Bounce on a p-Laplacian

Dimitri Mugnai*
Dipartimento di Matematica e Informatica
Via Vanvitelli 1
06123 Perugia, Italy
email: mugnai@dipmat.unipg.it

Abstract

The existence of nontrivial solutions for reversed variational inequalities involving p-Laplace operators is proved. The solutions are obtained as limits of solutions of suitable penalizing problems.

A. M. S. subject classification 2000: 49J40, 47J20, 47J30
Key words: Reversed variational inequalities

1 Introduction

Several interesting problems can be described by variational inequalities in which the sign of the inequality is opposite with respect to the sign of classical ones *à la* Lions–Stampacchia ([12]). This is the reason why they are called *reversed* variational inequalities. For instance the bounce problem gives rise to a reversed variational inequality (see [5], [7] and [13]), as well as the *jumping* problem (see [14]).

In this paper we consider a non-Hilbert version of the notion of reversed variational inequality, introduced for the Hilbert case in [15], in connection with a fourth order elliptic problem (see also [13]). In the present case the strategy used in [15] to prove the existence of solutions cannot be adapted directly and needs some refined tools of nonlinear analysis.

We study the existence of a solution u of the following problem:

$$(P) \quad \begin{cases} \exists\, u \in K_\phi := \left\{u \in W_0^{1,p}(\Omega) : u \geq \phi\right\} \text{ such that} \\ \displaystyle\int_\Omega |Du|^{p-2} Du \cdot D(v-u)\,dx - \alpha \int_\Omega |u|^{p-2}u(v-u)\,dx \leq 0 \\ \forall\, v \in K_\phi, \end{cases}$$

*Supported by MIUR, national project *Variational Methods and Nonlinear Differential Equations*

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $1 \leq N < p$, ϕ is a measurable function, with $\sup_\Omega \phi < 0$, and $\alpha < \lambda_1$, where

$$\lambda_1 = \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega |Du|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}. \tag{1}$$

By a *solution* we mean a function $u \in K_\phi$ which solves (P) and which is *not* a solution of the associated equation $\Delta_p u + \alpha |u|^{p-2} u = 0$ in Ω.

If $p = 2$ and $N = 1$ problem (P) admits bounce trajectories as solutions (see [7], [13] or [14]). In fact (P) is not a "problem with obstacle" (described by classical variational inequalities), but is a "bounce problem".

It is apparent that, taking $v = u + \psi$ in (P), with $\psi \in \mathcal{D}(\Omega)$ and $\psi \geq 0$, any solution of (P) is a nontrivial solution of the problem

$$\begin{cases} -\Delta_p u - \alpha |u|^{p-2} u \leq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ_p is the p–Laplace operator. Moreover, since $\alpha < \lambda_1$, any solution of (P) is nonpositive, so that $v = -u$ solves

$$-\Delta_p v - \alpha v^{p-1} \geq 0, \qquad v \geq 0 \quad \text{in } \Omega. \tag{2}$$

As a corollary of a result of Serrin and Zou ([16, Theorem I′]), if $\Omega \subset \mathbb{R}^N$, $1 \leq N < p$, is an exterior domain and v solves (2) with $\alpha = 1$, then $v \equiv 0$. Moreover for the problem

$$-\Delta_m v - v^{p-1} \geq 0, \qquad v \geq 0 \quad \text{in } \Omega \tag{3}$$

they prove that:

- if Ω is an exterior domain, inequality (3) has a nontrivial solution if and only if $m \in (1, N)$ and $p > \frac{m(N-1)}{N-m}$ ([16, Corollary I]);
- if $\Omega = \mathbb{R}^N$ and $m \in (1, N)$, then (3) has a nontrivial solution if and only if $p > \frac{m(N-1)}{N-m}$ ([16, Corollary II (iii)]).

Having in mind these results, we want to show that in bounded domains Ω problem (P), and so inequality (2), admits bounded solutions.

