Available online at www.sciencedirect.com

ScienceDirect

Procedia Engineering 199 (2017) 1411–1416

Procedia Engineering

www.elsevier.com/locate/procedia

X International Conference on Structural Dynamics, EURODYN 2017

Consequences of different definitions of bending curvature on nonlinear dynamics of beams

Enrico Babilio[a,*], Stefano Lenci[b]

[a]Department of Structures for Engineering and Architecture, University of Naples "Federico II", via Forno Vecchio 36, 80134 Naples, Italy
[b]Department of Civil and Building Engineering and Architecture, Polytechnic University of Marche, via Brecce Bianche 12, 60131 Ancona, Italy

Abstract

Beam theories may be grouped in two broad categories, namely induced or intrinsic theories. In the former, beam models are obtained as exact consequences of three-dimensional theory, making use either of asymptotic expansions in a slenderness parameter or projections of three-dimensional elasticity on certain function spaces, while beams are inherently one-dimensional bodies in the latter category. Although induced theories show a clear connection between three- and one-dimensional representations, they are often more demanding with respect to intrinsic ones, in which a finite number of strain parameters, depending on just one space variable, characterizes the motion of beams in an internally consistent way and without a direct linkage to three-dimensional material properties. Hence, as a consequence, intrinsic theories do not provide any structure for constitutive equations and, at least in principle, different choices can be allowed. A typical example of this fact is represented by the one-dimensional relationship between the bending moment and the beam curvature, since for this latter two notions are admissible. Indeed, both are adopted in the literature and, apparently, preferring one to the other is only related to the predictive capability of the ensuing model. The arising question is about possible differences in both static and dynamic responses of beams, when one or the other definition of curvature is selected.

Peer-review under responsibility of the organizing committee of EURODYN 2017.

Keywords: geometrically exact beam, curvature definition, dynamic response, nonlinear constitutive equation, hyperelastic material

1. Introduction

Strings and beams are ubiquitous among man-made products and even in nature. Consequently, models for their analysis are currently met in any field of civil and industrial engineering. Further, the application of beam models is not restricted to bodies having a prevalent dimension with respect to the others, but is extended to structures that can suitably resemble beams, at least at a proper "distance" of observation or at a certain level of investigation. An example is represented by the so-called beam-like lattice structures [1].

The analysis of behavior of beams with low, high or even very high slenderness, which is the ratio between the length of the beam and the smallest gyration radius, is an important field of application of any beam theory.

* Corresponding author. Tel.: +39-081-253-8985 ; fax: +39-081-253-8989.
E-mail address: enrico.babilio@unina.it

Peer-review under responsibility of the organizing committee of EURODYN 2017.
10.1016/j.proeng.2017.09.382

Noteworthy, those beam theories, which are completely decoupled from the three-dimensional theory, do not provide any structure for constitutive equations, as typically happens for models belonging to the class of Cosserat beam theories [2] (for a systematic treatment of beams as oriented bodies, see [3]). Thus, in such cases, constitutive equations must be either introduced on an axiomatic way or experimentally established. For considerations on how to obtain constitutive equations, see [4].

A key question, indeed treated in many papers and books, is which curvature is the more suitable measure to adopt in formulating a one-dimensional constitutive relationship for the bending moment especially in case of beams undergoing large deflections and rotations. We emphasize that, interestingly, in the scientific literature two different notions of curvature are used and, apparently, preferring one to the other is only related to the predictive capability of the ensuing model.

The present contribution, which is part of a larger research project aimed at understanding how one choice or the other influences the theoretical response of beams, is organized as follows. The kinematics of a geometrically exact nonlinear Timoshenko beam is introduced in Sec. 2 and dynamic equations are reported in Sec. 3. One-dimensional constitutive equations are indicated in Sec. 4, which is actually the main part of this contribution. The Sec. 5 reports some basic numerical examples. The contribution ends with some conclusive remarks (Sec. 6).

