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The classical Lenz vector and the two-dimensional quantum harmonic oscillator

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A B S T R A C T

Both the two-dimensional harmonic oscillator and the Newton potential allow particular solutions for the orbits which are ellipses with center of attraction in the center, in the first case, and in one focus, in the second. The same complex map which allows to go from Kepler's to Hooke's orbits, and back, is used to transform the Lenz vector, defined for the Kepler orbit, into two conserved quantities for the harmonic motion. Upon quantization, the resulting operators, together with the angular momentum L_z, are found to correspond to the generators of the SU(2) internal symmetry of the two-dimensional quantum oscillator and the connection to the Schwinger model of angular momentum is made apparent. We give a self-contained new look on this topic.

1. Introduction

This paper is a reflection on a subject which has been explored in depth in some few works in classical mathematical physics, but is barely known to the broader physics community. We accidently rediscovered certain results established in the past and present them here, highlighting the aspects which we found more interesting. We keep our presentation accessible to the general physicist reader while preserving the necessary mathematical rigor.

The section on the Kepler's problem in the Landau and Lifshitz book on Mechanics [1] is concluded showing that there is a further integral of motion *"which exists only in fields* $U = \alpha/r$*"*, attractive or repulsive. This is better known as the Lenz vector $\boldsymbol{A}$, directed along the major axis of the Kepler trajectory, from the focus to the perihelion.[1] Its constancy forbids the precession of the perihelion. Indeed the two fundamental frequencies $(\omega_r, \omega_\theta)$ in the coordinates of the plane of the orbit, coincide for $U = -1/r$— the motion is said to be 'completely degenerate'— and orbits are closed.[2]

The classical expression of $\boldsymbol{A}$ is the following. If m is the reduced mass, the Lenz vector is usually defined by

$$\boldsymbol{A} = -\frac{\kappa}{r}\boldsymbol{r} + \frac{1}{m}\boldsymbol{p}\times\boldsymbol{L} \tag{1}$$

Here $\boldsymbol{L}$ is the orbital angular momentum, which is orthogonal to the plane of the orbit and to $\boldsymbol{A}$.

In a renowned work, Pauli [2] introduced a quantum version of $\boldsymbol{A}$ (transforming (1) into an hermitean operator) and presented an algebraic derivation of the spectrum of hydrogen together with an explanation of the degeneracy of levels with the orbital quantum number ℓ, the archetypal application of symmetry methods in quantum physics.

It is less known however that there is a geometric relation connecting the Kepler elliptical orbits, with center of attraction in a focus, with those of the classical two-dimensions harmonic oscillator, with center of attraction in the center of the ellipse. This relation is better understood substituting vectors in the plane with complex numbers, thus writingformulating the equations of the orbits in the complex plane. Most importantly, the same complex map which relates Kepler and oscillator (Hooke) orbits, allows to derive the Newton force law from the elastic force law, a surprising result touched by Arnold [3] in some writings which inspired our discussion.

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[1] A particular solution of the motion in the $1/r$ potential is an ellipse with semimajor axis a, semiminor axis b, eccentricity $e = \sqrt{(a^2-b^2)/a^2}$, and center of attraction in one of the two focii: a Kepler orbit in the gravitational potential. The classical Lenz vector $\boldsymbol{A}$ has the center of attraction of the elliptic orbit as its point of application and is directed to the perihelion. The conservation of $\boldsymbol{A}$ ensures that the orbit is closed whereas the conservation of angular momentum, descending from the rotational symmetry of the Hamiltonian, ensures that the orbit is in a plane. The magnitude of $\boldsymbol{A}$ is proportional to the eccentricity of the elliptical orbit $|A| = \kappa e$, where the constant of proportionality $\kappa > 0$ defines the Newton potential κ/r.

[2] In absence of degeneracy the motion is not strictly periodic and the system does not return to some given state after a finite amount of time, even though it can pass arbitrarily close to it.

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What is the fate of the Lenz vector, which exists only for $1/r$ fields, when mapped with the same transformation allowing to go from Newton to Hooke and back?

