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A hybrid method to evaluate pure endowment policies: Crédit Agricole and ERGO Index linked policies

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HIGHLIGHTS

- A model to evaluate pure endowment policies as a financial claim is proposed.
- New choices of the time dependent Black Scholes model parameters are presented.
- A generalization of the geometric Brownian mean reverting Gompertz model is used.
- An expansion of the survival probability for small mortality risk volatility is deduced.
- The method is tested comparing the observed and theoretical policy values.

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ABSTRACT

An empirical method to evaluate pure endowment policies is proposed. The financial component of the policies is described using the time dependent Black Scholes model and making a suitable choice for its time dependent parameter functions. Specifically, the integral of the time dependent risk free interest rate is modeled using an extension of the Nelson and Siegel yield curve (see Dielbold and Li, 2006). The time dependent volatility is expressed using two different models. One of these is based on an extension of the Nelson and Siegel model (Dielbold and Li, 2006), while the other assumes that the volatility is a piecewise function with respect to the time variable. The demographic component is modeled using a generalization of the geometric Brownian mean reverting Gompertz model while an asymptotic formula for survival probability is derived when the mortality risk volatility is small. The method has been tested on two policies. In these the risk free interest rate parameters are calibrated using the one-month, three-month, six-month, one-year, three-year and five-year US treasury constant maturity yields and the parameters of the volatility are calibrated using the VSTOXX volatility indices. The choice of the data employed in the calibration depends on the policy to be evaluated. The performance of the method is established comparing the observed values of the policies with the values obtained using this method.

1. Introduction

This paper proposes an empirical method that evaluates pure endowment policies. The pure endowment contract guarantees the policy holder on maturity date, the choice between a pre-specified amount of money or a payment related to the price of a risky asset (or assets) specified in the contract. No payments are owed by the policy seller if the policy holder is deceased on the maturity date.

The life insurance market is increasing all over the world for a variety of reasons. Numerous companies offer life insurance policies as part of their benefit packages in addition to the private policies the companies issue. An increase in sales of guaranteed products may also be due to recent changes in labor laws and in life style. In 2010, guaranteed products confirmed the 2009 positive trend (i.e. +5%), and the Unit & Index linked contracts increased by 58%. The scale of this phenomenon becomes particularly significant in the life insurance market in China where the causes lie in the rise of a middle class, the increase of per capita income, and the impact of market liberalization. The renewed interest in these simple policies and the request for pricing transparency highlights the importance of having simple procedures to evaluate them.

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We illustrate the proposed procedure to evaluate two pure endowment policies: the index linked policy Azione Più Capitale Garantito Em.63 of the Crédit Agricole insurance company and the index linked policy proposed by ERGO Insurance group (ERGO Versicherungsgruppe AG Düsseldorf). These policies are single-premium index linked life insurance policies whose benefits are directly linked to the performance of the Dow Jones Euro Stoxx 50 Index.

The duration of the Azione Più Capitale Garantito Em.63 policy covers the period from March 31, 2010 (date of issue) to May 8, 2015 (expiration date). On the maturity date the company guarantees the insured, should he be living, the payment of the premium plus a variable bonus obtained by multiplying the premium by 50% of the relative difference of the Dow Jones Euro Stoxx 50 values between March 31, 2010 and July 27, 2015, in the case of a positive difference. In the case of a negative difference the variable bonus will be equal to zero.

The duration of the ERGO contract is three years plus two days with the date of issue being May 8, 2007, and the expiration date May 10, 2010. If the policyholder is living on the maturity date, the contract provides for payment of the nominal capital plus a bonus equal to 84.00% of the relative difference (if positive) of the reference index values between the date of issue and April 28, 2010 (expiration date). Should relative difference be negative the bonus will be zero.

We note that the first policy, which has expired, witnessed the collapse of Lehman Brothers while the second one, which has not expired, is being influenced by the advent of the new regulations and the EURO-zone crisis. The premiums of these policies are clearly influenced by these events, and consequently, their evaluation may be a challenging task.

Several other pure endowment policies can be evaluated with the proposed method. The choice of these two policies is based on the fact that their values are freely available. Note that acquiring life insurance policy data is not easy.

1.1. Research background

From the scientific literature on the evaluation of life insurance policies two predominant approaches emerge: actuarial (see Hardy (2000, 2002) and the reference therein) and financial (see Ballotta and Haberman (2006), Nteukam et al. (2011)) and the reference therein).

The "financial" approach relies on hedging the financial risk. The hedging is "perfect" in the case of an assumed complete market and only "risk-minimizing" in the more realistic case of an incomplete market. As a matter of fact, this approach is only valid if the underlying hedging is actually applied, which is not always the case in practice. This approach contrasts with the "actuarial" one that relies on the equivalence principle based on the law of large numbers and where the pure premium is determined as the mean of the future losses. In the financial approach risk management consists in hedging the position on financial markets while in the actuarial approach it implies reserving and raising capital in order to cover the future losses with a given probability. The fact that the financial risk is not completely diversifiable gives rise to large capital costs. The actuarial approach also received some attention in the literature, for example in papers by Hardy in which both approaches are compared particularly with regard to reserving and risk management (see Hardy (2000, 2002)).

This paper uses a financial approach in that the evaluation of the policy premiums is done like the evaluation of financial options. We model the risky asset specified in the contract under the risk-neutral measure and the mortality risk (i.e. the mortality rate) under the physical measure assuming that these two measures are independent. Moreover, we assume risk-neutrality of the insurer with respect to mortality risk and a complete financial market where a unique risk-neutral measure exists (see Bacinello and Persson (2002), Aase and Persson (1994), Møller (2001) and Bauer et al. (2010)). Under these assumptions the resulting procedures to evaluate pure endowment policies are very simple and easy to interpret.

The assumption that the insurer is risk-neutral with respect to mortality risk means that he does not receive any economic compensation for accepting risk. This assumption is motivated by the fact that the insurer can eliminate the risk by suitably increasing the number of identical and independent contracts in the portfolio (see Bacinello and Persson (2002)). When this is not possible a market price of risk must be introduced. In this case the evaluation of the policy premium using the expected value with respect to the physical measure does not correspond to the replication of the claim by a self-financing strategy. Roughly speaking, in this case, the insurer is not guaranteed to be hedged by the rise in future mortality rates.

However, for simplicity, we work under the assumptions mentioned above. We want to investigate how the use of the physical measure for the mortality risk and the risk neutral measure for the financial risk can provide satisfactory risk premium evaluations.

We model the risky asset using the geometric Brownian motion with time dependent coefficients and we propose two procedures to calibrate the model parameters. In the scientific literature the most common choice for evaluating index-linked policies is the classical geometric Brownian motion with constant coefficients (see, for example, Bélanger et al. (2009) and Nteukam et al. (2011)).

We model the mortality rate using a continuous time stochastic process. This approach is quite common in the current literature given that it permits one to determine two important features of the intensity of mortality: the time dependence and the probability of some future developments. The work of Milevsky and Promislow (2001) is an example of this approach. They propose a method to evaluate mortality-contingent claims as options on two underlying stochastic variables: the interest rate and the mortality risk. Since the options on interest rates are known, unlike those on mortality, they focus on the evaluation of options on mortality-contingent claims. They also model the hazard rate (force of mortality) stochastically using the mean reverting Gompertz model and model the interest rate using a Cox–Ingersoll–Ross process. The two stochastic variables are assumed to be independent. They show that both mortality and interest rate risks can be hedged.

The mean reverting Gompertz model for the mortality rate has also been used by Ballotta and Haberman (2006) to evaluate guaranteed annuity options. In Ballotta and Haberman (2006), the financial and the mortality risks are assumed to be independent and the term structure of interest rates is driven by the single-factor Heath–Jarrow–Morton model.

In the scientific literature we can also find affine models to describe the dynamics of the mortality rate. In Biffis (2005) asset prices and mortality dynamics are modeled using affine jump–diffusions. In this manner, the author is able to fully exploit the analytical tractability of affine processes in the context of both financial and mortality risks.

In Schrager (2006), a model for stochastic mortality based on the literature on affine term structure models is proposed. This model satisfies three requirements that enable it to be easily used: analytical tractability, clear factor interpretation and compatibility with financial option pricing models. This model has been tested using Dutch data on mortality rates and applied to the pricing of guaranteed annuity options.

Other recent works, Bertocchi et al. (2013) and Wong and Chan (2007), propose the use of multiscale stochastic volatility models to describe the behavior of contracts having long residual lives. The use of such models is motivated by the fact that one factor stochastic models do not provide satisfactory evaluations of contracts having long maturities. In Wong and Chan (2007) the multiscale stochastic volatility model is used to evaluate lookback options and dynamic fund protection and in Bertocchi et al. (2013) to evaluate pure endowment policies. Bertocchi et al. (2013) model mortality risk through the mean reverting Gompertz model as suggested in Milevsky and Promislow (2001) and they derive an asymptotic formula for the survival probability when the volatility risk goes to zero.

