International Scholarly Research Network
ISRN Geometry
Volume 2011, Article ID 694605, 6 pages
doi:10.5402/2011/694605

Research Article

On Twisted Tensor Product Group Embeddings and the Spin Representation of Symplectic Groups: The Case q Odd

Antonio Cossidente

Dipartimento di Matematica e Informatica, Università degli studi della Basilicata, Contrada Macchia Romana, 85100 Potenza, Italy

Correspondence should be addressed to Antonio Cossidente, antonio.cossidente@unibas.it

Received 29 March 2011; Accepted 2 May 2011

Academic Editors: M. Khalkhali and U. Lindström

The group $PSp_8(q)$, q odd, has a maximal subgroup isomorphic to $3.PSp_2(q^3)$ belonging to the Aschbacher class $\mathcal{C}_9$. It is the full stabilizer of a complete partial ovoid and of a complete partial 3-spread of $\mathcal{W}_7(q)$.

1. Introduction

Let G be a classical group associated with a finite dimensional vector space over $\mathrm{GF}(q)$, say V. In his celebrated paper [1], Aschbacher describes a family $\mathcal{C}$ of eight "geometric" classes of subgroups of G and shows that any subgroup of G either lies in one of these classes or has the form $H = N_G(X)$, for some quasisimple subgroup X of G satisfying some special conditions. Given such a group H not lying in one of the eight classes of $\mathcal{C}$, the main purpose is to determine whether or not H is maximal in G. If not, there exists a quasisimple subgroup K with $X < K < G$ and one wants to study such configurations, possibly from a geometric viewpoint. For more details, see [2].

Let $\mathbf{G}$ be a finite classical group with natural module V_0 of dimension $n \geq 2$ over the Galois field $\mathrm{GF}(q^t)$. Let $V_0^{\varphi^i}$ denote the $\mathbf{G}$-module V_0 with group action given by $g \cdot v = g^{\varphi^i}(v)$, where g^{φ^i} denotes the matrix g with its entries raised to the q^ith power, $i = 0, \ldots, t-1$. Then one can form the so-called *twisted tensor product module* $V_0 \otimes V_0^{\varphi} \otimes \cdots \otimes V_0^{\varphi^{t-1}}$. Such a module can be realized over the subfield $\mathrm{GF}(q)$ of $\mathrm{GF}(q^t)$. This gives rise to an absolutely irreducible representation of the group $\mathbf{G}$ on an n^t-dimensional natural module over $\mathrm{GF}(q)$. If $\mathbf{G}$ is a symplectic group, then under the twisted tensor product embedding $\mathbf{G}$ turns out to be again a subgroup of a symplectic group, and only when q is even, it is actually a subgroup of an orthogonal group, see [3].

Such representations are given by Steinberg [4] and further studied by Seitz [2]. See also [5]. We refer to this class of subgroups as $\mathcal{C}_9$, as suggested by Seitz.

In [6] we studied the geometry of two classes of twisted tensor product group embeddings: $PSp_2(q^t) \leq P\Omega^+_{2^t}(q)$, where $t \geq 2$ and q is even; and $PSp_{2m}(q^t) \leq P\Omega^\epsilon_{(2m)^t}(q)$ with q even. We will use $Sp_2(q)$ although some references will use $\mathrm{SL}_2(q)$. We found that our embedding of $PSp_2(q^t)$ is associated with an embedding of the projective line $PG(1,q^t)$ as a complete partial ovoid of a quadric in $PG(2^t-1,q)$ (i.e., a maximal set of pairwise nonorthogonal points of the quadric); if $t \geq 3$, then the quadric is hyperbolic. Such partial ovoids are of some interest because their size attains the Blokhuis-Moorhouse bound [7]. In particular, when $t = 3$ and $q \geq 4$, the embedding yields another description of the Desarguesian ovoid of the hyperbolic quadric of $PG(7,q)$ [8]. Similarly, the embedding of $PSp_{2m}(q^t)$, q even, in $P\Omega^\epsilon_{(2m)^t}(q)$ has a particular application when $m = 2$ in the embeddings of symplectic ovoids of $PG(3,q^t)$ as partial ovoids of hyperbolic quadrics of $PG(4^t-1,q)$ again with the size attaining the Blokhuis-Moorhouse bound.

