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Stability and Invariance Analysis of Approximate Explicit MPC based on PWA Lyapunov Functions

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Abstract: For piecewise affine (PWA) systems whose dynamics are only defined in a bounded and possibly non-invariant set $\mathcal{X}$, this paper proposes a numerical approach to analyze the stability of the origin and to find a region of attraction. The approach relies on introducing fake dynamics outside $\mathcal{X}$ and on synthesizing a piecewise affine and possibly discontinuous Lyapunov function on a larger bounded set containing $\mathcal{X}$ by solving a linear program. The existence of a solution proves that the origin is an asymptotically stable equilibrium of the original PWA system and determines a region of attraction contained in $\mathcal{X}$. The procedure is particularly useful in practical applications for analyzing a posteriori the stability properties of approximate explicit model predictive control laws defined over a bounded set $\mathcal{X}$ of states, and to determine whether, for a given set of initial states, the closed-loop system evolves within the domain $\mathcal{X}$ where the control law is defined.

1. INTRODUCTION

Model predictive control (MPC) is a well-known control technique to satisfy constraints on state and control variables in an optimized way. MPC has been widely adopted in the process industries, and deeply studied by the research community (see, e.g., Rawlings and Mayne (2009)). It is well known that the main drawback of MPC is the computation time for solving on-line an optimization problem, which prevents the application of MPC on fast-sampling processes, or when high hardware costs must be avoided.

Explicit MPC is a very effective way of simplifying on-line computations (Bemporad et al., 2002). The control law is computed off-line as an explicit function of the state vector by solving a multiparametric programming problem. For constrained linear and piecewise affine (PWA) systems, the resulting explicit function is in most cases a PWA function. In this way on-line computations reduce to searching and evaluating a lookup table of linear gains and affine terms.

However, in most practical applications of explicit MPC, the control law is only defined on a *bounded* set of states $\mathcal{X} \in \mathbb{R}^n$, and often does not enjoy a-priori stability guarantees. There are several reasons for such a situation. First, constraints on states are usually treated as *soft* constraints, while instead most stability proofs of classical linear or hybrid MPC schemes rely on *hard* state constraints (if any are enforced). Second, multiparametric solvers only solve the MPC problem on a bounded set $\mathcal{X}$ of states in order to guarantee termination and a finite number of partitions of $\mathcal{X}$. Third, the *exact* solution may be too complex, so *approximate* explicit MPC solutions are sought (Alessio and Bemporad, 2009; Bemporad and Filippi, 2003; Bemporad et al., 2011; Christophersen et al., 2007; Grieder and Morari, 2003; Johansen and Grancharova, 2003; Jones et al., 2007; Muñoz de la Peña et al., 2006). Especially in the hybrid MPC case the number of regions tends to grow quite considerably with the prediction horizon, and sub-optimal explicit MPC solutions based on switching among a set of explicit linear MPC controllers (based on linear time-invariant models) can be exploited. Such solutions, thanks to their reduced complexity, have been proposed in practical applications (see e.g. Bemporad et al. (2007); Di Cairano et al. (2010)) and can be implemented in low cost hardware solutions (Poggi et al., 2011). As a result in general (i) the bounded region $\mathcal{X}$ where the explicit control law is defined is not invariant, (ii) the origin may not be asymptotically stable, and (iii) its domain of attraction may be unknown.

To solve this problem without increasing the complexity of the controller, an a posteriori analysis must be carried out on the closed-loop system, see e.g. Christophersen et al. (2004, 2007). This problem is extensively treated in Biswas et al. (2005), where the use of different Lyapunov functions is discussed. The most used functions are quadratic, or piecewise quadratic (PWQ), Lyapunov functions (see Johansson and Rantzer (1998) for the continuous-time and Ferrari Trecate et al. (2002) for the discrete time cases). As highlighted also in Grieder et al. (2005), the search for a PWQ Lyapunov function can be overly conservative,

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even with the use of the so called S-procedure, which is not lossless in the considered cases (see e.g. Boyd et al. (1994)). A valid alternative are PWA Lyapunov functions, although usually the considered set where the system is defined is considered invariant by assumption, because, as remarked in Biswas et al. (2005), the notion of stability has no practical relevance if the state trajectory exits the defined set of states. In case the given set is not invariant, a possible approach is to perform a reachability analysis to find the maximum positively invariant set (see Raković et al. (2004), Blanchini and Miani (2008) and the references therein) to establish, using a recursive procedure, an invariant subset of the given set. However, this procedure can lead to very involved solutions due to the exponential explosion of the tree of one-step reachable subsets, and in many cases searching for the maximum invariant set requires an infinite number of steps.

