# Solution to an open problem about a transformation on the space of copulas*

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**Abstract**

We solve a recent open problem about a new transformation mapping the set of copulas into itself. The obtained mapping is characterized in algebraic terms and some limit results are proved.

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## 1 Introduction

Nowadays, the "copula industry" is actively producing a large number of methods in order to enlarge well–known families and/or construct novel copulas. Such investigations are usually motivated by the need to introduce more flexible stochastic models that go beyond traditional (and often unrealistic) assumptions related to the distribution of a multivariate random vector.

A large class of these methods is provided by transformations of copulas, i.e. mappings from the space of copulas (or some of its subsets) into itself that are usually employed to add parameters to some known families. Such mappings include, for instance, the distortions of copulas (see, e.g., [4, 9, 12, 16, 18, 22]), which represents one of the most extensively studied transformations, as well as other different constructions (see, e.g., [1, 5, 13, 14]).

In the current paper we are interested in the mapping transforming a function $C\colon [0,1]^2 \to [0,1]$ into another function $C_\lambda$ defined on $[0,1]^2$ by

$$C_\lambda(x,y) = \frac{C(x,y)}{1+\lambda-\lambda C^*(x,y)}, \tag{1.1}$$

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where $C^*$ is the so–called *dual* of $C$, $C^*(x, y) = x + y - C(x, y)$, and $\lambda \in (0, 1]$ ($\lambda = 0$ corresponds to the identity transformation). Analogously, Eq. (1.1) can be rewritten in the form

$$C_\lambda(x, y) = \frac{C(x, y)}{1 + \lambda \overline{C}(x, y)}, \tag{1.2}$$

where $\overline{C}(x, y) = 1 - x - y + C(x, y)$ is the survival function associated with $C$:

As noted in [15], if $C$ coincides with one of the Frechét–Hoeffding bounds $M(x, y) = \min(x, y)$ and $W(x, y) = \max(x + y - 1, 0)$ then $C_\lambda$ of (1.1) is a copula. Moreover, if $C(x, y) = \Pi(x, y) = xy$ is the independence copula, then $C_\lambda$ corresponds to the Ali–Mikhail–Haq family of copula with parameter $\lambda \in [-1, 0)$. The seemingly natural question is, therefore, whether Eq. (1.1) defines a copula for any initial copula $C$ (see [15, Problem 4.2] and also [5, section 4]). The answer to this question is affirmative as we show below, and the proof is based on a density argument in the space of copulas. Additionally, the new transformation is characterized in algebraic terms and some results concerning the limit behavior of the transformation (and a related transformation) are presented.

# 2 The main result

We start by considering how Eq. (1.1) acts in the class of quasi–copulas (for a definition, see [11]). Quasi–copulas are generalizations of copulas that are used, e.g., in finding bounds for several sub–classes of copulas [2, 20, 21]. The following result holds.

**Proposition 2.1.** *If $Q$ is a quasi–copula, then $Q_\lambda$ given by* (1.1) *is a quasi–copula for every $\lambda \in (0, 1]$.*

*Proof.* First, notice that $Q_\lambda$ is well-defined since the denominator of Eq. (1.1) is bounded from below by $1$ because every quasi–copula is bounded below by $W$. Moreover, $Q_\lambda$ satisfies the boundary conditions for a quasi–copula, i.e.

$$Q_\lambda(x, 1) = Q_\lambda(1, x) = x$$

for all $x \in [0, 1]$. It can be easily checked that $Q_\lambda$ is increasing in each variable (keeping the other fixed). Finally, in order to show that $Q_\lambda$ is 1–Lipschitz continuous, consider $x, y, y + h \in [0, 1]$. Then we have

$$\begin{aligned} Q_\lambda(x, y + h) - Q_\lambda(x, y) &= \frac{\lambda h Q(x, y) + (Q(x, y + h) - Q(x, y))(1 + \lambda(1 - x - y))}{(1 + \lambda(1 - x - y - h + Q(x, y + h)))(1 + \lambda(1 - x - y + Q(x, y)))} \\ &\leq \frac{h}{(1 + \lambda(1 - x - y - h + Q(x, y + h)))} \leq h, \end{aligned}$$

from which the assertion follows. □

Now, in order to prove that the transformation of Eq. (1.1) also maps copulas into copulas we need to show that it preserves the 2–increasing property, as is shown below by density arguments.