In [9] we investigated further these twisted tensor product group embeddings, but from a different perspective. We showed how the n^t-dimensional module over $\mathrm{GF}(q)$ for $\mathbf{G}$ may be viewed projectively as a subspace of the projective space $PG(\binom{nt}{t}-1,q)$ containing the Grassmannian of $(t-1)$-subspaces of $PG(nt-1,q)$. From this viewpoint $\mathbf{G}$ preserves the intersection of the subspace and the Grassmannian. When $n = 2m \leq 4$, this approach enabled us to address some questions on maximality. We proved that under the twisted tensor product group embedding of $PSp_{2m}(q^t)$, $m \leq 2$, q even, an intermediate embedding of type $\mathcal{C}_3$ occurs: $PSp_{2m}(q^t) < PSp_{2mt}(q) < P\Omega_{(2m)^t}(q)$. The partial ovoid referred to above lies on a unique quadric in $PG((2m)^t-1,q)$. It turns out that this quadric is precisely that arising from the spin representation of $Sp_{2mt}(q)$.

Note that class $\mathcal{C}_9$ has also been studied by Schaffer in [10], where he used representation theory techniques; his arguments rely on the Classification of Finite Simple Groups. He eliminated a number of possibilities, largely when t is composite and showed that the remaining subgroups in this class are maximal except in a small number of cases. The main exceptions are precisely $PSp_2(q^t) \leq P\Omega^+_{2^t}(q)$ and $PSp_4(q^t) \leq P\Omega^\epsilon_{4^t}(q)$ with q even.

In this paper we consider the twisted tensor product embedding of $PSp_2(q^t)$ inside $PSp_{2^t}(q)$ when q is odd, in the smallest case, that is, $t = 3$. The normalizer N of $PSp_2(q^3)$ in $PSp_8(q)$, that has structure $3.PSp_2(q^3)$, is maximal in $PSp_8(q)$ [10]. We study the action of N on points of $PG(7,q)$. It turns out that N is the full stabilizer in $PSp_8(q)$ of a complete partial ovoid and also of a complete partial spread of the symplectic space $\mathcal{W}_7(q)$. The partial ovoid is of some interest because of its connections with the generalized hexagons of type ${}^3D_4(q)$ and $G_2(q)$, see [11].

2. The Geometric Approach to the Twisted Tensor Product Embedding

In this section, specializing to the case $n = 2$ and $t = 3$, we recall the alternative perspective for at least some of the subgroups in the Aschbacher's class $\mathcal{C}_9$ given in [9].

Let E_i, $1 \leq i \leq 3$, be 2-dimensional vector spaces over $\mathrm{GF}(q^3)$ and let $E = E_1 \oplus E_2 \oplus E_3$. Suppose that for each i, e_{i1}, e_{i2} is a basis for E_i and suppose that $H \leq \mathrm{GL}(E_1)$. For $v = \sum_j \lambda_j e_{ij} \in E_i$ we write $v^\Psi = \sum_j \lambda_j^q e_{i+1j} \in E_{i+1}$ (with $i+1$ interpreted modulo 3), and for $h \in H$

we write h^{Ψ} for the matrix h with every entry raised to the power q. Hence, to any $v \in E_1$ there correspond "conjugate" vectors $v^{\Psi^{i-1}} \in E_i$ and H acts on E_i via $h^{\Psi^{i-1}}(v^{\Psi^{i-1}}) = (hv)^{\Psi^{i-1}}$. Therefore we have an action of H on E and H preserves a fibration of E into 3-dimensional subspaces of the form $\langle v, v^{\Psi}, v^{\Psi^2}\rangle$. In projective terms, E corresponds to a projective space $\Sigma = PG(5, q^3)$ and H preserves a partial 2-spread of Σ. We may regard Ψ as a semilinear map on E. The vectors in E fixed by Ψ are precisely the vectors $v + v^{\Psi} + v^{\Psi^2}$, where $v \in E_1$, and they form a 6-dimensional vector space V over $\mathrm{GF}(q)$ that spans E and is preserved by H. In Σ we have a set of points preserved by H forming a subgeometry $\Sigma_0 = PG(5, q)$ and on restriction, the partial 2-spread above becomes a 2-spread $\mathcal{S}$ of Σ_0 preserved by H. Suppose that H preserves a nondegenerate alternating form f_1 on E_1, then H preserves the alternating form f_i on E_i given by $f_i(u^{\Psi^{i-1}}, w^{\Psi^{i-1}}) = f_1(u, w)^{q^{i-1}}$ and an alternating form f on E in which $f_{|E_i}$ is f_i and in which $E_1 \oplus E_2 \oplus E_3$ is an orthogonal decomposition. Moreover the restriction of f to V is a nondegenerate alternating form on V. Thus H acts as a subgroup of $Sp_2(q^3)$ embedded in $Sp_6(q)$ on Σ_0 preserving a spread $\mathcal{S}$ consisting now of totally isotropic planes.

