Evaluation and interval approximation of fuzzy quantities

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Abstract

In this paper we present a general framework to face the problem of evaluate fuzzy quantities. A fuzzy quantity is a fuzzy set that may be non normal and/or non convex. This new formulation contains as particular cases the ones proposed by Fortemps and Roubens [7], Yager and Filev [12, 13] and follows a completely different approach. It starts with idea of "interval approximation of a fuzzy number" proposed, e.g., in [4, 8, 9].

Keywords: Fuzzy sets, fuzzy quantities, evaluation, interval approximation

1. Introduction

Several authors have faced the problem to evaluate fuzzy numbers in order to define ranking methods that are essential in optimization problems. The problem to associate a real number to a fuzzy set is crucial even for defuzzification problems, but in these cases we are up against fuzzy set that are not fuzzy numbers as they are usually not normal and not convex. We call these fuzzy sets, "fuzzy quantities". This problem has been debated by other authors [1, 5, 6, 7, 12, 13] following different approaches. Fortemps and Roubens in [7], propose a particular figure and a numerical result, without a general formula, but this result can be interpreted as a generalization of "area compensation method". Yager and Filev in [12, 13] propose general procedure for particular fuzzy sets defined by the union of subsets of an interval. Facchinetti and Pacchiarotti in [5] propose a geometrical approach that is coherent with Fortemps and Roubens particular results. These three ideas appear to be completely different. Anzilli and Facchinetti in [1] propose the introduction of "ambiguity" of a fuzzy quantity to have a more detailed evaluation. Another approach based on total variation of bounded variation function is introduced by Anzilli and Facchinetti in [2]. In this paper we try to find a general formulation in which the results obtained in [7, 12, 13] are particular cases and that offers the possibility to define other methods changing the parameters included in its formulation. The main idea we have followed is connected with methods of interval approximation of fuzzy numbers (see, e.g., [4, 8, 9]). We introduce a suitable functional and define the approximation interval for a fuzzy quantity by the nearest interval with respect to the chosen functional. Following this idea we introduce a new evaluation that let us the possibility either to find the weakness of the classical methods or to show its more generality and advantages. In Section 2 we give basic definitions and notations. In Section 3 we introduce our definition of fuzzy quantity. In Section 4 we present a review of the evaluation methods proposed by Fortemps and Roubens [7] and Yager and Filev [12, 13]. In section 5 we introduce our general framework.

2. Preliminaries and notation

Let X denote a universe of discourse. A fuzzy set A in X is defined by a membership function $\mu_A : X \to [0,1]$ which assigns to each element of X a grade of membership to the set A. The height of A is $h_A = \mathit{height}\, A = \sup_{x\in X} \mu_A(x)$. The support and the core of A are defined, respectively, as the crisp sets $\mathit{supp}(A) = \{x \in X; \mu_A(x) > 0\}$ and $\mathit{core}(A) = \{x \in X; \mu_A(x) = 1\}$. A fuzzy set A is normal if its core is nonempty. The union of two fuzzy set A and B is the fuzzy set $A \cup B$ defined by the membership function $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$, $x \in X$. The intersection is the fuzzy set $A \cap B$ defined by $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$. A fuzzy number A is a fuzzy set of the real line $\mathbb{R}$ with a normal, convex and upper-semicontinuous membership function of bounded support. From the definition given above there exist four numbers $a_1, a_2, a_3, a_4 \in \mathbb{R}$, with $a_1 \le a_2 \le a_3 \le a_4$, and two functions $f_A, g_A : \mathbb{R} \to [0,1]$ called the left side and the right side of A, respectively, where f_A is nondecreasing and g_A is nonincreasing, such that

$$\mu_A(x) = \begin{cases} 0 & x < a_1 \\ f_A(x) & a_1 \le x < a_2 \\ 1 & a_2 \le x \le a_3 \\ g_A(x) & a_3 < x \le a_4 \\ 0 & a_4 < x\,. \end{cases}$$

The α-cut of a fuzzy set A, $0 \le \alpha \le 1$, is defined as the crisp set $A_\alpha = \{x \in X; \mu_A(x) \ge \alpha\}$ if $0 < \alpha \le 1$ and as the closure of the support if $\alpha = 0$. Every α-cut of a fuzzy number is a closed interval $A_\alpha = [a_L(\alpha), a_R(\alpha)]$, for $0 \le \alpha \le 1$, where $a_L(\alpha) = \inf A_\alpha$ and $a_R(\alpha) = \sup A_\alpha$.

A fuzzy number A is said to be a trapezoidal fuzzy number if its membership function is given by

$$\mu_A(x) = \begin{cases} 0 & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & a_1 \leq x < a_2 \\ 1 & a_2 \leq x \leq a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & a_3 < x \leq a_4 \\ 0 & a_4 < x \,. \end{cases}$$

If $a_2 = a_3$ the trapezoidal fuzzy number reduces to a triangular fuzzy number.

3. Fuzzy quantities

The paper's aim is to evaluate a general (non-convex) fuzzy quantity with N humps, being N a positive integer. Such a fuzzy quantity can be obtained as the union of N convex fuzzy sets.