This paper provides a procedure to find a Lyapunov function for (approximate) explicit MPC closed-loop systems, and more generally for (possibly discontinuous) autonomous discrete-time PWA systems whose dynamics are defined only for a set $\mathcal{X}$ of states that may not be invariant, so as to prove stability and find domains of attraction. The idea is to synthesize a Lyapunov function on a set of states that is larger than the given region $\mathcal{X}$, by making use of a "fake" dynamics outside $\mathcal{X}$, and then find an invariant subset of $\mathcal{X}$. To minimize conservativeness, continuity of the Lyapunov function is not imposed on the boundaries of the PWA partitions.

The paper is organized as follows: Section 2 introduces the class of considered autonomous PWA (closed-loop) systems. The analysis problem is formulated for a generic PWA autonomous system. Section 3 shows the proposed solution for the stability and invariance analysis, while a simulation example is presented in Section 4. Conclusions are gathered in Section 5.

2. PROBLEM FORMULATION

Consider the autonomous discrete-time piecewise affine (PWA) system

$$x(k+1) = A_i x(k) + a_i \text{ if } x(k) \in \mathcal{X}_i \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a_i \in \mathbb{R}^n$, the sets $\mathcal{X}_i$, $i \in \mathcal{I} \triangleq \{1,...,s\}$, are (possibly non-closed) polyhedra defined such that $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$, $\forall\, i,j \in \mathcal{I}$ with $i \neq j$, and such that $\cup_{i=1}^{s}\mathcal{X}_i \triangleq \mathcal{X}$ is a bounded and closed polyhedron (polytope). The set of the vertices of the closure $\bar{\mathcal{X}}_i$ of $\mathcal{X}_i$ is denoted by $vert(\bar{\mathcal{X}}_i)$. The interior of each partition $\mathcal{X}_i$ can be described as

$$int(\mathcal{X}_i) = \{x : H_i x < h_i,\ i \in \mathcal{I}\} \tag{2}$$

with $int(\mathcal{X}_i) \neq \emptyset$, where H_i and h_i are constant matrices of suitable dimensions, and $<$ indicates a component-wise inequality. Note that the dynamics (1) may not be continuous with respect to x on the boundaries of the partitions [1].

Assumption 1. Given the PWA system (1), there exists an index $i \in \mathcal{I}$ such that $0 \in vert(\bar{\mathcal{X}}_i)$, $0 \in int(\mathcal{X})$, and $x = 0$ is an equilibrium point.

[1] In case the PWA mapping defined in (1) is continuous, the sets $\mathcal{X}_i$ can be treated as closed polytopes, i.e., $\mathcal{X}_i = \{x \in \mathbb{R}^n : H_i x \leq h_i\}$, $\forall i \in \mathcal{I}$, as no ambiguity arises on overlapping boundaries.

Note that, if the origin is not on a vertex of any polyhedron $\mathcal{X}_i$, it is always possible to further partition $\mathcal{X}$ to obtain a new set of partitions $\mathcal{X}_i$ which fulfills this assumption.

The problem addressed in this paper is the following: prove that the origin is an asymptotically stable equilibrium point, and find an invariant subset $\mathcal{P} \subseteq \mathcal{X}$ of its domain of attraction.

3. PWA LYAPUNOV ANALYSIS

Since the set $\mathcal{X}$ is not assumed to be invariant, when looking for a Lyapunov function we must take into account that trajectories may possibly exit $\mathcal{X}$. To this purpose, define the set

$$\mathcal{X}^1 \triangleq \mathcal{X} \cup \{A_i x + a_i :\ x \in \mathcal{X}_i,\ i \in \mathcal{I}\} \tag{3}$$