Consider the 3-fold alternating product of E, $\bigwedge^3(E)$, an H-module of dimension $\binom{6}{3}$ over $\mathrm{GF}(q^3)$. If $A \oplus B$ is any decomposition for E, then

$$\bigwedge^{3}(E) = \bigoplus_{i+j=3}\left(\bigwedge^{i}(A) \otimes \bigwedge^{j}(B)\right). \tag{2.1}$$

Thus $\bigwedge^3(E)$ has a subspace $\bigwedge^1(E_1) \otimes \bigwedge^2(E_2 \oplus E_3)$ and, by iteration, a subspace $\bigwedge^1(E_1) \otimes \bigwedge^1(E_2) \otimes \bigwedge^1(E_3)$, that is, $E_1 \otimes E_2 \otimes E_3$. This latter subspace is preserved by H. The 3-dimensional subspaces of E correspond to 1-dimensional subspaces of $\bigwedge^3(E)$. Each 3-dimensional $\mathrm{GF}(q)$-subspace of V determines a 3-dimensional $\mathrm{GF}(q^3)$-subspace of E and so $\bigwedge^3(V)$ may be regarded as a $\mathrm{GF}(q)$-subspace of $\bigwedge^3(E)$. For any $v \in E_1$, the 3-subspace $\langle v, v^{\Psi}, v^{\Psi^2}\rangle$ is mapped to the 1-dimensional subspace corresponding to $v \wedge v^{\Psi} \wedge v^{\Psi^2} \in E_1 \otimes E_2 \otimes E_3$. In projective terms $PG(\binom{6}{3} - 1, q^3)$ contains the Grassmannian $\mathcal{G}$ of planes of Σ and $E_1 \otimes E_2 \otimes E_3$ corresponds to a 7-dimensional subspace Δ of $PG(\binom{6}{3} - 1, q^3)$ containing the image of the partial spread and it is fixed by H. The planes of Σ_0 form a Grassmannian $\mathcal{G}_0$ lying in a projective space $PG(\binom{6}{3} - 1, q)$ that is a subgeometry of $PG(\binom{6}{3} - 1, q^3)$. Each of the subspaces of $\mathcal{S}$ is mapped into $\Delta \cap \mathcal{G}_0$.

As showed in [5, 2.4.1] the points of $PG(1, q^3)$ may be represented as points of $PG(7, q)$. Given that **G** preserves the set of all such points and that **G** acts irreducibly, these points must span $PG(7, q)$.

Let us return to $\mathcal{S}$ and its image in $\Delta \cap \mathcal{G}_0$. We have seen that these points in $\Delta \cap \mathcal{G}_0$ may be represented by $v \otimes v^{\Psi} \otimes v^{\Psi^2}$ as v varies in E_1. Moreover we may take H to be the group $\mathrm{SL}_2(q^3)$ acting absolutely irreducibly on E_1. Hence the points corresponding to $\mathcal{S}$ generate a $\mathrm{GF}(q)$-subspace Δ_0 of projective dimension 5. It follows that the $\mathrm{GF}(q^3)$-span of $\mathcal{S}$ is precisely Δ. Hence we see the twisted tensor product module for $\mathrm{SL}_2(q^3)$ as the subspace Δ_0 of $PG(\binom{6}{3} - 1, q)$.