Definition 3.1. Let N be a positive integer and let $a_1, a_2, \ldots, a_{4N}$ be real numbers with $a_1 < a_2 \leq a_3 < a_4 \leq a_5 < a_6 \leq a_7 < a_8 \leq a_9 < \cdots < a_{4N-2} \leq a_{4N-1} < a_{4N}$. We call *fuzzy quantity*

$$\begin{aligned} A =& (a_1, a_2, \ldots, a_{4N}; \\ & h_1, h_2, \ldots, h_N, \\ & h_{1,2}, h_{2,3}, \ldots, h_{N-1,N}) \end{aligned} \tag{1}$$

where $0 < h_j \leq 1$ for $j = 1, \ldots, N$ and $0 \leq h_{j,j+1} < \min\{h_j, h_{j+1}\}$ for $j = 1, \ldots, N-1$, the fuzzy set defined by a continuous membership function $\mu : \mathbb{R} \to [0,1]$, with $\mu(x) = 0$ for $x \leq a_1$ or $x \geq a_{4N}$, such that for $j = 1, 2, \ldots, N$

(i) μ is strictly increasing in $[a_{4j-3}, a_{4j-2}]$, with $\mu(a_{4j-3}) = h_{j-1,j}$ and $\mu(a_{4j-2}) = h_j$,
(ii) μ is constant in $[a_{4j-2}, a_{4j-1}]$, with $\mu \equiv h_j$,
(iii) μ is strictly decreasing in $[a_{4j-1}, a_{4j}]$, with $\mu(a_{4j-1}) = h_j$ and $\mu(a_{4j}) = h_{j,j+1}$,

and for $j = 1, 2, \ldots, N-1$

(iv) μ is constant in $[a_{4j}, a_{4j+1}]$, with $\mu \equiv h_{j,j+1}$,

where $h_{0,1} = h_{N,N+1} = 0$. Thus the height of A is

$$h_A = \max_{j=1,\ldots,N} h_j \,.$$

Remark 3.2. When $N = 1$ the fuzzy quantity $A = (a_1, a_2, a_3, a_4; h_1)$ defined in (1) is fuzzy convex, that is every α-cut A_α is a closed interval, with a continuous membership function of bounded support and with height $h_A = h_1$. Note that if $h_1 = 1$ then A is a fuzzy number.
When $N \geq 2$ the fuzzy quantity A defined in (1) is a non-convex fuzzy set with N humps and height $h_A = \max_{j=1,\ldots,N} h_j$. Such a fuzzy quantity can be obtained as the union of N convex fuzzy quantities.

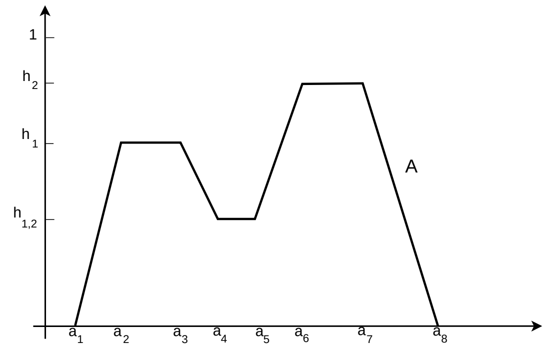

Figure 1: Fuzzy quantity with $N = 2$.

Definition 3.3. For $j = 1, \ldots, N$ we let

$$x_j(\alpha) = \mu^{-1}_{4j-3,4j-2}(\alpha) \qquad h_{j-1,j} \leq \alpha \leq h_j \,,$$

where $\mu_{4j-3,4j-2} = \mu_{|[a_{4j-3}, a_{4j-2}]}$ is the restriction of μ to the interval $[a_{4j-3}, a_{4j-2}]$, and

$$y_j(\alpha) = \mu^{-1}_{4j-1,4j}(\alpha) \qquad h_{j,j+1} \leq \alpha \leq h_j$$

where $\mu_{4j-1,4j} = \mu_{|[a_{4j-1}, a_{4j}]}$ is the restriction of μ to the interval $[a_{4j-1}, a_{4j}]$.

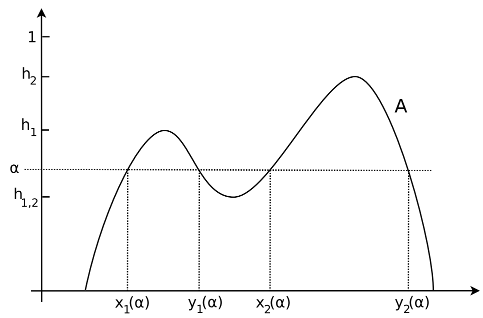

Figure 2: Example of α-cut.

If $N = 1$, that is if A is a convex fuzzy quantity with α-cuts $A_\alpha = [a_L(\alpha), a_R(\alpha)]$, we have $x_1(\alpha) = a_L(\alpha)$ and $y_1(\alpha) = a_R(\alpha)$ for $0 \leq \alpha \leq h_A$.