The components of the transformed vector are found to correspond to two of the five components of the classical quadrupole tensor in the coordinate-momentum representation and they come up with the right constants needed to make them constants of motion, as it can ben seen by the computation of Poisson brackets. Therefore two classical conserved quantities of the harmonic oscillator are nothing but the Hooke counterparts of the Lenz vector. The geometric meaning of the quadrupole components is less direct than that of the Lenz vector and it encodes the higher symmetry of the oscillator orbit with respect to the Kepler's one, where the minor axis of the ellipse is not a symmetry axis. These relations have been very much elaborated in classical mathematical physics [4].

Following the Pauli approach, what is the meaning of the mapped components, A_x and A_y, in quantum physics? Not much, when taken on their own, but, when considered together with the angular momentum, which has the component L_z only, the three form an SU(2) Lie algebra. We show that this is indeed the internal SU(2) symmetry of the two-dimensional harmonic oscillator and make a straightforward connection to the Schwinger model of angular momentum.

It is particularly interesting to observe that the operators $X_{ij} = a_i a_j^\dagger$, which allow to switch between the oscillator degenerate states, are themselves the generators of the SU(2) symmetry, $U = \exp(\varepsilon_{ij} X_{ji})$, with $\varepsilon_{ij} X_{ji}$ which are readily translated in terms of the mapped A_x and A_y, and of the angular momentum L_z.

The Pauli version of the classical Lenz vector explains the $\ell \leq n-1$ degeneracy of hydrogen. The mapped components of the classical Lenz vector, upon quantization, are two of the three generators of the internal SU(2) symmetry of the two-dimensional quantum oscillator, and this is in turn the reason for the degeneracy of states.[3]

Let us make a step back and present the complex map which allows to connect Kepler's to Hooke's orbits.

In the complex ζ-plane, a circle of radius r, with center in the origin, is represented by the formula $\zeta = re^{i\phi}$. The map

$$\zeta \to z = \zeta + \frac{1}{\zeta} \tag{4}$$

defines an ellipse z with semimajor axis $a = r + 1/r$ and semiminor axis $b = r - 1/r$. The position of focii is given by $x = \pm ae = \pm 2$. The map

$$z \to w = z^2 \tag{5}$$

shifts the focus located at $x = -2$ into the origin of axes, since $w = \zeta^2 + 1/\zeta^2 + 2$, thus giving an ellipse w with center of attraction[4] in a focus. Similarly it shifts an ellipse with center of attraction in the rightmost focus to an ellipse with center of attraction in the origin. Abandoning this specific example, it is easily seen that a generic ellipse $z = a\cos\theta + ib\sin\theta$, with center of attraction in the center, is mapped by $w = z^2$ into an ellipse with center of attraction in the leftmost focus. This is sometimes called a 'Bohlin map' [6].

Differently from the Kepler case (ellipse w), the center of attraction in the two-dimensional oscillator is located in the center (ellipse z). Therefore Kepler and Hooke orbits can be connected by the map (5), as is well known [3]. We will review all this in Section II.

The Lenz vector A can be written as a complex number A, whose real and imaginary parts define the conserved components of A in the plane of the orbit. The transformation (5) can be used to map the Lenz vector for the Kepler orbit into its counterpart for the Hooke orbit. This passage will be described in detail in Section III.

We show that the two components identifying the mapped Lenz vector correspond to two of the five entries of a (symmetric and traceless) quadrupole tensor $Q_{i,j}$, with $i, j = 1, 2, 3$, in space-momentum coordinates. These are known to be classical conserved quantities for the two-dimensional harmonic oscillator.

Introducing the quantization rules, we recover the SU(2) symmetry of the two-dimensional harmonic oscillator: two of the generators turn out to be the operators derived form the components of the mapped Lenz vector, whereas the remaining SU(2) generator is the angular momentum L_z orthogonal to the plane of the orbit.

The connection with the Schwinger model of angular momentum will then be apparent, as will be discussed in Section IV.

2. From Hooke to Kepler orbits

In the following we derive the rule connecting the Kepler and Hooke force laws and orbits. This will be illustrated making use of the variational principle.

The variational principle, in the Maupertuis form, states that the path γ taken by a particle in the potential $U(\boldsymbol{r}) < E$ is an extremum of the (reduced) action [5].

$$S = \int_\gamma ds \sqrt{E - U(\boldsymbol{r})} \tag{6}$$

where E is the conserved total energy and $v = ds/dt$ is the particle velocity and we discard the overall constant of $\sqrt{2m}$.