As a consequence of the previous discussion, it is clear that $\phi \leq 0$ is a necessary condition for existence of solutions to (P) when $\alpha < \lambda_1$. On the other hand, the assumptions $\alpha < \lambda_1$ and $\sup_\Omega \phi < 0$ are essential in all the proofs of the paper, while the case $\alpha \geq \lambda_1$ is still open. Indeed, if $\alpha \geq \lambda_1$, the geometrical structures of f_ω change and it seems hard to find critical points of f_ω. For example, the Palais-Smale condition (see Proposition 2.2) might not hold: in fact, if $\alpha = \lambda_1$ and $u_n = t_n e_1$, where $t_n \to \infty$ and e_1 is the eigenvalue associated to λ_1 (see the proof of Theorem 2.1), u_n has no converging subsequences.

Due to the lack of a general theory for "reversed" inequalities, as Stampacchia's Lemma for linear (classical) variational inequalities, a natural way to face problem (P) is to study the following family of approximating problems:

$$(P_\omega) \qquad \begin{cases} \Delta_p u + \alpha |u|^{p-2} u - \omega((u - \phi)^-)^{k-1} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ω is a positive parameter approaching $+\infty$, $k > p$ and k is subcritical (see Section 2). First the existence of a nontrivial solution u_ω of (P_ω) is established for each ω and for each $N \geq 1$ and then we show that the family $\{u_\omega \,|\, \omega \geq \omega_0 > 0\}$ of such nontrivial solutions u_ω is bounded in $W_0^{1,p}(\Omega)$. Finally, when $p > N$, any limit u of sequences of solutions u_{ω_n} of (P_{ω_n}) satisfies the reversed variational inequality (P). Moreover, the crucial part of the proof is that any solution u of (P) constructed by the limiting process satisfies $\Delta_p u + \alpha|u|^{p-2}u = \mu$ for a suitable nontrivial nonnegative Radon measure μ depending on u.

2 The approximating problems

Let Ω be a bounded and smooth domain of $\mathbb{R}^N$, $N \geq 1$, $\alpha < \lambda_1$, $\omega \geq \omega_0 > 0$, $1 < p < k$ (and $k < pN/(N-p)$ if $p < N$). Assume ϕ is a measurable function defined in Ω with $\sup_\Omega \phi < 0$ and consider the following problems:

$$(P_\omega) \qquad \begin{cases} \Delta_p u + \alpha|u|^{p-2}u - \omega((u-\phi)^-)^{k-1} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$, $u^- = \max\{-u, 0\}$ and $u \in W_0^{1,p}(\Omega)$. We endow $W_0^{1,p}(\Omega)$ with the standard norm $\|u\| = \left(\int_\Omega |Du|^p \, dx\right)^{1/p}$ and we use the standard notation p' to denote the real number such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Consider $f_\omega : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined as follows:

$$f_\omega(u) = \frac{1}{p}\int_\Omega |Du|^p \, dx - \frac{\alpha}{p}\int_\Omega |u|^p \, dx - \frac{\omega}{k}\int_\Omega ((u-\phi)^-)^k \, dx.$$

We observe that f_ω is a C^1 functional on $W_0^{1,p}(\Omega)$ and that its critical points are solutions of (P_ω).

Note that, in general, the function $g(x,s) = \alpha|s|^{p-2}s - \omega((s-\phi(x))^-)^{k-1}$ does not satisfy either the classical Ambrosetti-Rabinowitz condition (see [1]) or the generalized Ambrosetti-Rabinowitz condition introduced in [9] and [10], namely $\exists\, \Theta$ and $\exists\, s_0 > 0$ such that

$$0 < \Theta\, G(x,s) \leq s g(x,s) \qquad \forall\, s,\ |s| \geq s_0, \tag{4}$$

where $G(x,s) = \int_0^s g(x,\sigma)\, d\sigma$ and either $\Theta > 2$ ([1]) or $\Theta > p$ ([9],[10]).

In fact equation (4) reads as

$$0 < \Theta\frac{\alpha}{p}|s|^p + \Theta\frac{\omega}{k}((s-\phi(x))^-)^k \leq \alpha|s|^p - \omega((s-\phi(x))^-)^{k-1}s.$$

Of course, if $\alpha > 0$ and $\Theta > p$, the last inequality is not satisfied for any $s > 0$, and it is not satisfied for any $s > 0$ when $\alpha \leq 0$, whatever $\Theta > 0$ is. See also [11] for other classes of nonlinearities.

