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A note on some fundamental results in complete gauge spaces and application

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Abstract
We discuss the extension of some fundamental results in nonlinear analysis to the setting of gauge spaces. In particular, we establish Ekeland type and Caristi type results under suitable hypotheses for mappings and cyclic mappings. Our theorems generalize and complement some analogous results in the literature, also in the sense of ordered sets and oriented graphs. We apply our results to establishing the existence of solution to a second order nonlinear initial value problem.

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1 Introduction

The variational principle established by Ekeland in 1972, see [1, 2], is one of the most discussed and applied result in the context of nonlinear analysis. This principle plays a crucial role in establishing many theoretical results. Here, we refer to the statement below.

Definition 1.1 Let (X,d) be a metric space. A function $\varphi : X \to [0,+\infty)$ is lower semicontinuous at $x \in X$ if and only if, for every sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to +\infty$, $\liminf_{n\to+\infty} \varphi(x_n) \geq \varphi(x)$. Also, φ is lower semicontinuous if and only if it is lower semicontinuous at every $x \in X$.

Now, $L(y) := \{x \in X : \varphi(x) \leq y\}$ is called the lower counter set defined by a point $y \in [0,+\infty)$. Then the following results hold true.

Proposition 1.1 *Let (X,d) be a metric space. Let $\varphi : X \to [0,+\infty)$ be a function. Then ϕ is lower semicontinuous if and only if $L(y)$ is closed for every $y \in [0,+\infty)$.*

Theorem 1.1 ([2]) *Let (X,d) be a complete metric space and $\varphi : X \to [0,+\infty]$ be a proper and lower semicontinuous function. Then, for all $c > 0$, $\delta > 0$, and $x_0 \in X$ such that $\varphi(x_0) \leq \inf_{x\in X} \varphi(x) + c\delta$, there exists $x^* \in X$ such that*

(i) $\varphi(x^*) \leq \varphi(x_0)$;
(ii) $d(x_0,x^*) \leq \delta$;
(iii) $\varphi(x^*) < \varphi(x) + cd(x,x^*)$ *for all* $x \neq x^*$.

Of course, in the above statement we can, alternatively, consider a function $\varphi : X \to (-\infty, +\infty]$, by adding the hypothesis that it is bounded from below. The other fundamental theorem that we would like to discuss is the following fixed point result established by Caristi in 1976; see [3].

Theorem 1.2 ([3]) *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping not necessarily continuous. Assume that there exists a function $\varphi : X \to [0, +\infty)$, which is lower semicontinuous, such that*

$$d(x, fx) \leq \varphi(x) - \varphi(fx) \quad \text{for all } x \in X.$$

Then f has a fixed point z, that is, $z = fz$.

Also, f is called a Caristi mapping on (X, d). The above theorems are strongly related each other: it is well known that the results of Ekeland and Caristi are equivalent.

On the other hand, we notice that most of the spaces studied in mathematical analysis, share many algebraic and topological properties as well as metric properties. Consequently, there is no line separating clearly metric theory from the other topological or set-theoretic branches. In view of this fact, many authors considered the problem of establishing theoretic results of nonlinear analysis in a metric space (see, for instance, [4–6]). On the other hand, since several notions and theorems in the literature do not require that all the properties of a metric hold true, various definitions of generalized metrics were introduced (see, for example, [7, 8]). Here we are interested in the so-called gauge spaces that are characterized by the fact that the distance between two points of the space may be zero even if the two points are distinct. For instance, Frigon [9, 10], Chiş and Precup [11] gave generalizations of fixed point theorems and Ekeland's variational principle on gauge spaces (see also [12–15]). Consistent with this line of research, our aim is to further discuss the above fundamental theorems, by establishing new results under modified conditions in complete gauge spaces. In particular, we deal with ordered sets and oriented graphs. Then, to illustrate the usefulness of our theory, we apply our results to establishing the existence of a solution to a second order nonlinear initial value problem.

2 Preliminaries

We collect some preliminaries on gauge spaces and basic definitions.

Definition 2.1 Let X be a nonempty set. A function $d : X \times X \to [0, +\infty)$ is called a pseudo-metric in X whenever

(i) $d(x, x) = 0$ for all $x \in X$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Definition 2.2 Let X be a nonempty set endowed with a pseudo-metric d. The d-ball of radius $\varepsilon > 0$ centered at $x \in X$ is the set

$$B(x; d, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

Definition 2.3 A family $\mathcal{F} = \{d_\lambda \mid \lambda \in \mathcal{A}\}$ of pseudo-metrics is called separating if for each pair (x, y) with $x \neq y$, there is a $d_\lambda \in \mathcal{F}$ such that $d_\lambda(x, y) \neq 0$.