1.2. Main results of the paper

The methods proposed here to evaluate pure endowment policy premiums are based on few simple ideas. As previously mentioned, we use the time dependent Black Scholes model to evaluate the financial component of the policy and a generalization of the mean reverting Gompertz model to describe the demographic component of the policy. The latter model has been introduced and tested on the Italian human mortality database in Giacometti et al. (2011). We use the results proposed there to calibrate the model parameters in Section 4.

Let us describe the risky asset S_t, $t > 0$, specified in the policy and the mortality rate h_t, $t > 0$, associated to the human life with the following stochastic model:

$$dS_t = r(t)S_t + \sigma(t)S_t dW_t, \quad t > 0, \tag{1}$$

$$dh_t = \left(g + \frac{1}{2}(\sigma^*)^2 + b\ln(\hat{h}_0) + b\,g\,t + b\,\ln(h_t)\right) h_t\,dt + \sigma^* e^{at} h_t dQ_t, \quad t > 0, \tag{2}$$

where $r(t), \sigma(t), t > 0$ are given real functions, the quantities g, b, h_0, a, σ^* are real constants, $\ln(\cdot)$ denotes the natural logarithm of $\cdot$ and $W_t,\ Q_t, t > 0$, are standard Wiener processes that satisfy the following conditions:

$$\begin{aligned} &W_0 = Q_0 = 0,\\ &E(dW_t\, dW_t) = dt,\\ &E(dQ_t\, dQ_t) = dt,\\ &E(dW_t\, dQ_t) = 0, \quad t > 0. \end{aligned} \tag{3}$$

In (3) the symbol $E(\cdot)$ denotes the expected value of $\cdot$. Condition (3) implies that the financial and mortality risks are independent. This assumption is widely used in evaluating insurance contracts (see, for example, Ludkovski and Young (2008), Ballotta and Haberman (2006) and the references therein).

Since the two risks are assumed to be independent, the pricing of simple equity-linked contracts can be reduced to the evaluation of the following product:

$$E\left(e^{-\int_0^T r(\tau)d\tau} P(S_T)\right) E\left(e^{-\int_0^T h_u du}\right), \tag{4}$$

where P is the payoff function associated with the policy. The first expected value in (4) is the value of a European option in the time dependent Black Scholes model and the second expected value is the survival probability. We derive an asymptotic expansion for this expected value for small mortality risk volatility, σ^*. We use the first three terms of this expansion to approximate the survival probability. These terms are given by elementary functions of the model parameters. This expansion is inspired by the one proposed in Bertocchi et al. (2013) for the mean reverting Gompertz model. It allows us to express the demographic component of the policy as a second order degree polynomial in the variable σ^* (i.e. the mortality risk volatility).

We use formula (4) to value some insurance contracts traded in the life insurance market giving a practical procedure to select the unknown functions $r(t), \sigma(t),\ 0 < t < T$, while the unknown parameters h_0, g, b, σ^*, a are determined using the results proposed in Giacometti et al. (2011).

Note that we are using the risk-neutral measure to evaluate the financial component of the policy and the physical measure for the demographic component. This is meaningful under the assumption of a complete financial market and the risk-neutrality of an insurer with respect to mortality risk (see Aase and Persson (1994), Møller (2001) and Bauer et al. (2010), for further details).

That is, the main contribution of this paper is the development of two procedures to calibrate the model parameters of the financial component of the pure endowment policy (i.e. the first integral in the product appearing in formula (4)), the derivation of a perturbation expansion to approximate the survival probability (i.e. the second integral in the product appearing in formula (4)) for small mortality risk volatility and the numerical experiments on real data.

The two procedures used to evaluate the financial component of the policy model the integral of the risk free interest rate over the time period covered of the policy using the Nelson–Siegel yield curve (see Nelson and Siegel (1987)). The model parameters are calibrated through the least squares method which uses as data the one-month, three-months, six-months, one-year, three-years and five-years US treasury constant maturity yields (data downloadable at the website http://www.federalreserve.gov). The choice of this model and of these data for the calibration is motivated by a recent study of Dielbold and Li (2006) that shows the ability of the Nelson–Siegel model to describe these yields.

The two procedures differ in the evaluation of the integral of the square of volatility. The first procedure (P_1 for short) chooses the volatility to be constant in the time period covered by the policy. The value of this constant is the VSTOXX 24 months volatility index value available on the website http://www.stoxx.com/download/historical_values/h_vstoxx.txt.

The second procedure (P_2 for short) models the integral of the square of volatility through the Nelson–Siegel yield curve and calibrates the model parameters using as data the eight VSTOXX volatility indices (V6$_i$, $i = 1, 2, 3, 6, 9, 12, 24$ months to expiration) available at the website mentioned above.

The computation of these eight sub-indices per option expiry 1, 2, 3, 6, 9, 12, 18 and 24 months is based on the square-root of the implied variance. The VSTOXX indices are evaluated using EURO STOXX 50 realtime options prices. These indices reflect the market expectations of near term up to long term volatility.

This second procedure is suggested by the fact that we need market expectation on the future volatilities in order to know a kind of "forward volatility curve".

1.3. Outline of the paper

In Section 2 we derive the closed form formulae for the financial components of the two policies we are interested in. In Section 3 we present the asymptotic formula for the survival probability as the mortality risk volatility goes to zero. We show its performance in approximating the true survival probability and in reproducing the Italian survival probability observed in 2005 and 2010. In Section 4 we present two case studies and the calibration procedures used. We establish the performance of the proposed procedures comparing the observed values of the policies considered and the values obtained with the procedures themselves. We carried out an empirical analysis to understand how to calibrate the parameters of the time dependent Black Scholes model involved in the policy formula and how the choice of the cohort influences the policy values. In Section 5 some conclusions are drawn. Finally, in the Appendix the derivation of the asymptotic formula for the survival probability is presented.

2. A hybrid model for the financial components of the pure endowment policy

The hybrid model used to describe the financial component of the policy expressed by the asset price S_t is described by the time dependent log-normal process (1) where $r(t), \sigma(t),\ t < T, T > 0$, are the deterministic functions of the time variable t. The function $r(t),\ t < T$ expresses the risk free interest rate and the function $\sigma(t),\ t < T$ expresses the asset volatility. Using the dynamics (1) the following straightforward generalization of the well known Black and Scholes formula to price a European call option having maturity T and strike price E published by Black and Scholes (1973) holds (see Wilmott (1998)):

$$C(S,t) = S\,N(d_1) - Ee^{-\int_t^T r(\tau)d\tau}\,N(d_2) \tag{5}$$

where d_1, d_2 are given by:

$$d_1 = \frac{\log\left(\frac{S}{E}\right) + \int_t^T r(\tau)d\tau + \frac{1}{2}\int_t^T \sigma^2(\tau)d\tau}{\sqrt{\int_t^T \sigma^2(\tau)d\tau}}, \tag{6}$$

$$d_2 = \frac{\log\left(\frac{S}{E}\right) + \int_t^T r(\tau)d\tau - \frac{1}{2}\int_t^T \sigma^2(\tau)d\tau}{\sqrt{\int_t^T \sigma^2(\tau)d\tau}}. \tag{7}$$

Note that we have assumed that the dividend is zero.

Now we adapt formula (5) to evaluate the two index linked policies described previously.

On the maturity date Action Capital Piú Guaranteed Em.63 Crédit Agricole Vita SpA guarantees the insured, should he be living, the payment of the premium plus a variable bonus obtained by multiplying the premium by 50% of the relative difference $(S_T - S_r)/S_r$ of the Dow Jones Euro Stoxx 50 values between March 31, 2010 (i.e. $S_r = 2931.16$) and July 27, 2015, in the case of a positive difference. In the case of a negative difference the variable bonus will be equal to zero. The payoff of this policy is given by:

$$\begin{aligned} P_1\left(\frac{S_T}{S_r}, T\right) &= K + 0.5K\left[\frac{S_T - S_r}{S_r}\right] \\ &= K + 0.5K\ \max\left[\frac{S_T - S_r}{S_r}; 0\right] \\ &= K + 0.5\frac{K}{S_r}\ \max\left[\frac{S_T}{S_r} - 1; 0\right] \end{aligned} \tag{8}$$

where K is the nominal capital paid at the start of the contract. For this contract $K = 100$.