2. Kinematics

Let us consider an initially straight beam undergoing planar and twist-less deformed states and further assume that cross sections remain flat and undistorted, but not necessarily orthogonal to the beam axis during the deformation process. A slice of initial length dx is depicted in Fig. 1, to which we refer for notations, in the reference and deformed configurations: $u(x, t)$ and $v(x, t)$ stand for axial and transversal displacements of cross section center, $\varphi(x, t)$ and $\theta(x, t)$ are axis and cross section rotations, respectively. The shear angle id denoted by $\gamma(x, t)$. Displacements and forces are positive if concordant with the reference (see vectors in Fig. 1); rotations and moments are positive if counterclockwise oriented. The length of the deformed axis of the slice is given by

$$\mathrm{d}s = \sqrt{2\,\varepsilon + 1}\,\mathrm{d}x\,, \tag{1}$$

where ε is the axial strain written as

$$\varepsilon = u' + \frac{u'^2}{2} + \frac{v'^2}{2}\,, \tag{2}$$

the prime denoting derivative with respect to x. The rotation φ and the shear angle γ are given by

$$\varphi = -\arctan\left(\frac{v'}{1+u'}\right), \qquad \gamma = \theta - \varphi\,. \tag{3}$$

Finally, two curvature functions, $\kappa = \kappa(x, t)$ and $\hat{\kappa} = \hat{\kappa}(x, t)$, are introduced as

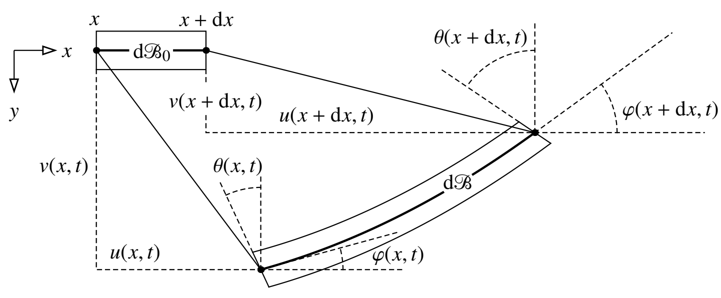

Fig. 1. Beam slice d$\mathscr{B}_0$ and d$\mathscr{B}$ in reference and deformed configurations, respectively.

$$\kappa = \theta' = \gamma' + \frac{v'u'' - \left(1 + u'\right) v''}{\left(1 + u'\right)^2 + v'^2} , \tag{4}$$

and

$$\hat{\kappa} = \theta' \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\kappa}{\sqrt{2\varepsilon + 1}} . \tag{5}$$

The former curvature is called material [5], normalized [6], flexural [7] or mechanical [8,9]. The latter, which differs from the other one whenever axial strain does not vanish, is indicated as the geometric curvature in [7] and in [9], while in [10] is reported either as geometric curvature and spatial measure of curvature.

We call, from now on, κ as the mechanical curvature and $\hat{\kappa}$ as the geometric curvature, in agreement with [11]. We emphasize that κ and $\hat{\kappa}$ are deformations that we expect to play a role in constitutive equations and hence we postulate that their dual entities exist. In this sense, we admit that both κ and $\hat{\kappa}$, in spite of names, have a mechanical meaning.

3. Equations of motion

On making use of virtual variations of Eqs.(2-4) and introducing properly conjugated generalized stresses N, T and M, the equations of motion can be derived from the Principle of Virtual Work as (see [11], for further details)

$$\rho A \ddot{u} + c \dot{u} = \left(N \frac{\mathrm{d}s}{\mathrm{d}x} (1 + u') - T v' \left(\frac{\mathrm{d}s}{\mathrm{d}x} \right)^{-1} \right)' , \tag{6}$$

$$\rho A \ddot{v} + c \dot{v} = \left(N \frac{\mathrm{d}s}{\mathrm{d}x} v' + T (1 + u') \left(\frac{\mathrm{d}s}{\mathrm{d}x} \right)^{-1} \right)' + q , \tag{7}$$

$$\rho I \ddot{\theta} = T \frac{\mathrm{d}s}{\mathrm{d}x} - M' , \tag{8}$$

where, in addition to already defined symbols, ρ and c are the mass density and the damping coefficient, A and I are the cross-sectional area and the second moment of area, $q = q(t)$ is the transversal load and the overdot denotes derivative with respect to time t.

Notice that Eqs. (6-8) are geometrically exact.

