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Nuclear and Particle Physics Proceedings 267–269 (2015) 131–139

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Scattering Amplitudes at LHC

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Abstract

I analyze the algebraic patterns underlying the structure of scattering amplitudes in quantum field theory. I focus on the decomposition of amplitudes in terms of independent functions and the systems of differential equations the latter obey. In particular, I discuss the key role played by unitarity for the decomposition in terms of master integrals, by means of generalized cuts and integrand reduction, as well as for solving the corresponding differential equations, by means of Magnus exponential series.

Keywords: Scattering Amplitudes, Feynman Diagrams

1. Introduction

High energy particle collisions are the ideal framework for accessing new informations on matter constituents and forces of nature. The higher the energy of the colliding particles, the richer the landscape of the produced ones. The discovery of new physics interactions cannot be disentangled from the discovery of massive, *heavy* particles, emerging from collisions of ever increasing energy. On the other side, by increasing energy, also the probability of producing *many light* particles is enhanced. Therefore, advances in High Energy Particle Physics necessarily depend on our ability to describe the scattering processes involving *many* light and heavy *particles* at very high accuracy.

Scattering amplitudes are numbers that represent the probabilities that a certain set of particles will turn into certain other particles upon colliding. Particle collisions are described by *Feynman diagrams*, representing the different ways particle shuffle during an interaction. In perturbation theory, the scattering between two colliding particles that produce n outgoing ones are described at leading order (LO) by tree diagrams, with $n + 2$ legs. Quantum corrections to this process receive contributions from diagrams containing either additional number of legs or closed loops. Tree-level diagrams represent rational functions of the kinematic variables, therefore they can be considered easy to compute. Loop diagrams, instead, represent very challenging integrals. The complexity of perturbative calculations grows with the number of loops, the number of legs, and the masses of the involved particles. In general, when a direct integration of Feynman integrals is prohibitive, the evaluation of scattering amplitudes beyond the leading order (LO) is addressed in two stages: *i)* the decomposition in terms of an integral basis, and *ii)* the evaluation of the elements of such a basis, called *master integrals* (MIs). In this contribution, I elaborate on the algebraic properties of Feynman integrals, which can be exploited for decomposing them in terms of MIs and for computing the latter. The techniques I discuss can be applied to generic amplitudes, and have a impact on high-accuracy prediction for collider physics, as well as for exploring the more formal aspect of quantum field theory.

What's the most natural way to decompose amplitudes? Amplitudes can be decomposed in terms of independent integrals, exactly like a vector can be decomposed along of basic directions. To achieve amplitudes decomposition, one needs a *basis*, and a *projection* technique. The latter is necessary to extract the coefficients of the linear combination (equivalent to a scalar prod-

http://dx.doi.org/10.1016/j.nuclphysbps.2015.10.093

uct). For generic multi-loop amplitudes, the basis is not known. Therefore, the projection becomes an operation that should not only determine the coefficients, but, also, identify the corresponding master diagrams.

Factorization is the basic idea we are going to elaborate on. Factorization is ubiquitous in the discovery of new mathematical and physical concepts. Complex numbers emerged from factorizing the simplest number we may think of, *i.e.* $1 = (-i)i$; quantum mechanics relies on factorization of the identity matrix, $\mathbb{I} = \sum_n |n\rangle\langle n|$; Dirac equation emerged by factorizing the d'Alambertian operator, *i.e.* $\Box = (-i\partial\!\!\!/)(i\partial\!\!\!/)$.
Let us discuss what can emerge when amplitudes factorize.

1.1. On-shell methods and Amplitudes decomposition

The *unitarity-cut* of internal lines is a suitable projection technique yielding amplitudes decomposition. Mathematically, it corresponds to apply a kinematic constraint to particles that are exchanged between two adjacent interaction vertices: a propagating particle carries a momentum whose squared value is different from its squared-mass ($p^2 \neq m^2$); cutting it, amounts to bring the particle on-shell ($p^2 = m^2$). The by now known as *on-shell* and *generalized-unitarity methods* exploit the ideas of "multiple-cuts", namely cutting more than one internal line.

Why multiple-cuts are important? First, because multiple-cuts yield functions identification. Any diagram is characterized by its internal lines, therefore, any master diagram is univocally identified by a cut-diagram where all internal particles are on-shell. Moreover, when applied to amplitudes, multiple-cuts behave like *high-pass filters*. In fact, since any amplitude a sum of diagrams, cutting simultaneously a certain number of internal line amounts to *isolating* only the diagrams that have those internal lines, while the others are automatically discarded. Therefore, by considering all possible cuts of an amplitude, in a top-down procedure, from the maximum number of cuts to the lowest one, it is possible to build a (triangular) system of equations from which *all* coefficients can be determined.

Elaborating on the concept of generalized cuts [1–5], unitarity has been inspiring a novel organization of the perturbative calculus, where Feynman diagrams are grouped according to their multi-particle cuts.