Using formulae (5) and (8), we obtain the following formula to evaluate the financial component of the policy:

$$\begin{aligned} V_1\left(\frac{S}{S_r}, t\right) &= Ke^{-\int_t^T r(\tau)d\tau} \\ &\quad + 0.5K\left\{\left(\frac{S_T}{S_r}\right)N(d_1^*) - e^{-\int_t^T r(\tau)d\tau}N(d_2^*)\right\}, \end{aligned} \tag{9}$$

where d_1^* and d_2^* are given by:

$$d_1^* = \frac{\log\left(\frac{S}{S_r}\right) + \int_t^T r(\tau)d\tau + \frac{1}{2}\int_t^T \sigma^2(\tau)d\tau}{\sqrt{\int_t^T \sigma^2(\tau)d\tau}}, \tag{10}$$

$$d_2^* = \frac{\log\left(\frac{S}{S_r}\right) + \int_t^T r(\tau)d\tau - \frac{1}{2}\int_t^T \sigma^2(\tau)d\tau}{\sqrt{\int_t^T \sigma^2(\tau)d\tau}}. \tag{11}$$

On the maturity date the second policy, that of ERGO insurance company, guarantees the insured, should he be living, the payment of the premium plus a bonus of 84% of the variation $(S_T - S_r)/S_r$ (if positive) of the reference index (i.e. Dow Jones Euro Stoxx 50 index). S_r is the value of this index on May 8, 2007 (i.e. $S_r = 4411.32$) and S_T is the value on April 28, 2010. In case of a negative variation the bonus is zero. Note that this policy has expired so the value S_T is known and equal to $S_T = 2788.54$. Since $S_T < S_r$ this policy did not pay any bonus on the maturity date. The payoff function of this policy is given by:

$$\begin{aligned} P_2\left(\frac{S_T}{S_r}, T\right) &= K + 0.84K\left[\frac{S_T - S_r}{S_r}\right] \\ &= K + 0.84K\ \max\left[\frac{S_T - S_r}{S_r}; 0\right] \\ &= K + 0.84\frac{K}{S_r}\ \max\left[\frac{S_T}{S_r} - 1; 0\right] \end{aligned} \tag{12}$$

where K is the nominal capital paid at the start of the contract. For this contract $K = 1000$.

Note that we can use formula (5) to evaluate the financial component of the policy at time $t < T$ interpreting the underlying asset as the ratio S/S_r and we have:

$$\begin{aligned} V_2\left(\frac{S}{S_r}, t\right) &= Ke^{-\int_t^T r(\tau)d\tau} \\ &\quad + 0.84\,K\left\{\left(\frac{S}{S_r}\right)N(d_1^*) - e^{-\int_t^T r(\tau)d\tau}N(d_2^*)\right\} \end{aligned} \tag{13}$$

where d_1^* and d_2^* are given in formulae (10) and (11).

The choice of the functions r and σ is a challenging task in evaluating these policies due to their long maturity. More specifically, we are interested in choosing their integrals: $\hat{r}(t,T) = \int_t^T r(\tau)d\tau$ and $\hat{\sigma}(t,T) = \int_t^T \sigma^2(\tau)d\tau$.

We use the Nelson–Siegel yield curve to estimate the quantity $\hat{r}(t,T)$. The Nelson–Siegel yield curve was proposed in Nelson and Siegel (1987) and used to forecast the term structure of government bond yields in Diebold and Li (2006). We make the following choice for $\hat{r}(t,T),\ t < T$:

$$\hat{r}(t,T) = \int_t^T r(\tau)d\tau = y_t(T-t)(T-t), \tag{14}$$

where $y_t(\tau)$ is the yield curve given by:

$$\begin{aligned} y_t(\tau) &= \beta_{1,t} + \beta_{2,t}\left(\frac{1 - e^{-\lambda_t\tau}}{\lambda_t\tau}\right) \\ &\quad + \beta_{3,t}\left(\frac{1 - e^{-\lambda_t\tau}}{\lambda_t\tau} - e^{-\lambda_t\tau}\right). \end{aligned} \tag{15}$$

The term $\hat{r}(t,T)$ should reflect the expectations of the financial markets on future interest rates and for this reason we calibrate model (15) using long term government bond yields.

In Dielbold and Li (2006) the parameters $\beta_{1,t}, \beta_{2,t}, \beta_{3,t}$ are interpreted as three latent dynamic factors and λ_t is a positive parameter that governs the exponential decay. The parameter $\beta_{1,t}$ can be interpreted as a long term factor since it is the limit of $y_t(\tau)$ as τ goes to $+\infty$ while the parameter $\beta_{2,t}$ can be interpreted as a short term factor since it only plays significant role in short term maturities while being negligible for long term maturities. Finally $\beta_{3,t}$ can be interpreted as a medium-term factor because the time function that multiplies this parameter is zero at time t zero, attains its maximum value for $t > 0$ and decays to zero when the time goes to infinity (see Dielbold and Li (2006) for further details).

The model (15) is calibrated using data from the US treasury yields available on the website: http://www.federalreserve.gov/releases/h15/.

These data are daily observations of the yields having maturity τ_1 = 1-month, τ_2 = 3-months, τ_3 = 6-months, τ_4 = 1-year, τ_5 = 3-years, τ_6 = 5-years. Let R^4 be the four dimensional real Euclidean space and N be a positive integer, $N \leq 6$. We denote the observed yields at time t by $\tilde{y}_t(\tau_j),\ j = 1, 2, \dots, N$, and we calibrate the model (15) solving the following optimization problem:

$$\min_{(\beta_{1,t},\beta_{2,t},\beta_{3,t},\lambda_t)\in R^4} L_t(\beta_{1,t}, \beta_{2,t}, \beta_{3,t}, \lambda_t) \tag{16}$$

where the objective function L_t and the quantities $\hat{y}_t(\tau_j),\ j = 1, 2, \dots, N$ are given by:

$$L_t(\beta_{1,t}, \beta_{2,t}, \beta_{3,t}, \lambda_t) = \sum_{j=1}^{N} \left[\tau_j \tilde{y}_t(\tau_j) - \hat{y}_t(\tau_j)\right]^2, \tag{17}$$

and

$$\hat{y}_t(\tau_j) = \tau_j\beta_{1,t} + \beta_{2,t}\left(\frac{1-e^{-\lambda_t\tau_j}}{\lambda_t}\right) + \beta_{3,t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t} - \tau_j e^{-\lambda_t\tau_j}\right), \quad j = 1, 2, \dots, N. \tag{18}$$

Two choices of volatility function are made. The first one is very simple and consists in choosing the volatility $\sigma(\tau),\ \tau \in [t, T]$ to be equal to the VSTOXX 24 months volatility index available at: http://www.stoxx.com/indices/index_information.html?symbol=V6I6.

That is, we assume that the volatility observed on the transaction day t remains unchanged until the expiration of the policy. This corresponds to assuming that the volatility is constant in the time interval $[t, T]$.

The second choice implies that the integral of the square of the volatility function has the following form:

$$\hat{\sigma}(t, T) \equiv \int_t^T \sigma^2(\tau)d\tau = \alpha_{1,t}(T-t) + \alpha_{2,t}\left(\frac{1-e^{-\lambda_t^*(T-t)}}{\lambda_t^*}\right) + (T-t)\alpha_{3,t}\left(\frac{1-e^{-\lambda_t^*(T-t)}}{\lambda_t^*(T-t)} - e^{-\lambda_t^*(T-t)}\right). \tag{19}$$

That is, we use a Nelson–Siegel model to estimate the quantity $\hat{\sigma}(t, T),\ t < T$.

This choice is made since we desire to know the market expectations of near-term up to long term volatilities.

The model (19) is calibrated using VSTOXX $\tau_1^* =$ 1-month, $\tau_2^* =$ 2-months, $\tau_3^* =$ 3-months, $\tau_4^* =$ 6-months, $\tau_5^* =$ 9-months, $\tau_6^* =$ 12-months, $\tau_7^* =$ 18-months, and $\tau_8^* =$ 24-months volatility indices available at http://www.stoxx.com/download/historical_values/h_vstoxx.txt. The computation of these eight indices is based on the square-root of the implied variance of the EURO STOXX 50 option prices.

The calibration procedure used is analogous to the one used to calibrate model (15). Let N_v be a positive integer, $N_v \le 8$ and let us denote by $\tilde{\sigma}_t(\tau_j^*),\ j = 1, 2, \dots, N_v$, the VSTOXX volatility indices at time t. We calibrate the model (19) solving the following optimization problem:

$$\min_{(\alpha_{1,t},\alpha_{2,t},\alpha_{3,t},\lambda_t^*)\in R^4} L_t^*(\alpha_{1,t}, \alpha_{2,t}, \alpha_{3,t}, \lambda_t^*) \tag{20}$$

where the objective function L_t^* and the quantities $\hat{\sigma}_t(\tau_j),\ j = 1, 2, \dots, N$ are defined as follows:

$$L_t^*(\alpha_{1,t}, \alpha_{2,t}, \alpha_{3,t}, \lambda_t^*) = \sum_{j=1}^{N} \left[\tau_j \tilde{\sigma}_t^*(\tau_j) - \hat{\sigma}_t(\tau_j^*)\right]^2, \tag{21}$$

and

$$\hat{\sigma}_t(\tau_j) = \tau_j\alpha_{1,t} + \alpha_{2,t}\left(\frac{1-e^{-\lambda_t^*\tau_j}}{\lambda_t^*}\right) + \alpha_{3,t}\left(\frac{1-e^{-\lambda_t^*\tau}}{\lambda_t^*} - \tau_j e^{-\lambda_t^*\tau_j}\right), \quad j = 1, 2, \dots, N. \tag{22}$$

Note that the parameters $\alpha_{1,t}, \alpha_{2,t}, \alpha_{3,t}$ can be interpreted as was done for $\beta_{1,t}$, $\beta_{2,t}$ and $\beta_{3,t}$. The parameter $\alpha_{1,t}$ can be interpreted as a long term factor since it is the limit of $\hat{\sigma}(t, T)$ as T goes to $+\infty$ while the parameter $\alpha_{2,t}$ can be interpreted as a short term factor since it plays a significant role only for small maturities and it is negligible for long maturities. Finally, $\alpha_{3,t}$ can be interpreted as a medium-term factor. As mentioned above, the parameters of model (22) are a "measure" of the market expectations of the short, medium and long term volatilities.