Definition 2.4 Let X be a nonempty set and $\mathcal{F} = \{d_\lambda \mid \lambda \in \mathcal{A}\}$ be a family of pseudo-metrics on X. The topology $\mathcal{T}(\mathcal{F})$ having as a sub-basis the family

$$\mathcal{B}(\mathcal{F}) = \{B(x; d_\lambda, \varepsilon) \mid x \in X, d_\lambda \in \mathcal{F}, \varepsilon > 0\}$$

of balls is called the topology in X induced by the family $\mathcal{F}$. The pair $(X, \mathcal{T}(\mathcal{F}))$ is called a gauge space. Notice that $(X, \mathcal{T}(\mathcal{F}))$ is Hausdorff if we require that $\mathcal{F}$ is separating.

Definition 2.5 Let $(X, \mathcal{T}(\mathcal{F}))$ be a gauge space with respect to the family $\mathcal{F} = \{d_\lambda \mid \lambda \in \mathcal{A}\}$ of pseudo-metrics on X. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(a) the sequence $\{x_n\}$ converges to x if and only if

$$\text{for all } \lambda \in \mathcal{A} \text{ and } \varepsilon > 0, \text{there exists } N \in \mathbb{N} \text{ such that } d_\lambda(x_n, x) < \varepsilon \text{ for all } n \geq N.$$

In this case, we denote $x_n \xrightarrow{\mathcal{F}} x$.

(b) The sequence $\{x_n\}$ is Cauchy if and only if

$$\text{for all } \lambda \in \mathcal{A} \text{ and } \varepsilon > 0, \text{there exists } N \in \mathbb{N} \text{ such that } d_\lambda(x_{n+p}, x_n) < \varepsilon, \forall n \geq N, p \in \mathbb{N}.$$

(c) $(X, \mathcal{T}(\mathcal{F}))$ is complete if and only if any Cauchy sequence in $(X, \mathcal{T}(\mathcal{F}))$ is convergent to an element of X.

(d) A subset of X is said to be closed if it contains the limit of any convergent sequence of its elements.

For complete reading on gauge spaces we suggest [16]. Notice that every metric space is a pseudo-metric space. Also, if a pseudo-metric d is not a metric, it is because there are at least two points $x \neq y$ for which $d(x, y) = 0$. In most situations this does not happen, which means that metrics come up in mathematics more often than pseudo-metrics; however, pseudo-metrics arise in a natural way in functional analysis and in the theory of hyperbolic complex manifolds [17].

Frigon in 2011, see [10], proved useful generalizations of the Ekeland variational principle and Caristi's fixed point theorem in complete gauge spaces. However, in establishing her results, she does not require that the family $\mathcal{F}$ is separating, but she uses a gauge structure $\{d_n \mid n \in \mathbb{N}\}$ satisfying the following condition:

$$d_1(x, y) \leq d_2(x, y) \leq \cdots \quad \text{for all } x, y \in X. \tag{1}$$

Theorem 2.1 ([10]) *Let X be endowed with a complete gauge structure $\{d_n \mid n \in \mathbb{N}\}$ satisfying condition* (1). *For every $n \in \mathbb{N}$, let $\varphi_n : X \to [0, +\infty]$ be a proper and lower semicontinuous function. Then, for all sequences of positive numbers $\{c_n\}$, $\{\delta_n\}$, and $x_0 \in X$ such that $\varphi_n(x_0) \leq \inf_{x \in X} \varphi_n(x) + c_n\delta_n$, there exists $x^* \in X$ such that*

(i) *$\varphi_n(x^*) \leq \varphi_n(x_0)$ for all $n \in \mathbb{N}$;*

(ii) *$d_n(x_0, x^*) \leq \delta_n$ for all $n \in \mathbb{N}$;*

(iii) *for all $x \neq x^*$, there exists $n \in \mathbb{N}$ such that $\varphi_n(x^*) < \varphi_n(x) + c_n d_n(x, x^*)$.*

Theorem 2.2 ([10]) *Let X be endowed with a complete gauge structure $\{d_n \mid n \in \mathbb{N}\}$ satisfying condition* (1). *Let $f : X \to X$ be a mapping. For every $n \in \mathbb{N}$, let $\varphi_n : X \to [0, +\infty)$ be a lower semicontinuous function such that*

$$d_n(x, fx) \le \varphi_n(x) - \varphi_n(fx) \quad \textit{for all } x \in X.$$

Then f has a fixed point z, that is, $z = fz$.