New approaches tackling the evaluation of one-loop multi-parton amplitudes have recently been under intense development (see [6–9] for reviews).

1.2. Scattering Amplitudes in the Complex Plane

Unitarity of scattering amplitudes [10, 11] has then been strengthened by the complementary classification of the mathematical structures present in the residues at the singularities. Fundamental results along this direction are the on-shell recurrence relation for tree-level amplitudes [12], its link to the leading singularity of one-loop amplitudes [2], and the discovery of a relation between numerator and denominators of one-loop Feynman integr*als* [13, 14]. These new insights, which stem from a reinterpretation of tree-level scattering within the twistor string theory [15], have catalyzed, on the one side, the study of novel mathematical frameworks in the more supersymmetric sectors of quantum field theories, such as dual conformal symmetries, grassmanians, Wilson-loops/gluon-amplitudes duality, color/kinematic, and gravity/gauge dualities (see the collection [16]), till the recent idea of the amplituhedron [17]. Unitarity-based methods [1–5, 18–27] are responsible for the breakthrough advances in automating the evaluation multi-particle one-loop scattering amplitudes, urgently demanded by the experimental programmes at hadron colliders. Novel ideas about the generalized-unitarity cutting techniques have been discussed in the contribution of William Torres Bobadilla [28].
Within the unitarity-based methods, the difficulty of loop-integration reduces to phase-space integration, where the components of the loop momenta not constrained by the multiple-cut conditions becomes integration variables. Since multiple-cut conditions, to be satisfied, require that the moment of the cut-particles have complex-valued components, Cauchy's residue theorem emerged as a fundamental tool for the decomposition of scattering amplitudes by means of generalized unitarity. The holomorphic anomaly [29, 30] and the spinor integration [3, 19], as well as, Cauchy's residue theorem [2, 12], the Laurent series expansion [22, 31–33], Stokes' Theorem [4], and the global residue theorem [34, 35] have been employed for carrying out the integration of the phase-space integrals, left over after applying the on-shell cut-conditions to the loop integrals.

Are we really sure that integration cannot be avoided? In this contribution, I would like to discuss the algebraic patterns beneath Feynman calculus, and where *integration* can be replaced by *algebraic* operations.

2. Tree-level amplitudes

Tree-level scattering amplitudes are found to obey a quadratic recurrence relation [12], depicted in Fig.1,

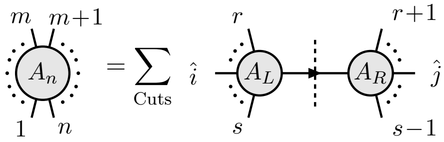

Figure 1: Tree-level recurrence relation. The n-point amplitude A_n is written in terms of products of two lower-point amplitudes, A_L and A_R. Momenta indicated with a *hat* are complex valued and on-shell.

whose derivation relies on Cauchy's residue theorem. By following Ref. [36], the on-shell recurrence can be understood as coming simply from a partial fractioning formula. In fact, let us consider the following integral,

$$\oint \frac{dz}{z(z-z_1)(z-z_2)\cdots(z-z_n)} = 0\,, \tag{1}$$

which has simple poles in the complex plane, and vanishing contribution when $z \to \infty$. Therefore one can relate the residue at the pole $z = 0$ to the (sum of the) residues at the finite poles $z = z_i$, as,

$$\begin{aligned}\frac{(-1)^n}{z_1 z_2\cdots z_n} &= \frac{1}{z_1(z_1-z_2)\cdots(z_1-z_n)}\\ &+ \frac{1}{(z_2-z_1)z_2\cdots(z_2-z_n)}\\ &+ \cdots\cdots\\ &+ \frac{1}{(z_n-z_1)(z_n-z_2)\cdots(z_n-z_{n-1})z_n}\,,\end{aligned} \tag{2}$$

which is just a partial fractioning formula. Let us now consider modified Feynman denominators, each depending on a complex z_i variable, which can be interpreted as the component along a massless reference momentum η. The value of z_i can be chosen to enforce the on-shell condition,

$$(q_i - z_i\eta)^2 - m_i^2 \;=\; 0\,, \qquad z_i = \frac{q_i^2 - m_i^2}{2\eta\cdot q_i}\,. \tag{3}$$