3. A formula for the survival probability in a generalized mean reverting geometric Gompertz model

Let us consider the stochastic process $h_t,\ t > 0$, defined by (2). This process can be described using the auxiliary process $y_t,\ t > 0$, as follows:

$$h_t = \hat{h}_0\, e^{g\,t}\, e^{\sigma^* y_t}, \quad t > 0, \tag{23}$$

where $y_t,\ t > 0$, satisfies the stochastic differential equation:

$$dy_t = -by_t dt + e^{at} dQ_t, \quad t > 0 \tag{24}$$

with initial condition:

$$y_0 = 0, \tag{25}$$

where $dQ_t,\ t > 0$, is the standard Wiener process appearing in (2), and g, σ^*, b, a and $\hat{h}_0$ are positive constants appearing in (2).

We derive an asymptotic formula for $E(e^{-\int_t^T h_\tau d\tau} \mid h_0 = \hat{h}_0)$, $T > 0$, as $\sigma^* \to 0$ when $h_\tau,\ \tau > 0$, is defined by (23), (24), (25) (i.e. it satisfies (2)). Note that when $t = 0$, the quantities $E(e^{-\int_t^T h_\tau d\tau} \mid h_0 = \hat{h}_0)$ coincide with the second term of (4). We express $h_\tau,\ \tau > 0$, as follows:

$$h_\tau = \hat{h}_0\, e^{g\tau}\, e^{\sigma^* y_\tau} = \hat{h}_0\, e^{g\tau} \sum_{m=0}^{+\infty} \frac{(\sigma^*)^m}{m!} y_\tau^m, \quad \tau > 0, \tag{26}$$

from (26) we have:

$$e^{-\int_t^T h_\tau d\tau} = e^{-\hat{h}_0 \sum_{m=0}^{+\infty} \frac{(\sigma^*)^m}{m!} \int_t^T y_\tau^m e^{g\tau} d\tau}, \quad t, T \ge 0,\ t < T. \tag{27}$$

Considering the first three terms of the expansion in powers of σ^* contained in (27) we have:

$$e^{-\int_t^T h_\tau d\tau} = e^{-\hat{h}_0 \int_t^T e^{g\tau} d\tau} \left\{1 - \hat{h}_0\sigma^* \int_t^T y_\tau\, e^{g\tau} d\tau + \frac{\hat{h}_0^2(\sigma^*)^2}{2}\left[\int_t^T y_\tau e^{g\tau} d\tau\right]^2 - \frac{\hat{h}_0(\sigma^*)^2}{2}\int_t^T y_\tau^2 e^{g\tau} d\tau\right\} + o((\sigma^*)^2), \quad t, T \ge 0,\ t < T,\ \sigma^* \to 0. \tag{28}$$

Substituting formula (28) in formula for the survival probability we obtain:

$$\begin{aligned} E&\left(e^{-\int_t^T h_\tau d\tau} \mid h_0 = \hat{h}_0\right) \\ &= e^{-\hat{h}_0 \int_t^T e^{g\tau} d\tau}\left\{1 - \hat{h}_0\sigma^* \int_t^T E(y_\tau \big| y_0 = 0)\, e^{g\tau} d\tau + \frac{\hat{h}_0^2(\sigma^*)^2}{2} E\left(\left[\int_t^T y_\tau e^{g\tau} d\tau\right]^2 \Big| y_0 = 0\right) - \frac{\hat{h}_0(\sigma^*)^2}{2}\int_t^T E(y_\tau^2 \mid y_0 = 0) e^{g\tau} d\tau\right\} \\ &\quad + o((\sigma^*)^2), \quad t, T \ge 0,\ t < T,\ \sigma^* \to 0. \end{aligned} \tag{29}$$

Table 1
Parameter values of the demographic component of the hybrid formula used to forecast the values of the Crédit Agricole and Ergo index linked policies.

Cohort	h_0	a	b	g	σ^*
1966	0.0001180	0.0018	0.3338	0.0761	0.0341
1977	0.0001175	0.0005	0.6315	0.0722	0.0311
1980	0.0001431	0.0068	0.5990	0.0653	0.0455

In the Appendix, under suitable assumptions on a, b and g and using (29), we deduce the following formula:

$$E\left(e^{-\int_t^T h_\tau d\tau} \mid h_0 = \hat{h}_0\right) = E^a\left(e^{-\int_t^T h_\tau d\tau} \mid h_0 = \hat{h}_0\right) + o((\sigma^*)^2),$$
$$t, T \geq 0,\ t < T,\ \sigma^* \to 0, \qquad (30)$$

where E^a is given by:

$$E^a\left(e^{-\int_t^T h_\tau d\tau} \mid h_0 = \hat{h}_0\right) = e^{-\hat{h}_0 e^{gt}(e^{g(T-t)}-1)/g} \cdot \Bigg\{1 - \frac{\hat{h}_0(\sigma^*)^2}{4(a+b)} \times \Bigg[\Bigg(\frac{e^{(g+2a)t}(e^{(g+2a)(T-t)}-1)}{g+2a} - \frac{e^{(g-2b)t}(e^{(g-2b)(T-t)}-1)}{g-2b}\Bigg)\Bigg] + \frac{\hat{h}_0^2(\sigma^*)^2}{2}\Bigg[-\frac{1}{2(a+b)}\left(\frac{e^{(g-b)t}(e^{(g-b)(T-t)}-1)}{g-b}\right)^2 + \frac{1}{(a+b)}\frac{1}{(g+b+2a)} \cdot \Bigg[\Bigg(\frac{e^{2(g+a)t}(e^{2(g+a)(T-t)}-1)}{2(g+a)} - \left(\frac{e^{(g-b)(T+t)+2(a+b)t} - e^{2(g+a)t}}{g-b}\right)\Bigg)\Bigg]\Bigg]\Bigg\}, \quad 0 \leq t < T. \quad (31)$$

We use formula (31) to evaluate the survival probability. The values of the parameters appearing in (31) are suggested by the data analysis presented in Giacometti et al. (2011) where the model (23), (24) has been calibrated using the Italian Human Mortality Database available at http://www.mortality.org/.

In the case studies proposed in Section 4 we use formula (31) and the parameters shown in Table 1.

Note that in order to use the same set of parameters for each cohort in the case studies of Section 4 we choose h_0 to be the hazard rate of an individual aged one year at time zero. This choice of h_0 implies that we must set $t = x$ and $T = x + 4$ in formula (31) to evaluate the probability that an individual aged x in the current year will survive the following four years (see Table 2).

We investigate whether the survival probabilities obtained with the model (23), (24), (25) and formula (31) reflect those in the ISTAT tables. To this end we compute the survival probability, $E(e^{-\int_t^T h_\tau d\tau} \mid h_0)$, using the Monte Carlo method and formula (31) for different choices of the model parameters. We compare the survival probabilities obtained with these two approaches with those of the 2005 and 2010 ISTAT tables.

Let us explain how we use the Monte Carlo method to evaluate the expected value, $E(e^{-\int_t^T h_\tau d\tau} \mid h_0)$, which defines the survival probability.

Let n be a positive integer and let $t_i = t + 0.5(2i-1)(T-t)/n$, $i = 1, 2, \ldots, n$, be the quadrature nodes of the midpoint rule. We approximate the survival probability as n goes to infinity as follows:

$$E\left(e^{-\int_t^T h_\tau d\tau} \mid h_0\right) \approx E\left(e^{-\frac{(T-t)}{n}\sum_{i=1}^{n} h_{t_i}} \mid h_0\right), \quad 0 \leq t \leq T. \qquad (32)$$

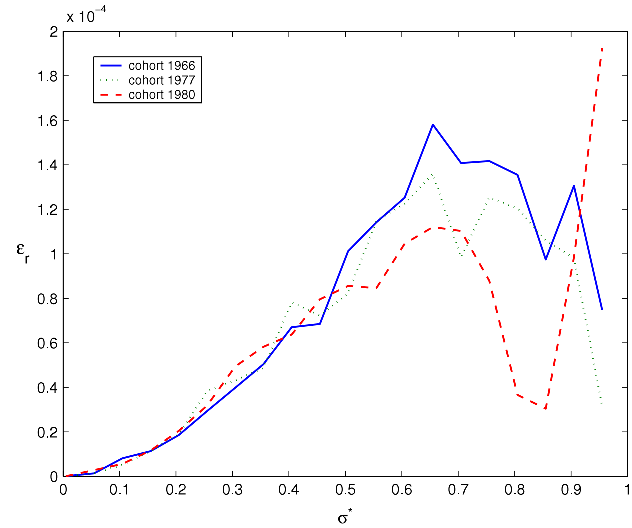

Fig. 1. Relative error ϵ_r between E^{MC} and E^a as a function of the mortality risk volatility σ^* for three cohorts 1966, 1977, 1980. The remaining parameter values are those employed in obtaining the results shown in Table 2 (ISTAT 2010).