4. Constitutive equations

Non-standard constitutive laws for axial and transversal forces N and T and for the bending moment M have been derived in [11] by comparing internal virtual works for a beam treated as both a one-dimensional directed body and a three-dimensional body, and restricting to a linear hyper elastic Green material, are detailed as

$$N = \frac{\sin^2 \gamma}{\sqrt{2\varepsilon + 1}} \chi G A + \frac{\varepsilon}{\sqrt{2\varepsilon + 1}} E A + \frac{1 + 2\cos^2 \gamma}{2\sqrt{2\varepsilon + 1}} \kappa^2 E I , \tag{9}$$

$$T = \sqrt{2\varepsilon + 1} \sin \gamma \cos \gamma \chi G A - \sqrt{2\varepsilon + 1} \sin \gamma \cos \gamma \kappa^2 E I , \tag{10}$$

$$M = \left(\varepsilon + (2\varepsilon + 1) \cos^2 \gamma \right) \kappa E I + \frac{1}{2} \kappa^3 E \mathbb{I} , \tag{11}$$

where, apart symbols already defined, χ is the shear correction factor, E is the Young's modulus, G is the (second) Lamé's constant and $\mathbb{I}$ is the fourth moment of area. By substituting κ with $\hat{\kappa}$ we get the alternative version of Eqs. (9-11). Both systems of constitutive laws are strongly coupled due to the geometric nonlinearity of the problem.

Since both curvatures, κ and $\hat{\kappa}$, are used in the scientific literature, with arguments in favor of the former in [2,6–8,10] or the latter in [12–16], the arising question is which is the most suitable one to be selected.

In order to argue an answer to such a question, in [11], in a slightly different notation, the dimensionless linearized (with respect to the curvature) version of Eq. (11), namely

$$m_1 = \frac{M}{\kappa EI} = \varepsilon + (2\varepsilon + 1)\cos^2\gamma\,, \tag{12}$$

is compared with the approximate expressions commonly reported in the literature, that is

$$m_2 = \frac{M}{\kappa EI} = 1\,, \qquad m_3 = \frac{M}{\kappa EI} = \frac{1}{\sqrt{2\varepsilon + 1}}\,, \tag{13}$$

and it is found that choosing m_2 appears better than choosing m_3, although none of these approximate expressions (with difference between them only of few percents) capture the increment of $M/(\kappa EI)$ with respect to ε (see Fig. 2). Hence, it can be concluded that, in one-dimensional models, it is better to use the mechanical curvature κ.

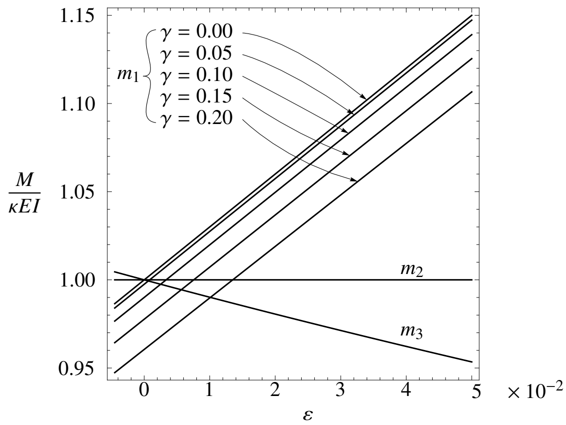

Fig. 2. Dimensionless functions m_1, m_2 and m_3 versus ε.

It is noteworthy that such an answer can't be directly obtained from intrinsic beam theories, in which the motion of beams is characterized in an internally consistent way, without a direct linkage to three-dimensional material properties. Hence, intrinsic theories do not provide any structure for constitutive equations and thus, in the case of moment-curvature relationships, different choices of curvature are, at least in principle, all equally allowed.

5. Numerical examples

Equations of motion (6-8), with constitutive relationships (9-11) are actually hard to handle. In the static case, they have been treated in [11] (to which we refer, since here details must be skipped for brevity) by applying a Finite Difference technique, with results in perfect agreement with classical findings reported in literature [17]. However, the dynamic case is found much more hard to face and some approximations must be considered.

We introduce a parameter of smallness ϵ [18] and assume that v' and θ' are small of order ϵ, u' is of order ϵ^2, γ is small enough (ϵ or even $\sqrt{\epsilon}$) to assume

$$\sin\gamma \simeq \gamma\,, \qquad \cos\gamma \simeq 1\,. \tag{14}$$

Expanding Eq. (10) and Eq. (11) in Taylor series with respect to ϵ leads to

$$T = \chi G A(\theta + v')\,, \qquad M = \kappa E I + \frac{1}{2}\kappa^3 E\mathbb{I}\,. \tag{15}$$