Proposition 3.4. *Let A be the fuzzy quantity defined in* (1) *with height h_A. Then each α-cut A_α, with $0 < \alpha \leq h_A$, is the union of a finite number of disjoint intervals. That is there exist an integer $n_\alpha = n^A_\alpha$, with $1 \leq n_\alpha \leq N$, and $A^\alpha_1, \ldots, A^\alpha_{n_\alpha}$ disjoint intervals such that*

$$A_\alpha = \bigcup_{i=1}^{n_\alpha} A^\alpha_i = \bigcup_{i=1}^{n_\alpha} [a^L_i(\alpha), a^R_i(\alpha)] \,. \tag{2}$$

Thus n_α is the number of intervals producing the α-cut A_α.

For example, in the case $N = 2$ with $h_1 < h_2$ (see Fig. 2)

- for $0 < \alpha \leq h_{1,2}$ we have $n_\alpha = 1$ and

$$A_\alpha = A^\alpha_1 = [a^L_1(\alpha), a^R_1(\alpha)] = [x_1(\alpha), y_2(\alpha)] \,,$$

- for $h_{1,2} < \alpha \leq h_1$ we have $n_\alpha = 2$ and

$$A_\alpha = A_1^\alpha \cup A_2^\alpha = [a_1^L(\alpha), a_1^R(\alpha)] \cup [a_2^L(\alpha), a_2^R(\alpha)] \\ = [x_1(\alpha), y_1(\alpha)] \cup [x_2(\alpha), y_2(\alpha)]\,,$$

- for $h_1 < \alpha \leq h_2$ we have $n_\alpha = 1$ and

$$A_\alpha = A_1^\alpha = [a_1^L(\alpha), a_1^R(\alpha)] = [x_2(\alpha), y_2(\alpha)]\,.$$

Proof. For each $0 < \alpha \leq h_A$ let

$$L^\alpha = \mu^{-1}(\{\alpha\}) \cap \bigcup_{j=1}^{N}]a_{4j-3}, a_{4j-2}] \\ R^\alpha = \mu^{-1}(\{\alpha\}) \cap \bigcup_{j=1}^{N} [a_{4j-1}, a_{4j}[\,. \tag{3}$$

Since μ in continuous, strictly increasing in $[a_{4j-3}, a_{4j-2}]$ and strictly decreasing in $[a_{4j-1}, a_{4j}]$ we have $card(L^\alpha) = card(R^\alpha)$ and we let

$$n_\alpha = card(L^\alpha) = card(R^\alpha)\,. \tag{4}$$

By defining for $i = 1\ldots, n_\alpha$

$$a_1^L(\alpha) = \min L^\alpha \\ a_i^L(\alpha) = \min\left(L^\alpha - \{a_1^L(\alpha), \ldots, a_{i-1}^L(\alpha)\}\right)$$

and

$$a_1^R(\alpha) = \min R^\alpha \\ a_i^R(\alpha) = \min\left(R^\alpha - \{a_1^R(\alpha), \ldots, a_{i-1}^R(\alpha)\}\right)$$

we have

$$L^\alpha = \{a_1^L(\alpha), \ldots, a_{n_\alpha}^L(\alpha)\} \\ R^\alpha = \{a_1^R(\alpha), \ldots, a_{n_\alpha}^R(\alpha)\}\,. \tag{5}$$

Taking into account the properties of the membership function μ it follows that the following inequalities must be satisfied $a_1^L(\alpha) \leq a_1^R(\alpha) < a_2^L(\alpha) \leq a_2^R(\alpha) < \cdots < a_{n_\alpha}^L(\alpha) \leq a_{n_\alpha}^R(\alpha)$ and, moreover,

$$A_\alpha = \bigcup_{i=1}^{n_\alpha} A_i^\alpha$$

where $A_i^\alpha = [a_i^L(\alpha), a_i^R(\alpha)]$. □

In the following we denote the middle point of the interval $A_i^\alpha = [a_i^L(\alpha), a_i^R(\alpha)]$ by

$$mid(A_i^\alpha) = \frac{1}{2}\left(a_i^L(\alpha) + a_i^R(\alpha)\right)$$

and the spread of A_i^α by

$$spr(A_i^\alpha) = \frac{1}{2}\left(a_i^R(\alpha) - a_i^L(\alpha)\right)\,.$$

4. Evaluation of fuzzy quantities

In this section we analyse the evaluation defined by Fortemps and Roubens [7] and Yager and Filev [12, 13] and propose a unique method that realizes to unify the two approaches even if they seem so different. In particular the Fortemps and Roubens evaluation is a weighted average of the arithmetic means of the midpoints of each interval that produces each α-cut where the weights are connected with the number of those intervals. The Yager and Filev evaluation is different and is the mean value of the weighted average of the midpoints of the intervals producing every α-cut with weights connected with their spreads. An example is furnished to show how the method works.