We evaluate the right hand side of formula (32) using the Monte Carlo method with 50 000 replications and $n = 183$ quadrature nodes.

Table 2 shows the survival probability computed using the Monte Carlo approach (E^{MC}), formula (31) (E^a) and ISTAT table (E^I). The last column of Table 2 shows the relative error between E^I and E^a. The values of the model parameters are shown in Table 1, while t and T are given respectively in column one and column two of Table 2. The choice of t and T made in Table 2 is motivated by the features of the policies we consider in the case studies. Table 2 shows that formula (31) works well in the time intervals we are interested in. Furthermore, formula (31) outperforms the Monte Carlo method in terms of computational cost. In fact, the time required by the Monte Carlo method is 117 s, while the time required to evaluate formula (31) is $1.09 \cdot 10^{-6}$ s.

Finally, we investigate the performance of formula (31) in approximating the survival probability as the mortality risk volatility increases.

Fig. 1 shows the relative error between the survival probability computed using the Monte Carlo method (E^{MC}) and formula (31) (E^a) as a function of the mortality risk volatility σ^*. More specifically, Fig. 1 shows the quantity $\epsilon_r = |E^a - E^{MC}|/E^{MC}$, for $\sigma^* = \sigma_i^* = 0.005 + 0.05(i-1)$, $i = 1, 2, \ldots, 20$, for the three different cohorts while t and T are chosen from Table 2 year 2010.

Fig. 1 and Table 2 show that the mortality model and formula (31) provide satisfactory values for the survival probability.

We expected this mortality model to perform well since it meets some criteria for being a good stochastic mortality model (see Cairns et al. (2006) for the list of criteria and Luciano and Vigna (2008)). In fact, it depends on a small number of parameters, it fits historic data and it allows us to derive elementary formulae to approximate the survival probability used to evaluate insurance derivatives with great savings in computer time. As underlined in Giacometti et al. (2011), the term e^{at}, $t > 0$, in (24) enables the mortality model to describe two contrasting situations: catastrophic events (wars, epidemics, climate disasters) where the value of the parameter a is a large positive number and improvements in medical treatments for widespread diseases (i.e., for example, oncological and genetic diseases) where a is a large negative number. In Bauer et al. (2008) these events have been considered in specifying the volatility structure of some stochastic mortality models. Their analysis shows that the generalization introduced here is a reasonable one for accounting for these events.

Table 2
Comparison of the survival probability values using the Monte Carlo approach (E^{MC}), formula (31) (E^a), 2005 and 2010 ISTAT tables (E^I) (http://demo.istat.it/). The values of the parameters involved in formula (31) and Monte Carlo approach are those shown in Table 1.

Cohort	t	T	E^{MC}	E^a	ISTAT 2005, E^I	Relative error, $\frac{\lvert E^I-E^a\rvert}{E^a}$
1966	40	44	0.98848	0.98847	0.99074	0.00229
1977	30	34	0.99525	0.99525	0.99557	0.00032
1980	25	29	0.99665	0.99665	0.99603	0.00062
Cohort	t	T	E^{MC}	E^a	ISTAT 2010, E^I	Relative error, $\frac{\lvert E^I-E^a\rvert}{E^a}$
1966	45	49	0.98319	0.98318	0.98674	0.00361
1977	35	39	0.99320	0.99320	0.99469	0.00149
1980	30	34	0.99537	0.99537	0.99618	0.00081

4. Case studies

We solve the optimization problems (16) and (20) using the steepest descent algorithm with variable step-size.

The initial guesses used to solve problem (16) are selected from a set of points uniformly distributed in the box $[-10, 10] \times [-10, 10] \times [-10, 10] \times [0, 10] \subset R^4$. This choice is motivated by the numerical results shown in Dielbold and Li (2006, Table 3 p. 351). The same procedure is used to select the initial guesses used to solve problem (20).

The main steps of the minimization procedure are:

Step 1 generate N_u points uniformly distributed in $[-10, 10] \times [-10, 10] \times [-10, 10] \times [0, 10] \subset R^4$. Select the N_b points where the objective function attains the smallest value. Choose the maximum number of iterations n_{iter}, the largest step-size h_{max} and the tolerance $\epsilon_{tol} > 0$. Set the initial guess counter j equal to one (i.e. set $j = 1$);

Step 2 if $j < N_b$ go to *Step* 3 otherwise go to *Step* 5;

Step 3 set the iteration counter k equal to zero, set the step-size h equal to h_{max} and the initial point equal to the jth guess. Compute the objective function (see formulae (17) and (21)) at this point;

- *Step* 3_1 if the iteration counter k is greater than n_{iter} or the value of the objective function is smaller than ϵ_{tol} go to *Step* 4 otherwise go to *Step* 3_2;
- *Step* 3_2 compute the gradient vector at the current iteration point;
- *Step* 3_3 compute the new iteration point moving along the direction of minus the gradient with step-size h. If the constraints are not satisfied or the value of the objective function at the current iteration point is larger than the value at the previous iteration reduce the step-size h and repeat *Step* 3_3;
- *Step* 3_4 increase the iteration counter k ($k \to k + 1$) go to *Step* 3_1;

Step 4 store the value of the objective function, select a new initial guess $j \to j + 1$ go to *Step* 2;

Step 5 select as the solution of the optimization problem the point where the objective function attains the smallest value.

Note that the procedure is suitable for a parallel implementation. In the numerical experiment we choose $N_u = 10\,000$, $N_b = 100$, $n_{iter} = 1000$ and $\epsilon_{tol} = 10^{-2}$.

4.1. Crédit Agricole Index linked Policy: a comparison between the observed and forecast policy values

In this subsection we consider the Azione Più Capitale Garantito Em.63 policy proposed by Crédit Agricole Vita S.p.A belonging to the insurance company Crédit Agricole Assurances Italia.

The date of issue of Azione Più Capitale Garantito Em.63 policy was March 31, 2010, and the expiration date is May 8, 2015. Should the insured be living on the date of maturity, the company guarantees the payment of the premium plus a variable bonus obtained by multiplying the premium by 50% of the performance of the Dow Jones Euro Stoxx 50 index as shown in formula (8). We evaluate the policy approximating the product (4) as follows:

$$V_A(S_t, t) = E^a\left(e^{-\int_t^T h_\tau d\tau} \mid h_0 = \hat{h}_0\right) \cdot V_1\left(\frac{S_t}{S_r}, t\right), \tag{33}$$

where V_1 is given in formula (9), E^a in formula (31) and S_r is the reference value, that is, the Dow Jones Euro Stoxx 50 index value on *March* 31, 2010 (i.e. $S_r = 2931.16$).

The parameters appearing in formula (31) depend on the cohort considered and are chosen as shown in Table 1.

The parameters appearing in formula (9) are calibrated using procedures P_1 and P_2. As previously mentioned both of these procedures calibrate model (15) solving the optimization problem (16). The data used in the calibration are shown in Table 3. The parameters $\beta_{i,t}$, $i = 1, 2, 3$ and λ_t resulting from the calibration are shown in Table 5.

The value of the Dow Jones on the calibration day is shown in the first column of Table 5. Note that we calibrate the model on the same day we want to evaluate the policy.

Procedure P_1 uses the volatility shown in the second column of Table 5, while procedure P_2 uses the values shown in Table 4 to calibrate model (20). The parameters $\alpha_{i,t}$, $i = 1, 2, 3$ and λ_t^* resulting from the calibration are shown in Table 6.

Finally, the term "relative error" in Tables 7–9 denotes the ratio $|V_A(S_t, t) - V_t|/|V_t|$ where V_t is the observed policy value on date t.

Looking at Tables 7 and 8, that correspond respectively to the 1966 and 1977 cohorts, we can see that the largest relative error between the observed and forecast policy values is one percent. This means that in the worst case the observed and the forecast values have the same first two digits. This implies that the forecast values can have an error of less than fifty cents for a policy value of about ninety euros.

Note that procedure P_2 only slightly improves the forecast values obtained using procedure P_1.

Finally, Table 6 shows that the long term volatility $\alpha_{1,t}$ fluctuates in the period September 2010–January 2012 achieving the smallest value in March 2011 (the earthquake on March 11, 2011 in Japan and the Monthly Economic Report Executive Summary at the end of March 2011) and the largest value in September 2011 (on September 20 Standard and Poor's downgraded its ratings on Italy and the Federal Reserve System Monthly Report announced the "operation twist". However, the financial markets did not consider the measure efficient to deal with debt).