Considering planar orbits γ, we rewrite the action in the complex z-plane in the form

$$S = \int_\gamma |dz| \sqrt{E - U(|z|)} \tag{7}$$

where we take the potential to be defined by

$$U(|z|) \equiv \frac{1}{4}\left|\frac{dw}{dz}\right|^2 \tag{8}$$

If w is given by

$$w = z^2 \tag{9}$$

then

$$U(|z|) = |z|^2 \tag{10}$$

corresponding to the harmonic potential in two dimensions. All physical (dimensional) parameters will be set to one for simplicity. The harmonic motion is in the z-plane.

The term $|dw/dz|^2$ can be factored out from the argument of the integral in (7), changing the integration measure from $|dz|$ to $|dw|$, thus giving

$$S = \int_\gamma |dw| \sqrt{4E\left|\frac{dz}{dw}\right|^2 - 1} \tag{11}$$

However

$$\left|\frac{dz}{dw}\right|^2 = \frac{1}{4|w|} \quad \text{and} \quad |z|^2 = |w| \tag{12}$$

so that

[3] The excited states of the harmonic oscillator can be written in the form of symmetric rank-n tensors$\Psi_n = a_{i_1}^\dagger \cdots a_{i_n}^\dagger \Psi_0$ (2)whose dimensionality, for $i_1, \ldots, i_n = 1, 2$, is given by$d = \binom{n+1}{n} = n + 1$ (3)corresponding to the degeneracy of E_n levels.

[4] The point from which the position vectors stem.

$$S = \int_\gamma |dw| \sqrt{E' - \left(-\frac{E}{|w|}\right)} \tag{13}$$

which is the Maupertius action for the motion of a particle in the complex w-plane under the action of the attractive Newton potential as expressed by

$$V(|w|) = -\frac{E}{|w|} \tag{14}$$

E', in our units, is $E' = -1$, the negative energy in the Newton attractive potential to be compared with the positive E in the Hooke potential U. The Kepler orbit is in the w-plane.

This shows how the $w = z^2$ map connects the harmonic (Hooke potential U in (10)) and Keplerian motions (potential V in (14)). The same result is described in more general terms in a theorem that can be found in Ref. [3].

We note however that one more conclusion can be drawn from the variational principle (7). Namely we may observe that expressing the kinetic energy

$$S = \int_\gamma \left|dz\right| \sqrt{E - U(|z|)} = \int_\gamma \left|dz\right| \sqrt{\frac{1}{2}\left(\frac{d}{dt}\left|z\right|\right)^2 + \frac{L^2}{2\left|z\right|^2}} \tag{15}$$

and

$$S = \int_\gamma \left|dw\right| \left|\frac{dz}{dw}\right| \sqrt{\frac{1}{2}\left(\frac{1}{2\left|w\right|^{1/2}} \frac{d}{dt}\left|w\right|\right)^2 + \frac{L^2}{2|w|}} = $$
$$= \frac{1}{2}\int_\gamma \left|dw\right| \sqrt{\frac{1}{2}\left(\frac{d}{d\tau}\left|w\right|\right)^2 + \frac{L^2}{2|w|^2}} \tag{16}$$

provided that

$$d\tau = 2\left|w\right|dt = 2\left|z\right|^2 dt \tag{17}$$

The Hooke orbits are changed into Kepler orbits by the complex map $w = z^2$, provided that an appropriate rescaling of the time variable is done.

The angular momentum of the Hooke orbit in the z-plane is by definition the conserved quantity

$$L_H = \left|z\right|^2 \frac{d\varphi}{dt} \tag{18}$$

from which the area law follows. On the other hand

$$L_H = \left|z\right|^2 \frac{d\varphi}{dt} = \left|w\right| \frac{d\varphi}{dt} = 2\left|w\right|^2 \frac{d\varphi}{2|w|dt} = 2\left|w\right|^2 \frac{d\varphi}{d\tau} = L_K \tag{19}$$

in terms of position and time of the Kepler orbit[5] (the two constant values for the area laws may also have different values). Therefore we may argue that the rule

$$(\text{Hooke}) \ \frac{d}{dt} \leftrightarrow (\text{Kepler}) \ \frac{d}{d\tau} \qquad \text{with} \qquad d\tau = 2\left|z\right|^2 dt \tag{23}$$

allows to connect Hooke and Kepler motions. Indeed it is observed that, see next Section, using (23), one obtains the Newton force law in the w-plane, corresponding to the gradient of (14)

$$\frac{d^2 w}{d\tau^2} = -E\frac{w}{\left|w\right|^3} \qquad \text{with} \qquad w = z^2 \tag{24}$$

where $E = 1/2\left|\dot{z}\right|^2 + 1/2\left|z\right|^2$ is the elastic conserved energy and physical constants are set to one for simplicity.