**Proposition 2.2.** *If $C$ is a copula, then $C_\lambda$ given by* (1.1) *is a copula for every $\lambda \in (0,1]$.*

*Proof.* Suppose that $C$ is an absolutely continuous copula that admits continuous mixed partial derivatives and set $\Sigma(x,y) = x + y$. Then the first derivative of $C_\lambda$ with respect to first variable is given by

$$D_1 C_\lambda = \frac{\lambda C(1 - D_1 C)}{(1 + \lambda - \lambda(\Sigma - C))^2} + \frac{D_1 C}{1 + \lambda - \lambda(\Sigma - C)}.$$

Thus, the mixed second partial derivative of $C_\lambda$ is given by

$$\begin{aligned} D_{12} C_\lambda =& \frac{2\lambda^2 C(1 - D_1 C)(1 - D_2 C)}{(1 + \lambda - \lambda(\Sigma - C))^3} + \frac{\lambda D_2 C(1 - D_1 C) - \lambda C D_{12} C}{(1 + \lambda - \lambda(\Sigma - C))^2} \\ &+ \frac{\lambda(1 - D_2 C) D_1 C}{(1 + \lambda - \lambda(\Sigma - C))^2} + \frac{D_{12} C}{1 + \lambda - \lambda(\Sigma - C)} \\ =& \frac{2\lambda^2 C(1 - D_1 C)(1 - D_2 C)}{(1 + \lambda - \lambda(\Sigma - C))^3} + \frac{\lambda D_2 C(1 - D_1 C) + \lambda D_1 C(1 - D_2 C)}{(1 + \lambda - \lambda(\Sigma - C))^2} \\ &+ \frac{D_{12} C(1 + \lambda - \lambda\Sigma)}{(1 + \lambda - \lambda(\Sigma - C))^2}, \end{aligned}$$

which is non-negative since the first partial derivatives of a copula are bounded above by 1 wherever they exist. Now, the transformation of type (1.1) maps quasi–copulas into quasi–copulas and it is continuous with respect to the $L^\infty$ norm. Taking into account that copulas with continuous mixed partial derivative are dense in $\mathscr{C}$ with respect to the same norm (consider, for instance, Bernstein copulas [17]) and that we have shown $C_\lambda \in \mathscr{C}$ for every such copula, the assertion now follows. □

As a consequence, equality (1.1) defines, for every $\lambda \in (0,1]$, a mapping $T_\lambda : \mathscr{C} \to \mathscr{C}$, where $\mathscr{C}$ is the class of bivariate copulas. In particular, for any $\lambda$, if $\mu_C$ denotes the measure induced by $C \in \mathscr{C}$, we have

$$T_\lambda(C)(x,y) = \frac{\mu_C([0,x] \times [0,y])}{1 + \lambda \mu_C([x,1] \times [y,1])} \tag{2.1}$$

For a fixed $\lambda$, the transformation $T_\lambda$ is injective, since $T_\lambda(C_1) = T_\lambda(C_2)$ implies $C_1 = C_2$.

**Remark 2.1.** It is immediate that, for the copula $W$, $T_\lambda(W) = W$ holds for any $\lambda \in (0,1]$. Moreover, $W$ is the only fixed point of $T_\lambda$. In fact, if $C = C_\lambda$ for some $\lambda \in (0,1]$, then, for every $(x,y) \in [0,1]^2$, we must have

$$\lambda C(x,y)(1 - x - y + C(x,y)) = 0,$$

from which it follows that $C = W$. Notice that, here, a key role is played by the pointwise bounds in the space of (quasi–)copulas. In fact, consider the semi–copula $S$ defined by $S(x,y) = 0$ on $(0,1)^2$, and $S(x,y) = \min(x,y)$, otherwise. Then $S = S_\lambda$ for every $\lambda \in (0,1]$.

Moreover, since $W$ is invariant under $T_\lambda$, it also follows that, if $C$ is a patchwork copula with basis copula given by $W$, then $T_\lambda(C)$ is also a copula of this type (see [8] for definition of patchwork).

