Hindawi
International Journal of Differential Equations
Volume 2018, Article ID 6259307, 6 pages
https://doi.org/10.1155/2018/6259307

Research Article

Uniqueness Results for Higher Order Elliptic Equations in Weighted Sobolev Spaces

Loredana Caso, Patrizia Di Gironimo, Sara Monsurrò, and Maria Transirico

University of Salerno, Via Giovanni Paolo II, No. 132, 84084 Fisciano, Italy

Correspondence should be addressed to Sara Monsurrò; smonsurro@unisa.it

Received 28 November 2017; Accepted 11 January 2018; Published 1 March 2018

Academic Editor: P. A. Krutitskii

We prove some uniqueness results for the solution of two kinds of Dirichlet boundary value problems for second- and fourth-order linear elliptic differential equations with discontinuous coefficients in polyhedral angles, in weighted Sobolev spaces.

1. Introduction

The Dirichlet problem for polyharmonic equations in bounded domains of $\mathbb{R}^n$ has been studied, among the first, by Sobolev in [1].

The problem was developed in various directions. For instance, Vekua in [2, 3] considers different boundary value problems in not necessarily bounded domains for harmonic, biharmonic, and metaharmonic functions. Successively, analogous problems in more general cases, for what concerns domains and operators, have been studied with different methods by many authors (see, e.g., [4–7]).

In particular, in [7], the author obtains a uniqueness result for the Dirichlet problem for polyharmonic operators of order $2m$ in polyhedral angles of $\mathbb{R}^n$. This result has been later on generalized, in [5], to the case of operators in divergence form of order $2m$ with discontinuous bounded measurable elliptic coefficients.

In [6] the authors study a boundary value problem for biharmonic functions in presence of nonregular points on the boundary of the domain. It is well known that in the neighborhood of these singular points (corners or edges) the solution of the problem presents a singularity that can be characterized by the presence of a suitable weight.

Uniqueness results for different Dirichlet problems in weighted Sobolev spaces for different classes of weights can be found in [8–12]. Studies of Dirichlet problems in the framework of weighted Sobolev spaces and in the case of unbounded domains can be found in [13–22].

In this paper, we extend the results of [5, 7] to the case of weighted Sobolev spaces. More precisely, we prove some uniqueness results for the solution of two kinds of Dirichlet boundary value problems for second- and fourth-order linear elliptic differential equations with discontinuous coefficients in the polyhedral angle $\mathbb{R}^n_l$, $0 \le l \le n-1$, $n \ge 2$, in weighted Sobolev spaces.

The first problem we consider is the following:

$$\sum_{i,j=1}^{n} \left(a_{ij} u_{x_i}\right)_{x_j} = f, \quad f \in L^2_{-s}\left(\mathbb{R}^n_l\right), \qquad u \in \overset{\circ}{W}{}^{1,2}_{s}\left(\mathbb{R}^n_l\right), \tag{1}$$

where, for $k \in \mathbb{N}_0$ and $s \in \mathbb{R}$, $W^{k,2}_s(\Omega)$ denotes a weighted Sobolev space where the weight is a power of the distance from the origin, $\overset{\circ}{W}{}^{k,2}_{s}(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^{k,2}_s(\Omega)$, and $W^{0,2}_s(\Omega) = L^2_s(\Omega)$; see Section 2 for details.

The second problem we study is

$$\sum_{i,j=1}^{n} \left(a_{ij} u_{x_i x_j}\right)_{x_i x_j} = f, \quad f \in L^2_{-s}\left(\mathbb{R}^n_l\right), \qquad u \in \overset{\circ}{W}{}^{2,2}_{s}\left(\mathbb{R}^n_l\right). \tag{2}$$

In both cases the coefficients a_{ij} belong to some weighted Sobolev spaces.

The main tool in our analysis is a generalization of the Hardy's inequality proved by Kondrat'ev and Olènik in [23].