3 Main results

3.1 Some consequences of Frigon's theorems

Inspired by the significant work of Frigon [10], we give some consequences of Theorem 2.2.

Theorem 3.1 *Let X be endowed with a complete gauge structure $\{d_n \mid n \in \mathbb{N}\}$ satisfying condition* (1). *Let $T, f : X \to X$ be two mappings with T continuous. For every $n \in \mathbb{N}$, let r_n be a negative real number such that*

$$d_n(x, Tfx) \le d_n(x, Tx) + r_n d_n(x, fx) \quad \textit{for all } x \in X. \tag{2}$$

Then f has a fixed point z, that is, $z = fz$.

Proof The continuity of T implies that the function $\varphi_n : X \to [0, +\infty)$ defined by

$$\varphi_n(y) := -\frac{1}{r_n} d_n(y, Ty) \quad \text{for all } y \in X,$$

is lower semicontinuous. From (2), we get

$$d_n(x, fx) \le \varphi_n(x) - \varphi_n(fx) \quad \text{for all } x \in X.$$

Thus, by Theorem 2.2, f has a fixed point. □

Theorem 3.2 *Let X be endowed with a complete gauge structure $\{d_n \mid n \in \mathbb{N}\}$ satisfying condition* (1). *Let $f : X \to X$ be a mapping. For every $n \in \mathbb{N}$, let $k_n \in [0, 1)$ be such that*

$$d_n\left(fx, f^2x\right) \le k_n d_n(x, fx) \quad \textit{for all } x \in X. \tag{3}$$

If one of the following conditions holds:

(i) *the function $h_n : X \to [0, +\infty)$ defined by $h_n(x) := d_n(x, fx)$ is lower semicontinuous*;

(ii) *the mapping f is continuous*;

then f has a fixed point in X.

Proof Note that (ii) implies (i). In fact, let $x \in X$ and $\{x_m\} \subset X$ such that $x_m \to x$ as $m \to +\infty$ and assume that f is continuous. From

$$\begin{aligned} h_n(x) = d_n(x, fx) &\le d_n(x, x_m) + d_n(x_m, fx_m) + d_n(fx_m, fx) \\ &= d_n(x, x_m) + h_n(x_m) + d_n(fx_m, fx), \end{aligned}$$

we get

$$h_n(x) \le \liminf_{m\to+\infty} h_n(x_m).$$

Now, we prove that f has a fixed point in X if (i) holds. By (3), we have

$$d_n(x,fx) - k_n d_n(x,fx) \le d_n(x,fx) - d_n(fx,f^2x) \quad \text{for all } x \in X.$$

This implies that

$$d_n(x,fx) \le \varphi_n(x) - \varphi_n(fx) \quad \text{for all } x \in X,$$

where $\varphi_n : X \to [0,+\infty)$ is defined by

$$\varphi_n(y) := \frac{1}{1-k_n} d_n(y,fy) \quad \text{for all } y \in X.$$

Now, by (i), the function φ_n is lower semicontinuous for all $n \in \mathbb{N}$. Thus, the existence of a fixed point follows by an application of Theorem 2.2. □

As consequences of Theorem 3.2, we give the following results, without proof. For the origin of Theorem 3.4 and different contractive conditions, see [18, 19].

Theorem 3.3 *Let X be endowed with a complete gauge structure $\{d_n \mid n \in \mathbb{N}\}$ satisfying condition* (1). *Let $f : X \to X$ be a mapping. For every $n \in \mathbb{N}$, let $k_n \in [0,1)$ be such that*

$$d_n(fx,fy) \le k_n d_n(x,y) \quad \textit{for all } x,y \in X. \tag{4}$$

Then f has a fixed point in X.

Theorem 3.4 *Let X be endowed with a complete gauge structure $\{d_n \mid n \in \mathbb{N}\}$ satisfying condition* (1). *Let $f : X \to X$ be a mapping. For every $n \in \mathbb{N}$, let $k_n \in [0,1)$ be such that*

$$d_n(fx,fy) \le k_n \max\left\{ d_n(x,y), d_n(x,fx), d_n(y,fy), \frac{d_n(x,fy) + d_n(y,fx)}{2} \right\} \quad \textit{for all } x,y \in X.$$

If one of the following conditions holds:

(i) *the function $h_n : X \to [0,+\infty)$ defined by $h_n(x) := d_n(x,fx)$ is lower semicontinuous*;

(ii) *the mapping f is continuous*;

then f has a fixed point in X.