which represents an extension of $\mathcal{X}$, including all the state values that can be reached in one time step starting from $\mathcal{X}$. As dynamics (1) are not defined outside $\mathcal{X}$, the proposed strategy consists in defining a "fake" dynamics in a region $\mathcal{X}^2$ covering $\mathcal{X}^1 \setminus \mathcal{X}$. First, a set $\mathcal{X}^2 \supseteq \mathcal{X}^1$ is defined as the bounding box of $\mathcal{X}^1$, i.e., the smallest closed hyper-rectangle containing $\mathcal{X}^1$. Let $x^{(i)}$ denote the i-th component of the state vector, $i = 1,...,n$, and define $\overline{x}^{(i)} \triangleq \sup_{\mathcal{X}^1} x^{(i)}$ and $\underline{x}^{(i)} \triangleq \inf_{\mathcal{X}^1} x^{(i)}$. Then

$$\mathcal{X}^2 \triangleq \left\{x \in \mathbb{R}^n : \underline{x}^{(i)} \leq x^{(i)} \leq \overline{x}^{(i)},\ i = 1,...,n\right\} \tag{4}$$

Consider the "fake" dynamics

$$x(k+1) = \rho x(k), \text{ if } x(k) \in \mathcal{X}^2 \setminus \mathcal{X} \tag{5}$$

where $\rho \in [0,1)$. The region $\mathcal{X}^2 \setminus \mathcal{X}$ can be divided in convex polyhedral regions as in (Bemporad et al., 2002, Th. 3). As a result, a number of new regions $\mathcal{X}_i$, $i = s+1,...,\tilde{s}$, is created. Let $\tilde{\mathcal{I}} \triangleq \{1,...,\tilde{s}\}$. The dynamics of the extended system on $\mathcal{X}^2$ is

$$x(k+1) = \begin{cases} A_i x(k) + a_i & \text{if } x(k) \in \mathcal{X}_i,\ i \in \mathcal{I}, \\ \rho x(k) & \text{if } x(k) \in \mathcal{X}^2 \setminus \mathcal{X}. \end{cases} \tag{6}$$

For convenience, we define $A_i = \rho I$, $a_i = 0$ for $i \in \tilde{\mathcal{I}} \setminus \mathcal{I}$.

Lemma 1. The set $\mathcal{X}^2$ is invariant for the PWA system (6).

Proof: If $x \in \mathcal{X}^2$, then either $x \in \mathcal{X}$ or $x \in \mathcal{X}^2 \setminus \mathcal{X}$. If $x \in \mathcal{X}$ then the successor state $A_i x + a_i \in \mathcal{X}^1 \subseteq \mathcal{X}^2$. If $x \in \mathcal{X}^2 \setminus \mathcal{X}$, the successor state $\rho x \in \mathcal{X}^2$, because $\mathcal{X}^2$ is a hyper-rectangle (convex set) including the origin. ■

Lemma 1 will be of fundamental importance when facing the problem of finding a Lyapunov function for system (1). The choice of defining $\mathcal{X}^2$ as a bounding box, and the dynamics in $\mathcal{X}^2 \setminus \mathcal{X}$ as in (5) is simplistic, yet we will prove its effectiveness. Other choices of $\mathcal{X}^2 \supseteq \mathcal{X}^1$ and of the dynamics (5) are clearly possible, provided that $\mathcal{X}^2$ is invariant.

3.1 Synthesis of the PWA Lyapunov function

By recalling classical results of stability of nonlinear discrete-time systems (see e.g. Vidyasagar (1993) and (Lazar, 2006, Chap.2)), we look for a function $V : \mathcal{X}^2 \to \mathbb{R}$ that satisfies the conditions

$$V(x) \geq \alpha\|x\| \tag{7a}$$

$$V(f(x)) - \lambda V(x) \leq 0 \tag{7b}$$

$\forall x \in \mathcal{X}^2$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the PWA state update function defined in (6), $\alpha > 0$, $\lambda \in (0, 1)$, and $\|\cdot\|$ denotes any linear vector norm (such as 1-norm or ∞-norm) [2]. Note that (7a) and (7b) imply the condition $V(0) = 0$.