2. Preliminary Results

Let Ω be an open subset of $\mathbb{R}^n$ with $n \geq 2$, whose boundary contains $x = 0$. For $k \in \mathbb{N}_0$ and $s \in \mathbb{R}$, $W_s^{k,2}(\Omega)$ denotes the space of all functions $u : \Omega \to \mathbb{R}$ such that $|x|^s D^\alpha u \in L^2(\Omega)$ for $|\alpha| \leq k$, normed by

$$\|u\|_{W_s^{k,2}(\Omega)} = \sum_{|\alpha|\leq k} \left\||x|^s D^\alpha u\right\|_{L^2(\Omega)},$$
$$\overset{\circ}{W}{}_s^{k,2}(\Omega) \text{ is the closure of } C_0^\infty(\Omega) \text{ in } W_s^{k,2}(\Omega), \tag{3}$$
$$W_s^{0,2}(\Omega) = L_s^2(\Omega).$$

From [24] and Propositions 6.3 and 6.5, we get the following.

Proposition 1. *If G is a bounded open subset in $\mathbb{R}^n$ with $0 \in \partial G$, then*

$$W_s^{k,2}(G) \hookrightarrow W^{k,2}(G) \quad \text{for } s \leq 0. \tag{4}$$

Furthermore, for each $q \in [1, 2[$ there exists $\epsilon_0 = \epsilon_0(q) > 0$ such that

$$W_s^{k,2}(G) \hookrightarrow W^{k,q}(G) \quad \text{for } 0 < s \leq \epsilon_0. \tag{5}$$

In the present paper we use the following notation:

(i) $V \subset \mathbb{R}^n$ is a cone with vertex in the origin of coordinates;

(ii) B_R, $R > 0$, is the open ball of center in the origin and radius R;

(iii) $V_R = V \cap B_R$;

(iv) for every $l \in \{0, \dots, n-1\}$,

$$\mathbb{R}_l^n = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i > 0,\ i = n - l, \dots, n\}, \tag{6}$$

is the "polyhedral angle" with vertex in the origin;

(v) $\mathbb{R}_+^n = \mathbb{R}_0^n$ is the half-space;

(vi) $Q_R = \mathbb{R}_l^n \cap B_R$.

To prove our main results, consisting in two uniqueness theorems, we will use the following inequality. We observe that this is a slightly modified version of a generalized Hardy's inequality that was proved by Kondrat'ev and Olènik in [23], adapted to our needs (see also [5]).

Lemma 2 (generalized Hardy's inequality). *Let $p > 1$ and $r \in \mathbb{R}$ be such that $r + n - p \neq 0$. Assume that for a sufficiently smooth function g the following condition is fulfilled:*

$$\int_{V_{R_2}\setminus V_{R_1}} |x|^r |\nabla g(x)|^p \, dx < +\infty, \tag{7}$$

where $\nabla g = (\partial g/\partial x_1, \dots, \partial g/\partial x_n)$ is the gradient of the function g and $0 < R_1 < R_2$. Then, there exist two constants $M, K > 0$ such that

$$\int_{V_{R_2}\setminus V_{R_1}} |x|^{r-p} |g(x) - M|^p \, dx < K \int_{V_{R_2}\setminus V_{R_1}} |x|^r |\nabla g(x)|^p \, dx, \tag{8}$$

where K does not depend on the function g, R_1, and R_2. If, in addition, $g(0) = 0$ then $M = 0$.

Remark 3. We remark that there are always important restrictions on the dimension n of the space, the order of "singularity" r, and the summability exponent p (see, e.g., [23, 25–29], where different variants of Hardy or Caffarelli-Kohn-Nirenberg type inequalities are proved).

3. Dirichlet Problem for Second-Order Elliptic Equations

We consider the following differential operator in divergence form in the polyhedral angle $\mathbb{R}_l^n$, $0 \leq l \leq n-1$:

$$\sum_{i,j=1}^n \left(a_{ij} u_{x_i}\right)_{x_j}, \tag{9}$$

where the coefficients a_{ij} are measurable functions such that there exist two positive constants λ and μ such that

$$\lambda |x|^{2s} |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\, \xi_i \xi_j \leq \mu |x|^{2s} |\xi|^2 \quad \text{a.e. in } \mathbb{R}_l^n,\ \forall \xi \in \mathbb{R}^n. \tag{10}$$

We study the Dirichlet problem

$$\sum_{i,j=1}^n \left(a_{ij} u_{x_i}\right)_{x_j} = f, \quad \text{a.e. in } \mathbb{R}_l^n, \qquad u \in \overset{\circ}{W}{}_s^{1,2}(\mathbb{R}_l^n), \tag{11}$$

where $f \in L_{-s}^2(\mathbb{R}_l^n)$.

