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Explicit Frequency Response Function of beams with Crack of Uncertain Depth

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Abstract

Detection of cracks in structural components and identification of their size for structures having beam form is of crucial importance in many engineering applications. Usually, the crack characteristics are assumed to be known. However they possess considerable scatter or uncertainty assumed in this paper by both a probabilistic and non-probabilistic model. In order to evaluate the main statistics as well the upper and lower bounds of the response, the *Frequency Response Function* of damaged beams with uncertain depth of the crack is derived in explicit approximate form.

Keywords: damaged beam; uncertain crack; frequency response function.

1. Introduction

For damaged structures the dynamic response changes with respect to the undamaged ones due to the changes produced on its mechanical properties by the presence of the crack [1,2]. In this framework an interesting issue is the effect of a single crack on the structural response [3-6]. In Structural Dynamics, the *Frequency Response Function* (*FRF*) is a complex function able to provide information about the behaviour of a structure over a range of frequencies. In this paper a novel procedure for deriving in explicit approximate form the *FRF* of damaged beams with uncertain depth of the crack, generalizing the procedure recently proposed by the authors [7], is presented. By adopting the approximate *FRF* and modelling the uncertainty by both probabilistic and interval approaches, a

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procedure is proposed to evaluate the bounds of the interval displacement and of the mean-value and variance of the random response. Finally, interval versus stochastic analysis results are derived and compared in a numerical application.

2. Analytical model of the cracked beam

The mathematical model adopted to analyze the damaged beam with a transverse on-edge non-propagating crack is based on the finite element model proposed in Refs. [1,2].

According to the Saint-Venant principle the presence of a crack in the beam modifies the stress field in the region adjacent to the crack. Such a perturbation of the stress field is relevant especially when the crack is open and determines a local reduction of the flexural rigidity affecting only the element that contains a central crack.

It follows that the element stiffness matrix, with the exception of the terms which represent the cracked element, may be regarded as unchanged under a certain limitation of the element size.

Undamaged parts of the beam are modelled by Euler type finite elements with two nodes and two degrees of freedom (transverse displacement and rotation) at each node.

The calculation of the additional stress energy introduced by the crack has been studied in fracture mechanics and the flexibility coefficients are expressed by a stress intensity factor in the linear elastic range, using Castigliano's theorem. The generic component $d_{ij}^{(0)}$ of the compliance (or flexibility) matrix $\mathbf{D}_{\mathrm{e}}^{(0)}$ of the undamaged element and the terms $d_{ij}^{(1)}$ of the additional flexibility matrix $\mathbf{D}_{\mathrm{e}}^{(1)}$ due to the crack can be derived respectively as

$$d_{ij}^{(0)} = \frac{\partial^2 W^{(0)}}{\partial P_i \partial P_j}, \quad d_{ij}^{(1)} = \frac{\partial^2 W^{(1)}}{\partial P_i \partial P_j}; \quad i,j = 1,2; \quad P_1 = P, \quad P_2 = M \tag{1}$$

being $W^{(0)}$ the strain energy of an element without a crack, whereas $W^{(1)}$ the additional energy. By the principle of virtual work the stiffness matrix of the undamaged and cracked element takes the following form:

$$\mathbf{K}_{\mathrm{e}} = \mathbf{T}\mathbf{D}_{\mathrm{e}}^{(0)-1}\mathbf{T}^T; \quad \mathbf{K}_{\mathrm{c,e}} = \mathbf{T}\mathbf{D}_{\mathrm{e}}^{-1}\mathbf{T}^T; \quad \mathbf{T}^T = \begin{bmatrix} -1 & -\ell & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{2}$$

where the apex *T* means transpose matrix. Once the stiffness matrices of the undamaged and cracked elements are defined, for the beam discretized in N_{e} finite elements the stiffness matrix $\mathbf{K}$ of order $n \times n$ with $n = 2N_{\mathrm{e}}$ can be straightforwardly evaluated following the classical assembly rules.

Moreover, in the framework of the finite element approximation, it is usually assumed that the crack does not modify the mass distribution.