The goal is now to synthesize a PWA Lyapunov function for system (6) satisfying (7). A state x and its successor $f(x)$ in (7b) may belong to the same region or to different regions, say $x \in \mathcal{X}_i$ and $f(x) \in \mathcal{X}_j$, $(i, j) \in \tilde{\mathcal{I}} \times \tilde{\mathcal{I}}$. Similarly to (Grieder et al., 2005), to characterize such transitions we define the *region transition map* $\mathcal{S}$

$$\mathcal{S}_{(i,j)} \triangleq \begin{cases} 1 \text{ if } \exists x \in \bar{\mathcal{X}}_i : \ A_i x + a_i \in \bar{\mathcal{X}}_j \\ 0 \text{ otherwise} \end{cases} \quad (8)$$

For any pair $(i, j) \in \tilde{\mathcal{I}} \times \tilde{\mathcal{I}}$, the subset of $\mathcal{X}_i$ that is mapped in one step into the region $\mathcal{X}_j$ is contained in the set

$$\mathcal{X}_{(i,j)} \triangleq \{x \in \mathcal{X}^2 : \ H_i x \le h_i, \ H_j(A_i x + a_i) \le h_j\} \quad (9)$$

that we refer to as *transition set*. Note that $\mathcal{X}_{(i,j)} = \emptyset$ if $\mathcal{S}_{(i,j)} = 0$.

We are ready to define the candidate PWA Lyapunov function $V : \mathcal{X}^2 \to \mathbb{R}$

$$V(x) = \max_{i \in \mathcal{N}(x)} V_i(x) \quad (10a)$$

where

$$\mathcal{N}(x) \triangleq \{i \in \tilde{\mathcal{I}} : \ x \in \bar{\mathcal{X}}_i\} \quad (10b)$$

and $V_i : \bar{\mathcal{X}}_i \to \mathbb{R}$ is defined as

$$V_i(x) \triangleq F_i x + g_i \quad (10c)$$

for $i \in \tilde{I}$, where in (10c) $F_i \in \mathbb{R}^{1 \times n}$ and $g_i \in \mathbb{R}$ are coefficients to be determined. The condition $V(0) = 0$ and the continuity of V in 0 are immediately obtained by requiring that $g_i = 0$ for all $i \in \tilde{\mathcal{I}}$ such that $0 \in \bar{\mathcal{X}}_i$. Note that simply $V(x) = F_i x + g_i$ for $x \in int(\mathcal{X}_i)$. The rationale for using maximization in (10a) is that for numerical reasons we want to consider closed sets $\bar{\mathcal{X}}_i$ and the affine terms $V_i(x)$, $V_j(x)$ may not coincide on common boundaries $\bar{\mathcal{X}}_i \cap \bar{\mathcal{X}}_j$, in alternative to imposing very conservative continuity conditions on such boundaries.

Since $\bar{\mathcal{X}}_i$ are convex sets and V_i are affine on $\bar{\mathcal{X}}_i$, it is enough to impose the Lyapunov conditions (7) only at the vertices $vert(\bar{\mathcal{X}}_i)$ of the $\bar{\mathcal{X}}_i$ and $vert(\bar{\mathcal{X}}_{(i,j)})$ of the $\mathcal{X}_{(i,j)}$:

$$F_i v_i^h + g_i \ge \alpha \|v_i^h\| \quad (11a)$$

for all m_i vertices $v_i^h \in vert(\bar{\mathcal{X}}_i)$, $i \in \tilde{\mathcal{I}}$, $h = 1, \ldots, m_i$, and

$$F_j(A_i v_{ij}^h + a_i) + g_j - \lambda(F_i v_{ij}^h + g_i) \le 0 \quad (11b)$$

for all $v_{ij}^h \in vert(\mathcal{X}_{(i,j)})$, $h = 1, \ldots, m_{ij}$, such that $\mathcal{S}_{(i,j)} = 1$, with $(i, j) \in \tilde{\mathcal{I}} \times \tilde{\mathcal{I}}$. Considering that all the vertices of the partitions are given, the resulting constraints (11) define a linear feasibility problem in the unknowns F_i, g_i, α, for a fixed decay rate λ, and a feasible solution can be determined by linear programming (LP). As for the computational burden, the LP (11) has a number of variables equal to $n_v = 1 + \tilde{s}(n + 1)$. One inequality is imposed for each vertex of each region $\mathcal{X}_i$, $i = 1, ..., \tilde{s}$ to fulfill (11a). Moreover, to fulfill (11b), for each vertex of each $\mathcal{X}_{(i,j)}$ one has to impose another inequality. Then, the overall number of scalar constraints is

$$n_c = \sum_{i=1}^{\tilde{s}} \left(m_i + \sum_{j \in \tilde{\mathcal{I}} : \mathcal{S}_{(i,j)} = 1} \text{card}\left(vert(\mathcal{X}_{(i,j)})\right) \right)$$

where $\text{card}\left(vert(\mathcal{X}_{(i,j)})\right)$ is the number of vertex of $\mathcal{X}_{(i,j)}$.