The transformation of (1.1) preserves the concordance order between copulas. In fact, it can be easily checked that $C_1 \leq C_2$ pointwise implies $T_\lambda(C_1) \leq T_\lambda(C_2)$, from which we directly get the following consequences:

– the range of the transformation $T_\lambda$ goes from $T_\lambda(W) = W$ to $T_\lambda(M) < M$. In other words, the $T_\lambda$–image of any family of copulas cannot describe perfect positive dependence.

– If $\nu$ denotes a measure of concordance (see [19]) between two random variables with copula $C$, then
$$\nu(T_\lambda(W)) = -1 \leq \nu(T_\lambda(C)) \leq \nu(T_\lambda(M)) \tag{2.2}$$
for every $\lambda \in (0, 1]$

The upper bound in (2.2) can be calculated explicitly for concordance measures like Kendall's $\tau$ and Spearman's $\rho$. In fact, consider that, for any $\lambda \in (0, 1]$ the transformation of the comonotonicity copula $M$ can be written as

$$T_\lambda(M)(x, y) = \frac{\min(x, y)}{1 + \lambda - \lambda \max(x, y)}, \tag{2.3}$$

from which it is apparent that it is a semilinear copula [6, 10] generated by $f_\lambda(t) = 1/(1+\lambda-\lambda t)$. Moreover, formulas for measures of association of semilinear copulas are given in [6, Theorem 4]. By using them and by doing little algebra, the following result follows.

**Proposition 2.3.** *Let* $\lambda \in (0, 1]$. *Then the following inequalities hold:*

$$-1 \leq \tau(C_\lambda) \leq \frac{4\lambda - \lambda^2 - 4\ln(1 + \lambda)}{\lambda^2},$$
$$-1 \leq \rho(C_\lambda) \leq \frac{-12\lambda - 18\lambda^2 - 3\lambda^3 + 12(1 + \lambda)^2 \ln(1 + \lambda)}{\lambda^3}.$$

In particular, $\tau(C_1) \leq 0.228$, while $\rho(C_1) \leq 0.272$.

**Remark 2.2.** In the case of Eq. (2.3), $T_\lambda$ has transformed a copula which is purely singular into a copula with a singular component and an absolutely continuous component. See Figure 1.

**Remark 2.3.** It should be mentioned that the mapping $T_\lambda(C)$ is decreasing with respect to $\lambda$ and the pointwise ordering between copulas. Thus, since $T_0(C) = C$, it follows that, for $\lambda \in (0, 1]$, $T_\lambda(C) \leq C$, so $T_\lambda(C)$ is at most as positively quadrant dependent as $C$.

**Remark 2.4.** The transformation of eq. (1.1) maps the diagonal section of $C$, $\delta_C$, into the diagonal section $\delta_C(t)/(1 + \lambda - \lambda(2t - \delta_C(t)))$. As such, it changes the tail dependence coefficients of the corresponding copulas, where they exist (see, e.g., [7]). In particular, $UTDC(C_\lambda) = (1 - \lambda)UTDC(C)$, while $LTDC(C_\lambda) = LTDC(C)/(1 + \lambda)$, where UTDC and LTDC denote the upper and lower tail dependence coefficients, respectively.