3. Problem formulation

Usually parameters relating to crack, namely depth and position of the crack, could be affected by a lack of knowledge and consequently modelled as uncertain parameters. In this section the case of uncertain depth of the crack is introduced. The equation of motion of a quiescent cracked beam discretized by N_{e} finite elements subjected to an external deterministic excitation $\mathbf{f}(t)$ can be written as:

$$\mathbf{M}\,\ddot{\mathbf{u}}(\alpha,t) + \mathbf{C}(a_0,\alpha)\,\dot{\mathbf{u}}(\alpha,t) + \mathbf{K}(a_0,\alpha)\,\mathbf{u}(\alpha,t) = \mathbf{f}(t) \tag{3}$$

where $\mathbf{M}$ is the $n \times n$ mass matrix of the structure, $\mathbf{C}(a_0,\alpha)$ is the $n \times n$ damping matrix, $\mathbf{K}(a_0,\alpha)$ is the $n \times n$ stiffness matrix and $\mathbf{f}(t)$ is deterministic vector function of order $n \times 1$; $\mathbf{u}(\alpha,t)$ is the *uncertain* vector of nodal displacements of order $n \times 1$ and a dot over a variable denotes differentiation with respect to time *t*.

The $n \times n$ *uncertain* stiffness matrix $\mathbf{K}(a_0,\alpha)$ is here expressed as a function of the uncertain structural parameter α as follows:

$$\mathbf{K}(a_0,\alpha) = \mathbf{K}_{\mathrm{C}}(a_0) + \alpha\,\mathbf{K}_1(a_0); \qquad \mathbf{K}_1(a_0) = \left.\frac{\partial \mathbf{K}(a_0,\alpha)}{\partial \alpha}\right|_{\alpha=0} \tag{4}$$

where α is the undimensional zero mean fluctuation of the uncertain crack depth $a = a_0(1+\alpha)$ with a_0 its *mean value*. In Eq.(4) $\mathbf{K}_{\mathrm{C}}(a_0)$ is the *mean stiffness matrix*. It is a positive definite symmetric matrix of order $n \times n$, while

$\mathbf{K}_1(a_0)$ is a symmetric matrix of order $n \times n$ and rank *r*. The Rayleigh model is herein adopted for the uncertain damping matrix, i.e.:

$$\mathbf{C}(a_0,\alpha) = c_0\,\mathbf{M} + c_1\,\mathbf{K}(a_0,\alpha) = c_0\,\mathbf{M} + c_1\,\mathbf{K}_\mathrm{C}(a_0) + c_1\,\alpha\,\mathbf{K}_1(a_0) = \mathbf{C}_\mathrm{C}(a_0) + \alpha\,\mathbf{C}_1(a_0) \tag{5}$$

where c_0 and c_1 are the Rayleigh damping constants having units s^{-1} and s, respectively. Hereafter we indicate $\mathbf{K}_\mathrm{C}(a_0) = \mathbf{K}_\mathrm{C}$, $\mathbf{K}_1(a_0) = \mathbf{K}_1$ and $\mathbf{C}_\mathrm{C}(a_0) = \mathbf{C}_\mathrm{C}$ for sake of notation compactness.

4. Frequency domain response

Performing the Fourier transform of both sides of Eq.(3) and taking into account relationships (4) and (5) the following set of algebraic frequency dependent equations governing the response in the frequency domain is obtained:

$$\mathbf{U}(\alpha,\omega) = \left[\mathbf{I}_n + \mathbf{H}_\mathrm{C}(\omega)\,\mathbf{S}(\alpha,\omega)\right]^{-1}\mathbf{H}_\mathrm{C}(\omega)\mathbf{F}(\omega) = \mathbf{H}(\alpha,\omega)\mathbf{F}(\omega) \tag{6}$$

where $\mathbf{H}(\alpha,\omega)$ is the *frequency response function (FRF) matrix* (referred to also as *transfer function matrix*) given as:

$$\mathbf{H}(\alpha,\omega) = \left[\mathbf{I}_n + \mathbf{H}_\mathrm{C}(\omega)\,\mathbf{S}(\alpha,\omega)\right]^{-1}\mathbf{H}_\mathrm{C}(\omega) \tag{7}$$