Definition 4. We say that a function u is a generalized solution of problem (11) if it satisfies the integral identity

$$\int_{Q_R} \sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_j} \, dx = -\int_{Q_R} f v \, dx, \tag{12}$$

for any $R > 0$ and any function $v \in \overset{\circ}{W}{}_s^{1,2}(Q_R)$.

Now we prove our first uniqueness result.

Theorem 5. *Let $u \in \overset{\circ}{W}{}_s^{1,2}(\mathbb{R}_l^n)$ be a generalized solution of problem (11), with $f = 0$. Then there exists $\epsilon_0 > 0$ such that if $s \leq \epsilon_0/2$ and $s \neq (2-n)/2$ one has $u \equiv 0$ in $\mathbb{R}_l^n$.*

Proof. Let $\Theta(t)$ be an auxiliary function in $C_0^\infty([0,\infty[)$ defined by

$$\Theta(t) \equiv \begin{cases} 1 & 0 \le t \le 1, \\ \theta(t) & 1 \le t \le 2, \\ 0 & t \ge 2, \end{cases} \tag{13}$$

where $\theta(t)$ is such that $0 \le \theta(t) \le 1$. Let us also assume that there exists a positive constant K_0 such that

$$\left|\Theta'(t)\right|^2 \le K_0 \Theta(t). \tag{14}$$

Set, for any $R > 0$,

$$\Theta_R(x) = \Theta\left(\frac{|x|}{R}\right). \tag{15}$$

Note that the function Θ_R is such that, for any $j = 1, \ldots, n$, one has

$$\left(\Theta_R\right)_{x_j}(x) = \Theta'\left(\frac{|x|}{R}\right)\frac{x_j}{R|x|}. \tag{16}$$

Let $u \in \mathring{W}_s^{1,2}(\mathbb{R}_l^n)$ be a generalized solution of problem (11), with $f = 0$. We put

$$v_R(x) = u(x)\,\Theta_R(x). \tag{17}$$

Clearly, by definition of Θ_R and as a consequence of our boundary condition, one has that $v_R \in \mathring{W}_s^{1,2}(Q_{2R})$.

Thus, using v_R as test function in (12), we get

$$\begin{aligned} &\int_{Q_{2R}} \sum_{i,j=1}^n a_{ij} u_{x_i} u_{x_j} \Theta_R(x)\,dx \\ &\quad + \int_{Q_{2R}\setminus Q_R} \sum_{i,j=1}^n a_{ij} u_{x_i} u \left(\Theta_R\right)_{x_j}(x)\,dx = 0. \end{aligned} \tag{18}$$

From (10), (16), and (18) we deduce that there exists a positive constant $K_1 = K_1(n,\mu)$ such that

$$\begin{aligned} &\int_{Q_{2R}} \sum_{i,j=1}^n a_{ij} u_{x_i} u_{x_j} \Theta_R(x)\,dx \\ &\quad = \left|\int_{Q_{2R}\setminus Q_R} \sum_{i,j=1}^n a_{ij} u_{x_i} u \Theta'\left(\frac{|x|}{R}\right)\frac{x_j}{R|x|}dx\right| \\ &\quad \le K_1 \int_{Q_{2R}\setminus Q_R} |x|^{2s} u_x \Theta'\left(\frac{|x|}{R}\right)\frac{|u|}{|x|}dx \\ &\quad = K_1 \int_{Q_{2R}\setminus Q_R} |x|^{s} u_x \Theta'\left(\frac{|x|}{R}\right)|x|^{s-1}|u|\,dx, \end{aligned} \tag{19}$$

where u_x denotes the modulus of the gradient of u.

By applying Young's inequality one gets that for any $\epsilon > 0$

$$\begin{aligned} &\int_{Q_{2R}} \sum_{i,j=1}^n a_{ij} u_{x_i} u_{x_j} \Theta_R(x)\,dx \\ &\quad \le \frac{\varepsilon}{2} K_1 \int_{Q_{2R}\setminus Q_R} |x|^{2s} u_x^2 \Theta'^2\left(\frac{|x|}{R}\right)dx \\ &\qquad + \frac{K_1}{2\varepsilon}\int_{Q_{2R}\setminus Q_R} |x|^{2s-2} u^2 dx. \end{aligned} \tag{20}$$