Example 3.1 Let $X = \mathbb{R}$ and, for any $n \in \mathbb{N}$, define

$$d_n(x,y) = \frac{n}{n+1} \left| x^2 - y^2 \right| \quad \text{for all } x,y \in X.$$

Clearly, $\{d_n \mid n \in \mathbb{N}\}$ is a complete gauge structure satisfying condition (1). Also, define $f : X \to X$ by $fx = \frac{x}{2}$ for all $x \in X$. Now, for every $n \in \mathbb{N}$, let $k_n \in [\frac{1}{4},1)$ so that condition (4) is satisfied for all $x,y \in X$. Therefore, f has a fixed point in X; here 0 is a unique fixed point of f.

3.2 Results for cyclic mappings

In [20], Kirk *et al.* obtained extensions of well-known fixed point theorems for cyclic mappings, by considering, for instance, a cyclical contractive condition as given by the next theorem.

Definition 3.1 Let A, B be two nonempty subsets of a metric space (X,d). Then $f : X \to X$ is called a cyclic mapping associated to (A,B) if the following conditions hold:
(i) $X = A \cup B$;
(ii) $f(A) \subseteq B$ and $f(B) \subseteq A$.

Theorem 3.5 ([20]) *Let A, B be two nonempty closed subsets of a metric space (X,d) and $f : X \to X$ be a cyclic mapping associated to (A,B). Let $k \in (0,1)$ be such that*

$$d(fx,fy) \le kd(x,y) \quad \text{for all } x \in A \text{ and } y \in B.$$

Then f has a unique fixed point in $A \cap B$.

Inspired by this result, other fixed point theorems with cyclical contractive conditions were obtained (see, for instance, [21–23]). Our aim in this section is to prove some fixed point theorems for cyclic mappings in complete gauge spaces. First, we state the extension of Theorems 3.3 and 3.5 for a cyclic mapping and a complete gauge structure.

Theorem 3.6 *Let X be endowed with a complete gauge structure $\{d_n \mid n \in \mathbb{N}\}$ satisfying condition* (1). *Let A, B be two nonempty closed subsets of X and $f : A \cup B \to A \cup B$ be a cyclic mapping associated to (A,B). For every $n \in \mathbb{N}$, let $k_n \in [0,1)$ be such that*

$$d_n(fx,fy) \le k_n d_n(x,y) \quad \text{for all } x \in A \text{ and } y \in B.$$

Then f has a fixed point in $A \cap B$.

Now, we prove the following theorem.

Theorem 3.7 *Let X be endowed with a complete gauge structure $\{d_n \mid n \in \mathbb{N}\}$ satisfying condition* (1). *Let A, B be two nonempty closed subsets of X and $f : A \cup B \to A \cup B$ be a cyclic mapping associated to (A,B). For every $n \in \mathbb{N}$, let $\varphi_n^1 : A \to [0,+\infty)$ and $\varphi_n^2 : B \to [0,+\infty)$ be lower semicontinuous functions such that*

$$d_n(x,fx) \le \varphi_n^1(x) - \varphi_n^2(fx) \quad \text{for all } x \in A \tag{5}$$

and

$$d_n(x,fx) \le \varphi_n^2(x) - \varphi_n^1(fx) \quad \text{for all } x \in B. \tag{6}$$

Then f has a fixed point in $A \cap B$.

Proof Let $x_1 \in A$ and let $x_{m+1} = fx_m$ for all $m \in \mathbb{N}$. From (5) and (6) we get

$$\varphi_n^1(x_1) \ge \varphi_n^2(x_2) \ge \cdots \ge \varphi_n^1(x_{2m-1}) \ge \varphi_n^2(x_{2m}) \ge \cdots .$$

This implies that the sequences $\{\varphi_n^1(x_{2m-1})\}$ and $\{\varphi_n^2(x_{2m})\}$ are non-increasing and have the same limit, say $r \geq 0$. Let $p > m$. Then

$$\begin{aligned} d_n(x_{2m-1}, x_{2p}) &\leq \sum_{k=2m}^{2p} d_n(x_{k-1}, x_k) \\ &\leq \varphi_n^1(x_{2m-1}) - \varphi_n^2(x_{2m}) + \varphi_n^2(x_{2m}) - \varphi_n^1(x_{2m+1}) + \cdots + \varphi_n^1(x_{2p-1}) - \varphi_n^2(x_{2p}) \\ &= \varphi_n^1(x_{2m-1}) - \varphi_n^2(x_{2p}) \to 0 \quad (\text{as } m \to +\infty). \end{aligned}$$

