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Research Article

From Fibonacci Sequence to the Golden Ratio

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We consider the well-known characterization of the Golden ratio as limit of the ratio of consecutive terms of the Fibonacci sequence, and we give an explanation of this property in the framework of the Difference Equations Theory. We show that the Golden ratio coincides with this limit not because it is the root with maximum modulus and multiplicity of the characteristic polynomial, but, from a more general point of view, because it is the root with maximum modulus and multiplicity of a restricted set of roots, which in this special case coincides with the two roots of the characteristic polynomial. This new perspective is the heart of the characterization of the limit of ratio of consecutive terms of all linear homogeneous recurrences with constant coefficients, without any assumption on the roots of the characteristic polynomial, which may be, in particular, also complex and not real.

In this paper, we consider a well-known property of the Fibonacci sequence, defined by

$$F_0 = F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \qquad n > 1, \tag{1}$$

namely, the fact that the limit of the ratio of consecutive terms (the sequence defined from the ratio between each term and its previous one) is Φ, the highly celebrated Golden ratio:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi. \tag{2}$$

Many proofs already exist and are well known since long time, and we do not wish to add one more to the repertory.

The Fibonacci sequence can be studied in the framework of the Difference Equations Theory (e.g., see [1] Chapter 3 Page 43), where, roughly speaking, the properties of the more general sequences (F_n) satisfying

$$F_n + a_{k-1}F_{n-1} + \cdots + a_0 F_{n-k} = 0 \quad \forall n \geq k, \tag{3}$$

where $a_0 \neq 0$, $a_1, \ldots, a_{k-1} \in \mathbb{C}$, are studied. The first step of the theory consists in considering the associated characteristic polynomial

$$p(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0, \tag{4}$$

whose complex roots $\lambda_1, \ldots, \lambda_h$, with respective multiplicity $k_1, \ldots, k_h$, $k_1 + \cdots + k_h = k$ permit to express explicitly the terms F_n by means of the representation, known as Binet's formula:

$$\begin{aligned} F_n = {} & c_{1,1}\lambda_1^n + c_{1,2}n\lambda_1^n + \cdots + c_{1,k_1}n^{k_1-1}\lambda_1^n + \cdots + c_{h,1}\lambda_h^n \\ & + c_{h,2}n\lambda_h^n + \cdots + c_{h,k_h}n^{k_h-1}\lambda_h^n, \end{aligned} \tag{5}$$

where $c_{i,j}$ are (complex) numbers uniquely determined by $F_0, \ldots, F_{k-1}$. In the case of the Fibonacci sequence, (3), (4), and (5) become, respectively,

$$F_n - F_{n-1} - F_{n-2} = 0, \tag{6}$$

$$p(\lambda) = \lambda^2 - \lambda - 1, \tag{7}$$

$$F_n = c_{1,1}\Phi^n + c_{2,1}(1-\Phi)^n = \frac{\Phi}{\sqrt{5}}\Phi^n - \frac{1-\Phi}{\sqrt{5}}(1-\Phi)^n, \tag{8}$$

where the last equality comes out taking into account the initial conditions $F_0 = F_1 = 1$. Well, the fact that the Golden ratio, limit of the ratio of adjacent terms, coincides with one of the roots of the characteristic polynomial (7) is not a coincidence. It can be very easily seen that this is always the case, also for general order-k linear homogeneous recurrences with constant coefficients; the limit of the ratio of adjacent terms (in the following, we will refer to this limit as the *Kepler* limit of (F_n)), if it exists, is always one of the roots of the characteristic polynomial. It suffices to write, starting from (3),

$$\frac{F_n}{F_{n-1}}\frac{F_{n-1}}{F_{n-2}}\cdots\frac{F_{n-k+1}}{F_{n-k}}+a_{k-1}\frac{F_{n-1}}{F_{n-2}}\cdots\frac{F_{n-k+1}}{F_{n-k}}+\cdots+a_0=0 \tag{9}$$

and to substitute each ratio with its limit.

The question is:

> *Q: if for an order-k linear homogeneous recurrence with constant coefficients, the limit of the ratio of adjacent terms exists, which root is it*?

In the case of the Fibonacci sequence, it is clear that the Golden ratio is the root of the characteristic polynomial with maximum modulus, and all the proofs of (2) use more or less implicitly this property. And the same machinery works with several other examples of recurrences (e.g., see the pioneering paper [2]).

