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## ABSTRACT

This brief contribution provides a quantification of the terms of the turbulent kinetic energy transport equation for a round steady turbulent free jet. The analysis is based on the assumption of flow self-similarity, and it is performed by means of a simple analytical asymptotic analysis. The results are in good agreement with the experimental findings of Panchapakesan and Lumley [J. Fluid Mech. **246**, 197–223 (1993)] and with the large eddy simulations of Bogey and Bailly [J. Fluid Mech. **627**, 129–160 (2009)], hence providing a theoretical interpretation of such findings.

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Freely-evolving turbulence is ubiquitous in nature;<sup>1–3</sup> jets,<sup>4</sup> mixing layers,<sup>5,6</sup> and wakes<sup>7</sup> are used as models for a wide range of natural flows.

We focus on the evolution of round steady turbulent free jets and highlight the relationships existing between mean flow and turbulence by means of energetic arguments. In particular, the various mechanisms of transportation of a specific turbulent kinetic energy  $k \equiv \langle v_i'^2 \rangle / 2$  are analyzed.

These are described by the following equation:

$$\underbrace{\frac{dk}{dt}}_{(0)} = \underbrace{-\langle v_i' v_j' \rangle \frac{\partial V_i}{\partial x_j}}_{(I)} - \underbrace{\frac{\partial}{\partial x_j} \left[ \left\langle \frac{v_j' v_i'^2}{2} \right\rangle + \left\langle \frac{v_j' p'}{\rho} \right\rangle \right]}_{(II)} + \underbrace{v \frac{\partial^2 k}{\partial x_j^2}}_{(III)} - \underbrace{v \left\langle \left( \frac{\partial v_i'}{\partial x_j} \right)^2 \right\rangle}_{(IV)}, \quad (1)$$

in which Einstein's index convention is used and each term has a well-known meaning:

- (I) = energy transfer from the mean flow to turbulence (production of  $k$ );
- (II) = redistribution of turbulence within a given fluid volume (typically expressed as the divergence of a tensor);

- (III) = viscous diffusion of  $k$ ;
- (IV) = viscous dissipation of  $k$  (a negative definite term), usually labeled as

$$\varepsilon \equiv v \left\langle \left( \frac{\partial v_i'}{\partial x_j} \right)^2 \right\rangle. \quad (2)$$

The analyses that follow, based on the computation of the order of magnitude of each term of the above equation, are performed with a standard approach (e.g., Tennekes and Lumley,<sup>1</sup> Townsend,<sup>2</sup> and Pope<sup>3</sup>) that relies on the assumption of self-similar flow and on use of known scales. In the case of round free jets,

$$U = U_m(x) f\left(\frac{r}{l(x)}\right), \quad V = V_m(x) g\left(\frac{r}{l(x)}\right), \quad (3)$$

$$-\langle v_i' v_j' \rangle = q^2(x) h_{ij}\left(\frac{r}{l(x)}\right),$$

$l$  = transversal length scale (distance from the axis at which

$$U \text{ is a given fraction of } U_m), \quad (4)$$

$L$  = longitudinal length scale (distance over which  $U$  undergoes a significant change). (5)

Hence, it is simple to gauge the size of each term, and this being the size of the largest contribution of the given term,

$$\underbrace{\left[\frac{U_m q^2}{L}\right]}_{(0)} \quad \underbrace{\left[\frac{U_m q^2}{l}\right]}_{(I)} \quad \underbrace{\left[\frac{q^3}{l}\right]}_{(II)} \quad \underbrace{\left[\frac{v q^2}{l^2}\right]}_{(III)} \quad \underbrace{[?]}_{(IV)}, \quad (6)$$

and multiplying by  $l/(U_m q^2)$  gives

$$\underbrace{\left[\frac{l}{L}\right]}_{(0)} \quad \underbrace{[1]}_{(I)} \quad \underbrace{\left[\frac{q}{U_m}\right]}_{(II)} \quad \underbrace{\left[\frac{1}{Re_l}\right]}_{(III)} \quad \underbrace{[?]}_{(IV)}, \quad (7)$$

where the jet Reynolds number  $Re_l$  and the turbulence-to-mean-flow ratio are, respectively,

$$Re_l \equiv \frac{U_m l}{\nu} \quad \text{and} \quad \frac{q}{U_m} = \left[ \left( \frac{l}{L} \right)^\beta \right] \quad \text{with } \beta > 0 \quad (\beta = 1/2 \text{ for jets}). \quad (8)$$