Since the Newton potential is a homogeneous function of degree $k = -1$, the similarity transformation [1].

$$w \to \beta w \qquad \tau \to \beta^{1-1/2\,k}\tau \tag{25}$$

leaves invariant the equations of motion (the Lagrangian gets multiplied by a constant scale factor $\beta^{-1} = \beta^2/\beta^{2-k}$). The angular momentum has to transform accordingly as

$$L \to \beta^{1/2} L \tag{26}$$

Therefore we can write

$$S = \frac{1}{2}\int_\gamma \left|dw\right| \sqrt{\frac{1}{2}\left(\frac{d}{d\tau'}\left|w\right|\right)^2 + \frac{(L')^2}{2\left|w\right|^2}} \tag{27}$$

with

$$L' = \frac{2}{\alpha} L \tag{28}$$

and

$$d\tau' = \alpha\left|w\right|dt = \alpha\left|z\right|^2 dt \tag{29}$$

in place of(17), having set

$$\beta = (2/\alpha)^2 \tag{30}$$

However this transformation of angular momentum and time scale is tantamount an inconsequential rescaling of S by

$$S \to \frac{2}{\alpha} S \tag{31}$$

This means that the arbitrary time rescaling (29), which could have been postulated in place of (17), giving two different values for the constant area laws as in (28), can be considered as part of a similarity transformation, which also changes accordingly L and w.

3. The transformation of the Lenz vector

We define

$$w \equiv x + iy \qquad p \equiv p_x + ip_y \tag{32}$$

and

$$\boldsymbol{p} \times \boldsymbol{L} = L\left(p_y - ip_x\right) = -iLp \tag{33}$$

From now on the indices $_K$ and $_H$ will refer accordingly to quantities either in the Kepler or the Hooke case. Then the Lenz vector in complex coordinates is

$$A_K = -iL_K p_K - \kappa\frac{w}{|w|} \tag{34}$$

where the physical constant κ has been explicitly left for reasons that will

[5] In the case of the Hooke orbit (in the z-plane)$z = a\cos\omega t + i\sin\omega t$ (20) and $\varphi_H = \arctan(\Im z\,/\Re z)$ which gives $\dot{\varphi}_H = ab\omega/\left|z\right|^2$ and $L_H = \left|z\right|^2\dot{\varphi}_H = ab\omega$. In the same way we could compute the $\varphi_K = \arctan(\Im w\,/\Re w)$ where $w = z^2$. This gives $\dot{\varphi}_K = 2ab\omega/\left|z\right|^2 = 2L_H/\left|z\right|^2$ (21)To replace $\dot{\varphi}_K$ with $d\varphi_K/\,d\tau$ we divide the previous equation by $2\left|z\right|^2$ giving $((\left|z^2\right|)^2 = \left|w\right|^2)$ $d\varphi_K/d\tau = L_H/\left|w\right|^2 = L_K/\left|w\right|^2$ (22) where the first equality comes form (21) and the second from the definition of L_K. It follows that $\varphi_K = 2\varphi_H = 2\varphi$ by comparing with (19) or by direct inspection of the ratio φ_K/φ_H.

be clear in a moment. This is similarity transformation invariant. Using the complex map and (23) we find

$$L_K = (xp_y - yp_x) = \frac{1}{2i}\left(w^*\frac{dw}{d\tau} - w\frac{dw^*}{d\tau}\right) = \frac{1}{2i}\left(z^*\frac{dz}{dt} - z\frac{dz^*}{dt}\right) = L_H \tag{35}$$

Similarly working on momentum we get

$$p_K = \frac{dw}{d\tau} = \frac{1}{2|z|^2}\frac{d}{dt}z^2 = \frac{1}{z^*}p_H \tag{36}$$