This fluctuation of the long term volatility can be explained by the financial market instability due to the global economic crisis.

4.2. Ergo index linked policy: a comparison between the observed and forecast policy values

In this section we consider the index linked policy proposed by the ERGO Insurance group belonging to the German insurance company ERGO Versicherungsgruppe AG Düsseldorf.

Table 3
Treasury constant maturity yields used to calibrate the Nelson–Siegel model for Crédit Agricole policy.

Date	1-month	3-months	6-months	1-year	3-years	5-years
April 6, 2010	0.0017	0.0017	0.0026	0.0049	0.010074	0.0271
September 20, 2010	0.0012	0.0017	0.0020	0.0026	0.0073	0.0143
March 28, 2011	0.0004	0.0011	0.0018	0.0030	0.0129	0.0223
September 26, 2011	0.00	0.0002	0.0003	0.001	0.0039	0.0092
January 23, 2012	0.0003	0.0004	0.0007	0.0012	0.0039	0.0093

Table 4
VSTOXX (1, 2, 3, 6, 9, 12, 18 and 24 months) volatility indices used to solve problem (20) for Crédit Agricole policy.

Date	1-month	2-months	3-months	6-months	9-months	12-months	18-months	24-months
April 6, 2010	0.1940	0.2082	0.2191	0.2374	0.2485	0.2529	0.2667	0.2683
September 20, 2010	0.2415	0.2329	0.2606	0.2706	0.3095	0.3120	0.3253	0.2882
March 28, 2011	0.2167	0.2298	0.2369	0.2550	0.2572	0.2557	0.2613	0.2673
September 26, 2011	0.5007	0.4765	0.4737	0.4512	0.4379	0.3869	0.4142	0.4086
January 23, 2012	0.2475	0.2740	0.2908	0.3077	0.3161	0.3177	0.3152	0.3135

Table 5
Parameter values of the financial component of the hybrid formula used to forecast the value of the Crédit Agricole policy (maturity $T = July\ 27, 2015$ – date of issue *March* 31, 2010 – Reference value $S_r = 2931.16$ Dow Jones value on *March* 31, 2010).

Date t	Dow Jones value	Volatility value	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$	λ_t
April 6, 2010	2989.49	0.2683	−0.05917	0.962597	0.2055355	1.9182057
September 20, 2010	2082.67	0.2882	0.04165	−0.017557	0.22741	3.90039
March 28, 2011	2914.76	0.2673	0.03984	−0.01231	0.47568	3.90194
September 26, 2011	2083.35	0.4086	0.00399	−0.00623	0.81065	4.59029
January 23, 2012	2441.44	0.3135	0.00463	−0.00696	0.83933	4.00043

Table 6
Parameter values of the volatility model of procedure P_2 used to forecast the value of the Crédit Agricole policy (maturity $T = July\ 27, 2015$ – date of issue *March* 31, 2010 – Reference value $S_r = 2931.16$ Dow Jones value on *March* 31, 2010).

Date t	$\alpha_{1,t}$	$\alpha_{2,t}$	$\alpha_{3,t}$	λ_t^*
April 6, 2010	0.02705228436861	0.03004783119973	0.30000436741861	0.06001809752248
September 20, 2010	0.03066811406861	0.03351477910046	0.30014870219233	0.06062324316651
March 28, 2011	0.00457163537133	0.03434756291845	0.30018799381210	0.05684217385743
September 26, 2011	0.08790658039494	0.03758445186625	0.30031217509593	0.05740040191495
January 23, 2012	0.04285280831334	0.03273655020640	0.30011278147555	0.05650669934812

Table 7
Forecast values of the Crédit Agricole policy (maturity $T = July\ 27, 2015$ – date of issue *March* 31, 2010 – Reference value $S_r = 2931.16$ Dow Jones value on *March* 31, 2010, $K = 100$) – 1966 Cohort – obtained using the two procedures P_1 and P_2.

Cohort	Date	Policy value	Forecast value P_1	Relative error P_1	Forecast value P_2	Relative error P_2	N_{mat}	N_v
1966	April 6, 2010	84.35	82.99	$1.61029 \cdot 10^{-2}$	84.02	$3.8601 \cdot 10^{-3}$	6	8
1966	September 20, 2010	84.90	84.01	$1.03991 \cdot 10^{-2}$	84.68	$2.61849 \cdot 10^{-3}$	6	8
1966	March 28, 2011	83.48	83.54	$7.325878 \cdot 10^{-4}$	83.28	$2.3999 \cdot 10^{-3}$	5	8
1966	September 26, 2011	84.42	84.82	$4.7854 \cdot 10^{-3}$	84.30	$1.4737 \cdot 10^{-3}$	5	8
1966	January 23, 2012	84.01	83.87	$1.6178 \cdot 10^{-3}$	83.90	$1.3685 \cdot 10^{-3}$	5	8

Table 8
Forecast values of the Crédit Agricole policy (maturity $T = July\ 27, 2015$ – date of issue *March* 31, 2010 – Reference value $S_r = 2931.16$ Dow Jones value on *March* 31, 2010, $K = 100$) – 1977 Cohort – obtained using the two procedures P_1 and P_2.

Cohort	Date	Policy value	Forecast value P_1	Relative error P_1	Forecast value P_2	Relative error P_2	N_{mat}	N_v
1977	April 6, 2010	84.35	83.24	$1.3195 \cdot 10^{-2}$	84.27	$9.1631 \cdot 10^{-4}$	6	8
1977	September 20, 2010	84.90	84.96	$7.9001 \cdot 10^{-3}$	84.89	$1.0001 \cdot 10^{-4}$	6	8
1977	March 28, 2011	83.48	83.40	$2.8528 \cdot 10^{-3}$	83.45	$2.8630 \cdot 10^{-4}$	5	8
1977	September 26, 2011	84.42	84.43	$6.5448 \cdot 10^{-3}$	84.44	$2.7468 \cdot 10^{-4}$	5	8
1977	January 23, 2012	84.01	84.12	$7.4255 \cdot 10^{-5}$	84.02	$1.7544 \cdot 10^{-4}$	5	8

Table 9
Forecast values of the Crédit Agricole policy (maturity $T = July\ 27, 2015$ – date of issue *March* 31, 2010 – Reference value $S_r = 2931.16$ Dow Jones value on *March* 31, 2010, $K = 100$) – 1980 Cohort – obtained using the two procedures P_1 and P_2.

Cohort	Date	Policy value	Forecast value P_1	Relative error P_1	Forecast value P_2	Relative error P_2	N_{mat}	N_v
1980	April 6, 2010	84.35	85.26	$1.0779 \cdot 10^{-2}$	86.32	$2.3357 \cdot 10^{-2}$	6	8
1980	September 20, 2010	84.90	86.04	$1.3441 \cdot 10^{-2}$	86.72	$2.1409 \cdot 10^{-2}$	6	8
1980	March 28, 2011	83.48	85.29	$2.1668 \cdot 10^{-2}$	85.02	$1.8470 \cdot 10^{-2}$	5	8
1980	September 26, 2011	84.42	86.34	$2.2799 \cdot 10^{-2}$	85.81	$1.6428 \cdot 10^{-2}$	5	8
1980	January 23, 2012	84.01	85.23	$1.4572 \cdot 10^{-2}$	85.26	$1.4825 \cdot 10^{-2}$	5	8

Table 10
Treasury constant maturity yields used to calibrate the parameters of the Nelson–Siegel model for the ERGO policy.

Date	1-month	3-months	6-months	1-year	3-years
November 5, 2007	0.0386	0.0371	0.0394	0.0386	0.0371
May 5, 2008	0.0134	0.0154	0.0176	0.0198	0.0262
November 10, 2008	0.0011	0.0029	0.0091	0.0116	0.0178
May 11, 2009	0.0016	0.0019	0.0030	0.0053	0.0134
November 9, 2009	0.0006	0.0007	0.0017	0.0034	0.0140

Table 11
VSTOXX (1, 2, 3, 6, 9, 12, 18 and 24 months) volatility indices used to solve problem (20) for ERGO policy.

Date	1-month	2-months	3-months	6-months	9-months	12-months	18-months	24-months
November 5, 2007	0.2473	0.2333	0.2220	0.2307	0.2362	0.2246	0.2405	0.2430
May 5, 2008	0.1938	0.2002	0.2033	0.2088	0.2141	0.2156	0.2195	0.2246
November 10, 2008	0.5889	0.5525	0.4921	0.4896	0.4513	0.3915	0.4238	0.3504
May 11, 2009	0.3685	0.3816	0.3681	0.3588	0.3641	0.3459	0.3496	0.3439
November 9, 2009	0.2566	0.2643	0.2638	0.2827	0.2930	0.2993	0.3027	0.2985

Table 12
Parameter values of the financial component of the hybrid formula used to forecast the value of the Ergo policy (maturity T = *April* 28, 2010 – date of issue *May* 8, 2007 – Reference value $S_r = 4411.32$ Dow Jones value on *May* 8, 2007).