In the previous equation, $\mathbf{I}_n$ denotes the identity matrix of order *n*, $\mathbf{H}_\mathrm{C}(\omega)$ is the *FRF* matrix of the nominal structural system referred to the *mean stiffness matrix* and $\mathbf{S}(\alpha,\omega)$ is a complex matrix of order $n \times n$ accounting for the fluctuations of the uncertain parameter. $\mathbf{H}_\mathrm{C}(\omega)$ and $\mathbf{S}(\alpha,\omega)$ are given respectively by:

$$\begin{aligned}\mathbf{H}_\mathrm{C}(\omega) &= \left[-\omega^2\,\mathbf{M} + \mathrm{i}\,\omega c_0\,\mathbf{M} + \mathrm{i}\,\omega c_1\,\mathbf{K}_\mathrm{C} + \mathbf{K}_\mathrm{C}\right]^{-1};\\ \mathbf{S}(\alpha,\omega) &= \mathrm{i}\,\omega\,\alpha c_1\,\mathbf{K}_1 + \alpha\,\mathbf{K}_1\end{aligned} \tag{8}$$

In the previous equations the matrices $\mathbf{K}_\mathrm{C}$ and $\mathbf{K}_1$ have been defined in Eq.(4), while $\mathbf{U}(\alpha,\omega)$ and $\mathbf{F}(\omega)$ are the vectors collecting the Fourier transforms of $\mathbf{u}(\alpha,t)$ and $\mathbf{f}(t)$, respectively. Notice that since the Rayleigh model has been adopted for the damping, the *FRF* matrix $\mathbf{H}_\mathrm{C}(\omega)$ can be evaluated in closed form as:

$$\mathbf{H}_\mathrm{C}(\omega) = \mathbf{\Phi}_\mathrm{C}\,\mathbf{H}_\mathrm{C,m}(\omega)\,\mathbf{\Phi}_\mathrm{C}^T = \mathbf{\Phi}_\mathrm{C}\left[-\omega^2\mathbf{I}_m + \mathrm{i}\,\omega\left(c_0\,\mathbf{I}_m + c_1\,\mathbf{\Omega}_\mathrm{C}\right) + \mathbf{\Omega}_\mathrm{C}^2\right]^{-1}\mathbf{\Phi}_\mathrm{C}^T \tag{9}$$

where $\mathbf{\Phi}_\mathrm{C}$ is the modal matrix, of order $n \times m$, pertaining to the *mean* configuration. Specifically, the modal matrix $\mathbf{\Phi}_\mathrm{C}$, collecting the first *m* eigenvectors normalized with respect to the mass matrix $\mathbf{M}$, is evaluated as solution of the following eigenproblem:

$$\mathbf{K}_\mathrm{C}\mathbf{\Phi}_\mathrm{C} = \mathbf{M}\,\mathbf{\Phi}_\mathrm{C}\,\mathbf{\Omega}_\mathrm{C}^2; \quad \mathbf{\Phi}_\mathrm{C}^T\,\mathbf{M}\,\mathbf{\Phi}_\mathrm{C} = \mathbf{I}_m; \quad \mathbf{\Phi}_\mathrm{C}^T\,\mathbf{K}_\mathrm{C}\,\mathbf{\Phi}_\mathrm{C} = \mathbf{\Omega}_\mathrm{C}^2 \tag{10}$$

$\mathbf{\Omega}_\mathrm{C}^2$ being the spectral matrix listing the squares of the natural circular frequencies of the structure referred to the mean value of the uncertain parameter and $\mathbf{I}_m$ the identity matrix of order *m*. It is worth noting that in Eq.(9) $\mathbf{H}_\mathrm{C,m}(\omega)$ is the modal transfer function matrix of the “mean” structure that for classically damped structural systems is a diagonal matrix.