The size of the dissipative term (IV) has been left unspecified because a naive order-of-magnitude-analysis would give a wrong estimate for such contribution. Hence, a number of arguments are used to determine the size of (IV):

- in asymptotic conditions,  $l/L \rightarrow 0$  and  $Re_l \rightarrow \infty$ ; only the production terms (I) in (7) does not vanish. Hence, the request that (1) and (7) be balanced leads to the dissipation (IV) being of the same size of production (I), i.e.,

$$(I) \sim (IV) \iff -\langle v'_i v'_j \rangle \frac{\partial V_i}{\partial x_j} \sim \varepsilon, \quad (9)$$

- energy cascading arguments suggest that the turbulence energy, of size  $\mathcal{O}(q^2)$ , extracted at the size of energy-containing eddies (here  $l$ ) cascades toward the smallest scales of the flow at a frequency  $U_m/l$ .

The above suggests that

$$\varepsilon = \mathcal{O}\left(\frac{U_m q^2}{l}\right), \quad (10)$$

and this result has been confirmed by various independent analyses.<sup>8-10</sup>

A much more complete illustration of all the dynamics that characterize the evolution of a round steady turbulent jet is achieved if higher order terms in the energy balance are retained. We, thus, focus on those terms which are  $\mathcal{O}(l/L)$  and  $\mathcal{O}[\sqrt{(l/L)}]$  smaller than production and dissipation, in particular on the transport terms (0) and (II) of Eqs. (1) and (7). For steady mean flow conditions and upon use of the continuity equation, Eq. (1) reads as

$$\underbrace{U \frac{\partial k}{\partial x}}_{c_1} + \underbrace{V \frac{\partial k}{\partial r}}_{c_2} + \underbrace{\langle u'v' \rangle \frac{\partial U}{\partial r}}_{p_1} + \underbrace{(\langle u'^2 \rangle - \langle v'^2 \rangle) \frac{\partial U}{\partial x}}_{p_2} + \underbrace{\frac{1}{r} \frac{\partial}{\partial r} \left\{ r \left[ \left\langle \frac{v'(u'^2 + v'^2 + w'^2)}{2} \right\rangle + \frac{\langle v'p' \rangle}{\rho} \right] \right\}}_{\tau} + \varepsilon = 0. \quad (11)$$

We now describe the balance of Eq. (11) on the basis of self-similar solutions like those of Eq. (3). Such solutions can be better specified on use of the following assumption required for the existence of self-similar solutions,

$$q \sim U_m = Bx^{-1}, \quad \text{with } B \text{ constant}, \quad (12)$$

$$l(x) = Ax, \quad \text{with } A \text{ constant and such that } A \ll 1, \quad (13)$$

and of the typical Gaussian shape for the cross-flow dependence of all variables<sup>3,11</sup> (this allows for more tractable analytical calculations than the power law profile also used in self-similarity studies),

$$f(\eta) = e^{-\left(\frac{\eta}{\sqrt{2}}\right)^2} = e^{-\eta^2}, \quad (14)$$

$$U(x, \eta) = U_m(x)f(\eta), \quad V(x, \eta) = U_m(x)g(\eta),$$

$$-\langle v'_i v'_j \rangle(x, \eta) = U_m^2(x)h_{ij}(\eta) \quad \text{with } g(\eta) = \frac{A}{2\eta}[-1 + (1+2\eta^2)f(\eta)],$$

$$h_{12}(\eta) = h_{21}(\eta) = \frac{f'(\eta)}{Re_{l_T}}, \quad (15)$$

$$h_{11}(\eta) \sim h_{22}(\eta) \sim \frac{Af(\eta)(1-2\eta^2)}{Re_{l_T}}, \quad \text{and } Re_{l_T} \equiv \frac{U_m l}{\nu_T} \sim \frac{1}{A} = \text{const}, \quad (16)$$

where the pressure-induced turbulent energy flux has been taken as smaller than that due to velocity fluctuations<sup>9,12</sup> and where  $g(\eta)$  has been obtained upon integration of the continuity equation, while  $h_{ij}(\eta)$  from a Boussinesq-type closure for turbulence.