Observe that in one setting we have H acting as a subgroup of $\mathrm{GL}_2(q^3)$ on $PG(5, q)$, so here it is an Aschbacher $\mathcal{C}_3$ group. In a second setting it is a subgroup of $\mathrm{GL}_8(q)$ and lies in Aschbacher class $\mathcal{C}_9$.

3. The Embedding $PSp_2(q^3) < PSp_8(q)$, q Odd

We consider a vector space V of dimension 6 and the corresponding projective space $\Sigma_0 = PG(5,q)$.

Let $\mathcal{I}$ be the set of all totally isotropic planes of Σ_0 with respect to a nondegenerate alternating form f and let $\mathcal{S}$ be a regular spread of Σ_0 (with elements in $\mathcal{I}$). Then the Grasmannian, $\mathcal{G}_0$, of planes of Σ_0 has dimension $\binom{6}{3} - 1$ and the image of $\mathcal{I}$ in $\mathcal{G}_0$ spans a subspace F_3 of dimension $\binom{6}{3} - \binom{6}{1} - 1$. The vector space equivalent of F_3 is the Weyl module of $Sp_6(q)$ for the fundamental weight λ_3. When q is even, F_3 has a unique maximal subspace fixed by $PSp_6(q)$, denoted N_3. The quotient space $M_3 = F_3/N_3$ has dimension 7 and corresponds to the spin module for $Sp_6(q)$. For more details, see [3, 12–16].

When q is odd, M_3 is the direct sum of the three twists of E_1 and their twisted tensor product. The symplectic form on $\bigwedge^3(V)$ is given by the wedge product

$$\bigwedge^3(V) \times \bigwedge^3(V) \longrightarrow \bigwedge^6(V) = \mathrm{GF}(q). \tag{3.1}$$

The restriction of this alternating form to F_3 must be nonsingular since F_3 is a simple module. By projection, we get an embedding of $PSp_2(q^3)$ in $PSp_8(q)$, that is, $PSp_2(q^3)$ in its twisted tensor product group representation.

In [9] we proved the following theorem.

Theorem 3.1. *Under the twisted tensor product group embedding $PSp_2(q^3) < P\Omega_8^+(q)$, $q \geq 4$ even, an intermediate $\mathcal{C}_3$-embedding occurs: $PSp_2(q^3) < PSp_6(q) < P\Omega_8^+(q)$.*

Moreover, $N_{P\Omega_8^+(q)}(PSp_2(q^3))$ is the stabilizer of $\mathcal{O}$ in $P\Omega_8^+(q)$ and it is a maximal subgroup of $PSp_6(q)$.

Remark 3.2. It is a consequence of [17, Theorem I] that $Sp_6(q)$, q even, in its spin representation, is a maximal subgroup of $P\Omega_8^+(q)$. See also [18, 19].

Now, we focus on the case q odd.

It is easy to see that under the twisted tensor product embedding, $PSp_2(q^3)$, q odd, turns out to be a subgroup of $PSp_8(q)$ rather than a subgroup of $P\Omega_8^+(q)$, and it fixes a partial ovoid $\mathcal{O}$ of $\mathcal{W}_7(q)$ of size q^3+1, that is, a set of q^3+1 points no two of them conjugate with respect to f.

Lemma 3.3. *The normalizer N of $PSp_2(q^3)$ in $PSp_8(q)$ stabilizes $\mathcal{O}$.*

Proof. Let F be the stabilizer in $PSp_2(q^3)$ of a point of the projective line $PG(1,q^3)$. Then F can also be described as the normalizer in $PSp_2(q^3)$ of a Sylow p-subgroup S of $PSp_2(q^3)$ and there is a cyclic subgroup C of F such that $C \cap S = 1$ and $F = CS$. Exactly one point of $PG(1,q^3)$ is fixed by F and exactly one point of $\mathcal{O}$ is fixed by the image of F under the twisted tensor product embedding, say $\widetilde{F}$. Suppose that $\widetilde{F}$ fixes exactly one point P of $PG(7,q)$ (necessarily P will be in $\mathcal{O}$). If $h \in N$, then $h^{-1}Fh$ fixes Ph, but $h^{-1}\widetilde{F}h$ is the normalizer in $PSL_2(q^3)$ of a Sylow p-subgroup so it fixes a point of $\mathcal{O}$. Hence $Ph \in \mathcal{O}$. It follows that $\mathcal{O}h = \mathcal{O}$. □