Since $d_n(x_m, x_{m+1}) \to 0$ as $m \to +\infty$, we see that $\{x_m\}$ is a Cauchy sequence and $A \cap B \neq \emptyset$. Now, we have the following:

$$d_n(x, fx) \leq \min\{\varphi_n^1(x) - \varphi_n^2(fx), \varphi_n^2(x) - \varphi_n^1(fx)\} \quad \text{for all } x \in A \cap B.$$

Thus,

$$d_n(x, fx) \leq \varphi_n(x) - \varphi_n(fx) \quad \text{for all } x \in A \cap B,$$

where $\varphi_n : A \cap B \to [0, +\infty)$ is defined by $\varphi_n(x) := \frac{1}{2}(\varphi_n^1(x) + \varphi_n^2(x))$ for all $x \in A \cap B$. Clearly, φ_n is lower semicontinuous and, hence, the conclusion follows from Theorem 2.2. □

Example 3.2 Let $A = B = X = [0, +\infty)$ and define

$$d_1(x, y) = \begin{cases} 0 & \text{if } x = y \text{ or } x, y \in [0, 1], \\ 1 & \text{otherwise,} \end{cases}$$

and, for any $n \in \mathbb{N} \setminus \{1\}$,

$$d_n(x, y) = \begin{cases} d_{n-1}(x, y) & \text{if } x, y \in [0, n], \\ n & \text{otherwise.} \end{cases}$$

Clearly, $\{d_n \mid n \in \mathbb{N}\}$ is a complete gauge structure satisfying condition (1). Also, let $f : X \to X$ be defined by

$$fx = \begin{cases} 0 & \text{if } x \notin [1, 2], \\ 1 & \text{if } x \in [1, 2]. \end{cases}$$

It follows that

$$d_1(x, fx) = \begin{cases} 0 & \text{if } x \in [0, 1], \\ 1 & \text{otherwise,} \end{cases}$$

and

$$d_n(x, fx) = \begin{cases} d_{n-1}(x, fx) & \text{if } x \in [0, n], \\ n & \text{otherwise.} \end{cases}$$

Notice that $\varphi : X \to X$, defined by

$$\varphi(x) = \begin{cases} 0 & \text{if } x \in [0,1], \\ x & \text{otherwise,} \end{cases}$$

is a lower semicontinuous function such that $d_n(x,fx) \leq \varphi(x) - \varphi(fx)$, for all $x \in X$. Thus, we can apply Theorem 3.7, with $\varphi_n^1 = \varphi_n^2 = \varphi$, to conclude that f has a fixed point; here 0 and 1 are fixed points of f.

The following result uses a nice contractive condition introduced by Geraghty in 1973; see [24].

Theorem 3.8 *Let X be endowed with a complete gauge structure $\{d_n \mid n \in \mathbb{N}\}$ satisfying condition* (1). *Let A, B be two nonempty closed subsets of X and $f : A \cup B \to A \cup B$ be a cyclic mapping associated to (A,B). For every $n \in \mathbb{N}$, let $\alpha_n : [0,+\infty) \to [0,1)$ be such that $\alpha_n(t_m) \to 1$ implies $t_m \to 0$. Assume that, for every $n \in \mathbb{N}$, the following condition holds*:

$$d_n(fx,fy) \leq \alpha_n\big(d_n(x,y)\big)d_n(x,y) \quad \textit{for all } x \in A \textit{ and } y \in B. \tag{7}$$

Then f has a fixed point in $A \cap B$.

Proof Let $x_1 \in A$ and let $x_{m+1} = fx_m$ for all $m \in \mathbb{N}$. From (7), we deduce that

$$d_n(x_{m+1},x_{m+2}) \leq \alpha_n\big(d_n(x_m,x_{m+1})\big)d_n(x_m,x_{m+1}) \leq d_n(x_m,x_{m+1})$$

and so the sequence $\{d_n(x_m,x_{m+1})\}$ is non-increasing and bounded from below. This implies that there exists $r_n \geq 0$ such that $d_n(x_m,x_{m+1}) \to r_n$ as $m \to +\infty$. If $r_n > 0$. Then, by (7), we obtain

$$\frac{d_n(x_{m+1},x_{m+2})}{d_n(x_m,x_{m+1})} \leq \alpha_n\big(d_n(x_m,x_{m+1})\big).$$