In order to understand the role of κ in the transformation we compute

$$\frac{d^2w}{d\tau^2} = \frac{1}{4}\frac{1}{|z|^2}\frac{d}{dt}\left(\frac{1}{|z|^2}\frac{d}{dt}z^2\right) = -E_H\frac{1}{z(z^*)^3} = -E_H\frac{w}{|w|^3} \tag{37}$$

where we have used the equation of motion of the harmonic oscillator

$$\frac{d^2z}{dt^2} + z = 0 \tag{38}$$

and recognized the expression for the harmonic oscillator energy

$$E_H \equiv \frac{1}{2}|\dot{z}|^2 + \frac{1}{2}|z|^2 \tag{39}$$

Comparing (37) with the Keplerian equation of motion

$$\frac{d^2w}{d\tau^2} = -\kappa\frac{w}{|w|^3} \tag{40}$$

we understand that in mapping (34), κ must replaced by E_H. Using (35), (36) and (39) we thus see that the Lenz vector is equal to

$$A_K = -iL_K p_K - \kappa\frac{w}{|w|} = -\left(\frac{1}{2}z^2 + \frac{1}{2}\dot{z}^2\right) \equiv -A_H \tag{41}$$

where A_H is the vector corresponding to the Lenz vector in the Hooke problem. It is straightforward to verify that A_H is a conserved quantity of the Hooke motion, since from (38)

$$\frac{d}{dt}\left(\frac{1}{2}\dot{z}^2 + \frac{1}{2}z^2\right) = 0 \tag{42}$$

We can expand A_H as (from now on to the variables x, y, p_x and p_y will refer to the Hooke motion)

$$A_H = \frac{1}{2}(x^2 - y^2) + \frac{1}{2}\left(p_x^2 - p_y^2\right) + i(xy + p_x p_y) \tag{43}$$

Since A_H is conserved, both the real and the imaginary parts of it are separately conserved. From the fact that E_H and $\mathrm{Re}(A_H)$ are both conserved it follows that the one dimensional harmonic oscillator energy is separately conserved on both the x and y axis. We successively found that this result was also discussed in Refs. [7], although in a slightly different form, whereas part of what follows in the next section was addressed in Ref. [8].

We will proceed in the next section to describe the role of $A_H \equiv A$ in the quantum harmonic oscillator problem.

4. SU(2) and the Schwinger model

We found in the previous Section that the mapped classical Lenz vector in complex coordinates is given by

$$A = \frac{1}{2}(p_x + ip_y)^2 + \frac{1}{2}(x + iy)^2 \tag{44}$$

This can be rewritten as

$$A = Q_2 + iQ_{xy} \tag{45}$$

where the Q's are two of the five components of the classical quadrupole tensor in the coordinate-momentum representation[6]

$$Q_{xy} = xy + p_x p_y \tag{46}$$

and

$$Q_2 = \frac{1}{2}(x^2 - y^2) + \frac{1}{2}\left(p_x^2 - p_y^2\right) \tag{47}$$

The general expression of the classical quadrupole tensor in coordinates and momenta will be given in the form

$$Q_{ij} = A\left(x_i x_j - 1/3\,\delta_{ij}\boldsymbol{x}^2\right) + B\left(p_i p_j - 1/3\,\delta_{ij}\boldsymbol{p}^2\right) \qquad i,j = 1,2,3 \tag{48}$$

where the constants A and B are chosen appropriately as can be seen in footnote (7). Since Q is defined as a traceless tensor, there are only two independent diagonal entries: we choose Q_{xx} and Q_{yy}. These can be combined into $Q_{xx} \pm Q_{yy} = Q_{1/2}$. Q_1 turns out to be the energy, whereas Q_2 is given above, with appropriate normalization.

Let us promote x and p to quantum operators with the quantization rules

$$[x_i, p_j] = i\delta_{ij} \tag{49}$$

using $\hbar = 1$ natural units. Computing explicitly the commutators, one can easily show that L_z, given by

$$L_z = xp_y - yp_x \tag{50}$$

together with Q_{xy} and Q_2, form an SU(2) Lie algebra. Indeed the following correspondence holds

$$\begin{aligned} Q_{xy} &= \sigma^1 \\ L_z &= \sigma^2 \\ Q_2 &= \sigma^3 \end{aligned} \tag{51}$$

More generally it is known that the quadrupole components in coordinate-momentum representation are in correspondence with the Gell-Mann matrices.[7] The correspondence in (51) reflects[8]

$$[\lambda^i, \lambda^j] = 2i\varepsilon_{ijk}\lambda^k \quad \text{for} \quad i,j = 1,2,3 \tag{53}$$

We want to show now that the internal symmetry of the two-dimensional quantum oscillator is generated indeed from Q_{xy}, L_z, Q_2.