As in the case q even, the normalizer N of $PSp_2(q^3)$ in $PSp_8(q)$ has structure $3.PSp_2(q^3)$. It should be noted that N stabilizes also a partial spread of $\mathcal{W}_7(q)$ of size q^3+1 consisting of maximal totally isotropic subspaces of $\mathcal{W}_7(q)$ tangent to $\mathcal{O}$. The action of N on $\mathcal{O}$ and on $\mathcal{S}$ is 2-transitive, see [11, Lemma 4.4(a)].

Proposition 3.4. *The group N has four orbits on points of $PG(7,q)$: $\mathcal{O}$ of size q^3+1, $\mathcal{O}_1$ of size $q^3(q^3+1)(q-1)/2$ consisting of points on secant lines to $\mathcal{O}$; $\mathcal{O}_2$ of size $(q^3+1)(q^3+q^2+q)$ consisting of points on members of the partial spread $\mathcal{S}$ and $\mathcal{O}_3$ of size $q^3(q^3-1)(q+1)/2$.*

Proof. It is sufficient to prove that N has three orbits on $PG(7,q)\setminus\mathcal{O}$. Take 2 points P_1 and P_2 on $\mathcal{O}$. The line L joining P_1 and P_2 is hyperbolic. The stabilizer of P_1, P_2 in N acts transitively on $L\setminus\{P_1,P_2\}$. Since N acts 2-transitively on $\mathcal{O}$, we get the orbit $\mathcal{O}_1$ of size $q^3(q^3+1)(q-1)/2$. If $P\in\mathcal{O}$, the stabilizer of P in N has order $3(q-1)q^3(q^2+q+1)/2$. As we have seen, there is a unique member S_P of $\mathcal{S}$ on P. Moreover, $\mathrm{Stab}_N(P)$ acts transitively on $S_P\setminus\{P\}$. This way, we obtain the orbit $\mathcal{O}_2$ of size $(q^3+1)(q^3+q^2+q)$. To determine the fourth N-orbit $\mathcal{O}_3$ we need some information on the twisted tensor product embedding of a Singer cyclic group S of $PSp(2,q^3)$. In $PSp_2(q^6)$ a Singer cycle has the diagonal representation $\mathrm{diag}(\omega,\omega^{q^3})$, where ω is a primitive element of $\mathrm{GF}(q^6)$ over $\mathrm{GF}(q)$. The $(q+1)$th power T of the twisted tensor product embedding of S has the diagonal representation $\mathrm{diag}(\omega^a,\omega^{aq},\omega^{aq^2},\omega^{aq^3},\omega^{aq^4},\omega^{aq^5},\omega^b,\omega^{bq})$, where $a=q^3+2q^2+2q+1$ and $b=q^5+q^4+q^3+q^2+q+1$. It turns out that T fixes a line ℓ pointwise and a projective 5-space X setwise inducing a unitary Singer cyclic group of order q^2-q+1. In particular, from the diagonal representation of T, we see that each T-orbit not on ℓ or X (that has size q^2-q+1, generates a projective 6-space and its stabilizer in N has order $3(q^2-q+1)$. This way we get the N-orbit $\mathcal{O}_3$ of size $q^3(q^3-1)(q+1)/2$. □

Proposition 3.5. *The partial ovoid $\mathcal{O}$ is complete.*

Proof. From the previous proposition it follows that $\mathcal{O}$ is a complete partial ovoid of $\mathcal{W}_7(q)$. Indeed, if $\mathcal{H}$ is a hyperplane of $PG(7,q)$ then $\mathcal{H}=\mathcal{P}^{\perp}$, for some point P, where $\perp$ denotes the symplectic polarity. There exist pairs $P\in\mathcal{O}$, $Q\in\mathcal{O}_1$ with PQ totally isotropic, $P\in\mathcal{O}$, $R\in\mathcal{O}_2$ with PR totally isotropic, $P\in\mathcal{O}$, $T\in\mathcal{O}_3$ with PT totally isotropic. For, if $P\in\mathcal{O}$, then $P^{\perp}$ is a hyperplane of $PG(7,q)$ which must meet any secant line L to $\mathcal{O}$, necessarily at a point Q of $\mathcal{O}_1$, if L is not on P; and it contains S_P, and so a point R of $\mathcal{O}_2$; finally, the remaining points of $P^{\perp}$ (which must exist, by counting) lie in $\mathcal{O}_3$. Hence, since $\mathcal{O}_1,\ \mathcal{O}_2,\ \mathcal{O}_3$ are orbits of N, each point of $\mathcal{O}_i$ is collinear with a point of $\mathcal{O}$ (in a totally isotropic line) and so cannot be added to $\mathcal{O}$ to obtain a partial ovoid. Thus $\mathcal{O}$ is complete. □