These results immediately provide some insight. For example, the cross-flow profile of  $V$ , illustrated in Fig. 1 and remarkably similar to that obtained by more detailed and expensive calculations [see Fig. 16(b) of Ref. 12], clarifies some of the modalities of fluid entrainment in the jet; the negative cross-flow velocity occurring far from the jet axis ( $\eta \gg 1$ ) characterizes the entrainment of the environmental fluid within the jet.

Other useful results are the cross-flow profiles of various turbulence statistics, many of them well described by the relationships of

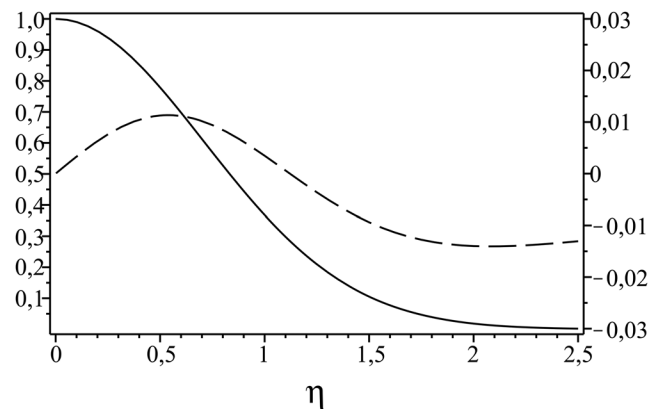
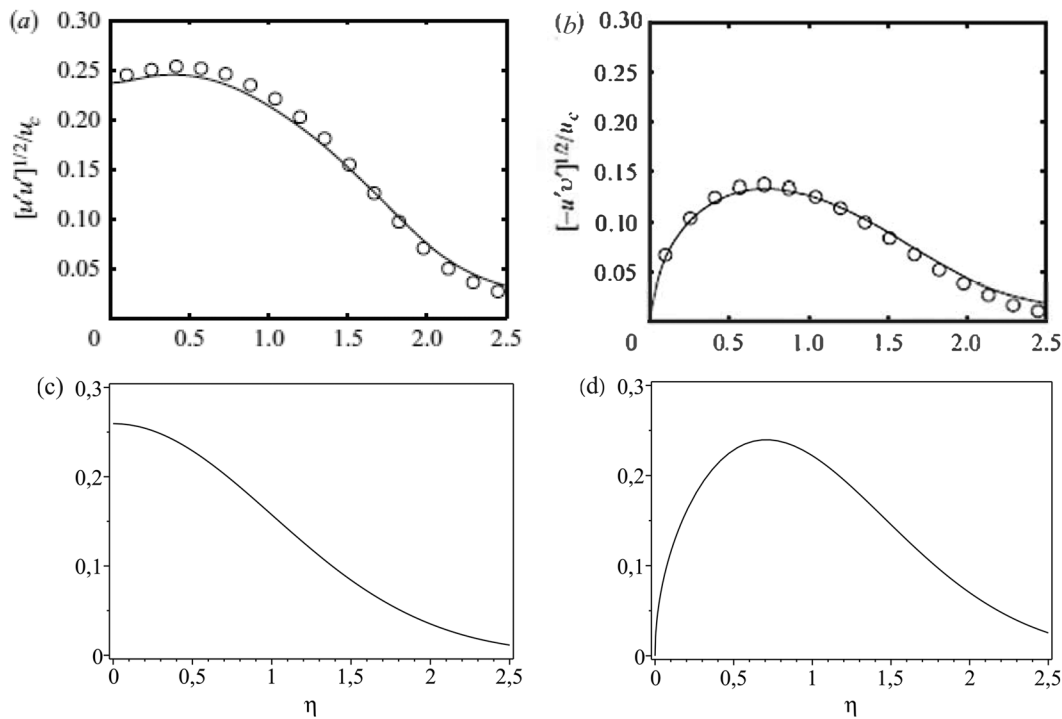


FIG. 1. Cross-flow profiles of normalized  $U$  (solid line and left ordinates) and normalized  $V$  (dashed line and right ordinates) of Eqs. (14) and (15).