Surprisingly, the correct answer to the question is *not* that the limit of the ratio of adjacent terms is the root with maximum modulus. The fact that the Golden ratio is the unique root of maximum modulus is just a *coincidence*! In fact, consider, for instance, the sequence

$$\begin{aligned} &F_0=1, \qquad F_1=1-\Phi, \\ &F_n=F_{n-1}+F_{n-2}, \qquad n>1, \end{aligned} \tag{10}$$

with different initial conditions and same characteristic polynomial (7). Here, Binet's formula reads $F_n=(1-\Phi)^n$ (the term containing Φ^n is in some sense "killed" by the initial conditions). In this case, the sequence is geometric, and the Kepler limit is the other root of the characteristic polynomial: the smaller. This sequence (along with those obtained multiplying each term by a nonzero constant) is known as an "exception", and much attention is devoted in the literature to the roots of maximum modulus; up to some assumptions which should exclude the exceptions, it is known that the Kepler limit exists if, among the roots of maximum modulus of (4), there exists a unique root of maximal multiplicity (the "dominant" root; see [3] and references therein).

Another surprise: the existence of the dominant root is independent of the existence of the Kepler limit, and also the value of the limit is not necessarily the dominant root. This is shown by the following examples. Consider the sequence

$$\begin{aligned} &F_0=3, \qquad F_1=1, \\ &F_2=-1, \qquad F_3=1, \\ &F_n=3F_{n-1}-3F_{n-2}+3F_{n-3}-2F_{n-4}, \qquad n>3; \end{aligned} \tag{11}$$

the characteristic polynomial is $p(\lambda)=\lambda^4-3\lambda^3+3\lambda^2-3\lambda+2$, whose roots are $2,\ 1,\ -i,\ i$; Binet's formula is

$$F_n=1^n+(-i)^n+i^n. \tag{12}$$

In this case, the dominant root exists, while the Kepler limit does not. In the case

$$\begin{aligned} &F_0=1, \qquad F_1=1, \qquad F_2=1, \\ &F_n=F_{n-1}-4F_{n-2}+4F_{n-3}, \qquad n>2, \end{aligned} \tag{13}$$

the characteristic polynomial is $p(\lambda)=\lambda^3-\lambda^2+4\lambda-4$, whose roots are $-2i,\ 2i, 1$. Binet's formula is

$$F_n=1^n. \tag{14}$$

Here, the characteristic polynomial has *not* a dominant root, but there exists the Kepler limit, which is the root 1 (note that 1 is not among the roots of maximum modulus).

These phenomena attracted several researchers since Poincaré, who proved in a more general context (e.g., see [4]) the following result.

Theorem A. *Let $\lambda_1,\ldots,\lambda_k$ be the roots of the characteristic equation*

$$\lambda^k+a_{k-1}\lambda^{k-1}+\cdots+a_0=0 \tag{15}$$

of (3)*, and suppose that $|\lambda_i|\neq|\lambda_j|$ for $i\neq j$. Then, either $F_n=0$ for all large n or there exists an index $i\in\{1,\ldots,k\}$ such that*

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\lambda_i. \tag{16}$$

The condition about the pairwise distinct moduli is optimal (namely, necessary and sufficient) if one requires the existence of the Kepler limit for all possible initial conditions; in other words, if there exist two distinct roots with the same modulus, then it is always possible to consider initial conditions such that the Kepler limit does not exist.

What can be said about the general case, also when the moduli of the roots are not necessarily distinct? Is there a necessary and sufficient condition for the existence of the Kepler limit?

Well, for a given linear recurrence with constant coefficients (F_n), consider the "essential Binet's formula representation"; namely, write it in the form (5) where the coefficients $c_{i,j}$ have been computed starting from the initial conditions, and where only the nonzero coefficients $c_{i,j}$ appear:

$$F_n=\sum_l c_{i_l,j_l}n^{j_l-1}\lambda_{i_l}^n. \tag{17}$$

In this representation, one can see only a *subset* of the k addends of the right hand side of (5). In some sense, the addends not appearing in the right hand side of (17) can be considered as "killed" by the initial conditions. In this representation, among the addends containing the "survived" roots, count how many, among them, have maximum modulus and contain the maximum power of n (let us call them "leading"). For instance, the classical Fibonacci sequence (1) admits the Golden Ratio Φ as unique leading root (see (8)); the sequence (10) has the same characteristic polynomial of (1), but the unique leading root is $1-\Phi$; in the case of (11), the leading roots are $1, -i, i$ (see (12)). Moreover, if we consider the sequence

$$F_0 = 1, \qquad F_1 = -3, \qquad F_2 = -11, \\ F_n = 6F_{n-1} - 11F_{n-2} + 6F_{n-3}, \quad n > 2, \tag{18}$$

the characteristic polynomial is $p(\lambda) = (\lambda - 3)(\lambda - 2)(\lambda - 1)$, and Binet's formula is

$$F_n = 5 \cdot 1^n + (-4) \cdot 2^n, \tag{19}$$

the unique leading root being 2.

In [3], it is proved that if you count exactly one leading term, then the Kepler limit exists, and in such case, the Kepler limit is exactly the corresponding root ("the unique leading root"). When this happens, we say that the initial conditions are *in agreement* with the characteristic polynomial.