Further, by neglecting the longitudinal inertia and the longitudinal damping [19], in the hypothesis of vanishing axial force, Eq. (6) allows rewriting Eq. (7) as

$$\rho A\,\ddot{v} + c\,\dot{v} = \left(T\frac{\mathrm{d}s}{\mathrm{d}x}\frac{1}{1+u'}\right)'\,, \tag{16}$$

from which we attain

$$\theta' = -v'' + \frac{\rho A\ddot{v} + c\dot{v} - q}{\chi GA} . \tag{17}$$

Finally, by inserting the second time derivative of Eq. (17) in the first space derivative of Eq. (8) and discarding negligible terms, we get

$$\rho A\ddot{v} + c\dot{v} + \rho I\ddot{v}'' + EIv'''' - \frac{\rho r^2}{\chi G}\left(\rho A\ddddot{v} + c\dddot{v}\right) - \frac{Er^2}{\chi G}\left(\rho A\ddot{v}'' + c\dot{v}'\right) = q - 3E\mathbb{I}v''\left(v'''^2 + \frac{v''v''''}{2}\right) , \tag{18}$$

being $r = \sqrt{I/A}$ the cross sectional gyration radius.

By introducing the linear wave speed of longitudinal waves p along the beam and the slenderness λ of the beam as

$$p = \sqrt{\frac{E}{\rho}} , \qquad \lambda = \frac{L}{r} , \tag{19}$$

and by taking L as characteristic length scale, we define rescaled space and time variables respectively as

$$\xi = \frac{x}{L} , \qquad \tau = \frac{t}{t_0} = \frac{tp}{L\lambda} . \tag{20}$$

Rescaled displacement $V = V(\xi, \tau)$ and its partial derivatives are related to their dimensional counterparts by

$$\frac{\partial^j}{\partial \tau^j}\frac{\partial^i}{\partial \xi^i}V = t_0^j L^{i-1}\frac{\partial^j}{\partial t^j}\frac{\partial^i}{\partial x^i}v , \tag{21}$$

where i and j are integers spanning from 0 up to the requested order and we accept that the zeroth order derivatives of V and v stand for the functions themselves.

On substituting Eq. (21) in Eq. (18) and considering a rectangular cross section, after some algebra, we attain

$$\ddot{V} + 2\pi^2\zeta\dot{V} + \frac{\ddot{V}''}{\lambda^2} + V'''' + \frac{27}{10}\frac{V''}{\lambda^2}\left(2V'''^2 + V''V''''\right) = \frac{\pi P}{4}\cos\Omega\tau + \frac{12}{5\lambda^4}\left(\lambda^2\ddot{V}'' + 2\pi^2\zeta\left(\ddot{V} + \lambda^2\dot{V}''\right) + V''''\right) , \tag{22}$$

where dimensionless loading function and damping ratio are set as

$$P\cos\Omega\tau = \frac{4\lambda^2}{\pi EA}qL , \qquad \zeta = \frac{cLp\lambda}{2\pi^2 EA} , \tag{23}$$

and overdot and prime, in analogy with what we did before, stand for derivative with respect to τ and ξ.

We consider a single-mode approximation of Eq. (22) by choosing the tentative solution

$$V(\xi, \tau) = \upsilon(\tau)\sin(\pi\xi) , \tag{24}$$

with υ a generalized coordinate representing the transversal dimensionless displacement of the middle span of the beam. Substituting the solution (24) in Eq. (22), multiplying everything by $\sin(\pi\xi)$ and integrating over a unit interval, after a little rearrangement, we get

$$\left(1 + \frac{7}{5}\frac{\pi^2}{\lambda^2}\right)\ddot{\upsilon} + 2\pi^2\zeta\left(1 + \frac{12}{5}\frac{\pi^2}{\lambda^2}\right)\dot{\upsilon} + \pi^4\upsilon + \frac{27}{40}\frac{\pi^8}{\lambda^2}\upsilon^3 = P\cos\Omega\tau + \frac{12}{5}\left(\frac{\ddddot{\upsilon} + 2\pi^2\zeta\dddot{\upsilon}}{\lambda^4}\right) . \tag{25}$$

Frequency responses reported in Fig. 3, corresponding to a beam for which L is 30 times the cross section height ($\lambda = 103.92$), have been obtained by applying a brute force integration procedure, for six different values of P from 50 up to 100 (with an increasing step of 10), while frequency Ω varies, forward and backward. The integration time covers 500 cycles of loading function. From Eq. (25), it can be recognized that last term in the right-hand side becomes smaller and smaller in comparison with other terms for increasing λ. It is actually not surprising, since for large values of λ the effect of shear and rotary inertia becomes negligible. For the investigated cases, the squared difference between results from Eq. (25) and from its counterpart, obtained by dropping the last term in the right-hand side, was of the order 10^{-14} up to 10^{-12}.