4.1. The Fortemps and Roubens evaluation

Definition 4.1. Let us consider a fuzzy quantity A defined in (1). We denote

$$S_1 = \sum_{j=1}^{N} \int_{h_{j-1,j}}^{h_j} x_j(\alpha)\, d\alpha \\ S_2 = \sum_{j=1}^{N} \int_{h_{j,j+1}}^{h_j} y_j(\alpha)\, d\alpha\,.$$

We define the value of A as

$$V_1(A) = \frac{S_1 + S_2}{2\left(\sum_{j=1}^{N} h_j - \sum_{j=1}^{N-1} h_{j,j+1}\right)}\,. \tag{6}$$

Applying (6) to the particular fuzzy quantity considered in [7] we obtain the same result.

Proposition 4.2. *Let A be the fuzzy quantity defined in* (1) *with α-cuts given by* (2). *Then*

$$V_1(A) = \frac{1}{\int_0^{h_A} n_\alpha\, d\alpha} \int_0^{h_A} \sum_{i=1}^{n_\alpha} mid(A_i^\alpha)\, d\alpha \tag{7}$$

where $h_A = \max_{j=1,\ldots,N} h_j$.

Proof. Let L^α and R^α be as defined in (3). Recalling that $\mu(a_{4j-3}) = h_{j-1,j}$, $\mu(a_{4j-2}) = h_j$, $\mu(a_{4j-1}) = h_j$ and $\mu(a_{4j}) = h_{j,j+1}$, we have

$$L^\alpha = \bigcup_{j=1}^{N} \{x_j(\alpha);\ \alpha \in]h_{j-1,j}, h_j]\} \\ R^\alpha = \bigcup_{j=1}^{N} \{y_j(\alpha);\ \alpha \in]h_{j,j+1}, h_j]\}\,. \tag{8}$$

Then, since μ is strictly increasing in $[a_{4j-3}, a_{4j-2}]$, from (4) it follows that

$$n_\alpha = card(L^\alpha) = \sum_{j=1}^{N} \chi_{]h_{j-1,j}, h_j]}(\alpha)\,,$$

where $\chi_{]h_{j-1,j},h_j]}$ is the characteristic function of the interval $]h_{j-1,j}, h_j]$. Then, taking into account that $h_{0,1} = 0$, we obtain

$$\int_0^{h_A} n_\alpha \, d\alpha = \sum_{j=1}^{N} (h_j - h_{j-1,j})$$

and thus

$$\int_0^{h_A} n_\alpha \, d\alpha = \sum_{j=1}^{N} h_j - \sum_{j=1}^{N-1} h_{j,j+1} \,. \tag{9}$$

Furthermore, from (5) and (8) we get

$$\sum_{i=1}^{n_\alpha} a_i^L(\alpha) = \sum_{j=1}^{N} x_j(\alpha)\, \chi_{]h_{j-1,j},h_j]}(\alpha)$$

and thus

$$\begin{aligned} &\int_0^{h_A} \sum_{i=1}^{n_\alpha} a_i^L(\alpha)\, d\alpha \\ &= \int_0^{h_A} \sum_{j=1}^{N} x_j(\alpha)\, \chi_{]h_{j-1,j},h_j]}(\alpha)\, d\alpha \\ &= \sum_{j=1}^{N} \int_{h_{j-1,j}}^{h_j} x_j(\alpha)\, d\alpha = S_1 \,. \end{aligned}$$

In a similar way, from (5) and (8) we obtain

$$\sum_{i=1}^{n_\alpha} a_i^R(\alpha) = \sum_{j=1}^{N} y_j(\alpha)\, \chi_{]h_{j,j+1},h_j]}(\alpha)$$

and thus

$$\int_0^{h_A} \sum_{i=1}^{n_\alpha} a_i^R(\alpha)\, d\alpha = \sum_{j=1}^{N} \int_{h_{j,j+1}}^{h_j} y_j(\alpha)\, d\alpha = S_2 \,.$$

Then from (6)

$$\begin{aligned} V_1(A) &= \frac{S_1 + S_2}{2\left(\sum_{j=1}^{N} h_j - \sum_{j=1}^{N-1} h_{j,j+1}\right)} \\ &= \frac{\int_0^{h_A} \sum_{i=1}^{n_\alpha} a_i^L(\alpha)\, d\alpha + \int_0^{h_A} \sum_{i=1}^{n_\alpha} a_i^R(\alpha)\, d\alpha}{2 \int_0^{h_A} n_\alpha \, d\alpha} \\ &= \frac{1}{\int_0^{h_A} n_\alpha \, d\alpha} \int_0^{h_A} \sum_{i=1}^{n_\alpha} \left(\frac{a_i^L(\alpha) + a_i^R(\alpha)}{2} \right) d\alpha \,. \end{aligned}$$

□

Remark 4.3. From (7) we get

$$V_1(A) = \frac{1}{\int_0^{h_A} n_\alpha \, d\alpha} \int_0^{h_A} V_1(A_\alpha)\, n_\alpha \, d\alpha$$

where

$$V_1(A_\alpha) = \frac{1}{n_\alpha} \sum_{i=1}^{n_\alpha} mid(A_i^\alpha) \,.$$

Thus the evaluation $V_1(A)$ is a weighted average of α-cuts values $V_1(A_\alpha)$, where the weights are connected with the number of intervals producing every α-cut. Furthermore the value $V_1(A_\alpha)$ of each α-cut A_α is the arithmetic mean of the midpoints of its intervals.