Nevertheless we still apply the Mountain Pass Theorem to get a nontrivial critical point u_ω for f_ω, i.e. a solution of problem (P_ω).

We first recall the following definition.

Definition 2.1. *Let $c \in \mathbb{R}$. We say that a C^1 functional $f : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ satisfies the Palais-Smale condition at level c, or that $(PS)_c$ holds, if every sequence $(u_n)_n$ in $W_0^{1,p}(\Omega)$ such that $f(u_n) \to c$ and $f'(u_n) \to 0$, has a strongly converging subsequence.*

Sequences $(u_n)_n$ such that $f(u_n) \to c$ and $f'(u_n) \to 0$ are called Palais-Smale sequences at level c, or $(PS)_c$–sequences.

As in the Hilbert case, in order to prove $(PS)_c$, it is sufficient to check that a Palais-Smale sequence has a bounded subsequence. In fact, one can show that the following property holds (see [4]).

PROPOSITION 2.1 ((S_+) property). *Let* $\Phi : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ *be defined as* $\Phi(u) = \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}$. *If a sequence* $(u_n)_n$ *weakly converges in* $W_0^{1,p}(\Omega)$ *to* u *and*

$$\limsup_{n\to\infty} < \Phi'(u_n), u_n - u > \leq 0,$$

then $u_n \to u$ *strongly in* $W_0^{1,p}(\Omega)$.

Thus, in order to prove that f_ω satisfies $(PS)_c$, we first prove that $(PS)_c$ holds if any $(PS)_c$–sequence has a bounded subsequence, and then we prove that any $(PS)_c$–sequence is bounded.

LEMMA 2.1. *Let* $c \in \mathbb{R}$ *and let* $(u_n)_n$ *be a* $(PS)_c$*–sequence for* f_ω *which has a bounded subsequence. Then* $(u_n)_n$ *admits a strongly convergent subsequence.*

Proof. The proof reminds the proof given in [9] for some nonlinear problems in presence of the p–Laplace operator, under the generalized Ambrosetti Rabinowitz condition.

We recall that $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ compactly $\forall\, q < p^*$, where $p^* = \frac{pN}{N-p}$ if $p < N$ and $p^* = \infty$ if $p \geq N$.

Therefore $\int |u_n|^{p-2}u_n(u_n - u) \to 0$ and $\int ((u_n - \phi)^-)^{k-1}(u_n - u) \to 0$. Then we get

$$< \Phi'(u_n), u_n - u >= \int_\Omega |Du_n|^{p-2}Du_n \cdot (Du_n - Du)\, dx = f'_\omega(u_n)(u_n - u)$$

$$+\alpha \int_\Omega |u_n|^{p-2}u_n(u_n - u)\, dx - \omega \int_\Omega ((u_n - \phi)^-)^{k-1}(u_n - u)\, dx \to 0.$$

By the (S_+) property, we conclude that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. □

PROPOSITION 2.2 (Palais-Smale). *Suppose* $\sup_\Omega \phi < 0$ *and* $\alpha < \lambda_1$. *Then* f_ω *satisfies* $(PS)_c$ *for every* $c \in \mathbb{R}$.

Proof. Let u_n be a $(PS)_c$ sequence. By the previous Lemma, it is enough to show that $(u_n)_n$ has a bounded subsequence. Thus suppose by contradiction that $\|u_n\|$ is unbounded. Then we can suppose that there exists v in $W_0^{1,p}(\Omega)$ such that $v_n = u_n/\|u_n\|$ weakly converges to a function v in $W_0^{1,p}(\Omega)$.

Of course $\dfrac{f'_\omega(u_n)(u_n)}{\|u_n\|} \to 0$, where

$$\frac{f'_\omega(u_n)(u_n)}{\|u_n\|} = \frac{1}{\|u_n\|}\left\{\int_\Omega |Du_n|^p\, dx - \alpha \int_\Omega |u_n|^p\, dx\right.$$

$$\left. +\omega \int_\Omega ((u_n - \phi)^-)^{k-1}u_n\, dx\right\} = \frac{1}{\|u_n\|}\left\{pf_\omega(u_n)\right.$$

$$\left.+\left(\frac{p}{k} - 1\right)\omega \int_\Omega ((u_n - \phi)^-)^k\, dx + \omega \int_\Omega ((u_n - \phi)^-)^{k-1}\phi\, dx\right\}.$$