Date t	Dow Jones value	Volatility value	$\beta_{1,t}$	$\beta_{2,t}$	$\beta_{3,t}$	λ_t
November 5, 2007	4392.80	0.2430	0.04315	−0.01385	−0.00824	2.00031
May 5, 2008	3872.15	0.2246	0.04087	0.02720	−0.02023	2.00032
November 10, 2008	2625.84	0.3504	0.01499	−0.21366	0.43031	2.79009
May 11, 2009	2433.59	0.3439	0.02363	−0.21221	0.32572	2.79002
November 9, 2009	2860.11	0.2985	0.02144	−0.01222	−0.01556	1.00000

As mentioned in the Introduction, the date of issue of this policy is May 8, 2007 and the expiration date is April 28, 2010. On maturity, should the insured be living, the company guarantees the payment of the premium plus a variable bonus obtained by multiplying the premium by 84% of the performance of the Dow Jones Euro Stoxx 50 index as shown in formula (12). We evaluate the policy using the following approximation of the product (4):

$$V_E(S_t, t) = E^a \left(e^{-\int_t^T h_\tau d\tau} \mid h_0 = \hat{h}_0\right) \cdot V_2\left(\frac{S_t}{S_r}, t\right), \tag{34}$$

where V_2 is given in formula (13), E^a in formula (31) and S_r is the reference value, that is, the Dow Jones Euro Stoxx 50 index value on *May* 8, 2007 (i.e. $S_r = 4411.32$).

The parameters appearing in formula (31) depend on the cohort considered and are chosen as shown in Table 1.

As mentioned in Section 4.1, the parameters appearing in formula (9) are calibrated using procedures P_1 and P_2. The parameters relative to the risk free interest rate model are obtained solving the optimization problem (16). The data used in the calibration are shown in Table 10 and the results of the calibration (i.e. the parameters $\beta_{i,t}$, $i = 1, 2, 3$ and λ_t) are shown in Table 12.

The value of the index on the calibration day is shown in the first column of Table 12. Note that we calibrate the model on the same day we want to evaluate the policy.

Procedure P_1 uses the volatility shown in the second column of Table 12, while procedure P_2 uses the value shown in Table 11 to calibrate model (20). The parameters $\alpha_{i,t}$, $i = 1, 2, 3$ and λ_t^* resulting from the calibration are shown in Table 13.

Finally, as in Section 4.1 the term "relative error" in Tables 14–16 denotes the ratio $|V_E(S_t, t) - V_t|/|V_t|$.

Note that Tables 14–16 show forecast values that are more satisfactory than those obtained in Tables 7–9. The results on the Ergo policy confirm that the 1977 cohort seems to be the reference cohort in the policy evaluation. The relative errors show in Tables 14 and 15 guarantee that, in the worst case, the forecast values can have an error of less than one euro for a policy value of one thousand euros.

We note that procedure P_2 slightly improves the forecast values obtained with procedure P_1 except on November 5, 2007 and May 5, 2008. Moreover, the quality of forecast values deteriorates going from April 2007 to November 2008 and later improves (see Tables 14 and 15). This is probably due to changes that affected the volatilities (see Table 11) from November 2007 to November 2008, which was the period that culminated in the collapse of the Lehman Brothers on September 2008.

Table 13 shows that the parameter $\alpha_{1,t}$, that must to be interpreted as the long term volatility, shows significant changes in the period November 2007 to November 2008. The movement of the parameter $\alpha_{2,t}$, that is the short term volatility, is sizable from May 2008 to November 2008, while the remaining parameters remain substantially unchanged. Table 12 shows that the parameters of model (15) change significantly from May 2008 to May 2009. This can be explained by the fact that after the collapse of Lehman Brothers the financial markets needed time to find a new equilibrium.

We conclude this section highlighting two facts on the parameter values resulting from the calibration of procedure P_2. The first one concerns the parameters $\alpha_{j,t}$, $j = 1, 2, \ldots, 4$, shown in Tables 6 and 13. In fact the parameters concerning the short and medium term volatility remains substantially unchanged while the long term volatility parameter mutates significantly. It seems that the "volatility structure" of the two periods considered (April 2010–January 2012 and November 2007–November 2009) is similar. On the contrary, the parameters $\beta_{j,t}$, $j = 1, 2, \ldots, 4$, shown in Tables 5 and 11 are very different. Tables 5 and 11 show that the long term interest rate factor $\beta_{1,t}$ has been steadily reduced since November 2007 and that it may be further reduced in the near future.

5. Conclusions

In this paper two procedures for evaluating pure endowment policies are proposed and two case studies are presented to illustrate their performance.

Table 13
Parameter values of the volatility model of procedure P_2 used to forecast the value of the Ergo policy (maturity $T = April$ 28th, 2010 – date of issue May 8, 2007 – Reference value $S_r = 4411.32$ Dow Jones value on May 8, 2007).

Date t	$\alpha_{1,t}$	$\alpha_{2,t}$	$\alpha_{3,t}$	λ_t^*
November 5, 2007	0.01107091655956	0.03092886822081	0.30003829879652	0.05577315697222
May 5, 2008	0.00709112876650	0.03008981743826	0.30000138308351	0.05560673972022
November 10, 2008	0.07600731324011	0.03677730539915	0.30022414548394	0.05662581345016
May 11, 2009	0.08584779915678	0.02891120861765	0.29993890336330	0.05532688135876
November 9, 2009	0.07477507909707	0.02500195453560	0.29978046342505	0.05461038526222

Table 14
Forecast values of the Ergo policy (maturity $T = April$ 28, 2010 – date of issue May 8, 2007 – Reference value $S_r = 4411.32$ Dow Jones value on May 8, 2007) – 1966 Cohort – procedures P_1 and P_2.

Cohort	Date	Policy value	Forecast value P_1	Relative error P_1	Forecast value P_2	Relative error P_2	N_{mat}	N_v
1966	November 5, 2007	1019.02	1018.10	$9.0252 \cdot 10^{-4}$	1018.06	$9.3708 \cdot 10^{-4}$	5	8
1966	May 5, 2008	959.93	958.95	$1.0155 \cdot 10^{-3}$	959.50	$4.4363 \cdot 10^{-4}$	5	8
1966	November 10, 2008	910.98	910.11	$9.5588 \cdot 10^{-4}$	910.24	$8.0942 \cdot 10^{-4}$	5	8
1966	May 11, 2009	946.39	945.88	$5.4129 \cdot 10^{-4}$	946.23	$1.6861 \cdot 10^{-4}$	5	8
1966	November 9, 2009	987.99	987.51	$4.8682 \cdot 10^{-4}$	988.15	$1.6258 \cdot 10^{-4}$	5	8

Table 15
Forecast values of the Ergo policy (maturity $T = April$ 28, 2010 – date of issue May 8, 2007 – Reference value $S_r = 4411.32$ Dow Jones value on May 8, 2007) – 1977 Cohort – procedures P_1 and P_2.

Cohort	Date	Policy value	Forecast value P_1	Relative error P_1	Forecast value P_2	Relative error P_2	N_{mat}	N_v
1977	November 5, 2007	1019.02	1019.06	$3.9456 \cdot 10^{-5}$	1019.025	$5.1239 \cdot 10^{-6}$	5	8
1977	May 5, 2008	959.93	959.63	$3.1706 \cdot 10^{-4}$	960.17	$2.5519 \cdot 10^{-4}$	5	8
1977	November 10, 2008	910.98	910.55	$4.7184 \cdot 10^{-4}$	910.68	$3.2530 \cdot 10^{-4}$	5	8
1977	May 11, 2009	946.39	946.16	$2.4391 \cdot 10^{-4}$	946.51	$1.2888 \cdot 10^{-4}$	5	8
1977	November 9, 2009	987.99	987.64	$3.5029 \cdot 10^{-4}$	988.28	$2.9919 \cdot 10^{-4}$	5	8

Table 16
Forecast values of the Ergo policy (maturity $T = April$ 28, 2010 – date of issue May 8, 2007 – Reference value $S_r = 4411.32$ Dow Jones value on May 8, 2007) – 1980 Cohort – procedures P_1 and P_2.