Letting $m \to +\infty$, we deduce that $\alpha_n(d_n(x_m,x_{m+1})) \to 1$ and so $d_n(x_m,x_{m+1}) \to 0$. To show that the sequence $\{x_m\}$ is Cauchy, we suppose the contrary. Assume that, given $k \in \mathbb{N}$, there exist $m > p > k$ such that

$$d_n(x_{2p-1},x_{2m}) \geq \varepsilon.$$

From

$$d_n(x_{2p-1},x_{2m}) \leq d_n(x_{2p-1},x_{2p}) + d_n(x_{2p},x_{2m+1}) + d_n(x_{2m+1},x_{2m}),$$

we get

$$\begin{aligned}\big[1 - \alpha_n\big(d_n(x_{2p-1},x_{2m})\big)\big]\varepsilon &\leq \big[1 - \alpha_n\big(d_n(x_{2p-1},x_{2m})\big)\big]d_n(x_{2p-1},x_{2m}) \\ &\leq d_n(x_{2p-1},x_{2p}) + d_n(x_{2m+1},x_{2m}).\end{aligned}$$

Letting $p, m \to +\infty$, we deduce that $\alpha_n(d_n(x_{2p-1}, x_{2m})) \to 1$ and so $d_n(x_{2p-1}, x_{2m}) \to 0$. This implies that the sequence $\{x_m\}$ is Cauchy. Then $A \cap B \neq \emptyset$ and $x_m \to z \in A \cap B$. Since $f : A \cap B \to A \cap B$ is continuous, we get $z = fz$, that is, z is a fixed point of f in $A \cap B$. □

4 Ordered sets and oriented graphs

In this section, we adapt the ideas in [25] to get further theorems in complete gauge spaces.

4.1 Fixed points of monotone non-decreasing mappings

Let $(X, \preceq)$ be a partially ordered set and $f : X \to X$ be a mapping. Then f is said to be a monotone non-decreasing mapping if the following condition holds:

$$x \preceq y \quad \Longleftrightarrow \quad fx \preceq fy \quad \text{for all } x, y \in X.$$

Also, two elements $x, y \in X$ such that $x \preceq y$ are said to be comparable.

Theorem 4.1 *Let $(X, \preceq)$ be a partially ordered set endowed with a complete gauge structure $\{d_n \mid n \in \mathbb{N}\}$ satisfying condition (1). Let $f : X \to X$ be a continuous and monotone non-decreasing mapping. For every $n \in \mathbb{N}$, let $\varphi_n : X \to [0, +\infty)$ be a lower semicontinuous function such that*

$$d_n(x, fx) \leq \varphi_n(x) - \varphi_n(fx) \quad \textit{for all } x \in X \textit{ with } fx \preceq x.$$

Then f has a fixed point if and only if there exists $x_0 \in X$ with $fx_0 \preceq x_0$.

Proof Let $x_0 \in X$ with $fx_0 \preceq x_0$ and let $x_m = fx_{m-1}$ for all $m \in \mathbb{N}$. Since f is monotone non-decreasing, then $x_{m+1} \preceq x_m$ for every $m \in \mathbb{N}$. Therefore, for every $m, n \in \mathbb{N}$, we have

$$d_n(x_m, x_{m+1}) \leq \varphi_n(x_m) - \varphi_n(x_{m+1}).$$

This implies that the sequence $\{\varphi_n(x_m)\}$ is non-increasing and so there exists $r_n \geq 0$ such that $\varphi_n(x_m) \to r_n$ as $m \to +\infty$. For every $m, n, p \in \mathbb{N}$, we get

$$\begin{aligned} d_n(x_m, x_{m+p}) &\leq \sum_{i=0}^{p-1} d(x_{m+i}, x_{m+i+1}) \\ &\leq \varphi_n(x_m) - \varphi_n(x_{m+p}). \end{aligned}$$

This implies that $\{x_m\}$ is a Cauchy sequence in X. Now, by completeness, there exists $z \in X$ such that $x_m \to z$ as $m \to +\infty$. Finally, by continuity of f we conclude that $fz = z$, that is, z is a fixed point of f.

On the other hand, if x_0 is a fixed point of f, then $x_0 = fx_0$ and so the order relation $fx_0 \preceq x_0$ is trivially satisfied. This completes the proof. □

Remark 4.1 The novelty of the last theorem over the corresponding theorem without ordering is due to the fact that the contractive behavior of f is restricted to the elements $x \in X$ which are comparable to fx.