Passing to the limit we get, since $p < k$ and $\phi < 0$,

$$\lim_{n\to\infty} \frac{\int_\Omega ((u_n - \phi)^-)^k\, dx}{\|u_n\|} = 0 \text{ and } \lim_{n\to\infty} \frac{\int_\Omega ((u_n - \phi)^-)^{k-1}\phi\, dx}{\|u_n\|} = 0. \tag{5}$$

Since $\frac{f'_\omega(u_n)(u_n)}{\|u_n\|^p} \to 0$, one has

$$1 - \alpha \int_\Omega |v_n|^p \, dx + \frac{\omega \displaystyle\int_\Omega ((u_n - \phi)^-)^{k-1} u_n \, dx}{\|u_n\|^p} \to 1 - \alpha \int_\Omega |v|^p \, dx = 0.$$

In this way we get an immediate contradiction if $\alpha \le 0$. If $\alpha > 0$, we would have

$$0 = 1 - \alpha \int_\Omega |v|^p \, dx \ge 1 - \frac{\alpha}{\lambda_1} \int_\Omega |Dv|^p \, dx \ge 1 - \frac{\alpha}{\lambda_1} > 0.$$

□

We now observe that (P_ω) admits the trivial solution for every α and ω. We now want to prove the existence of a particular nontrivial solution, namely a solution which is below ϕ on a set of positive measure.

Definition 2.2. *We say that u in $W_0^{1,p}(\Omega)$ is a forcing solution of problem (P_ω) if it is a solution such that $meas(\{x \in \Omega \,|\, (u(x) - \phi(x))^- \ne 0\}) > 0$.*

The reason for this definition lies in the fact that if u_{ω_n} is a solution of (P_{ω_n}) and $u_{\omega_n} \rightharpoonup u$, then $u \ge \phi$ (see Lemma 3.1) and if, moreover, u_{ω_n} is a forcing solution of (P_{ω_n}), $u_{\omega_n} \to u$ uniformly and ϕ is continuous, then the coincidence set, or "contact set", $\{x \in \Omega \,\big|\, u(x) = \phi(x)\}$ is not empty (see Proposition 3.1), so u is "forced" by the sequence u_{ω_n} to be over ϕ and to touch it somewhere. In this way ϕ works as a bounce wall.

Remark 1. *A solution u of (P_ω) is a forcing solution if and only if $f_\omega(u) > 0$. In fact*

$$0 = f'_\omega(u)(u) = p f_\omega(u)$$
$$+ \left(\frac{p}{k} - 1\right) \omega \int_\Omega ((u - \phi)^-)^k \, dx + \omega \int_\Omega ((u_\omega - \phi)^-)^{k-1} \phi \, dx.$$

An application of Sobolev's inequality and the fact $\phi < 0$ give the following lemma.

Lemma 2.2. *Suppose $\sup_\Omega \phi < 0$. Then*

$$\int_\Omega ((u - \phi)^-)^k \, dx = o(\|u\|^k)$$

as $u \to 0$.

Theorem 2.1 (Existence Theorem). *Let $\alpha < \lambda_1$ and $\sup_\Omega \phi < 0$. Then for every ω there exists a forcing solution u_ω of problem (P_ω) such that $\sup\limits_{\omega \ge \omega_0} f_\omega(u_\omega) < +\infty$.*

Proof. By Lemma 2.2, given $\varepsilon < 1 - \alpha/\lambda_1$, there exists $\rho > 0$ such that

$$\inf_{\|u\| = \rho} f_\omega(u) \ge \frac{1}{p} \left(1 - \varepsilon - \frac{\alpha}{\lambda_1}\right) \rho^p > 0.$$

Let e_1 be the function which minimizes the Rayleigh quotient of (1). We can suppose $e_1 > 0$ in Ω (see [2]).