4.2. The Yager and Filev evaluation

Yager and Filev [12, 13] define the value of a fuzzy quantity A

$$V_2(A) = \frac{1}{h_A} \int_0^{h_A} \frac{\sum_{i=1}^{n_\alpha} mid(A_i^\alpha) spr(A_i^\alpha)}{\sum_{j=1}^{n_\alpha} spr(A_\alpha^j)} \, d\alpha \,. \tag{10}$$

This evaluation can be also expressed as

$$V_2(A) = \frac{1}{h_A} \int_0^{h_A} V_2(A_\alpha)\, d\alpha$$

where

$$V_2(A_\alpha) = \frac{\sum_{i=1}^{n_\alpha} mid(A_i^\alpha) spr(A_i^\alpha)}{\sum_{j=1}^{n_\alpha} spr(A_\alpha^j)} \,.$$

Thus the evaluation $V_2(A)$ is the mean value of $V_2(A_\alpha)$ that are a weighted average of the midpoints of intervals producing every α-cut, where the weights are connected with the interval spreads.

4.3. An application

We now apply the above methods to evaluate a fuzzy quantity as in Fig. 3 that is the typical output of a fuzzy control system.

Example 4.4. Let T and S be two symmetric triangular fuzzy numbers with centers t, s and spreads d_1, d_2, respectively. Thus the α-cus of T and S are, respectively,

$$\begin{aligned} T_\alpha &= [t_L(\alpha), t_R(\alpha)] \\ &= [t - (1-\alpha)d_1, t + (1-\alpha)d_1] \qquad 0 \le \alpha \le 1 \\ S_\alpha &= [s_L(\alpha), s_R(\alpha)] \\ &= [s - (1-\alpha)d_2, s + (1-\alpha)d_2] \qquad 0 \le \alpha \le 1 \,. \end{aligned}$$

Let us consider the fuzzy quantity A shown in Fig. 3 defined by the membership function

$$\mu_A(x) = \max\{\min\{\mu_T(x), h_1\}, \min\{\mu_S(x), h_2\}\}$$

where μ_T and μ_S are the membership functions of T and S, respectively, and $h_1 \le h_2$. Then $h_A = h_2$ and the α-cuts of A are given by, for $0 \le \alpha \le h_2$,

$$A_\alpha = \begin{cases} [t_L(\alpha), s_R(\alpha)] & 0 \le \alpha \le h_{1,2} \\ [t_L(\alpha), t_R(\alpha)] \cup [s_L(\alpha), s_R(\alpha)] & h_{1,2} \le \alpha \le h_1 \\ [s_L(\alpha), s_R(\alpha)] & h_1 \le \alpha \le h_2 \,. \end{cases}$$

Now we evaluate $V_1(A)$ and $V_2(A)$. Observing that from (4)

$$\int_0^{h_A} n_\alpha \, d\alpha = h_1 + h_2 - h_{1,2}$$

we obtain from (7)

$$V_1(A) = \frac{1}{\int_0^{h_A} n_\alpha \, d\alpha} \int_0^{h_A} \sum_{i=1}^{n_\alpha} mid(A_i^\alpha)\, d\alpha =$$
$$= \frac{1}{h_1 + h_2 - h_{1,2}} \left\{ \int_0^{h_{1,2}} \frac{t + s + (d_2 - d_1)(1-\alpha)}{2} \, d\alpha + \right.$$
$$\left. + \int_{h_{1,2}}^{h_1} (t+s)\, d\alpha + \int_{h_1}^{h_2} s \, d\alpha \right\}$$
$$= \frac{1}{h_1 + h_2 - h_{1,2}} \left\{ th_1 + sh_2 - \frac{t+s}{2} h_{1,2} + \right.$$
$$\left. + \frac{d_2 - d_1}{4}(2h_{1,2} - h_{1,2}^2) \right\}$$

and thus

$$V_1(A) = t\sigma_1 + s\sigma_2 - \left(\frac{t + s + d_1 - d_2}{2} + \frac{d_2 - d_1}{4} h_{1,2} \right) \sigma_{1,2} \tag{11}$$

where $\sigma_1 = \frac{h_1}{h_1 + h_2 - h_{1,2}}$, $\sigma_2 = \frac{h_2}{h_1 + h_2 - h_{1,2}}$ and $\sigma_{1,2} = \frac{h_{1,2}}{h_1 + h_2 - h_{1,2}}$. These coefficients are the same found in [5]. Note that $\sigma_1 + \sigma_2 - \sigma_{1,2} = 1$.
Furthermore, since $h_A = h_2$, we get from (10)

$$V_2(A) = \frac{1}{h_A} \int_0^{h_A} \frac{\sum_{i=1}^{n_\alpha} mid(A_i^\alpha) spr(A_i^\alpha)}{\sum_{j=1}^{n_\alpha} spr(A_\alpha^j)} \, d\alpha =$$
$$= \frac{1}{h_2} \int_0^{h_{1,2}} \frac{t + s + (d_2 - d_1)(1-\alpha)}{2} \, d\alpha +$$
$$+ \frac{1}{h_2} \int_{h_{1,2}}^{h_1} \frac{td_1 + sd_2}{d_1 + d_2} \, d\alpha + \frac{1}{h_2} \int_{h_1}^{h_2} s \, d\alpha$$

and thus

$$V_2(A) = t\pi + s(1-\pi) + - \left(\frac{(s - t - d_1 - d_2)(d_2 - d_1)}{2(d_1 + d_2)} + \frac{d_2 - d_1}{4} h_{1,2} \right) \frac{h_{1,2}}{h_2} \tag{12}$$

where $\pi = \frac{d_1 h_1}{(d_1 + d_2) h_2}$.