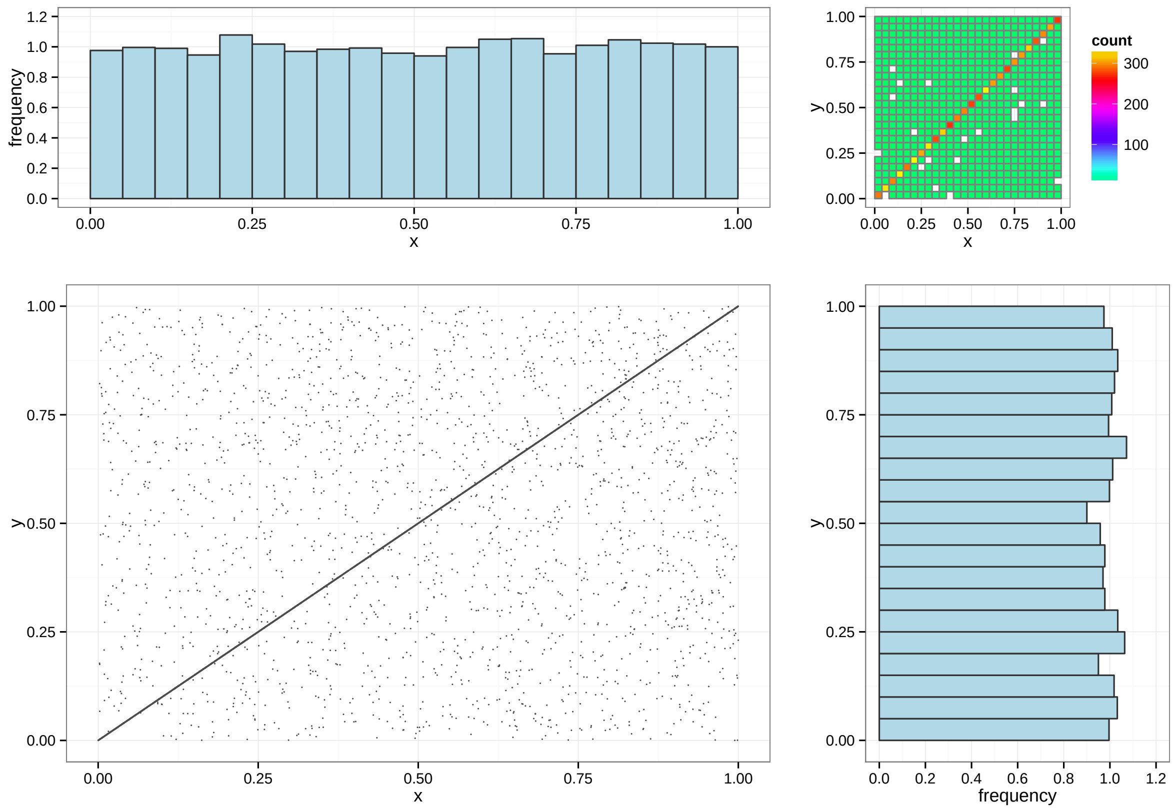


Figure 1: Sample of size 10.000 from $(X, Y) \sim T_\lambda(M)$ for $\lambda = 1/2$, its two dimensional histogram and the corresponding marginal histograms.

Now, following the idea of linear constructions of copulas [13], we would like to characterize the transformation of type (1.1) in algebraic terms.

**Proposition 2.4.** *Let $F : \mathscr{C} \to \mathscr{C}$ be a mapping given by*

$$F(C)(x, y) = \frac{a_0 + a_1 x + a_2 y + a_3 C(x, y)}{b_0 + b_1 x + b_2 y + b_3 C(x, y)},$$

*with $a_i, b_i \in \mathbb{R}$ for $(i = 0, \ldots, 3)$. Then $F$ is given by* (1.1).

*Proof.* Since $F(C)$ must be a copula, the boundary conditions imply that for every $(x, y) \in [0, 1]^2$ we have

$$\frac{a_0 + (a_1 + a_3)x + a_2}{b_0 + (b_1 + b_3)x + b_2} = x, \qquad \frac{a_0 + (a_2 + a_3)y + a_1}{b_0 + (b_2 + b_3)y + b_1} = y,$$

from which it is easily derived that

$$a_0 + a_2 = a_0 + a_1 = 0, \qquad b_1 + b_3 = b_2 + b_3 = 0.$$

In particular, since $C(0,0) = 0$, $a_0 = 0$, which, together with $C(1,1) = 1$ and previous constraints, implies $a_1 = a_2 = 0$, and $a_3 = b_0 + b_1$. Thus, $F$ is given by the formula

$$F(C)(x,y) = \frac{(b_0 + b_1)C(x,y)}{b_0 + b_1 x + b_1 y - b_1 C(x,y)},$$

and the assertion follows by dividing numerator and denominator by $b_0 + b_1 \neq 0$. □

In other words, transformations of type (1.1) are the only transformations that can be expressed as ratio of two linear functions involving the variables $x, y \in [0,1]$ as well as $z = C(x,y)$.

# 3 The induced transformation and its iterations

The transformation of (1.1) induces a mapping $T_\lambda$ in $\mathscr{C}$. It would be hence of interest to see what happens when the transformation is iterated. Interestingly, the iterations converge to the fixed point of $T_\lambda$, which is the countermonotonicity copula. Here, $T_\lambda^2 := T_\lambda \circ T_\lambda$ and, by recursion, for each $n \geq 3$, $T_\lambda^n := T_\lambda \circ T_\lambda^{n-1}$.