But $f_\omega(0) = 0$ and $\lim f_\omega(-te_1) = -\infty$. Therefore there exists $t_\omega > 0$ such that $f_\omega(-t_\omega e_1) < 0$. By the Mountain Pass Theorem ([1]) there exists a nontrivial critical point u_ω of f_ω for every ω. Moreover the following estimates hold:

$$\frac{\rho^p}{p}\left(1-\varepsilon-\frac{\alpha}{\lambda_1}\right) \leq f_\omega(u_\omega) \leq \sup_{t\in[0,t_\omega]} f_\omega(-te_1) \leq \sup_{t\geq 0} f_{\omega_0}(-te_1) < +\infty$$

for every $\omega \geq \omega_0$. □

REMARK 2. *If $p > N$ we can take $\varepsilon = 0$ in the inequalities above. In fact in this case $W_0^{1,p}(\Omega) \hookrightarrow C_0^0(\Omega)$, so there exists $\overline{\rho}$ such that, if $\rho \leq \overline{\rho}$ and $\|u\| \leq \rho$, then $u - \phi \geq 0$ (since $\sup_\Omega \phi < 0$). Moreover in such a case $\liminf\limits_{\omega\to+\infty} f_\omega(u_\omega) > 0$.*

3 Bounce equation

The aim of this section is to study the problem obtained from (P_ω) when ω tends to $+\infty$. We will essentially follow the approach of [15], but now there are some complications due to the Banach setting. In particular, in order to prove the following Theorem 3.2, we will need a theorem by Boccardo and Murat (see [3]).

From now on we will consider sequences of real positive numbers $(\omega_n)_n$ such that $\omega_n \to +\infty$ as $n \to \infty$ and we will investigate the asymptotic behaviour of forcing solutions $(u_{\omega_n})_n$ of $(P_{\omega_n})_n$ as n goes to infinity. For the sake of simplicity we will write ω in place of ω_n and $\omega \to +\infty$ in place of "$\omega_n \to +\infty$ as $n \to \infty$".

THEOREM 3.1. *If $\alpha < \lambda_1$, $\sup_\Omega \phi < 0$ and u_ω is a solution of (P_ω) such that $\sup\limits_\omega f_\omega(u_\omega) < +\infty$, then*

a) $\sup\limits_\omega\, \omega \displaystyle\int_\Omega ((u_\omega - \phi)^-)^{k-1}\, dx < +\infty$;

b) $(u_\omega)_\omega$ *is bounded.*

Proof. Suppose by contradiction that there exists a subsequence, which we still denote by $(u_\omega)_\omega$, such that $\|u_\omega\|$ diverges. Then, up to a subsequence, there exists v in $W_0^{1,p}(\Omega)$ such that $v_\omega = u_\omega/\|u_\omega\| \rightharpoonup v$ in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω. Observe that $f'_\omega(u_\omega)(u_\omega) = 0$, that is

$$\begin{aligned}
&\int_\Omega |Du_\omega|^p\, dx - \alpha\int_\Omega |u_\omega|^p\, dx + \omega\int_\Omega ((u_\omega-\phi)^-)^{k-1}u_\omega\, dx = pf_\omega(u_\omega)\\
&+\left(\frac{p}{k}-1\right)\omega\int_\Omega ((u_\omega-\phi)^-)^k\, dx + \omega\int_\Omega ((u_\omega-\phi)^-)^{k-1}\phi\, dx = 0.
\end{aligned} \tag{6}$$

Since $\sup_\Omega \phi < 0$, dividing by $\|u_\omega\|$ and passing to the limit, we get

$$\lim_{\omega\to+\infty}\frac{\omega\displaystyle\int_\Omega ((u_\omega-\phi)^-)^k\, dx}{\|u_\omega\|} = 0 \text{ and } \lim_{\omega\to+\infty}\frac{\omega\displaystyle\int_\Omega ((u_\omega-\phi)^-)^{k-1}\phi\, dx}{\|u_\omega\|} = 0. \tag{7}$$

In this way, since $\dfrac{f'(u_\omega)(u_\omega)}{\|u_\omega\|^p} = 0$, we get

$$1-\alpha\int_\Omega |v_\omega|^p\, dx + \frac{\omega\displaystyle\int_\Omega ((u_\omega-\phi)^-)^{k-1}u_\omega\, dx}{\|u_\omega\|^p} \to 1-\alpha\int_\Omega |v|^p\, dx = 0,$$

and one concludes as at the end of Proposition 2.2.