By inspection of Eqs.(7) and (9), it is observed that the evaluation of the *FRF* $\mathbf{H}(\alpha,\omega)$ matrix involves the inversion of a matrix expressed as sum of a diagonal matrix plus a deviation given by the full matrix $\mathbf{S}(\alpha,\omega)$. It follows that the *uncertain FRF* matrix can be determined in approximate explicit form by applying the *Rational Series Expansion* (*RSE*) [7], herein truncated to first-order terms, i.e.:

$$\mathbf{H}(\alpha,\omega) \approx \mathbf{H}_\mathrm{C}(\omega) - \sum_{j=1}^{r}\frac{p(\omega)\lambda_j\,\alpha}{1+p(\omega)\alpha\,\lambda_j\,b_j(\omega)}\,\mathbf{B}_j(\omega) = \mathbf{H}_\mathrm{C}(\omega) + \sum_{j=1}^{r}\eta_j\left(\alpha,\omega\right)\mathbf{B}_j(\omega) \tag{11}$$

with $p(\omega) = 1 + \mathrm{i}\,\omega\,c_1$ and

$$b_j(\omega)=\mathbf{v}_j^T\mathbf{H}_C(\omega)\,\mathbf{v}_j;\quad \mathbf{B}_j(\omega)=\mathbf{H}_C(\omega)\,\mathbf{v}_j\mathbf{v}_j^T\mathbf{H}_C(\omega);\quad \eta_j(\alpha,\omega)=-\frac{p(\omega)\lambda_j\alpha}{1+p(\omega)\alpha\lambda_j\,b_j(\omega)} \tag{12}$$

In the previous equations the following position has been made: $\mathbf{v}_j=\mathbf{K}_C\boldsymbol{\psi}_j$, with $\boldsymbol{\psi}_j$ and λ_j the j-th eigenvector $(j=1,\ldots,r)$ and the associated eigenvalue, solutions of the following eigenproblem:

$$\mathbf{K}_1\boldsymbol{\psi}_j=\lambda_j\mathbf{K}_C\boldsymbol{\psi}_j;\quad \boldsymbol{\Psi}^T\mathbf{K}_C\boldsymbol{\Psi}=\mathbf{I}_r;\quad \boldsymbol{\Psi}=\begin{bmatrix}\boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 & \cdots & \boldsymbol{\psi}_r\end{bmatrix} \tag{13}$$

Notice that only $r<n$ eigenvalues are different from zero and the generic term of the summation in Eq.(11) turns out to be a rank-one matrix. It follows that the evaluation of the *FRF* matrix involves the inversion of a matrix expressed as sum of a *mean FRF* matrix plus a deviation given as superposition of rank-one matrices.

In the following sub-sections the uncertain depth of the crack is modelled first by a random variable and then by an interval variable. For the first case a second-moment analysis, namely computing the evolution of mean values and covariances of response quantities, is performed employing a method developed in the frequency domain. For the interval case the bounds of the dynamic response are evaluated through a procedure which derives explicit expressions of the interval *FRF*.

4.1. Statistic functions of the response

Let us first consider the case of the stochastic crack depth $a=a_0\left(1+\tilde{\alpha}\right)$ where $\tilde{\alpha}$ is the undimensional zero mean random fluctuation of the uncertain crack depth and a_0 its mean value.

Taking into account Eq.(11), specifying the random fluctuation of the uncertain parameter, and by substituting Eq.(11) into Eq.(6) it follows that the *k*-th element of the vector $\tilde{\mathbf{U}}(\omega)$ can be expressed as:

$$\tilde{U}_k(\omega)\approx U_{k,C}(\omega)-\sum_{j=1}^{r}\frac{p(\omega)\lambda_j\tilde{\alpha}}{1+p(\omega)\tilde{\alpha}\lambda_j\,b_j(\omega)}\,d_{k,j}(\omega)=U_{k,C}(\omega)+\sum_{j=1}^{r}\tilde{\eta}_{k,j}(\alpha,\omega) \tag{14}$$

with

$$\begin{gathered}U_{k,C}(\omega)=\left\{\mathbf{H}_C(\omega)\,\mathbf{F}(\omega)\right\}_k;\quad d_{k,j}(\omega)=\left\{\mathbf{B}_j(\omega)\,\mathbf{F}(\omega)\right\}_k;\\ \tilde{\eta}_{k,j}(\alpha,\omega)=\tilde{\eta}_{k,j}(\omega)=-\frac{p(\omega)\lambda_j\tilde{\alpha}}{1+p(\omega)\,\tilde{\alpha}\,\lambda_j\,b_j(\omega)}d_{k,j}(\omega)\end{gathered} \tag{15}$$