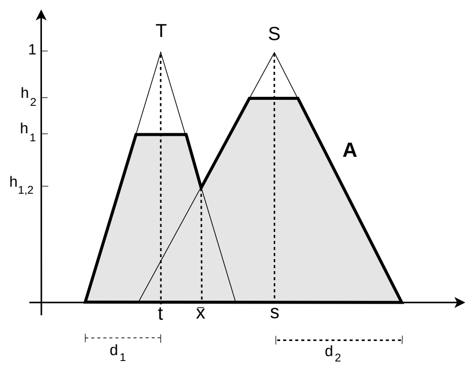

Figure 3: The fuzzy quantity A.

Remark 4.5. From (11) we can see that if we move only S to the right (only T to the left) $h_{1,2}$ goes to zero. This fact produces that $V_1(A)$ goes to $\sigma_1 t + \sigma_2 s$. This evaluation has totally forgotten d_1 and d_2. This weakness does not happen using the evaluation $V_2(A)$.

Remark 4.6. In the case when $d_1 = d_2$ the evaluation $V_2(A)$ expressed by (12) depends only on the ratio h_1/h_2, since $V_2(A) = s - (s-t)(h_1/h_2)/2$. This means that fuzzy quantities having different flat heights h_1, h_2 but the same ratio h_1/h_2 will have the same evaluation. This weakness does not occur in $V_1(A)$, since $V_1(A) = tw + s(1-w)$ where $w = (2\sigma_1 - \sigma_{1,2})/2 = (2h_1 - h_{1,2})/(2(h_1 + h_2 - h_{1,2}))$.

Remark 4.7. The relationship between these two valuations is shown by the following equation

$$V_1(A) = V_2(A)\, \sigma_2 + \bar{x}\,(1 - \sigma_2)$$

where $\bar{x} = \frac{s\, d_1 + t\, d_2}{d_1 + d_2}$ is such that $h_{1,2} = \mu_A(\bar{x})$.

5. A more general evaluation framework

In this section we propose a general formulation for the evaluation of a fuzzy quantity and show that the evaluation methods presented above are particular cases of our approach. To this end we define the approximation interval of a fuzzy quantity as the interval which is the nearest to the fuzzy quantity with respect to a suitable functional. We show that the evaluation we have proposed is the middle point of the approximation interval. Finally, we use the previous results to introduce a new evaluation of a fuzzy quantity and give an example to show how our evaluation works.

Definition 5.1. Let A be a fuzzy quantity with height h_A and α-cuts given by (2). We define the value of A as

$$V(A) = \int_0^{h_A} \sum_{i=1}^{n_\alpha} mid(A_i^\alpha)\, p_A^i(\alpha)\, \varphi_A(\alpha)\, d\alpha \tag{13}$$

where for each α the weights $p_A(\alpha) = (p_A^i(\alpha))_{i=1,\dots,n_\alpha}$ satisfy

$$\sum_{i=1}^{n_\alpha} p_A^i(\alpha) = 1$$

and the weight function $\varphi_A : [0,1] \to [0, +\infty[$ satisfies

$$\int_0^{h_A} \varphi_A(\alpha)\, d\alpha = 1 \,.$$

Thus our general method performs a horizontal aggregation, level by level, with weights p and a vertical aggregation using a weight function φ_A.

Remark 5.2. Note that if we choose

$$\begin{cases} p_A^i(\alpha) = \frac{1}{n_\alpha}\,, \\ \varphi_A(\alpha) = \frac{n_\alpha}{\int_0^{h_A} n_s \, ds} \end{cases}$$

we obtain the evaluation $V_1(A)$ (see (7)) and if we choose

$$\begin{cases} p_A^i(\alpha) = \frac{spr(A_i^\alpha)}{\sum_{j=1}^{n_\alpha} spr(A_j^\alpha)}, \\ \varphi_A(\alpha) = \frac{1}{h_A} \end{cases}$$

we obtain the evaluation $V_2(A)$ (see (10)).