Lemma 2. If the LP (11) associated with the autonomous PWA dynamics (6) and the candidate Lyapunov function (10) is feasible, then the origin is an asymptotically stable equilibrium with domain of attraction $\mathcal{X}^2$.

Proof: Since the $V_i(x)$ are affine functions defined on convex partitions $\mathcal{X}_i$, the satisfaction of (11a) for all $v_i^h \in vert(\bar{\mathcal{X}}_i)$, with $i \in \tilde{\mathcal{I}}$, $h = 1, \ldots, m_i$, for $x \in \bar{\mathcal{X}}_i$ leads to $\alpha\|x\| = \alpha\|\sum_{h=1}^{m_i} \beta_i^h v_i^h\| \le \sum_{h=1}^{m_i} \beta_i^h \alpha \|v_i^h\| \le \sum_{h=1}^{m_i} \beta_i^h (F_i v_i^h + g_i) = F_i(\sum_{h=1}^{m_i} \beta_i^h v_i^h) + g_i \sum_{h=1}^{m_i} \beta_i^h = F_i x + g_i$, where $\beta_i^h \ge 0$, $\sum_{h=1}^{m_i} \beta_i^h = 1$ are a set of coefficients defining x as a convex combination of the vertices of $\mathcal{X}_i$. For this reason, for $x \in int(\mathcal{X}_i)$, since $V_i(x) = F_i x + g_i$, (7a) holds. Moreover, on the boundaries of $\bar{\mathcal{X}}_i$, according to (10a), one has $\alpha\|x\| \le F_i x + g_i$ for all $i \in N(x)$, and therefore $\alpha\|x\| \le \max_{i \in \mathcal{N}(x)}\{F_i x + g_i\} = V(x)$. This implies that (7a) holds for all $x \in \mathcal{X}^2$, since $\mathcal{X}^2 = \bigcup_{i \in \tilde{\mathcal{I}}} \bar{\mathcal{X}}_i$. Following a similar procedure, it is possible to show that (7b) holds for all $x \in \mathcal{X}^2$. As a result, (7) hold for all $x \in \mathcal{X}^2$, which proves the lemma. ■

3.2 Feasibility

In case the LP (11) is infeasible, besides increasing the value of λ, a possibility is to increase the number of partitions of $\mathcal{X}^2$, therefore providing more flexible PWA Lyapunov functions.

PWQ Lyapunov approaches assume that V is quadratic on each cell $\mathcal{X}_i$. On the other hand, assuming that V is affine on each $\mathcal{X}_i$ may not provide enough degrees of freedom. Therefore, for each polyhedron $\mathcal{X}_i$ one can compute its Delaunay triangulation (Yepremyan and Falk, 2005) $\{\mathcal{X}_{i,1}^D, \ldots, \mathcal{X}_{i,n_i}^D\}$, $i \in \tilde{s}$. The PWA Lyapunov synthesis procedure is performed by replacing the sets $\mathcal{X}_i$ with the elements of the simplicial partition $\{\mathcal{X}_{1,1}^D, \ldots, \mathcal{X}_{1,n_1}^D, \mathcal{X}_{2,1}^D, \ldots, \mathcal{X}_{2,n_2}^D, \ldots, \mathcal{X}_{\tilde{s},1}^D, \ldots, \mathcal{X}_{\tilde{s},n_{\tilde{s}}}^D\}$, and consequently by setting $\tilde{\mathcal{I}} = \{1, \ldots, \sum_{i=1}^{\tilde{s}} n_i\}$, and $\mathcal{I}$ the subset of $\tilde{\mathcal{I}}$ of indices for which $\mathcal{X}_i \subseteq \mathcal{X}$.

Another possible way is to consider the $\mathcal{X}_{(i,j)}$ as the new $\mathcal{X}_i$ and restart the procedure. Alternatively, in case the $\mathcal{X}_i$ are simplices, one can split each of them into $n + 1$ new simplices by considering the barycenter $\bar{v} = \frac{1}{n+1} \sum_{i=0}^{n} v_i$ as a new vertex. Note that by iterating such procedures, the complexity of the LP (11) may grow quite fast. On the other hand, we underline that the complexity of the explicit MPC controller which leads to the closed-loop system (1) remains unchanged.