4.2 Fixed points of *G*-edge preserving mappings

Definition 4.1 A graph G is an ordered pair (V,E), where V is a set and $E \subseteq V \times V$ is a binary relation. We say that V is the vertex set and E is the edge set.

We refer the reader to [26] for a more detailed background on this topic.

Definition 4.2 Let $G = (V,E)$ be a graph and D be a subset of V. We say that D is G-directed if for every $x,y \in D$, there exists $z \in V$ such that $(x,z),(y,z) \in E$.

Example 4.1 Let $V = F([0,1],\mathbb{R})$ be the set of functions $u : [0,1] \to \mathbb{R}$ and define $E \subseteq V \times V$ by

$$(u,v) \in E \quad \Longleftrightarrow \quad u(t) \leq v(t) \quad \text{for all } t \in [0,1].$$

Then $G = (V,E)$ is a graph. Let $D = M([0,1],\mathbb{R})$ be the set of measurable functions $u : [0,1] \to \mathbb{R}$. Then D is G-directed. Indeed, for every $u,v \in D$, the function $z = \max\{u,v\}$ satisfies $(u,z),(v,z) \in E$.

Let (V,d) be a metric space. We consider a family $G = \{G_i : 1 \leq i \leq q\}$ of $q \geq 1$ graphs such that $G_i = (V,E_i)$, $E_i \subseteq V \times V$, $i = 1,2,\ldots,q$.

Definition 4.3 Let $f : V \to V$ be a given mapping. We say that f is G-monotone if for all $i = 1,2,\ldots,q$, we see that $(x,y) \in E_i$ implies that $(fx,fy) \in E_{i+1}$, with $E_{q+1} = E_1$. Consequently, $(f^{kq}x,f^{kq}y) \in E_i$ for each nonnegative integer number k if f is G-monotone.

Remark 4.2 If $q = 1$ ($G = G_1$), we say that f is G-edge preserving; see [27].

Building on Theorem 4.1, we give a further generalization of Caristi type, by substituting the ordering relation with an oriented graph.

Theorem 4.2 *Let X be a complete gauge structure $\{d_n \mid n \in \mathbb{N}\}$ satisfying condition* (1). *Let G be an oriented graph on X such that $(x,x) \in E(G)$ for all $x \in X$. Let $f : X \to X$ be a continuous and G-edge preserving mapping. For every $n \in \mathbb{N}$, let $\varphi_n : X \to [0,+\infty)$ be a lower semicontinuous function such that*

$$d_n(x,fx) \leq \varphi_n(x) - \varphi_n(fx) \quad \textit{for all } x \in X \textit{ with } (fx,x) \in E(G).$$

Then f has a fixed point if and only if there exists $x_0 \in X$ with $(fx_0,x_0) \in E(G)$.

Proof Let $x_0 \in X$ such that $(fx_0,x_0) \in E(G)$ and let $x_m = fx_{m-1}$ for all $m \in \mathbb{N}$. Since f is G-edge preserving, by Definition 4.3 and in view of Remark 4.2, we deduce that $(x_{m+1},x_m) \in E(G)$ for every $m \in \mathbb{N}$. Then, for every $m,n \in \mathbb{N}$, we get

$$d_n(x_m,x_{m+1}) \leq \varphi_n(x_m) - \varphi_n(x_{m+1}).$$

This implies that the sequence $\{\varphi_n(x_m)\}$ is non-increasing and so there exists $r_n \geq 0$ such that $\phi_n(x_m) \to r_n$ as $m \to +\infty$. For every $m, n, p \in \mathbb{N}$, we have

$$\begin{aligned} d_n(x_m, x_{m+p}) &\leq \sum_{i=0}^{p-1} d(x_{m+i}, x_{m+i+1}) \\ &\leq \varphi_n(x_m) - \varphi_n(x_{m+p}). \end{aligned}$$

This implies that $\{x_m\}$ is a Cauchy sequence in X. Now by completeness, there exists $z \in X$ such that $x_m \to z$ as $m \to +\infty$. Finally, by continuity of f we conclude that $fz = z$, that is, z is a fixed point of f.