Thus, taking into account (14) and applying the generalized Hardy's inequality (8) (with $p = 2$ and $r = 2s$) to the second term in the right-hand side of (20), we deduce that if $s \ne (2-n)/2$,

$$\begin{aligned} &\int_{Q_{2R}} \sum_{i,j=1}^n a_{ij} u_{x_i} u_{x_j} \Theta_R(x)\,dx \\ &\quad \le \frac{\varepsilon}{2} K_1 K_0 \int_{Q_{2R}\setminus Q_R} |x|^{2s} u_x^2 \Theta_R dx \\ &\qquad + \frac{K_1}{2\varepsilon}\int_{Q_{2R}\setminus Q_R} |x|^{2s-2} u^2 dx \\ &\quad \le \frac{\varepsilon}{2} K_1 K_0 \int_{Q_{2R}\setminus Q_R} |x|^{2s} u_x^2 \Theta_R dx \\ &\qquad + \frac{K_1 K}{2\varepsilon}\int_{Q_{2R}\setminus Q_R} |x|^{2s} u_x^2 dx. \end{aligned} \tag{21}$$

From the ellipticity condition in (10) and for $\epsilon = \lambda/K_1K_0$, we have

$$\int_{Q_{2R}} |x|^{2s} u_x^2 \Theta_R dx \le K_2 \int_{Q_{2R}\setminus Q_R} |x|^{2s} u_x^2 dx, \tag{22}$$

where the constant $K_2 = K_2(n,\lambda,\mu,K_0,K)$.

Thus for any $P > 0$ and for any $R > P$ we obtain

$$\int_{Q_P} |x|^{2s} u_x^2 dx \le K_2 \int_{Q_{2R}\setminus Q_R} |x|^{2s} u_x^2 dx. \tag{23}$$

Since u is a generalized solution of problem (11), with $f = 0$, and the constant K_2 does not depend on the radius R and on the solution u, the right-hand side of (23) tends to zero when $R \to +\infty$ and then

$$\int_{Q_P} |x|^{2s} u_x^2 dx = 0 \quad \forall P > 0. \tag{24}$$

This implies that

$$|x|^{2s} u_x^2 = 0 \quad \text{a.e. in } Q_P\ \forall P > 0; \tag{25}$$

therefore

$$u_x = 0 \quad \text{a.e. in } Q_P\ \forall P > 0. \tag{26}$$

By Proposition 1 we deduce that if the solution $u \in W_s^{1,2}(Q_P)$ with $s \le 0$, then $u \in W^{1,2}(Q_P)$, for any $P > 0$. On the other hand, if $s > 0$ for any $q \in [1,2[$ there exists $\epsilon_0 = \epsilon_0(q) > 0$ such that if $0 < s \le \epsilon_0/2$, then $u \in W^{1,q}(Q_P)$ for any $P > 0$. Thus, by (26) the function $u(x)$ is a constant in $\mathbb{R}_l^n$, and since $u \in \mathring{W}_s^{1,2}(\mathbb{R}_l^n)$ one concludes that $u = 0$ in $\mathbb{R}_l^n$. □

4. Dirichlet Problem for 4th-Order Elliptic Equations

Let us now consider the following differential operator of 4th order in the polyhedral angle $\mathbb{R}_l^n$, $0 \le l \le n-1$,

$$\sum_{i,j=1}^{n} \left(a_{ij} u_{x_i x_j}\right)_{x_i x_j}, \tag{27}$$

where a_{ij} are measurable symmetric coefficients and there exist two positive constants λ and μ such that

$$\lambda |x|^{2s} \le a_{ij}(x) \le \mu |x|^{2s} \quad \text{a.e. in } \mathbb{R}_l^n,\ i,j = 1,\dots n. \tag{28}$$

We want to prove a uniqueness result for the solution of the Dirichlet problem

$$\sum_{i,j=1}^{n} \left(a_{ij} u_{x_i x_j}\right)_{x_i x_j} = f, \quad \text{a.e. in } \mathbb{R}_l^n, \qquad u \in \mathring{W}_s^{2,2}\left(\mathbb{R}_l^n\right), \tag{29}$$

where $f \in L^2_{-s}(\mathbb{R}_l^n)$.

Definition 6. We say that a function u is a generalized solution of problem (29) if it satisfies the integral identity

$$\int_{Q_R} \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} v_{x_i x_j} dx = \int_{Q_R} f v \, dx, \tag{30}$$

for any $R > 0$ and any function $v \in \mathring{W}_s^{2,2}(Q_R)$.