Remark 1. The procedure described so far to find a PWA Lyapunov function given an invariant set is analogous to that in Grieder et al. (2005), with two important differences. The first is that we explicitly handle the case of discontinuous PWA Lyapunov functions. The second is that in Grieder et al. (2005) the analogous of the set $\mathcal{X}$ is

[2] Condition (7b) could be replaced by $V(f(x)) - V(x) \le -\gamma\|x\|$, where $\gamma = (1 - \lambda)\alpha > 0$. In fact by (7) it follows that $V(f(x)) - V(x) \le -(1 - \lambda)V(x) \le -(1 - \lambda)\alpha\|x\|$. Moreover an upperbound $\beta\|x\|$ on $V(x)$ can always be found here, as V is defined over a bounded set.

defined as the set of initial conditions for which the optimal control problem is feasible, and is assumed a priori to be invariant, which simplifies the approach. This is instead not assumed in our setup.

Since we analyze the stability and invariance of a system that we partially defined in an arbitrary way, we cannot get conclusions at this point about the invariance and stability of the "true" system (1) defined in $\mathcal{X}$, which is tackled in the next section.

3.3 Invariance analysis

Consider again system (6) in $\mathcal{X}^2$, assume that a feasible solution to (11) exists, define

$$\bar{V} \triangleq \inf_{x\in\mathcal{X}^2\setminus\mathcal{X}} V(x) \tag{12}$$

and consider the subset $\mathcal{P}$ of $\mathcal{X}$

$$\mathcal{P} \triangleq \{x \in \mathcal{X} : \ V(x) < \bar{V}\} \tag{13}$$

Note that the set $\mathcal{P}$ may not be convex, not even connected.

The stability of the augmented system (6) proved in Lemma 2 and the definition of $\mathcal{P}$ in (13) are exploited now to state the main result of the paper.

Theorem 1. Consider system (1), whose dynamics are only defined in $\mathcal{X}$. Assume that the dynamics are extended in $\mathcal{X}^2 \setminus \mathcal{X}$ as in (5) and that a Lyapunov function for system (6) is found by solving the LP (11). Then, the set $\mathcal{P} \subseteq \mathcal{X}$ defined in (13) is invariant with respect to the dynamics (1), the origin is asymptotically stable, and moreover $\lim_{k\to+\infty} x(k) = 0$ for any initial condition $x(0) \in \mathcal{P}$.

Proof: The proof consists of showing that the PWA Lyapunov function $V^{\mathcal{P}} : \mathbb{R}^n \to \mathbb{R}$

$$V^{\mathcal{P}}(x) \triangleq V(x), \ \forall x \in \mathcal{P} \tag{14}$$

where $V(x)$ is found as in Lemma 2 for (6) in $\mathcal{X}^2$, is a Lyapunov function for (1) over the set $\mathcal{P}$. First of all, considering that $\mathcal{P}$ is invariant for (6) in $\mathcal{X}^2$, one can note that the state update $x(k+1) \in \mathcal{P}$ for $x(k) \in \mathcal{P}$ is always calculated using the dynamics in (1). Then, $\mathcal{P}$ is an invariant set for (1), because dynamics (5) are never active. Considering that $\mathcal{P} \subseteq \mathcal{X}^2$, (7) hold for any point $x \in \mathcal{P}$, since it is already proved that (7) are satisfied for all $x(k) \in \mathcal{X}^2$. We conclude then that $V^{\mathcal{P}}(x)$ is a Lyapunov function for system (1) in $\mathcal{P}$. ■

As a practical procedure to represent the set $\mathcal{P}$, one can define the polyhedra

$$\mathcal{X}_i^{\mathcal{P}} \triangleq \{x \in \mathcal{X}_i : \ F_i x + g_i < \bar{V}\}, \ i = 1, ..., \mathcal{I}, \tag{15}$$

and define the invariant set $\mathcal{P}$ as

$$\mathcal{P} = \bigcup_{i=1}^{\mathcal{I}} \mathcal{X}_i^{\mathcal{P}} \tag{16}$$