**Proposition 3.1.** *Let* $\lambda \in (0,1]$. *For every* $C \in \mathscr{C}$ *we have* $\lim_{n\to\infty} d_\infty(T_\lambda^n C, W) = 0$.

*Proof.* First, we prove that

$$d_\infty(T_\lambda C, W) \leq \frac{d_\infty(C,W)}{1 + \lambda d_\infty(C,W)} \tag{3.1}$$

and distinguish two cases: (i) If $x + y - 1 \leq 0$ then we have $W(x,y) = 0$ so we get

$$\begin{aligned} T_\lambda C(x,y) - W(x,y) &= T_\lambda C(x,y) = \frac{C(x,y) - W(x,y)}{1 + \lambda(1 - x - y + C(x,y) - W(x,y))} \\ &\leq \frac{C(x,y) - W(x,y)}{1 + \lambda(C(x,y) - W(x,y))} \leq \frac{d_\infty(C,W)}{1 + \lambda d_\infty(C,W)}. \end{aligned}$$

(ii) If $x + y - 1 > 0$ then we have $W(x,y) = x + y - 1$ so we get

$$\begin{aligned} T_\lambda C(x,y) - W(x,y) &= \frac{C(x,y)}{1 + \lambda(1 - x - y + C(x,y))} - W(x,y) = \\ &= \frac{(C(x,y) - W(x,y))(1 - \lambda W(x,y))}{1 + \lambda(C(x,y) - W(x,y))} \leq \frac{d_\infty(C,W)}{1 + \lambda d_\infty(C,W)}. \end{aligned}$$

Finally, we simply iterate Eq. (3.1) to get

$$\begin{aligned} d_\infty(T_\lambda^n C, W) &\leq \frac{d_\infty(T_\lambda^{n-1} C, W)}{1 + \lambda d_\infty(T_\lambda^{n-1} C, W)} \leq \frac{\frac{d_\infty(T_\lambda^{n-2} C, W)}{1 + \lambda d_\infty(T_\lambda^{n-2} C, W)}}{1 + \lambda \frac{d_\infty(T_\lambda^{n-2} C, W)}{1 + \lambda d_\infty(T_\lambda^{n-2} C, W)}} = \frac{d_\infty(T_\lambda^{n-2} C, W)}{1 + 2\lambda d_\infty(T_\lambda^{n-2} C, W)} \\ &\leq \cdots \leq \frac{d_\infty(T_\lambda^{n-3} C, W)}{1 + 3\lambda d_\infty(T_\lambda^{n-3} C, W)} \leq \cdots \leq \frac{d_\infty(C,W)}{1 + n\lambda d_\infty(C,W)}, \end{aligned}$$