Definition 5.3. We say that $\widehat{C} = [\widehat{c}_L, \widehat{c}_R]$ is an approximation interval of the fuzzy quantity A with respect to $p_A = (p_A^i)_{i=1,\dots,n_\alpha}$ and φ_A if it minimizes

$$\begin{aligned} &I(C;A) = \\ &= \int_0^{h_A} \sum_{i=1}^{n_\alpha} (mid(C) - mid(A_i^\alpha))^2\, p_A^i(\alpha)\, \varphi_A(\alpha) d\alpha \\ &+ \theta \int_0^{h_A} \sum_{i=1}^{n_\alpha} (spr(C) - spr(A_i^\alpha))^2\, p_A^i(\alpha)\, \varphi_A(\alpha) d\alpha \end{aligned}$$

among all the intervals $C = [c_L, c_R]$, where $\theta \in]0, 1]$ is a parameter indicating the relative importance of the spreads against the mids ([9, 11]).

Theorem 5.4. *The approximation interval* $\widehat{C} = [\widehat{c}_L, \widehat{c}_R]$ *of the fuzzy quantity* A *with respect to* p_A *and* φ_A *is given by*

$$\widehat{c}_L = \int_0^{h_A} \sum_{i=1}^{n_\alpha} a_i^L(\alpha)\, p_A^i(\alpha)\, \varphi_A(\alpha)\, d\alpha$$

$$\widehat{c}_R = \int_0^{h_A} \sum_{i=1}^{n_\alpha} a_i^R(\alpha)\, p_A^i(\alpha)\, \varphi_A(\alpha)\, d\alpha$$

Observe that $\widehat{C}$ *doesn't depend on* θ.
Moreover, the evaluation $V(A)$ *defined in* (13) *is the middle point of the approximation interval* $\widehat{C}$.

Proof. We have to minimize the function

$$\begin{aligned} &g(c_L, c_R) = I(C;A) = \\ &= \frac{1}{4} \int_0^{h_A} \sum_{i=1}^{n_\alpha} \left(c_L + c_R - a_i^L(\alpha) - a_i^R(\alpha)\right)^2\, p_A^i(\alpha)\, \varphi_A(\alpha) d\alpha \\ &+ \frac{\theta}{4} \int_0^{h_A} \sum_{i=1}^{n_\alpha} \left(c_R - c_L - a_i^R(\alpha) + a_i^L(\alpha)\right)^2\, p_A^i(\alpha)\, \varphi_A(\alpha) d\alpha \end{aligned}$$

with respect to c_L and c_R. By solving $\frac{\partial g}{\partial c_L}(c_L, c_R) = \frac{\partial g}{\partial c_R}(c_L, c_R) = 0$ we get

$$c_L + c_R = \int_0^{h_A} \sum_{i=1}^{n_\alpha} (a_i^L(\alpha) + a_i^R(\alpha))\, p_A^i(\alpha)\, \varphi_A(\alpha)\, d\alpha$$

$$c_R - c_L = \int_0^{h_A} \sum_{i=1}^{n_\alpha} (a_i^R(\alpha) - a_i^L(\alpha))\, p_A^i(\alpha)\, \varphi_A(\alpha)\, d\alpha$$

and thus the solution is

$$\widehat{c}_L = \int_0^{h_A} \sum_{i=1}^{n_\alpha} a_i^L(\alpha)\, p_A^i(\alpha)\, \varphi_A(\alpha)\, d\alpha$$

$$\widehat{c}_R = \int_0^{h_A} \sum_{i=1}^{n_\alpha} a_i^R(\alpha)\, p_A^i(\alpha)\, \varphi_A(\alpha)\, d\alpha\,.$$

Since $\frac{\partial^2 g}{\partial c_L^2}(c_L, c_R) = \frac{\partial^2 g}{\partial c_R^2}(c_L, c_R) = \frac{1+\theta}{2}$ and $\frac{\partial^2 g}{\partial c_R \partial c_L}(c_L, c_R) = \frac{\partial^2 g}{\partial c_L \partial c_R}(c_L, c_R) = \frac{1-\theta}{2}$ we obtain

$$\det \begin{bmatrix} \frac{\partial^2 g}{\partial c_L^2}(c_L, c_R) & \frac{\partial^2 g}{\partial c_R \partial c_L}(c_L, c_R) \\ \frac{\partial^2 g}{\partial c_L \partial c_R}(c_L, c_R) & \frac{\partial^2 g}{\partial c_R^2}(c_L, c_R) \end{bmatrix} = \theta > 0$$

and $\frac{\partial^2 g}{\partial c_L^2}(c_L, c_R) = \frac{1+\theta}{2} > 0$. Then the solution $(\widehat{c}_L, \widehat{c}_R)$ minimizes $g(c_L, c_R)$. $\square$

We now use the above results to introduce an evaluation method that takes into account both the number of intervals of A_α and the spread of each interval.