The result is the following.

Theorem 7. *Let $u \in \mathring{W}_s^{2,2}(\mathbb{R}_l^n)$ be a generalized solution of problem (29), with $f = 0$. Then there exists $\epsilon_0 > 0$ such that if $s \le \epsilon_0/2$ and $s \ne (2-n)/2, (4-n)/2$ one has $u \equiv 0$ in $\mathbb{R}_l^n$.*

Proof. We shall rely on the methods developed in [5, 7]. We consider the function $\Theta_R(x)$ defined in (13) and satisfying (14). Furthermore, we assume that there exists a positive constant K_1 such that

$$\left|\Theta''(t)\right|^2 \le K_1 \Theta(t). \tag{31}$$

Note that the function Θ_R is such that, for any $i, j = 1, \dots, n$, one has (16) and

$$\left(\Theta_R\right)_{x_i x_j}(x) = \Theta''\left(\frac{|x|}{R}\right) \frac{x_i x_j}{R^2 |x|^2} + \Theta'\left(\frac{|x|}{R}\right) \frac{|x|^2 \delta_{ij} - x_i x_j}{R |x|^3}, \tag{32}$$

where δ_{ij} denotes the Kronecker delta.

Again we put

$$v_R(x) = u(x)\,\Theta_R(x), \tag{33}$$

where $u \in \mathring{W}_s^{2,2}(\mathbb{R}_l^n)$ is a generalized solution of problem (29), with $f = 0$.

Observe that the definition of Θ_R together with the boundary condition satisfied by u gives that $v_R \in \mathring{W}_s^{2,2}(Q_{2R})$. Hence, by the symmetry of a_{ij}, if we take v_R as test function in (30) we get

$$\begin{aligned}&\int_{Q_{2R}} \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j}^2 \Theta_R(x)\, dx \\ &\quad + 2 \int_{Q_{2R}\setminus Q_R} \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} u_{x_i} \left(\Theta_R\right)_{x_j}(x)\, dx \\ &\quad + \int_{Q_{2R}\setminus Q_R} \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} u \left(\Theta_R\right)_{x_i x_j}(x)\, dx = 0.\end{aligned} \tag{34}$$

From (28) and (34) we deduce that

$$\begin{aligned}&\lambda \int_{Q_{2R}} |x|^{2s} \sum_{i,j=1}^{n} u_{x_i x_j}^2 \Theta_R(x)\, dx \\ &\quad \le 2\mu \int_{Q_{2R}\setminus Q_R} |x|^{2s} \sum_{i,j=1}^{n} \left| u_{x_i x_j} u_{x_i} \left(\Theta_R\right)_{x_j}(x) \right| dx \\ &\quad + \mu \int_{Q_{2R}\setminus Q_R} |x|^{2s} \sum_{i,j=1}^{n} \left| u_{x_i x_j} u \left(\Theta_R\right)_{x_i x_j}(x) \right| dx.\end{aligned} \tag{35}$$

By applying (16), (32), and Young's inequality one gets that there exist two positive constants $K_2 = K_2(n, \lambda, \mu, K_0)$ and $K_3 = K_3(n, \lambda, \mu, K_0, K_1)$ such that for any $\varepsilon, \varepsilon_1 > 0$

$$\begin{aligned}&\int_{Q_{2R}} |x|^{2s} \sum_{i,j=1}^{n} u_{x_i x_j}^2 \Theta_R(x)\, dx \\ &\quad \le \varepsilon \int_{Q_{2R}\setminus Q_R} |x|^{2s} \sum_{i,j=1}^{n} u_{x_i x_j}^2 \Theta_R(x)\, dx \\ &\quad + \frac{\varepsilon_1}{2} \int_{Q_{2R}\setminus Q_R} |x|^{2s} \sum_{i,j=1}^{n} u_{x_i x_j}^2 \Theta_R(x)\, dx \\ &\quad + \frac{K_2}{\varepsilon} \int_{Q_{2R}\setminus Q_R} |x|^{2s-2} \sum_{i=1}^{n} u_{x_i}^2 dx \\ &\quad + \frac{K_3}{2\varepsilon_1} \int_{Q_{2R}\setminus Q_R} |x|^{2s-4} u^2 dx.\end{aligned} \tag{36}$$