from which we directly deduce the desired result. □

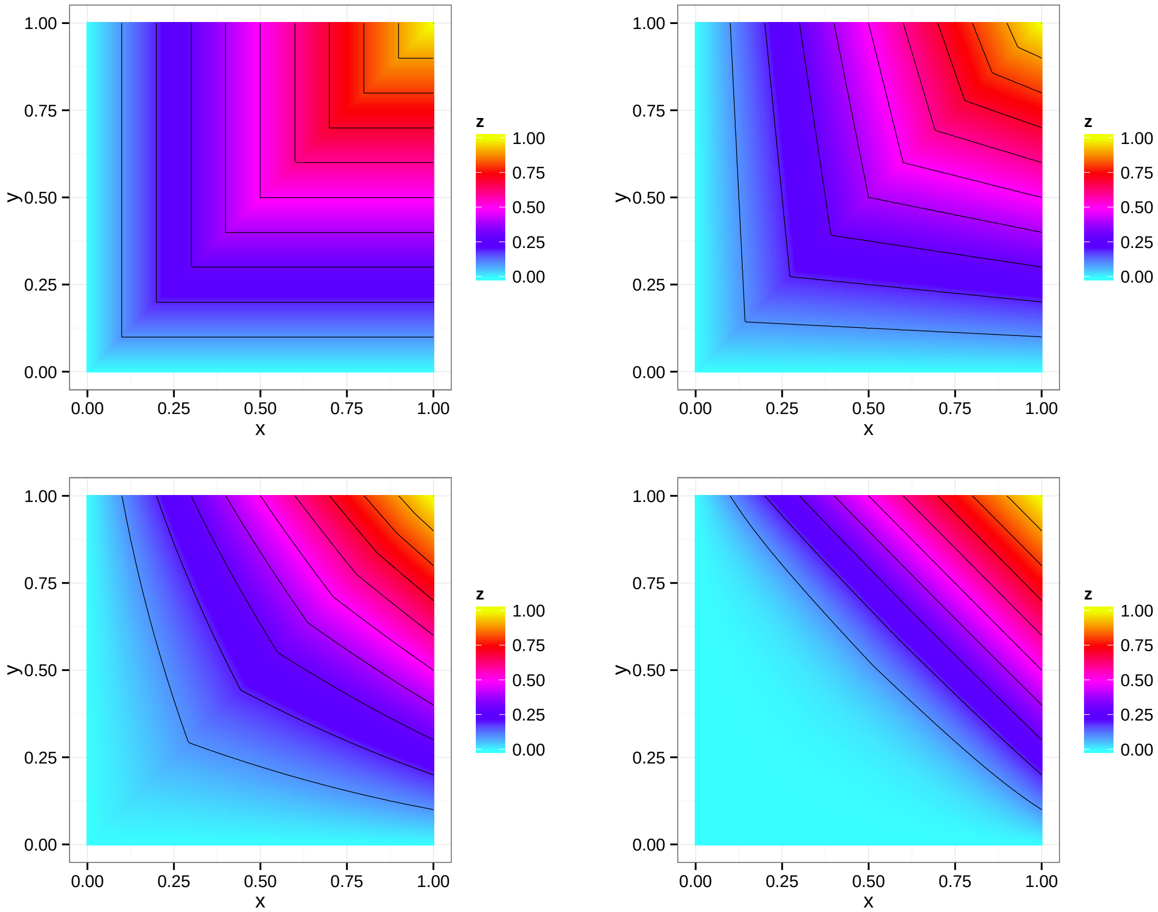


Figure 2: Image plot of $T^n_\lambda(M)$ for $n \in \{0, 1, 5, 20\}$, whereby $\lambda = 1/2$.

There is another analytically very simple transformation $S_\lambda : \mathscr{C} \to \mathscr{C}$ that fulfills the analogous limit result with $W$ replaced by $M$, i.e. $\lim_{n\to\infty} d_\infty(S^n_\lambda C, M) = 0$ for every $C \in \mathscr{C}$. In fact, defining

$$S_\lambda C(x, y) = \frac{C(x, y)(1 - \lambda x) + \lambda xy}{1 + \lambda(y - C(x, y))} \tag{3.2}$$

for every $\lambda \in (0, 1]$, the following result holds.

**Proposition 3.2.** *For every $\lambda \in (0, 1]$, the transformation $S_\lambda$ given by* (3.2) *maps $\mathscr{C}$ into $\mathscr{C}$. Moreover, it is bijective and continuous, has $M$ as unique fixed point and fulfills*

$$\lim_{n\to\infty} d_\infty(S^n_\lambda C, M) = 0 \quad \textit{for every } C \in \mathscr{C}.$$

*Proof.* We prove the result by expressing $S_\lambda$ in terms of $T_\lambda$. To do so, first define the isometry $\Phi : [0, 1]^2 \to [0, 1]^2$ by $\Phi(x, y) = (1 - y, x)$. Obviously the push-forward $\mu^\Phi_C$ of a doubly stochastic measure $\mu_C$ via $\Phi$ is again doubly stochastic, so we can view $\Phi$ also as a transformation on $\mathscr{C}$