It is now easy to prove **a)** from (6). □

It is also easy to prove the following lemma.

LEMMA 3.1. *If u_ω solves (P_ω) and $u_\omega \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, then $u - \phi \geq 0$ a.e. in Ω.*

It is readily seen that the following Proposition holds.

PROPOSITION 3.1. *Suppose ϕ is continuous, u_ω is a forcing solution of (P_ω) and $u \in W_0^{1,p}(\Omega)$ is such that u_ω converges to u uniformly in Ω. Then $\{x \in \Omega \,|\, u(x) = \phi(x)\} \neq \emptyset$.*

If u_ω is a forcing solution of (P_ω), let us define $\mathcal{A}_\omega$ as (a set equivalent to)

$$\mathcal{A}_\omega = \left\{ x \in \Omega \,\middle|\, u_\omega(x) < \phi(x) \right\},$$

and

$$\mathcal{A} = \left\{ x \in \Omega \,\middle|\, \exists \text{ a ngbrhd } U \text{ of } x, \exists\, \omega_0 \text{ s. t. } \forall\, \omega \geq \omega_0 \; m(U \cap \mathcal{A}_\omega) = 0 \right\}.$$

We observe that $\mathcal{A}$ is an open subset of Ω, and so its complementary set

$$\mathfrak{B} = \left\{ x \in \Omega \,\middle|\, \forall \text{ ngbrhd } U \text{ of } x, \forall\, \omega_0,\ \exists\, \omega \geq \omega_0 \text{ s. t. } m(U \cap \mathcal{A}_\omega) > 0 \right\}$$

is closed. We also remark that $\mathfrak{B}$ is, in some sense, the set of points in which u touches ϕ, or the *contact set*; actually, if u and ϕ are continuous, $\mathfrak{B}$ is the set of points x's of Ω where $u(x) = \phi(x)$.

THEOREM 3.2. *Suppose $(u_\omega)_\omega$ is a sequence of solutions of $(P_\omega)_\omega$ and $u_\omega \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Then there exists a nonnegative Radon measure μ such that*

$$\begin{cases} \Delta_p u + \alpha |u|^{p-2} u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense of distributions.

Such a measure μ is supported in $\mathfrak{B}$, that is $\mu(\mathcal{A}) = 0$.

Moreover, if $p > N$, $\sup \phi < 0$ and $\mathfrak{B} = \left\{ x \in \Omega \,\middle|\, u(x) = \phi(x) \right\} \neq \emptyset$ (for example if u_ω is a forcing solution $\forall\, \omega$) and ϕ is continuous, then $\mu(\mathfrak{B}) > 0$.

Proof. Consider the following linear and continuous functionals on $W_0^{1,p}(\Omega)$:

$$L_\omega(v) = -\int_\Omega |Du_\omega|^{p-2} Du_\omega \cdot Dv\, dx + \alpha \int_\Omega |u_\omega|^{p-2} u_\omega v\, dx.$$

We want to show that $\forall\, v$ in $W_0^{1,p}(\Omega)$, $L_\omega(v)$ converges to $L(v)$ defined as

$$L(v) = -\int_\Omega |Du|^{p-2} Du \cdot Dv\, dx + \alpha \int_\Omega |u|^{p-2} uv\, dx.$$

To this aim we apply the following theorem by Boccardo and Murat (see [3]).

Theorem 3.3 (Boccardo-Murat). *Suppose* $1 < p < \infty$, $v_\omega \rightharpoonup v$ *weakly in* $W^{1,p}(\Omega)$, *strongly in* $L^p_{loc}(\Omega)$ *and a.e. in* Ω, $h_\omega \to h$ *in* $W^{-1,p'}(\Omega)$ *and* g_ω *weakly* $- *$ *converges to* g *in the space* $\mathfrak{M}(\Omega)$ *of Radon measures. If*

$$-\Delta_p v_\omega = h_\omega + g_\omega \qquad \text{in } \mathcal{D}'(\Omega),$$

then

$$Dv_\omega \to Dv \qquad \text{strongly in } L^q(\Omega, \mathbb{R}^N) \text{ for any } q < p.$$

Moreover u *solves*

$$-\Delta_p v = h + g \qquad \text{in } \mathcal{D}'(\Omega).$$

Actually the authors proved this theorem for a more general operator of Leray-Lions type, but for our goals this version is enough.