On the other hand, if x_0 is a fixed point of f, then $(fx_0, x_0) \in E(G)$, since by hypothesis $(x, x) \in E(G)$ for all $x \in X$. This completes the proof. □

5 Application to ordinary differential equation

A typical application of fixed point methods is in establishing sufficient conditions for the existence of solution of integro-differential problems. Referring to [14], we consider the following second order nonlinear initial value problem:

$$\begin{cases} x''(t) = k(t, x(t)), & t > 0, \\ x(0) = \alpha, \\ x'(0) = \beta, \end{cases} \tag{8}$$

where $k : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. It is well known that the above problem is equivalent to the following integral equation:

$$x(t) = \int_0^t (t - s)k\big(s, x(s)\big)\,ds + \beta t + \alpha, \quad t \geq 0.$$

Let $X = C([0, +\infty), \mathbb{R}^n)$ be the set of continuous functions defined on $[0, +\infty)$. Then, for every $n \in \mathbb{N}$, we consider the semi-norm $\|\cdot\|_n : X \to [0, +\infty)$ given by

$$\|x\|_n = \max_{t \in [0,n]} \big|x(t)\big| \quad \text{for all } x \in X,$$

where $|\cdot|$ denote the norm in $\mathbb{R}^n$. Also, for every $n \in \mathbb{N}$, let

$$d_n(x, y) = \|x - y\|_n \quad \text{for all } x, y \in X.$$

Clearly, $\mathcal{F} = \{d_n \mid n \in \mathbb{N}\}$ is a family of pseudo-metrics on X satisfying condition (1). Also, $(X, \mathcal{T}(\mathcal{F}))$ is a complete gauge space.

We shall prove the following theorem.

Theorem 5.1 *For every $n \in \mathbb{N}$, assume that the following condition holds*:

$$\big|k\big(s, x(s)\big) - k\big(s, y(s)\big)\big| \leq \gamma(s)\big[1 - e^{-\min\{d_n(x,y),1\}}\big] \quad \textit{for each } s \in [0, n] \textit{ and for all } x, y \in X,$$

and $\gamma : [0,+\infty) \to [0,+\infty)$ *is such that the function* $t \mapsto \int_0^t (t-s)\gamma(s)\,ds$ *is bounded on* $[0,+\infty)$ *and*

$$\sup_{t\geq 0}\int_0^t (t-s)\gamma(s)\,ds \leq 1.$$

Then the second order nonlinear initial value problem (8) *has a solution* $x^* \in C([0,+\infty), \mathbb{R}^n)$.

Proof Consider the operator $f : X \to X$ defined by

$$fx(t) = \int_0^t (t-s)k\big(s,x(s)\big)\,ds + \beta t + \alpha, \quad t \geq 0, x \in X,$$

which is well defined, since k is a continuous function.

It is immediate that x^* is a solution of (8) if and only if x^* is a fixed point of f. Then we need to show that Theorem 3.8 is applicable to the operator f to conclude the proof of Theorem 5.1.

Let $n \in \mathbb{N}$ and let $x, y \in X$, then for all $t \in [0,n]$ we write

$$\begin{aligned}\big|fx(t) - fy(t)\big| &\leq \int_0^t (t-s)\big|k\big(s,y(s)\big) - k\big(s,x(s)\big)\big|\,ds \\ &\leq \int_0^t (t-s)\gamma(s)\big[1 - e^{-\min\{d_n(x,y),1\}}\big]\,ds \\ &\leq \big[1 - e^{-\min\{d_n(x,y),1\}}\big]\int_0^t (t-s)\gamma(s)\,ds \\ &\leq 1 - e^{-\min\{d_n(x,y),1\}}.\end{aligned}$$

Then, for all $n \in \mathbb{N}$, we get

$$d_n(fx,fy) \leq 1 - e^{-\min\{d_n(x,y),1\}} \quad \text{for all } x, y \in X,$$

which further gives us

$$d_n(fx,fy) \leq \frac{1 - e^{-\min\{d_n(x,y),1\}}}{\min\{d_n(x,y),1\}} d_n(x,y).$$

Notice that, for every $n \in \mathbb{N}$, the function $\alpha_n : [0,+\infty) \to [0,1)$ given by

$$\alpha_n(\tau) = \begin{cases} \frac{e^{-\min\{\tau,1\}}-1}{-\min\{\tau,1\}} & \text{if } \tau > 0, \\ \frac{1}{2} & \text{if } \tau = 0, \end{cases}$$

is such that $\alpha_n(t_m) \to 1$ implies $t_m \to 0$, as $m \to +\infty$. Thus, by an application of Theorem 3.8 with $A = B = X$, we see that f has a fixed point $x^* \in X$, that is, $x^* \in C([0,+\infty), \mathbb{R}^n)$ is a solution of (8). □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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