Thus, applying repeatedly the generalized Hardy's inequality (8) (with $p = 2$ and $r = 2s$ to the third integral on the right-hand side and with $p = 2$ and $r = 2s - 2$ to the last integral on the right-hand side and then again with $p = 2$ and $r = 2s$), we deduce that if $s \ne (2-n)/2, (4-n)/2$,

$$\begin{aligned}&\int_{Q_{2R}} |x|^{2s} \sum_{i,j=1}^{n} u_{x_i x_j}^2 \Theta_R(x)\, dx \\ &\quad \le K_4 \int_{Q_{2R}\setminus Q_R} |x|^{2s} \sum_{i,j=1}^{n} u_{x_i x_j}^2 dx,\end{aligned} \tag{37}$$

where the constant $K_4 = K_4(n, \lambda, \mu, K_0, K_1, K)$.

Thus for any $P > 0$ and for any $R > P$ we obtain

$$\int_{Q_P} |x|^{2s} \sum_{i,j=1}^{n} u_{x_i x_j}^2 dx \le K_4 \int_{Q_{2R} \setminus Q_R} |x|^{2s} \sum_{i,j=1}^{n} u_{x_i x_j}^2 dx. \quad (38)$$

Now, arguing as in the proof of Theorem 5, since u is a generalized solution of problem (29), with $f = 0$, the right-hand side of (38) tends to zero when $R \to +\infty$ and then

$$\int_{Q_P} |x|^{2s} \sum_{i,j=1}^{n} u_{x_i x_j}^2 dx = 0 \quad \forall P > 0. \quad (39)$$

This implies that

$$|x|^{2s} \sum_{i,j=1}^{n} u_{x_i x_j}^2 = 0 \quad \text{a.e. in } Q_P \ \forall P > 0; \quad (40)$$

therefore

$$u_{xx} = 0 \quad \text{a.e. in } Q_P \ \forall P > 0. \quad (41)$$

In view of Proposition 1 we obtain that if the solution $u \in W_s^{2,2}(Q_P)$ with $s \le 0$, then $u \in W^{2,2}(Q_P)$, for any $P > 0$, while if $s > 0$ for any $q \in [1, 2[$ there exists $\epsilon_0 = \epsilon_0(q) > 0$ such that if $0 < s \le \epsilon_0/2$, then $u \in W^{2,q}(Q_P)$ for any $P > 0$. Thus, by (41) the function u_x is constant a.e. in Q_P, and since $u \in \mathring{W}_s^{2,2}(\mathbb{R}_l^n)$ one concludes that $u_x = 0$ a.e. in Q_P, for any $P > 0$. The thesis follows then as the one of Theorem 5. □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. L. Sobolev, "Applications of functional analysis in mathematical physics," in *Translations of Mathematical Monographs*, vol. 7, American Mathematical Society, 1963, Izdat. Leningrad. Gos. Univ., Leningrad, 1950.

[2] I. N. Vekua, "On metaharmonic functions," *Lecture Notes of Tbilisi International Centre of Mathematics and Informatics Tbilisi Univ. Press*, vol. 14, 2013 (Russian).

[3] I. N. Vekua, *New Methods for Solving Elliptic Equations*, OGIZ, Moscow-Leningrad, 1948.

[4] F. Gazzola, H. Grunau, and G. Sweers, *Polyharmonic Boundary Value Problems*, vol. 1991 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2010.

[5] S. Monsurrò, I. Tavkhelidze, and M. Transirico, "Uniqueness results for the Dirichlet problem for higher order elliptic equations in polyhedral angles," *Boundary Value Problems*, vol. 2014, article 232, pp. 1–8, 2014.

[6] O. A. Olèinik, G. A. Iosifyan, and I. Tavkhelidze, "Asymptotic behavior of the solution of a biharmonic equation in the neighborhood of non regular points of the boundary of the domain at infinity," *Trudy Moskovskogo Matematičeskogo Obščestva*, vol. 4, pp. 166–175, 1981 (Russian).

[7] I. Tavkhelidze, "On some properties of solutions of polyharmonic equation in polyhedral angles," *Georgian Mathematical Journal*, vol. 14, no. 3, pp. 565–580, 2007.

[8] L. Caso, "Uniqueness results for elliptic problems with singular data," *Boundary Value Problems*, vol. 2006, Article ID 98923, 2006.

[9] L. Caso and R. D'Ambrosio, "Some uniqueness results for singular elliptic problems," *International Journal of Mathematics*, vol. 26, no. 3, Article ID 1550026, 2015.