**FIG. 2.** Top: cross-flow profiles of normalized  $\sqrt{\langle u'u' \rangle}$  (a) and  $\sqrt{\langle -u'v' \rangle}$  (b) from Fig. 17 of Ref. 12. Adapted with permission from Boogey and Bailly, *J. Fluid Mech.* **627**, 129–160 (2009). Copyright 2009, Cambridge University Press. Bottom: cross-flow profiles of normalized  $\sqrt{\langle u'u' \rangle}$  (c) and  $\sqrt{\langle -u'v' \rangle}$  (d) of Eqs. (14) and (15).

Eqs. (14)–(16) (see, for example, Figs. 17 and 18 of Ref. 12). In particular, Fig. 2 illustrates the close similarities between two second-order correlations of fundamental importance for the analysis of the transport equation for  $k$ .

It is now possible to determine the self-similar structure of all the terms that appear in Eq. (11).

Such a structure has been determined for three asymptotic regimes, which correspond to three positions in the cross-flow direction, namely, “far field” ( $\eta \gg 1$ ), “intermediate field” ( $\eta \sim 1$ ), and “near field” ( $\eta \ll 1$ ). The value of  $\varepsilon$  is obtained by difference.

Table I, which represents the main contribution of this brief note, summarizes all the results. In the “far field,” the exponential decay of  $f = e^{-\eta^2}$  controls the overall balance; hence, only the powers of  $f$  are given for this regime. In the “intermediate field,” where

$\eta = \mathcal{O}(1)$  and  $f \sim 1$ , the asymptotic behavior is controlled by the small parameter  $A \sim 1/Re_{lr} \ll 1$ , whose powers are provided in the table. Finally, the term balance in the “near field” is controlled by the size of  $\eta$ , whose powers are reported in the table.

In the same table

$$\Phi(x) \equiv \frac{U_m^3}{lRe_{lr}} \sim x^{-4}, \quad \text{and} \quad A \sim 0.067 \quad (17)$$

clarifies the streamwise flow dependence.

The above results lead to the following interpretations:

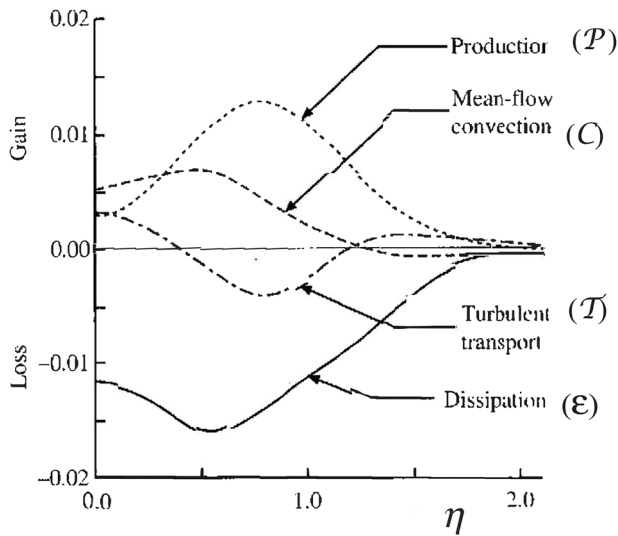
“far field”: because  $\mathcal{P}_1$ ,  $\mathcal{P}_2$ , and  $\mathcal{C}_1$  exponentially decay like  $f^2$  for  $\eta \gg 1$ , while  $\mathcal{C}_2$  and  $\mathcal{T}$  are proportional to  $f$ , far from the symmetry axis, i.e., at the interface with the external fluid, the balance is between  $\mathcal{C}_2$  and  $\mathcal{T}$ ;