The harmonic oscillator Hamiltonian in two dimensions (with $m = \omega = 1$)

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}(x^2 + y^2) = \sum_{i=1}^{2}\left(a_i^\dagger a_i + \frac{1}{2}\right) \tag{54}$$

has an internal SU(2) symmetry under unitary transformations of the creation/annihilation

[6] In the Kepler orbits the minor axis is not a symmetry axis. The two dimensional oscillator orbit is more symmetric. One can rotate the ellipse by $\pi/2$ around the center and exchange $a \leftrightarrow b$ obtaining the same ellipse. We can substitute the equation of such an ellipse in $Q_{xy} = xy$ in two different ways: by substituting $x = a\cos\theta\cos\eta - b\sin\theta\sin\eta$, $y = a\cos\theta\sin\eta + b\sin\theta\cos\eta$ or by substituting the same equations with $a \leftrightarrow b$ and $\eta \rightarrow \eta \pm \pi/2$. The sum of the two alternatives gives the symmetric $Q_{xy} = 1/2(a^2 - b^2)\sin 2\eta$.

[7] By choosing the normalizations of the coordinate and momentum terms, the Correspondence with Gell-Mann matrices is $Q_{xy} = xy + p_x p_y = \lambda^1$ $L_z = \lambda^2$ $Q_1 = \frac{1}{2\sqrt{3}}((x^2 + y^2 - 2z^2) + (p_x^2 + p_y^2 - 2p_z^2)) = \lambda^3$ $Q_{zx} = zx + p_z p_x = \lambda^4$ $L_y = -\lambda^5$ $Q_{yz} = yz + p_y p_z = \lambda^6$(52).$L_x = \lambda^7$ $Q_2 = \frac{1}{2}(x^2 - y^2) + \frac{1}{2}(p_x^2 - p_y^2) = \lambda^8$

[8] The SU(3) f^{abc} structure constants coincide with ε_{ijk} if $a,b,c = i,j,k = 1,2,3$.

$$U^{-1}\, a_i\, U = \mathscr{U}_{ij}\, a_j \tag{55}$$

$$U^{-1}\, a_i^\dagger\, U = \mathscr{U}^*_{ij}\, a_j^\dagger \tag{56}$$

where the $\mathscr{U}$ are special unitary matrices. The latter equation in infinitesimal form is

$$U^{-1}\, a_i^\dagger\, U = \left(\delta_{ij} - \varepsilon_{ji}\right) a_j^\dagger \tag{57}$$

where ε_{ij} is an anti-hermitian and traceless matrix

$$\varepsilon^*_{ij} = -\varepsilon_{ji} \tag{58}$$

which can be written in tems of Pauli matrices as

$$\varepsilon = i\theta^1 t^1 + i\theta^2 t^2 + i\theta^3 t^3 \tag{59}$$

where $t^a = \sigma^a/2$. To obtain (57) we have to require (infinitesimal transformation)

$$U = 1 + \mathrm{Tr}(\varepsilon \cdot X) \tag{60}$$

where

$$X_{ij} \equiv a_i a_j^\dagger \qquad i,j = 1,2 \tag{61}$$

as can easily be verified; we use the notation of [9]. The X operators, the generators of the U transformation in the Hilbert space of quantum states, allow to switch through the states in the space of degenerate states associated to a certain energy value.[9]

From (59) and (61) we get

$$U = 1 + i\theta^1\, J^1 + i\theta^2\, J^2 + i\theta^3\, J^3 \tag{64}$$

where we used the definitions of creation/annihilation operators thus finding that the generators J are given by

$$\begin{aligned} J^1 &= \frac{Q_{xy}}{2} = \frac{A_y}{2} \\ J^2 &= \frac{L_z}{2} \\ J^3 &= \frac{Q_2}{2} = \frac{A_x}{2} \end{aligned} \tag{65}$$

Therefore the x and y components of the mapped Lenz vector, which are Q_2 and Q_{xy} respectively, correspond to the symmetry generators J^3 and J^1 of the two-dimensional harmonic oscillator.