Definition 5.5. Let A be a fuzzy quantity with height h_A and α-cuts given by (2). We call $V_3(A)$ the evaluation of A obtained by (13) with

$$\begin{cases} p_A^i(\alpha) = \frac{spr(A_i^\alpha)}{\sum_{j=1}^{n_\alpha} spr(A_j^\alpha)}, \\ \varphi_A(\alpha) = \frac{n_\alpha}{\int_0^{h_A} n_\alpha\, d\alpha}. \end{cases}$$

Then

$$V_3(A) = \frac{\displaystyle\int_0^{h_A} \frac{\sum_{i=1}^{n_\alpha} mid(A_i^\alpha) spr(A_i^\alpha)}{\sum_{j=1}^{n_\alpha} spr(A_j^\alpha)} n_\alpha\, d\alpha}{\int_0^{h_A} n_\alpha\, d\alpha}. \quad (14)$$

Remark 5.6. If $A = (a_1, a_2, a_3, a_4; h_1)$ is a convex fuzzy quantity with height $h_A = h_1$ and α-cuts $A_\alpha = [a_L(\alpha), a_R(\alpha)]$, $0 \le \alpha \le h_A$, we obtain

$$V_1(A) = V_2(A) = V_3(A) = \frac{1}{h_A} \int_0^{h_A} \frac{a_L(\alpha) + a_R(\alpha)}{2}\, d\alpha\,.$$

Example 5.7. Let A be the fuzzy quantity of Example 4.4 shown in Fig. 3. Since from (4)

$$\int_0^{h_A} n_\alpha\, d\alpha = h_1 + h_2 - h_{1,2}$$

we get from (14)

$$\begin{aligned} &V_3(A) = \\ &= \frac{1}{\int_0^{h_A} n_\alpha\, d\alpha} \int_0^{h_A} \frac{\sum_{i=1}^{n_\alpha} mid(A_i^\alpha) spr(A_i^\alpha)}{\sum_{j=1}^{n_\alpha} spr(A_j^\alpha)} n_\alpha\, d\alpha = \\ &= \frac{1}{h_1 + h_2 - h_{1,2}} \left\{ \int_0^{h_{1,2}} \frac{t + s + (d_2 - d_1)(1 - \alpha)}{2}\, d\alpha + \right. \\ &\left. + \int_{h_{1,2}}^{h_1} \frac{t d_1 + s d_2}{d_1 + d_2}\, 2\, d\alpha + \int_{h_1}^{h_2} s\, d\alpha \right\} \end{aligned}$$

and thus

$$\begin{aligned} V_3(A) &= \left(2 \frac{t d_1 + s d_2}{d_1 + d_2} - s\right) \sigma_1 + s \sigma_2 \\ &- \left(\frac{3}{2} \frac{t d_1 + s d_2}{d_1 + d_2} - \frac{1}{2} \frac{t d_2 + s d_1}{d_1 + d_2} - \frac{d_2 - d_1}{2} + \right. \quad (15) \\ &\left. + \frac{d_2 - d_1}{4} h_{1,2}\right) \sigma_{1,2} \end{aligned}$$

where $\sigma_1 = \frac{h_1}{h_1 + h_2 - h_{1,2}}$, $\sigma_2 = \frac{h_2}{h_1 + h_2 - h_{1,2}}$ and $\sigma_{1,2} = \frac{h_{1,2}}{h_1 + h_2 - h_{1,2}}$.

Remark 5.8. From (15) and (11) we obtain

$$V_3(A) = V_1(A) + \frac{(s-t)(d_2-d_1)}{d_1+d_2}(1-\sigma_2)\,. \quad (16)$$

Equation (16) shows that the weakness of $V_1(A)$ described in Remark 4.5 does not happen using the evaluation $V_3(A)$.
Moreover, from (15) and (12) we get

$$V_3(A) = V_2(A)\sigma_2 + \frac{td_1+sd_2}{d_1+d_2}(1-\sigma_2)\,. \quad (17)$$

Equation (17) shows that the weakness of $V_2(A)$ described in Remark 4.6 does not occur for the evaluation $V_3(A)$.

6. Conclusion

Following the words of Grzegorzewski in [9] that, in his introduction, underlines the importance to develop interval approximation for general fuzzy sets, in this paper we introduce, for the first time, the interval approximation of a fuzzy quantity. Working with fuzzy numbers, this type of operation means to find the interval nearest to the original fuzzy number respect some type of metric. In fuzzy quantities' case, that are the typical outputs of a fuzzy control system, the introduction of a metrics is not so trivial, so in this first paper we have spoken of the interval nearest to the original fuzzy quantity respect a general functional. The functional we use is suggested by the distance proposed by Bertoluzza et al. [3] and generalized by Trutshnig et al. [11]. This distance depends by a parameter that modifies the weight of the "spread" part, but the nearest interval founded doesn't depends by it. Even in the formulation for fuzzy quantities this happens so our following study will be in the direction to understand why this fact happens and how to modify this situation. Another direction is to modify the functional we start and to try to find a sort of functional that should be a distance. As we have use this approach so as to evaluate a fuzzy quantity, having in mind a defuzzification problem, we have compared our results with other previous methods introduced by other authors finding a unifying view, we have left behind the method proposed by Facchinetti and Pacchiarotti [5]. This happens as their formulation is a geometrical view of Fortemps and Roubens idea in a more general case. Their proposal differs for two reasons. They suppose that the midpoint is not an imposed choice, but that it is possible to select a point of the interval depending by optimistic or pessimistic attitude of decision maker and that the measure that appear in the evaluation is not necessarily a Lebesgue measure but may be more general. Even in this direction we will try to find a unifying view.

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