References

[1] M. Aschbacher, "On the maximal subgroups of the finite classical groups," *Inventiones Mathematicae*, vol. 76, no. 3, pp. 469–514, 1984.

[2] G. M. Seitz, "Representations and maximal subgroups of finite groups of Lie type," *Geometriae Dedicata*, vol. 25, no. 1–3, pp. 391–406, 1988.

[3] P. B. Kleidman and M. W. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, LMS Lecture Note Series 129, Cambridge University Press, New York, NY, USA, 1990.

[4] R. Steinberg, "Representations of algebraic groups," *Nagoya Mathematical Journal*, vol. 22, pp. 33–56, 1963.

[5] A. Cossidente and O. H. King, "Embeddings of finite classical groups over field extensions and their geometry," *Advances in Geometry*, vol. 2, no. 1, pp. 13–27, 2002.

[6] A. Cossidente and O. H. King, "Twisted tensor product group embeddings and complete partial ovoids on quadrics in $PG(2^t - 1, q)$," *Journal of Algebra*, vol. 273, no. 2, pp. 854–868, 2004.

[7] A. Blokhuis and G. E. Moorhouse, "Some p-ranks related to orthogonal spaces," *Journal of Algebraic Combinatorics*, vol. 4, no. 4, pp. 295–316, 1995.

[8] W. M. Kantor, "Ovoids and translation planes," *Canadian Journal of Mathematics*, vol. 34, no. 5, pp. 1195–1207, 1982.

[9] A. Cossidente and O. H. King, "On twisted tensor product group embeddings and the spin representation of symplectic groups," *Advances in Geometry*, vol. 7, no. 1, pp. 55–64, 2007.

[10] M. Schaffer, "Twisted tensor product subgroups of finite classical groups," *Communications in Algebra*, vol. 27, no. 10, pp. 5097–5166, 1999.

[11] G. Lunardon, "Partial ovoids and generalized hexagons," in *Finite Geometry and Combinatorics*, LMS Lecture Note Series 191, pp. 233–248, Cambridge University Press, New York, NY, USA, 1993.

[12] A. E. Brouwer, "The composition factors of the Weyl modules with fundamental weights for the symplectic group," preprint.

[13] R. Gow, "Contraction of exterior powers in characteristic 2 and the spin module," *Geometriae Dedicata*, vol. 64, no. 3, pp. 283–295, 1997.

[14] N. Bourbaki, *Groupes et Algébres de Lie*, chapters 7 and 8, Hermann, Paris, France, 1975.

[15] M. W. Liebeck, "On the orders of maximal subgroups of the finite classical groups," *Proceedings of the London Mathematical Society*, vol. 50, no. 3, pp. 426–446, 1985.

[16] A. A. Premet and I. D. Suprunenko, "The Weyl modules and the irreducible representations of the symplectic group with the fundamental highest weights," *Communications in Algebra*, vol. 11, no. 12, pp. 1309–1342, 1983.

[17] W. M. Kantor, "Subgroups of classical groups generated by long root elements," *Transactions of the American Mathematical Society*, vol. 248, no. 2, pp. 347–379, 1979.

[18] W. M. Kantor and R. A. Liebler, "The rank 3 permutation representations of the finite classical groups," *Transactions of the American Mathematical Society*, vol. 271, no. 1, pp. 1–71, 1982.

[19] M. W. Liebeck, "The affine permutation groups of rank three," *Proceedings of the London Mathematical Society*, vol. 54, no. 3, pp. 477–516, 1987.