“intermediate field”: here,  $\eta$  is not an asymptotic variable, and the only  $\mathcal{O}(1)$  term, i.e., not proportional to  $A \ll 1$ , is  $\mathcal{P}_1$  which, approximately, balances  $\varepsilon$ , as also prescribed by the turbulent kinetic energy transport equation at  $\mathcal{O}(1)$  [see (9)], while  $\mathcal{C}$  approximately balances  $\mathcal{T}$  (both of order  $A^2$ ). From a physical viewpoint,  $\mathcal{P}_1$  dominates in this regime because it is independent from the jet lateral growth rate  $dl/dx (\sim A)$ , i.e., from derivatives in the streamwise direction. It is to be noted that terms  $\mathcal{C}_2$  and  $\mathcal{T}$ , apparently not including such derivatives, implicitly include them through  $V$  and  $\langle u'u' \rangle$ , respectively;

“near field”:  $\mathcal{P}_1 \propto \eta^2$  is much smaller than  $\mathcal{P}_2$ , i.e.,  $\mathcal{P} \sim \mathcal{P}_2$ .

**TABLE I.** Dependence of the contributions to (11) as function of the cross-flow position.

Position	$\frac{\mathcal{C}_1}{\Phi(x)} \propto$	$\frac{\mathcal{C}_2}{\Phi(x)} \propto$	$\frac{\mathcal{P}_1}{\Phi(x)} \propto$	$\frac{\mathcal{P}_2}{\Phi(x)} \propto$	$\frac{\mathcal{T}}{\Phi(x)} \propto$
$\eta \gg 1$	$f^2$	$f$	$f^2$	$f^2$	$f$
$\eta \sim 1$	$A^2$	$A^2$	1	$A^2$	$A^2$
$\eta \ll 1$	1	$\eta^2$	$\eta^2$	1	1



**FIG. 3.** Balance between the terms of (11). All variables are made dimensionless by means of  $U_m$  and  $l$ . From Pope, *Turbulent Flows*. Copyright 2000 Cambridge University Press. Adapted with permission from Cambridge University Press.

The above theoretical deductions are confirmed by and help explain the experimental data of Fig. 3 (see Pope<sup>3</sup> and Panchapakesan and Lumley<sup>9</sup>).

In particular, turbulence dissipation occurs all over the turbulent region, while the maximum of the production  $\mathcal{P}$  occurs in correspondence of the region where the flow deformation ( $\partial U/\partial r$ ) is the largest, i.e., at  $\eta \sim 1$ . In fact, recalling definitions (15),

$$\begin{aligned} 0 &= \frac{\partial}{\partial r} \left( \frac{\partial U}{\partial r} \right) = \frac{\partial}{\partial r} \left( U_m \frac{\partial f}{\partial r} \right) = \frac{1}{l} \frac{\partial}{\partial \eta} \left( \frac{U_m}{l} f' \right) = \frac{U_m}{l^2} f'' \\ &= \frac{U_m}{l^2} [-2f(1-2\eta^2)] \implies \eta \sim \frac{\sqrt{2}}{2}, \end{aligned} \quad (18)$$

which quantifies the exact location of the maximum turbulence production (as shown by Fig. 3).

Other notable results are (a) the “far field” balance between  $\mathcal{C}_2$  and  $\mathcal{T}$  (see region  $\eta > 2$  in Figs. 3) and (b) the “intermediate field” balances between  $\mathcal{P}_1$  and  $\varepsilon$  and between  $\mathcal{C}$  and  $\mathcal{T}$  (see region  $0.5 < \eta < 1.5$  in Fig. 3).

Using simple analytical tools, i.e., straightforward asymptotic arguments coupled with the self-similarity assumption, a quantification of the specific kinetic energy balance for a turbulent, round, free jet is made with minimal efforts. The same analysis provides theoretical support to similar literature results and allows insight into some basic dynamics of such jets.

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#### DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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