The symbol $\{\bullet\}_k$ means *k*-th element of the vector into curly parentheses. In order to evaluate in the time domain the mean value and variance of the *k*-th nodal response, $\mu_{\tilde{U}_k}(t)$ and $\sigma^2_{\tilde{U}_k}(t)$ respectively, the inverse Fourier Transform is applied to Eq.(14) obtaining:

$$\tilde{U}_k(t)=F^{-1}\left[\tilde{U}_k(\omega)\right]\approx U_{k,C}(t)+\sum_{j=1}^{r}\tilde{N}_{k,j}(t) \tag{16}$$

where $U_{k,C}(t)=F^{-1}\left[U_{k,C}(\omega)\right]$ and $\tilde{N}_{k,j}(t)=F^{-1}\left[\tilde{\eta}_{k,j}(\alpha,\omega)\right]$. Starting from Eq.(16), the mean value and variance functions of the stochastic variables $\tilde{U}_k(t)$ can be evaluated as follows, respectively:

$$\mu_{\tilde{U}_k}(t)=U_{k,C}(t)+\sum_{j=1}^{r}\mathrm{E}\left\langle\tilde{N}_{k,j}(t)\right\rangle\equiv U_{k,C}(t)+\sum_{j=1}^{r}e_{k,j}(t);\quad \sigma^2_{\tilde{U}_k}(t)=\sum_{j=1}^{r}e^2_{k,jj}(t)+2\sum_{j=1}^{r-1}\sum_{\ell=j+1}^{r}e_{k,j\ell}(t)-\left(\sum_{j=1}^{r}e_{k,j}(t)\right)^2 \tag{17}$$

where $e_{k,j\ell}(t)$ is the time domain stochastic average for the generic cross-term with respect to the zero-mean stochastic variable $\tilde{\alpha}$ which can be evaluated as:

$$e_{k,j\ell}(t)\equiv\mathrm{E}\left\langle\tilde{N}_{k,j}(t)\tilde{N}_{k,\ell}(t)\right\rangle=\frac{1}{4\pi}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}\tilde{\eta}_{k,j}(\alpha,\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\times\int_{-\infty}^{+\infty}\tilde{\eta}_{k,\ell}(\alpha,\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\right\}p_{\tilde{\alpha}}(\alpha)\,\mathrm{d}\alpha \tag{18}$$

4.2. Interval response in frequency and time domain

Let us now consider the case in which the uncertainty is modeled as an interval variable. In this case, according to the *interval analysis* [8], the bounded real number $\alpha^I \triangleq [\underline{\alpha}, \overline{\alpha}] \in \mathbb{IR}$ such that $\underline{\alpha} \le \alpha \le \overline{\alpha}$, with $\underline{\alpha}$ and $\overline{\alpha}$ denoting the lower and upper bound of α, is introduced. Referring to the so-called *extra symmetric unitary interval* (*EUI*) variable [9] $\hat{e}_\alpha^I \triangleq [-1,1]$ therefore $a = a_0\left(1+\Delta\alpha\,\hat{e}_\alpha^I\right)$, it follows that the FRF function can be rewritten as [7]:

$$\mathbf{H}^I(\omega) \approx \mathbf{H}_\mathrm{C}(\omega) - \sum_{j=1}^{r} \frac{p(\omega)\lambda_j\,\Delta\alpha\,\hat{e}_\alpha^I}{1+p(\omega)\Delta\alpha\,\hat{e}_\alpha^I\,\lambda_j\,b_j(\omega)}\,\mathbf{B}_j(\omega) = \mathbf{H}_\mathrm{mid}(\omega)+\mathbf{H}_\mathrm{dev}^I(\omega) \tag{19}$$