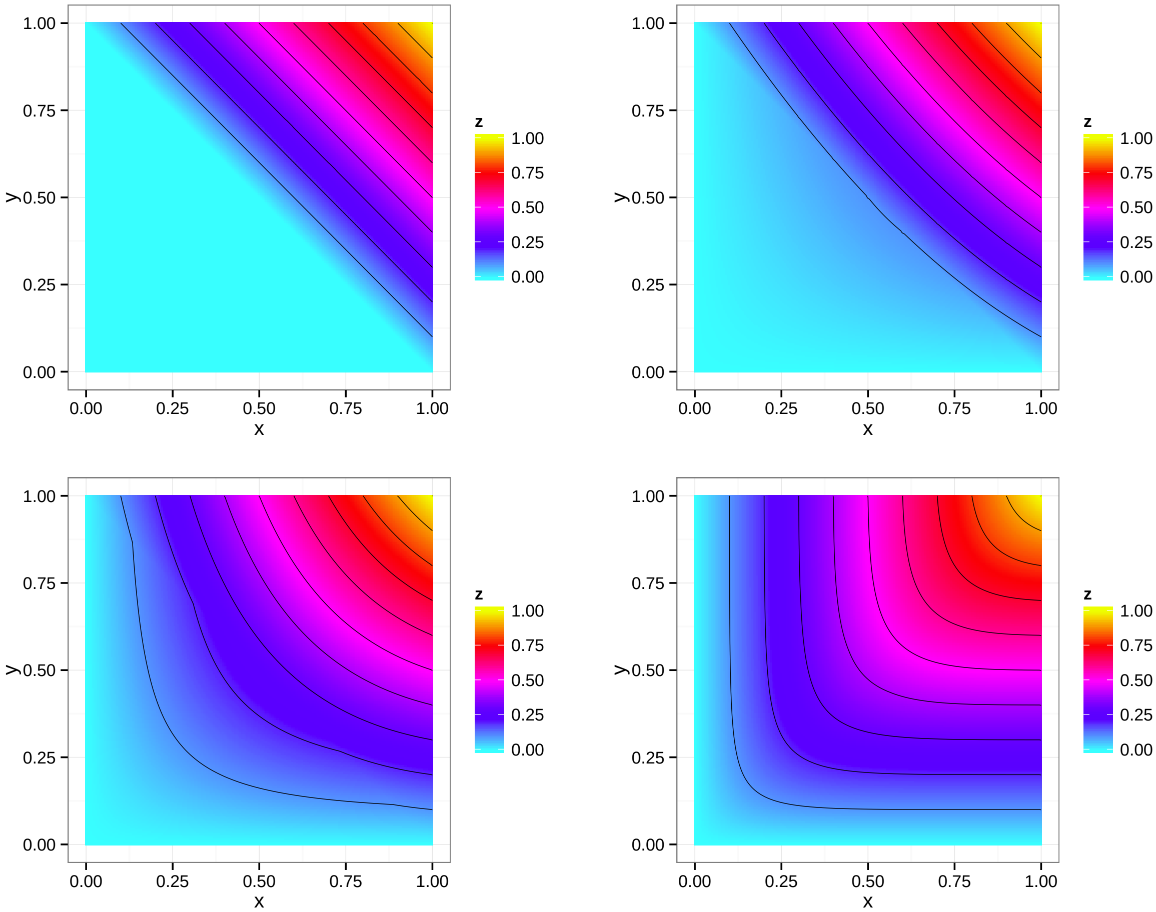


Figure 3: Image plot of $S_\lambda^n(W)$ for $n \in \{0, 1, 5, 20\}$, whereby $\lambda = 1/2$.

and write $\Phi(C)$ for the copula corresponding to $\mu_C^\Phi$. Actually, this transformation corresponds to the flipping operation studied in [3].

It is straightforward to verify that $\Phi(C)(x, y) = y - C(y, 1-x)$ for all $x, y \in [0, 1]$, $\Phi(M) = W$, and $\Phi : \mathscr{C} \to \mathscr{C}$ is bijective and continuous. The inverse $\Phi^{-1}$ is given by $\Phi^{-1}(C)(x, y) = x - C(1-y, x)$. Now, it holds that

$$\Phi^{-1} \circ T_\lambda \circ \Phi(C)(x, y) = x - T_\lambda \circ \Phi(C)(1-y, x) = x - \frac{x - C(x, y)}{1 + \lambda(y - C(x, y))} = S_\lambda C(x, y)$$

from which all the assertions follow immediately. □

Figures 2 and 3 illustrate an example of iteration of the transformations $T_\lambda$ and $S_\lambda$, respectively.

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