Cohort	Date	Policy value	Forecast value P_1	Relative error P_1	Forecast value P_2	Relative error P_2	N_{mat}	N_v
1980	November 5, 2007	1019.02	1029.06	$9.8490 \cdot 10^{-3}$	1029.021	$9.8141 \cdot 10^{-3}$	5	8
1980	May 5, 2008	959.93	966.95	$7.3203 \cdot 10^{-3}$	967.51	$7.8970 \cdot 10^{-3}$	5	8
1980	November 10, 2008	910.98	915.63	$5.1052 \cdot 10^{-3}$	915.76	$5.2526 \cdot 10^{-3}$	5	8
1980	May 11, 2009	946.39	946.58	$3.3784 \cdot 10^{-3}$	949.94	$3.7524 \cdot 10^{-3}$	5	8
1980	November 9, 2009	987.99	989.38	$1.4140 \cdot 10^{-3}$	990.02	$2.0647 \cdot 10^{-3}$	5	8

The policy premium is expressed as the product of two expected values. One models the financial component of the policy and the other one the survival probability. The financial component of the policy is prescribed using a time dependent Black Scholes model. The integral of the time dependent risk free interest rate and the volatility are modeled using an extension of the Nelson and Siegel yield curve (see Dielbold and Li (2006)). The insurance component of the policy is described using a generalization of the geometric Brownian mean reverting Gompertz model. A perturbation expansion of the survival probability for small mortality risk volatility is proposed. The sum of the first three terms of this expansion provides a formula for approximating the survival probability. This formula involves elementary functions and its performance is measured comparing the survival probabilities obtained with this formula to those provided by ISTAT Tables.

The main advantages of the proposed formula for the policy premium are its simplicity, its dependence on a small number of parameters and its negligible computational cost compared with the cost required by the Monte Carlo method.

The proposed calibration procedures use data from the US treasury constant maturity yields and the VSTOXX volatility indices to estimate the parameters of the financial component and the Italian human mortality database in Giacometti et al. (2011) to estimate the insurance component. The case studies considered show that the calibration procedures provide estimated values of the policies with two or three correct significant digits when compared with the observed values.

Further research should investigate the calibration of the proposed model to different data sets: historical mortality data and mortality bond prices. The comparison of model parameter values obtained by these two calibrations could be a way of measuring the market price of risk. This is a challenging problem which has been recently studied, for example, in Bauer et al. (2008).

Appendix

Let us prove formula (31) of Section 3. To this end we evaluate the three expected values appearing in formula (29), that is:

$$
\begin{aligned}
&E\left(e^{-\int_t^T h_\tau d\tau} \mid h_0 = \hat{h}_0\right) \\
&\quad = e^{-\hat{h}_0 \int_t^T e^{g\tau} d\tau} \left\{ 1 - \hat{h}_0 \sigma^* \int_t^T E(y_\tau \mid y_0 = 0)\, e^{g\tau} d\tau \right. \\
&\qquad + \frac{\hat{h}_0^2 (\sigma^*)^2}{2} E\left(\left[\int_t^T y_\tau e^{g\tau} d\tau\right]^2 \Big| y_0 = 0\right) \\
&\qquad \left. - \frac{\hat{h}_0 (\sigma^*)^2}{2} \int_t^T E(y_\tau^2 \mid y_0 = 0) e^{g\tau} d\tau \right\} + o((\sigma^*)^2), \\
&\quad t, T \geq 0,\ t < T,\ \sigma^* \to 0.
\end{aligned} \tag{35}
$$

Using (24), (25), it is easy to derive the following formulae:

$$E(y_\tau \mid y_0 = 0) = 0,$$
$$E(y_\tau^2 \mid y_0 = 0) = e^{2a\tau}\frac{1 - e^{-2(a+b)\tau}}{2(a+b)}, \quad \tau > 0,\ b > 0 \tag{36}$$

and

$$E(y_\tau y_s \mid y_0 = 0) = \frac{e^{-b(\tau+s)}}{2(a+b)}\left(e^{2(a+b)\min\{s,\tau\}} - 1\right),$$
$$s,\tau > 0,\ b > 0. \tag{37}$$

Substituting (36) into (29) when $g - 2b \neq 0$ we have:

$$E\left(e^{-\int_t^T h_\tau d\tau} \mid h_0 = \hat{h}_0\right)$$
$$= e^{-\hat{h}_0(e^{gT}-e^{gt})/g} \cdot \left\{1 - \frac{\hat{h}_0(\sigma^*)^2}{4(a+b)}\left[\left(\frac{e^{(g+2a)T}}{g+2a} - \frac{e^{(g-2b)T}}{g-2b}\right)\right.\right.$$
$$\left.\left. - \left(\frac{e^{(g+2a)t}}{g+2a} - \frac{e^{(g-2b)t}}{g-2b}\right)\right] + \frac{\hat{h}_0^2(\sigma^*)^2}{2}\right.$$
$$\left. \times\ E\left(\left[\int_t^T y_\tau e^{g\tau} d\tau\right]^2 \middle| y_0 = 0\right)\right\} + o((\sigma^*)^2)$$
$$t, T \geq 0,\ t < T,\ \sigma^* \to 0. \tag{38}$$

Let us evaluate $E\left(\left[\int_t^T y_\tau e^{g\tau} d\tau\right]^2 \mid y_0 = 0\right)$. Using (37) we have:

$$E\left(\left[\int_t^T y_\tau e^{g\tau} d\tau\right]^2 \middle| y_0 = 0\right)$$
$$= E\left(\left[\int_t^T e^{g\tau} d\tau \int_t^T e^{gs} ds\, y_\tau y_s\right] \middle| y_0 = 0\right)$$
$$= \int_t^T d\tau\, e^{g\tau} \int_t^T ds\, e^{gs} E(y_\tau y_s \mid y_0 = 0)$$
$$= \frac{1}{2(a+b)} \int_t^T d\tau\, e^{g\tau} \int_t^T ds\, e^{gs} e^{-b(s+\tau)}\left(e^{2(a+b)\min\{s,\tau\}} - 1\right)$$
$$= -\frac{1}{2(a+b)} \int_t^T d\tau\, e^{(g-b)\tau} \int_t^T ds\, e^{(g-b)s}$$
$$+ \frac{1}{2(a+b)} \int_t^T d\tau\, e^{(g-b)\tau} \int_t^T ds\, e^{(g-b)s} e^{2(a+b)\min\{s,\tau\}},$$
$$t, T \geq 0,\ t < T. \tag{39}$$

Finally, under the assumption $g - b \neq 0$ we obtain:

$$E\left(\left[\int_t^T y_\tau e^{g\tau} d\tau\right]^2 \middle| y_0 = 0\right)$$
$$= -\frac{1}{2(a+b)}\left(\frac{e^{(g-b)T} - e^{(g-b)t}}{g-b}\right)^2$$
$$+ \frac{1}{2(a+b)} \cdot \frac{1}{(g+b+2a)}\left[\left(\frac{e^{2(g+a)T} - e^{2(g+a)t}}{2(g+a)}\right)\right.$$
$$\left. - e^{(g+b+2a)t}\left(\frac{e^{(g-b)T} - e^{(g-b)t}}{g-b}\right)\right],$$
$$+ \frac{1}{2(a+b)}\frac{1}{(g-b)}\left[e^{(g-b)T}\left(\frac{e^{2(g+b+2a)T} - e^{2(g+b+2a)t}}{g+b+2a}\right)\right.$$
$$\left. - \left(\frac{e^{2(g+a)T} - e^{(g+a)t}}{2(g+a)}\right)\right], \quad t, T \geq 0,\ t < T. \tag{40}$$

Substituting (40) into (38) we obtain:

$$E\left(e^{-\int_t^T h_\tau d\tau} \mid h_0 = \hat{h}_0\right) = e^{-\hat{h}_0(e^{gT}-e^{gt})/g} \cdot \left\{1 - \frac{\hat{h}_0(\sigma^*)^2}{4(a+b)}\right.$$
$$\times \left[\left(\frac{e^{(g+2a)T}}{g+2a} - \frac{e^{(g-2b)T}}{g-2b}\right) - \left(\frac{e^{(g+2a)t}}{g+2a} - \frac{e^{(g-2b)t}}{g-2b}\right)\right]$$
$$+ \frac{\hat{h}_0^2(\sigma^*)^2}{2}\left[-\frac{1}{2(a+b)}\left(\frac{e^{(g-b)T} - e^{(g-b)t}}{g-b}\right)^2\right.$$
$$+ \frac{1}{2(a+b)}\frac{1}{(g+b+2a)}\left[\left(\frac{(e^{2(g+a)T} - e^{2(g+a)t})}{2(g+a)}\right)\right.$$
$$\left. - e^{(g+b+2a)t}\left(\frac{e^{(g-b)T} - e^{(g-b)t}}{g-b}\right)\right]$$
$$+ \frac{1}{2(a+b)}\frac{1}{(g-b)}\left[e^{(g-b)T}\left(\frac{e^{2(g+b+2a)T} - e^{2(g+b+2a)t}}{g+b+2a}\right)\right.$$
$$\left.\left.\left. - \left(\frac{e^{2(g+a)T} - e^{(g+a)t}}{2(g+a)}\right)\right]\right]\right\} + o((\sigma^*)^2),$$
$$t, T \geq 0,\ t < T,\ \sigma^* \to 0. \tag{41}$$

Note that when $g = 2b$ or $g = b$ formula (41) should be slightly modified. Formula (41) can be rewritten as done in formulae (30) and (31).

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