In addition, in order to check if a given set $\mathcal{P}_0 \subseteq \mathcal{X}$ of initial states of interest is contained in $\mathcal{P}$, which proves that all trajectories starting in $\mathcal{P}_0$ live in $\mathcal{X}$ and converge to 0, it is enough to check if

$$\mathcal{P}_0 \cap \mathcal{P} = \mathcal{P}_0 \tag{17}$$

or, equivalently,

$$\bigcup_{i=1}^{\mathcal{I}} \left(\mathcal{X}_i^{\mathcal{P}} \cap \mathcal{P}_0\right) = \mathcal{P}_0 \tag{18}$$

The overall stability and invariance procedure proposed in this paper is summarized in Algorithm 1.

Algorithm 1 Stability and invariance procedure

Given the (closed-loop) PWA system (1)

REPEAT

1. If necessary, split the existing regions in subsets, obtaining a new set of $\mathcal{X}_i$;
2. Compute $\mathcal{X}^1$ in (3) and find a bounding box $\mathcal{X}^2$;
3. Define the "fake" dynamics (5);
4. Find the transition map $\mathcal{S}$ in (8) and the transition sets $\mathcal{X}_{(i,j)}$ in (9);
5. Solve the LP feasibility problem (11)

UNTIL the LP has a solution, or a given maximum number of iterations has exceeded

6. If the LP was feasible, find the region of attraction $\mathcal{P} \subseteq \mathcal{X}$ in (13)

4. SIMULATION EXAMPLE

The proposed stability and invariance analysis procedure is tested on the closed-loop system composed by a discrete-time PWA system and a switched explicit linear MPC controller. The PWA system is defined by

$$x(k+1) = \tilde{A}_i x(k) + \tilde{B}_i u(k) \ \text{ if } x(k) \in \mathcal{X}_i,$$

with $i \in \{1, 2\}$, $\mathcal{X}_1 = \{x \in \mathbb{R}^2 : H_1 x \leq h_1\}$, $\mathcal{X}_2 = \{x \in \mathbb{R}^2 : H_2 x \leq h_2\} \setminus \mathcal{X}_1$, and

$$\tilde{A}_1 = \begin{bmatrix} 0.8 & 0.8 \\ 0 & 0.8 \end{bmatrix}, \ \tilde{A}_2 = \begin{bmatrix} 0.7 & 0.7 \\ 0 & 0.7 \end{bmatrix},$$

$$\tilde{B}_1 = \tilde{B}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0 & 0.1 \\ 0 & -0.1 \\ -0.1 & 0 \\ 1 & 0 \end{bmatrix}, \ H_2 = \begin{bmatrix} 0 & 0.1 \\ 0 & -0.1 \\ -1 & 0 \\ 0.1 & 0 \end{bmatrix},$$

$$h_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \ h_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The switched explicit linear MPC controller is defined by computing an explicit MPC control law $u_i(x)$ for each *linear* system $(\tilde{A}_i, \tilde{B}_i)$, and by setting

$$u(k) = u_i(k) \ \text{ if } x(k) \in \mathcal{X}_i, \ i \in \{1, 2\}. \tag{19}$$

The overall closed-loop system does not have any a priori stability properties. Moreover, it is easy to check that $\mathcal{X}$ is not invariant. Then, we find $\mathcal{X}^1$ and $\mathcal{X}^2$ according to (3) and (6) with $\rho = 0.99$. In this case, the set $\mathcal{X}^2 \setminus \mathcal{X}$ is convex

$$\mathcal{X}^2 \setminus \mathcal{X} = \left\{ x \in \mathbb{R}^2 : \begin{array}{c} \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 12 \\ 10 \end{bmatrix} \\ [1 \ 0]\, x < -10 \end{array} \right\}.$$

The regions obtained using the switched explicit MPC, together with the extension given by $\mathcal{X}^2 \setminus \mathcal{X}$, are shown in Fig. 1.