In the problem under investigation, it results $h_\omega := \alpha|u_\omega|^{p-2}u_\omega$ and $g_\omega := -\omega((u_\omega-\phi)^-)^{k-1}$, which weakly$- *$ converges to a suitable measure μ in $\mathfrak{M}(\Omega)$.

The rest of the proof of Theorem 3.2 can be made following the proof of Theorems 7.1 and 8.7 in [15]. Anyway, for the convenience of the reader, we give a sketch of it.

If $x_0 \in \mathcal{A}$, there exists a neighborhood U of x_0 and ω_0 such that $u_\omega \geq \phi$ in U for any $\omega \geq \omega_0$. Take $\psi \in C^\infty_C(U)$; then

$$\int_\Omega |Du_\omega|^{p-2}Du_\omega \cdot D\psi \, dx - \alpha \int_\Omega |u_\omega|^{p-2}u_\omega \psi \, dx = 0,$$

and passing to the limit, we get that x_0 doesn't belong to the support of μ.

Now assume $p > N$ and $\mathfrak{B} \neq \emptyset$. Assume by contradiction that $\mu \equiv 0$. Then u would solve $\Delta_p u + \alpha|u|^{p-2}u = 0$, and since $\alpha < \lambda_1$, we would have $u \equiv 0$. But $\mathfrak{B} = \{x \in \Omega : u(x) = \phi(x)\} \neq \emptyset$ and a contradiction arises, since $\sup_\Omega \phi < 0$. □

We now want to show that the limit of a weakly convergent sequence has some finer properties.

Proposition 3.2. *If* $p > N$, $\sup_\Omega \phi < 0$ *and* u_ω *is a solution of* (P_ω) *such that* $u_\omega \rightharpoonup u$ *in* $W^{1,p}_0(\Omega)$, *then* $u_\omega \to u$ *strongly in* $W^{1,p}_0(\Omega)$.

Proof. We have $f'_\omega(u_\omega)(u - u_\omega) = 0$, that is

$$\begin{aligned}\int_\Omega |Du_\omega|^{p-2}Du_\omega \cdot D(u-u_\omega)\, dx - \alpha \int_\Omega |u_\omega|^{p-2}u_\omega(u-u_\omega)\, dx \\ +\omega \int_\Omega ((u_\omega - \phi)^-)^{k-1}(u-u_\omega)\, dx = 0.\end{aligned} \tag{8}$$

But $p > N$, so $u_n \to u$ uniformly and then, by **a)** of Theorem 3.1,

$$\omega \int_\Omega ((u_\omega - \phi)^-)^{k-1}(u-u_\omega)\, dx \longrightarrow 0.$$

Then (8) gives

$$\int_\Omega |Du_\omega|^{p-2}Du_\omega \cdot D(u-u_\omega)\, dx \to 0,$$

and then $u_\omega \to u$ strongly in $W^{1,p}_0(\Omega)$ by the (S_+) property. □

Theorem 3.4. *Assume $p > N$, $\sup_\Omega \phi < 0$ and u_ω is a solution of (P_ω) which (weakly) converges to u in $W_0^{1,p}(\Omega)$. Then $\forall\, v \in K_\phi$*

$$\int_\Omega |Du|^{p-2} Du \cdot D(v-u)\, dx - \alpha \int_\Omega |u|^{p-2} u(v-u)\, dx \le 0. \tag{9}$$

Proof. Let v belong to $W_0^{1,p}(\Omega)$ and evaluate $f'_\omega(u_\omega)(v-u_\omega)$. We get

$$\int_\Omega |Du_\omega|^{p-2} Du_\omega \cdot D(v-u_\omega)\, dx - \alpha \int_\Omega |u_\omega|^{p-2} u_\omega (v-u_\omega)\, dx$$

$$+\omega \int_\Omega ((u_\omega - \phi)^-)^{k-1}(v-u_\omega)\, dx = 0.$$

But $((u_\omega - \phi)^-)^{k-1}(v-u_\omega) = ((u_\omega - \phi)^-)^{k-1}(v-\phi) + ((u_\omega - \phi)^-)^k \ge 0$.
The thesis follows from Proposition 3.2. □