Thus, Theorem A is, of course, an immediate corollary; it is trivial that in the case of pairwise distinct moduli, independently of the "killed" roots, the set of the "survived" roots, whatever it is, must contain a unique leading root.

On the other hand, for a general order-k linear homogeneous recurrence with constant coefficients, it may happen that (e.g., see (11)) there are more than one leading root. What can be said, in this case, about the existence of the Kepler limit? A third surprise, in this case, is that the Kepler limit does not exist; the condition given earlier (i.e., the uniqueness of the leading root) is also necessary.

In the end, the notion of *agreement* gives us the answer to the original question Q: *the unique leading root.*

The importance of this study relies upon the fact that the ratio of consecutive terms tells that all linear recurrences in agreement behave at infinity like geometric sequences, and the first terms (which give the initial conditions), along with the law (which can be conjectured after a reasonable number of tests), permit to predict the ratio. This may be very important for all applications, where linear recurrences represent mathematical models.

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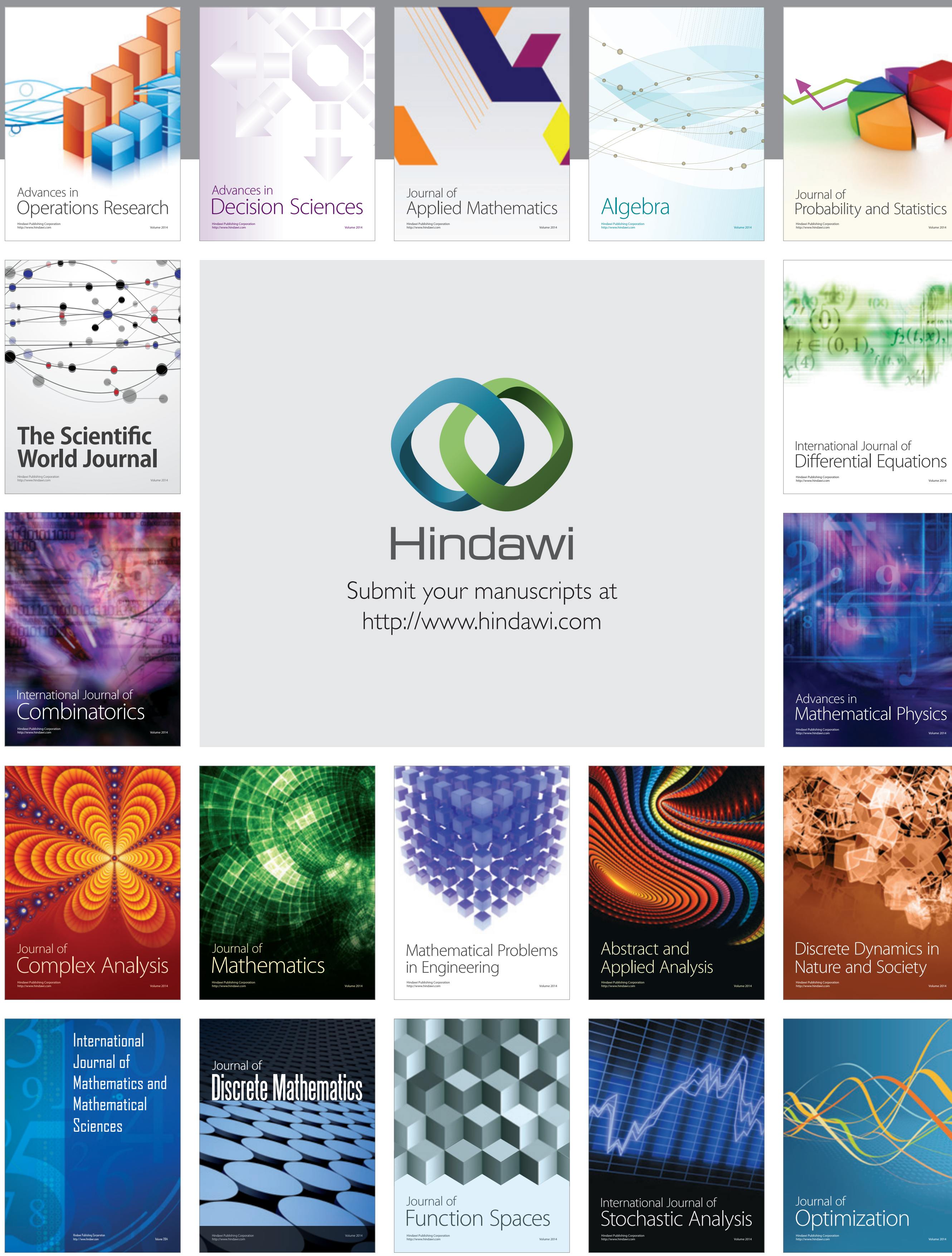

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