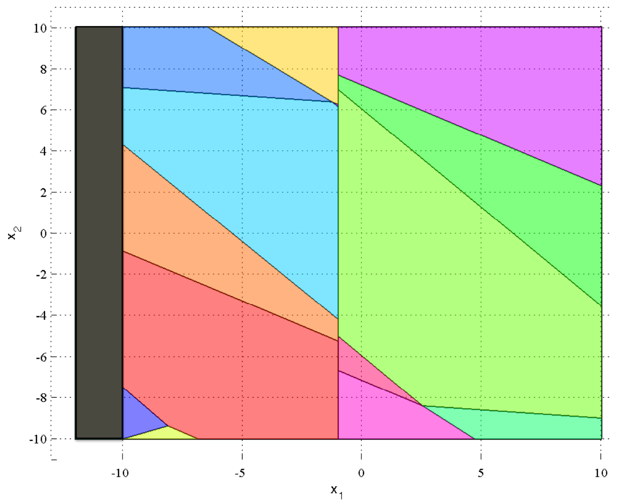

Fig. 1. The (invariant) bounding box $\mathcal{X}^2$ is constituted by the union of the regions of the explicit MPC (in color) and the box $\mathcal{X}^2 \setminus \mathcal{X}$ (in dark grey)

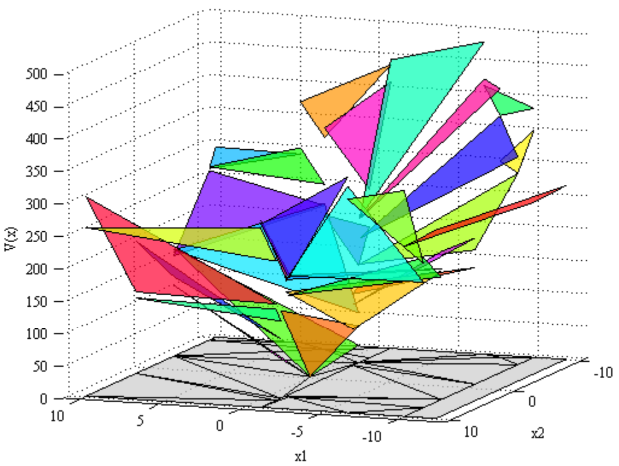

Fig. 2. The Lyapunov function obtained for $\lambda = 0.99$

According to Algorithm 1, we found the transition map $\mathcal{S}$ in (8), the transition sets $\mathcal{X}_{(i,j)}$ in (9), and solved the LP in (11) with $\lambda = 0.99$. The LP is composed of 435 constraints on $\tilde{s} = 31$ polytopes (i.e., regions $\mathcal{X}_i$). In order to solve it using the LINPROG solver of MATLAB, it took 0.10s on a 2.4 GHz processor. The corresponding Lyapunov function is shown in Figure 2 and the invariant set $\mathcal{P}$ is shown in Figure 3.

5. CONCLUSIONS

This paper has addressed the problem of determining the stability of (possibly discontinuous) discrete-time PWA systems using (possibly discontinuous) PWA Lyapunov functions, and to determine invariant sets. The problem is particularly relevant when dealing with approximations of explicit MPC, where a priori guarantees on stability and invariance are not available. The approach provides a valid alternative to the use of PWQ Lyapunov functions. Its main limitation is the possible growth of complexity of the LP problem (11) that may occur if, to increase the number of degrees of freedom of the PWA Lyapunov function, step 3 of Algorithm 1 needs to be executed several times.

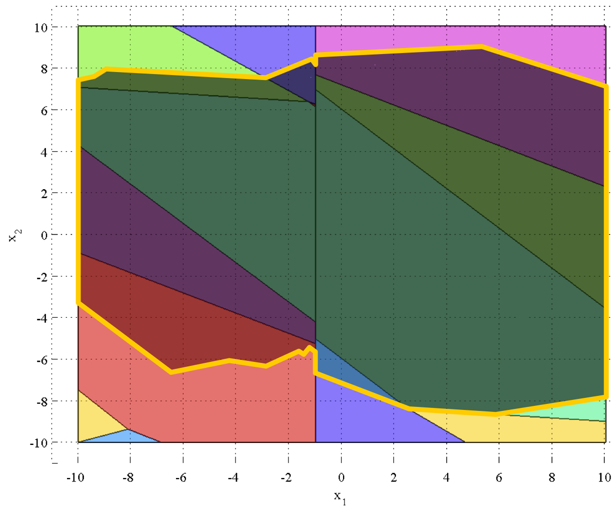

Fig. 3. The invariant set $\mathcal{P}$ obtained for $\lambda = 0.99$

Current research is devoted to extending the results of this paper to stability and invariance analysis of uncertain PWA systems.

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