Theorem 3.5 (Reversed Variational Inequality). *Suppose that $p > N$, $\sup_\Omega \phi < 0$ and $\alpha < \lambda_1$. Then there exists at least one nontrivial solution of problem (P).*

Proof. Theorem 2.1 + Theorem 3.1 + Lemma 3.1 + Proposition 3.1 + Theorem 3.2 + Theorem 3.4. □

Remark 3. *We remark again that (9) has quite a different nature with respect to classical variational inequalities (see [12]): in the latter case we would have $-\Delta_p +$ l.o.t. ≥ 0 in $\mathcal{D}'(\Omega)$, while in the former case we get $-\Delta_p +$ l.o.t. ≤ 0 in $\mathcal{D}'(\Omega)$.*

We also note that Theorem 3.4 states that the u is an "upper critical point" (see for example [6] or [8]) for the functional f defined on $W_0^{1,p}(\Omega)$ as

$$f(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |Du|^p\, dx - \dfrac{\alpha}{p}\displaystyle\int_\Omega |u|^p\, dx & \text{if } u \in K_\phi, \\ -\infty & \text{otherwise,} \end{cases}$$

and so Theorem 3.5 shows the existence of at least one nontrivial critical point of f which belongs to ∂K_ϕ. Such a result seems quite interesting: in fact f is upper semicontinuous and it is quite difficult to find critical points for it directly, since it is not clear how to define curves of steepest descent for such a functional.

Acknowledgement. The author wishes to thank the referee for the helpful comments, which improved the presentation of the paper.

References

[1] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[2] G. Barles, *Remarks on uniqueness results of the first eigenvalue of the p-Laplacian*, Ann. Fac. Sci. Toulouse Math. (5) **9**, No. 1, 65–75 (1988).

[3] L. Boccardo and F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal. T.M.A., **19**, No. 6, 581–597 (1992).

[4] F. .E. Browder, *Existence theorems for nonlinear partial differential equations*, Global Analysis (Proc. Sympos. Pure Math., Vol. XVI) Amer. Math. Soc., Providence, R. I. 1–60 (1970).

[5] G. Buttazzo and D. Percivale, *On the approximation of the elastic bounce problem on Riemannian manifolds*, J. Differential Equations **47** (1983), no. 2, 227–245.

[6] E. De Giorgi, A. Marino and M. Tosques, *Problems of evolution in metric spaces and maximal decreasing curve*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **68** (1980), no. 3, 180–187.

[7] M. Degiovanni, *Multiplicity of solutions for the bounce problem*, J. Differential Equations **54** (1984), 414–428.

[8] M. Degiovanni, A. Marino and M. Tosques, *Evolution Equations with Lack of Convexity*, Nonlinear Anal. T.M.A. (1985), Vol. **9**, No. 12, 1401–1443.

[9] G. Dinca, P. Jebelean and J. Mawhin, *A result of Ambrosetti-Rabinowitz type for p-Laplacian*, in "Qualitative problems for differential equations and control theory" (ed. C. Corduneanu), World Scientific, Singapore (1995).

[10] G. Dinca, P. Jebelean and J. Mawhin, *Variational and topological methods for Dirichlet problems with p-Laplacian*, Port. Math. (N.S.) **58** (2001), no. 3, 339–378.

[11] X. Fian, Y. Zhao and G. Huang, *Existence of solutions for the p-Laplacian with crossing nonlinearity*, Discrete Contin. Dyn. Syst. 8 (2002), no. 4, 1019–1024.

[12] J. L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math. **20** (1967), 493–519.

[13] A. Marino and D. Mugnai, *Asymptotical multiplicity and some reversed variational inequalities*, Topol. Methods Nonlinear Anal. **20**, No. 1 (2002), pp. 43-62.

[14] D. Mugnai, *Asymptotical critical points and reversed variational inequalities*, Ph. D. thesis.

[15] D. Mugnai, *On a "reversed" variational inequality*, Topol. Methods Nonlinear Anal., **17**, No. 2 (2001), 321–358.

[16] J. Serrin and H. Zou, *Cauchy–Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math. **189** (2002), 79–142.