[10] L. Caso, R. D'Ambrosio, and M. Transirico, "Solvability of the Dirichlet problem in $W^{2,p}$ for a class of elliptic equations with singular data," *Acta Mathematica Sinica*, vol. 30, no. 5, pp. 737–746, 2014.

[11] L. Caso, R. D'Ambrosio, and M. Transirico, "Well-posedness in weighted Sobolev spaces for elliptic equations of Cordes type," *Collectanea Mathematica*, vol. 67, no. 3, pp. 539–554, 2016.

[12] P. Di Gironimo, "Weighted system of elliptic equations: an existence and uniqueness theorem," *Far East Journal of Mathematical Sciences*, vol. 26, pp. 319–329, 2007.

[13] L. Caso, P. Cavaliere, and M. Transirico, "Existence results for elliptic equations," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 554–563, 2002.

[14] L. Caso and M. Transirico, "A priori estimates for elliptic equations in weighted Sobolev spaces," *Mathematical Inequalities & Applications*, vol. 13, no. 3, pp. 655–666, 2010.

[15] P. Di Gironimo, "ABP inequality and weak Harnack inequality for fully nonlinear elliptic operators with coefficients in weighted spaces," *Far East Journal of Mathematical Sciences*, vol. 64, pp. 1–21, 2012.

[16] P. Di Gironimo, "Harnack inequality for fully nonlinear elliptic equations with coefficients in weighted spaces," *International Journal of Analysis and Applications*, vol. 15, no. 1, pp. 1–19, 2017.

[17] P. Di Gironimo and A. Vitolo, "Elliptic equations with discontinuous coefficients in weighted Sobolev spaces on unbounded domains," *Journal of Mathematical Analysis and Applications*, vol. 253, no. 1, pp. 297–309, 2001.

[18] P. Di Gironimo and A. Vitolo, "Existence and uniqueness results for elliptic equations with coefficients in locally Morrey spaces," *International Journal of Pure and Applied Mathematics*, vol. 4, pp. 181–194, 2003.

[19] S. Monsurrò and M. Transirico, "Dirichlet problem for divergence form elliptic equations with discontinuous coefficients," *Boundary Value Problems*, vol. 2012, article 67, 2012.

[20] S. Monsurrò and M. Transirico, "A priori bounds in L^p for solutions of elliptic equations in divergence form," *Bulletin des Sciences Mathematiques*, vol. 137, no. 7, pp. 851–866, 2013.

[21] S. Monsurrò and M. Transirico, "A $W^{2,p}$-estimate for a class of elliptic operators," *International Journal of Pure and Applied Mathematics*, vol. 83, no. 3, pp. 489–499, 2013.

[22] S. Monsurrò and M. Transirico, "Noncoercive elliptic equations with discontinuous coefficients in unbounded domains," *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, vol. 163, pp. 86–103, 2017.

[23] V. A. Kondrat'ev and O. A. Olèinik, "Boundary value problems for the system of elasticity theory in unbounded domains. Korn's inequalities," *Akademiya Nauk SSSR i Moskovskoe Matematicheskoe Obshchestvo. Uspekhi Matematicheskikh Nauk*, vol. 43, article 5, pp. 55–98, 1988 (Russian).

[24] A. Kufner, *Weighted Sobolev Spaces*, vol. 31 of *Teubner-Texte zur Mathematik*, 1980.

[25] B. Bojarski and P. Hajlasz, "Pointwise inequalities for Sobolev functions and some applications," *Studia Mathematica*, vol. 106, pp. 77–92, 1993.

[26] H. Brezis and M. Marcus, "Hardy's inequalities revisited," *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze*, vol. 25, pp. 217–237, 1997.

[27] F. Catrina and Z.-Q. Wang, "On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions," *Communications on Pure and Applied Mathematics*, vol. 54, no. 2, pp. 229–258, 2001.

[28] J. Dávila and L. Dupaigne, "Hardy-type inequalities," *Journal of the European Mathematical Society*, vol. 6, no. 3, pp. 335–365, 2004.

[29] S. Filippas, A. Tertikas, and J. Tidblom, "On the structure of Hardy-Sobolev-Maz'ya inequalities," *Journal of the European Mathematical Society*, vol. 11, no. 6, pp. 1165–1185, 2009.