Equation (19) provides the *interval FRF* matrix as sum of the midpoint matrix, $\mathbf{H}_\mathrm{mid}(\omega)$, plus the interval deviation matrix, $\mathbf{H}_\mathrm{dev}^I(\omega)$, given, respectively, by:

$$\mathbf{H}_\mathrm{mid}(\omega) = \mathbf{H}_\mathrm{C}(\omega) + \sum_{j=1}^{r} a_{0,j}(\omega)\,\mathbf{B}_j(\omega); \quad \mathbf{H}_\mathrm{dev}^I(\omega) = \hat{e}_\alpha^I \sum_{j=1}^{r} \Delta a_j(\omega)\,\mathbf{B}_j(\omega) \tag{20}$$

with

$$a_{0,j}(\omega) = \frac{\left[p(\omega)\lambda_j\,\Delta\alpha\right]^2 b_j(\omega)}{1-\left[p(\omega)\lambda_j\,\Delta\alpha\,b_j(\omega)\right]^2}; \quad \Delta a_j(\omega) = \frac{p(\omega)\lambda_j\,\Delta\alpha}{1-\left[p(\omega)\lambda_j\,\Delta\alpha\,b_j(\omega)\right]^2} \tag{21}$$

where the argument $\Delta\alpha$ in the functions $a_{0,i\ell}(\omega)$ and $\Delta a_{i\ell}(\omega)$ as well as in the matrix functions $\mathbf{H}_\mathrm{mid}(\omega)$ and $\mathbf{H}_\mathrm{dev}^I(\omega)$ is omitted for the sake of conciseness. Upon substitution of the approximate *FRF* the response can be expressed as sum of the midpoint value plus the deviation as well, i.e.:

$$\mathbf{U}^I(\omega) = \mathbf{U}_\mathrm{mid}(\omega) + \mathbf{U}_\mathrm{dev}^I(\omega) = \left[\mathbf{H}_\mathrm{mid}(\omega) + \hat{e}_\alpha^I \sum_{j=1}^{r} \Delta a_j(\omega)\,\mathbf{B}_j(\omega)\right]\mathbf{F}(\omega) \tag{22}$$

The k-th element of the vector $\mathbf{U}(\alpha,\omega)$ can be written as:

$$U_k^I(\alpha,\omega) \approx U_{k,\mathrm{mid}}(\omega) + \hat{e}_\alpha^I \sum_{j=1}^{r} \Delta a_j(\omega)\,d_{k,j}(\omega) = U_{k,\mathrm{mid}}(\omega) + \hat{e}_\alpha^I \sum_{j=1}^{r} \gamma_{k,j}(\omega) \tag{23}$$

with

$$U_{k,\mathrm{mid}}(\omega) = \left\{\mathbf{H}_\mathrm{mid}(\omega)\,\mathbf{F}(\omega)\right\}_k; \quad d_{k,j}(\omega) = \left\{\mathbf{B}_j(\omega)\,\mathbf{F}(\omega)\right\}_k; \quad \gamma_{k,j}(\omega) = \Delta a_j(\omega)\,d_{k,j}(\omega) \tag{24}$$

The symbol $\{\bullet\}_k$ means k-th element of the vector into curly parentheses.

In the time domain the lower and upper bounds of dynamic response can evaluated once inverse Fourier transforms of midpoint and deviation functions are performed, that is:

$$u_{k,\mathrm{mid}}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_{k,\mathrm{mid}}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega; \quad u_{k,\mathrm{dev}}^I(t) = \hat{e}_\alpha^I \sum_{j=1}^{r} \frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma_{k,j}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \hat{e}_\alpha^I\,g_k(t) \tag{25}$$

It follows that the lower and upper bounds of dynamic response in the time domain can be written as:

$$\begin{aligned}\underline{u}_k(t) &= \min\left\{u_{k,\mathrm{mid}}(t) - \left|g_k(t)\right|,\, u_{k,\mathrm{mid}}(t) + \left|g_k(t)\right|\right\};\\ \overline{u}_k(t) &= \max\left\{u_{k,\mathrm{mid}}(t) - \left|g_k(t)\right|,\, u_{k,\mathrm{mid}}(t) + \left|g_k(t)\right|\right\}\end{aligned} \tag{26}$$