Writing x and p back in terms of the creation and annihilation operators, we obtain the generators as written in the Schwinger model of angular momentum

$$J^1 = \frac{a_1 a_2^\dagger + a_2 a_1^\dagger}{2} \qquad J^2 = \frac{i\left(a_1 a_2^\dagger - a_2 a_1^\dagger\right)}{2} \qquad J^3 = \frac{a_1 a_1^\dagger - a_2 a_2^\dagger}{2} \tag{66}$$

with the usual commutation relations and $[a_1, a_2] = [a_1, a_2^\dagger] = \cdots = 0$.

The mapped Lenz vector components $A_x = Q_2, A_y = Q_{xy}$ form, together with L_z, the Casimir operator

$$C = L_z^2 + Q_{xy}^2 + Q_2^2 \tag{67}$$

and give, by explicit calculation

$$C = H^2 - 1 \tag{68}$$

On the other hand it is known that the Casimir of the $\boldsymbol{n}+1$ representation (for $\boldsymbol{n} = 1, 2, 3...$) of SU(2) is given by the formula

$$C_{\boldsymbol{n}+1} = \left(n^2 + 2n\right) \qquad \text{for} \qquad n = 1, 2, 3, \ldots \tag{69}$$

so that, setting back $\hbar$ and ω

$$H = (n+1)\hbar\omega \tag{70}$$

with degeneracy

$$d(\boldsymbol{n}+1) = n + 1 \tag{71}$$

In a similar way we can obtain the energy from the sum of the squares of the three generators

$$J^2 = \frac{N}{2}\left(\frac{N}{2} + 1\right) = \frac{H^2 - 1}{4} \tag{72}$$

where the total number operator is

$$N = a_1^\dagger a_1 + a_2^\dagger a_2 = N_1 + N_2 \tag{73}$$

with

$$[N,\, J_i] = 0 \qquad \left[J^2,\, J^i\right] = 0 \tag{74}$$

which is another way of stating that the transformed Lenz vector components are conserved.

Thus we showed that the mapped Lenz vector generates the internal SU(2) symmetry of the harmonic oscillator, which is the origin of the quantum degeneracy of states.

5. Conclusions

We have shown that the generators of the SU(2) internal symmetry of the quantum harmonic oscillator can be written in terms of the L_z angular momentum and of two operators which, under a specific transformation, defined in (23), correspond to the components of the classical Lenz vector.

In the case of the $1/r$ potential, Pauli showed [2] that the hermitean quantum version of the Lenz vector can be considered on the same footing of the angular momentum by introducing the operator $L_{ij} = x_i\, p_j - x_j\, p_i$ with $i, j = 1, 2, 3, 4$ and $L_{i4} = A_i$, appropriately rescaled[10], x_4, p_4 being fictitious coordinates. The resulting Lie algebra is that of SO(4) which does not represent a geometrical symmetry of the hydrogen atom in the same way as SU(2) is not a geometrical symmetry of the two-dimensional quantum oscillator. Similarly, in the case of the two-dimensional harmonic oscillator, the angular momentum L_z, when taken together with the mapped components of the Lenz vector, generates an SU(2) algebra.

Author contribution

I wrote this paper in collaboration with three students of my lecture course on quantum mechanics at Sapienza University of Rome. I assigned the study of this topic to F. Sciotti, for his bachelor thesis. The question arose: what happens to the Lenz vector when mapped with the Bohlin map? All we report in this paper, comes out from our discussions, with very little knowledge of the specialized literature on this topic.

[9] For example$X_{12}\Psi_{3,1} \propto \Psi_{2,2}$ (62)where $\Psi_{3,1}$ and $\Psi_{2,2}$ have the same energy $E_4\sim$ [9]. Also observe that generators of the symmetry annihilate the 'vacuum' state $X_{ij}\Psi_0 = 0 \quad i \neq j$ (63) where $\Psi_0 = |0, 0\rangle$ is the ground state of the harmonic oscillator. The combination with $i = j$ from is $a_1 a_1^\dagger - a_2 a_2^\dagger$ which again annihilates Ψ_0.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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