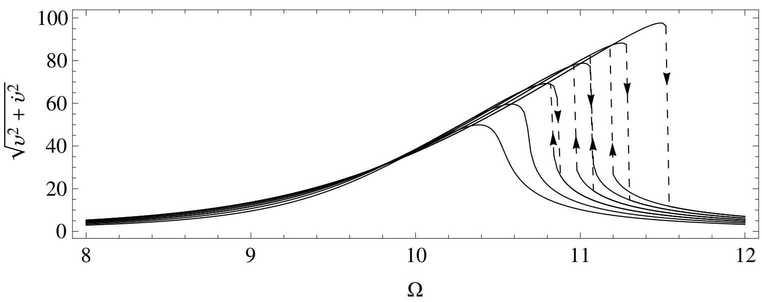

Fig. 3. Frequency response curves for $\lambda = 103.92$ and P from 50 to 100 (increasing step: 10).

6. Concluding remarks

The present communication briefly reports results concerning which is the more suitable notion of bending curvature to adopt in constitutive equations in a kinematically consistent beam theory. The theory has been developed on assuming that the beams undergo planar deformations, with large displacements and rotations under the main kinematic hypothesis of rigid cross sections. To represent specific constitutive equations in explicit form, a hyper elastic Green material, with a linear relationship between stress and strain, has been chosen. The one dimensional constitutive equations are nonlinear and strongly coupled. Some preliminary frequency response curves are reported to briefly illustrate the dynamical behavior of the beam.

References

[1] A.K. Noor, M.S. Anderson, W.H. Greene, Continuum models for beam- and platelike lattice structures, AIAA J. 16 (1978) 1219–1228.
[2] S.S. Antman, Nonlinear Problems of Elasticity, 2nd ed., Springer, 2005.
[3] P. Villaggio, Mathematical Models for Elastic Structures, Cambridge University Press, 1997.
[4] E. Reissner, On one-dimensional finite-strain beam theory: The plane problem, J. Appl. Math. Phys. ZAMP 23 (1972) 795–804.
[5] M. Géradin, A. Cardona, Flexible Multibody Dynamics: A Finite Element Approach, John Wiley, 2001.
[6] A.H. Nayfeh, P.F. Pai, Linear and Nonlinear Structural Mechanics, Wiley, 2004.
[7] W. Lacarbonara, Nonlinear Structural Mechanics: Theory, Dynamical Phenomena and Modeling, Springer US, 2013.
[8] A. Luongo, D. Zulli, Mathematical Models of Beams and Cables, ISTE-Wiley, 2013.
[9] S. Lenci, F. Clementi, G. Rega, Comparing nonlinear free vibrations of Timoshenko beams with mechanical or geometric curvature definition, Procedia IUTAM 20 (2017) 34–41.
[10] J. Gerstmayr, H. Irschik, On the correct representation of bending and axial deformation in the absolute nodal coordinate formulation with an elastic line approach, J. Sound Vib. 318 (2008) 461–487.
[11] E. Babilio, S. Lenci, On the notion of curvature and its mechanical meaning in a geometrically exact plane beam theory, Int. J. Mech. Sci. 128-129 (2017) 277–293.
[12] D.H. Hodges, Proper definition of curvature in nonlinear beam kinematics, AIAA J. 22 (1984) 1825–1827.
[13] M. Berzeri, A.A. Shabana, Development of simple models for the elastic forces in the absolute nodal co-ordinate formulation, J. Sound Vib. 235 (2000) 539–565.
[14] E.B. Magrab, Vibrations of Elastic Systems: With Applications to MEMS and NEMS, Springer, 2012.
[15] E. Babilio, Dynamics of an axially functionally graded beam under axial load, Eur. Phys. J.-Spec. Top. 222 (2013) 1519–1539.
[16] S. Lenci, F. Clementi, G. Rega, A comprehensive analysis of hardening/softening behaviour of shearable planar beams with whatever axial boundary constraint, Meccanica 51 (2016) 2589–2606.
[17] K. Mattiasson, Numerical results from large deflection beam and frame problems analysed by means of elliptic integrals, Int. J. Numer. Meth. Eng. 17 (1981) 145–153.
[18] E. Babilio, S. Lenci, The role of parameters of smallness in deduction of approximated theories for deterministic dynamics of beams, MATEC Web Conf. 83 (2016) 01001.
[19] A.H. Nayfeh, D.T. Mook, Nonlinear Oscillations, Wiley, 1979.