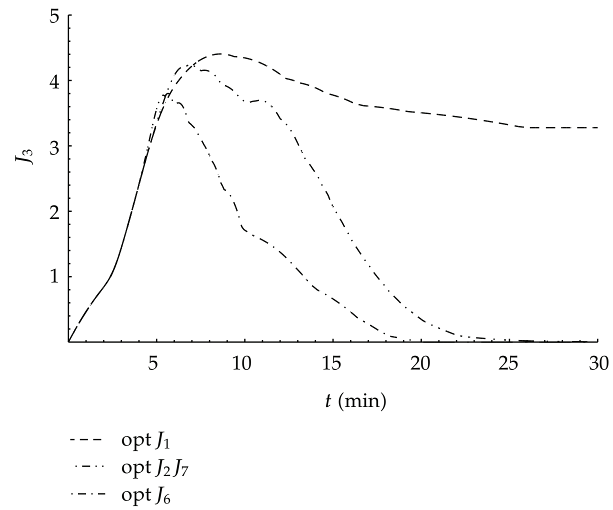


Figure 9: Behaviour of the cost functional J_3.

To complete the discussion, we focus the attention on the characteristics of dynamic random simulations (Figure 11). As proved in [8], the dynamic random simulation is similar to a fixed simulation with all right of way parameters equal to 0.5. In fact, dynamic random choices fit well the optimizations obtained with J_2 and J_7, since in the considered case the optimal value is 0.5 for each junction of 2×1 type. Unlike the case presented in [6, 8], optJ_1 does not guarantee an average optimal right of way parameter equal to 0.5, and this justifies the dissimilarities among the optimal performances obtained using optJ_1, opt$J_2 J_7$ and the global effects due to dynamic random simulations.

From all the previous observations, it is clear that for the case study and with the chosen initial data, the best performances on the whole network are given by the optimal algorithm for J_1. Such algorithm is preferable for maximizing traffic flows, since it allows not only the locally optimization of each node of 2×1 type, but also the global improvement

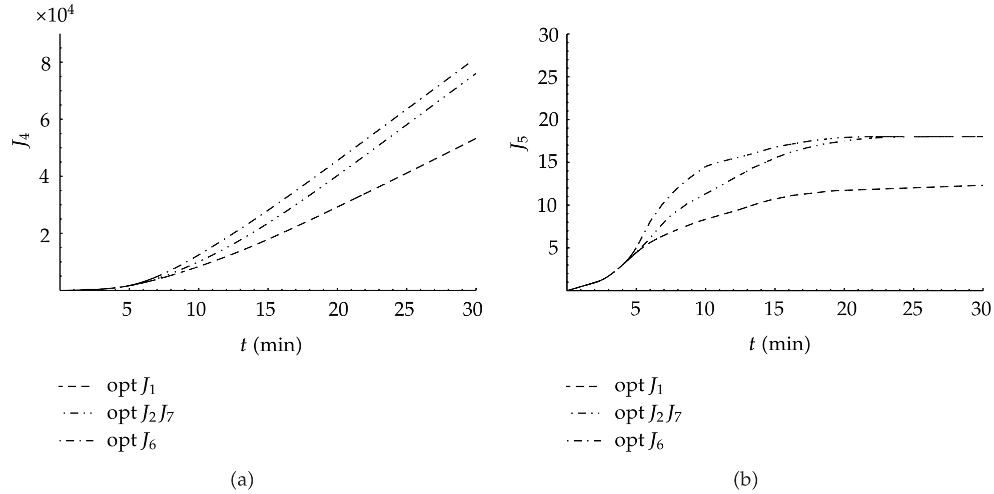


Figure 10: Behaviour of the optimal cost functionals J_4 (a) and SGW (b).

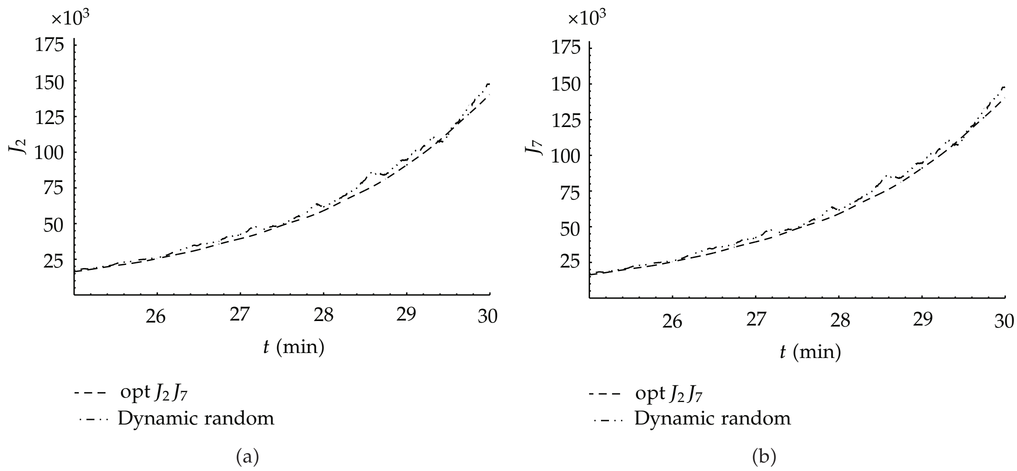


Figure 11: Behaviour of optimal J_2 (a) and J_7 (b) versus dynamic random simulations.

of traffic conditions over the whole network. The other algorithms are able only to improve locally traffic viability.

Acknowledgment

This work is partially supported by MIUR-FIRB Integrated System for Emergency (InSyEme) project under the Grant RBIP063BPH.

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