5. Numerical application

A damaged cantilever steel beam studied in [1] subjected to a transverse deterministic action $f(t)$ applied to the free-end is examined. The beam has length L=200mm and a rectangular cross-section with width b=1mm and height h=7.8mm. The Young's modulus and the material mass density are assumed $E = 207000\,\mathrm{N/mm^2}$ and $\rho = 7860\,\mathrm{Kg/m^3}$, respectively. The Rayleigh damping constants in Eq.(5) are evaluated as c_0= 83.927s^{-1} and c_1=0.0000138s in such a way that the modal damping ratio for the first and second modes of the structure referred to the mean stiffness matrix is $\zeta_0 = 0.05$. The considered acting load is expressed by a combination of decaying sinusoidal functions in the form $f(t) = \sum_{j=1}^{4} \exp(-jt/10)\sin(5jt)$.
The cantilever beam has been modelled by $N_e = 5$ finite elements with the crack supposed to be located in the middle of the second element.
The mean value of the crack depth is assumed to be $a_0 = 0.4h$. Moreover for the stochastic model, the zero mean random fluctuation $\tilde{\alpha}$ is uniformly distributed on the interval [-0.3,0.3] while in the interval model the deviation amplitude is fixed in $\Delta\alpha = 0.3$. Efficiency and accuracy of the presented procedures have been confirmed by the comparison with exact result and MonteCarlo simulation for the two cases of uncertainty. In this application, results provided by the two described approaches for the probabilistic and non-probabilistic crack depth model are compared in terms of bounds of the vertical displacement $u(t)$ of the free end of the cantilever beam, namely $u_9(t)$.
Figs.1a and 1b show lower $\underline{u}_9(t)$ and upper $\overline{u}_9(t)$ bounds of the interval free-end displacement compared with the bounds calculated as $\mu_{\tilde{U}_9}(t) \pm \sigma_{\tilde{U}_9}(t)$ and $\mu_{\tilde{U}_9}(t) \pm 3\sigma_{\tilde{U}_9}(t)$ being $\mu_{\tilde{U}_9}(t)$ the *mean value* and $\sigma_{\tilde{U}_9}(t)$ the *standard deviation*, respectively, of the stochastic response. As evident by the figures the intervals obtained via the stochastic model for the uncertain crack depth include the *interval* displacement response.

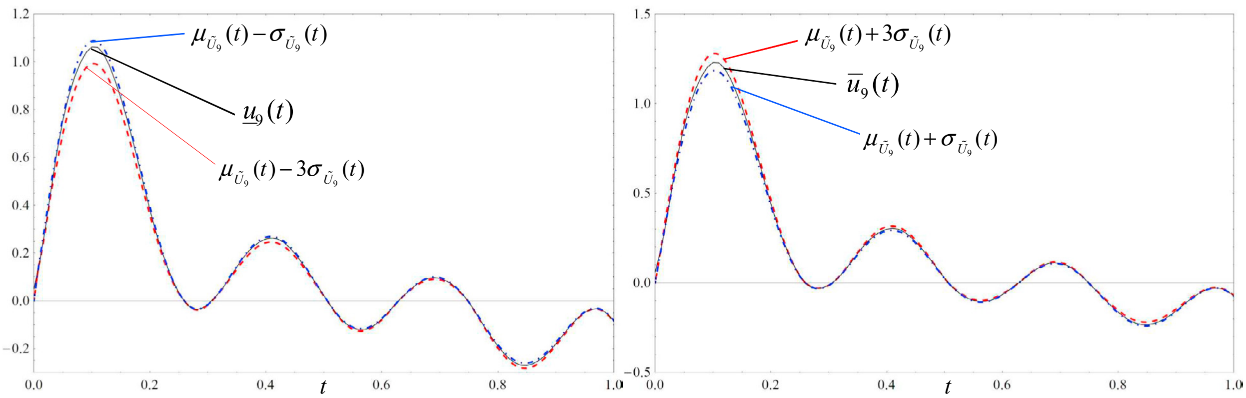

Fig. 1. (a) Lower and (b) upper bound of the free-end displacement of the damaged cantilever beam with interval and random crack depth

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