For $i \neq j$, one has,

$$(q_i - z_j\eta)^2 - m_i^2 \;=\; 2\eta\cdot q_i(z_i - z_j)\,, \tag{4}$$

which can be used to determine $(z_i - z_j)$. Therefore, by using the expression of z_i and $(z_i - z_j)$, Eq.(2) reads,

$$\begin{aligned}\frac{(-1)^n}{q_1^2-m_1^2}\frac{1}{q_2^2-m_2^2}\cdots\frac{1}{q_n^2-m_n^2} &=\\ = \frac{1}{q_1^2-m_1^2}\frac{1}{(q_2-z_1\eta)^2-m_2^2}&\cdots\frac{1}{(q_n-z_1\eta)^2-m_n^2}\\ +\frac{1}{(q_1-z_2\eta)^2-m_1^2}\frac{1}{q_2^2-m_2^2}&\cdots\frac{1}{(q_n-z_2\eta)^2-m_n^2}\\ +\cdots\cdots&\\ +\frac{1}{(q_1-z_n\eta)^2-m_1^2}\frac{1}{(q_2-z_n\eta)^2-m_2^2}&\cdots\frac{1}{q_n^2-m_n^2}\end{aligned} \tag{5}$$

which is a partial fractioning formula for the denominator of tree-level amplitudes. Scattering amplitudes, at tree-level, can be decomposed in terms independent building blocks, identified by a given Feynman denominator, $q_i^2 - m_i^2$. Each denominator identifies a singularity that corrisponds to a physical factorization channel, while the other terms multiplying it contribute to the residue. In case of scattering between (massless) spin-1 particles and fermions, while Feynman denominators go on-shell, the corresponding numerators factorize as well, according to the completeness relations,

$$-g^{\mu\nu} + \frac{q_i^\mu\eta^\nu + q_i^\nu\eta^\mu}{(q_i\cdot\eta)} \;=\; \sum_\lambda \epsilon_\mu^\lambda(q_i,\eta)\;\epsilon_\nu^{\lambda*}(q_i,\eta)\,, \tag{6}$$

$$(\not{q}_i + m) \;=\; \sum_s u_s(q_i)\,\bar{u}_s(q_i)\,. \tag{7}$$

Therefore, the amplitude factorize in product of simpler on-shel amplitudes, as shown in Fig.1. This result can be used to build scattering amplitudes (l.h.s.) from products of simpler amplitudes (r.h.s.) with complex, on-shell momenta.

Is that just accidental or partial fractioning can be exploited also at higher orders?

3. Higher-order amplitudes

The *integrand reduction algorithm* [13] had a dramatic impact on our ability of computing one-loop amplitudes. The basic idea lies in the existence of a relation between numerators and denominators of scattering amplitudes which can be used to decompose the integrands of one-loop amplitudes in terms of integrands of MIs. The amplitude decomposition in terms of MIs is then achieved after integrating the integrand decomposition. The coefficients of the MIs are a subset of the coefficients appearing in the decomposition of the integrands. Therefore, within the integrand reduction algorithm, coefficients can be determined simply by algebraic manipulation, with the great advantage of bypassing any integration. Applications of methods and tools based on the integrand reduction technique to phenomenlogy are discussed in the contribution of Giovanni Ossola [37].

The extension of the integrand decomposition beyond one-loop has been proposed in [38], and refined in [39–41], where the unitarity-based decomposition of multi-loop integrands has been addressed as a polynomial decomposition problem, and systematized within the *multivariate polynomial division algorithm*. Let us consider a multi-loop integral with n denominators,

$$\mathcal{I}_{12\ldots n} = \int d^d q_1 \cdots d^d q_m \, I_{12\ldots n} \,, \tag{8}$$

whose integrand reads,

$$I_{12\ldots n} \equiv \frac{N_{12\ldots n}}{D_1 \ldots D_n} \,. \tag{9}$$

By means of the polynomial division modulo Gröbner basis, one can perform the division between the numerator and the denominators, such that

$$N_{12\ldots n} = \sum_{i=1}^{n} N_{1\ldots(i-1)(i+1)\ldots n} \, D_i + \Delta_{12\ldots n} \,. \tag{10}$$

The first term on the r.h.s. is the quotient and the second term is the remainder. In the above results, the coefficient of the denominator D_i is a polynomial, suitably defined as $N_{1\ldots(i-1)(i+1)\ldots n}$. In fact, by substituting (10) in (9), one obtains,

$$I_{12\ldots n} = \sum_{i=1}^{n} I_{1\ldots(i-1)(i+1)\ldots n} + \frac{\Delta_{12\ldots n}}{D_1 \ldots D_n} \,, \tag{11}$$

namely, a relation expressing the n-denominator integrand as a combination of $(n-1)$-denominator integrands plus a term that corresponds to the residue of the n-denominator cut, $D_1 = \ldots = D_n = 0$. This result is depicted in Fig. 2. By iterating the division algorithm on the new lower-denominators integrands, one achieves the complete decomposition of the integrand $I_{12\ldots n}$ in terms of independent integrands,

$$\begin{aligned} I_{12\ldots n} \;=\;& \frac{\Delta_{12\ldots n}}{D_1 \ldots D_n} + \\ & + \frac{\Delta_{2\ldots n}}{D_2 \ldots D_n} + \ldots + \frac{\Delta_{12\ldots n-1}}{D_1 \ldots D_{n-1}} + \\ & + \ldots + \frac{\Delta_{n-1\, n}}{D_{n-1} D_n} + \ldots + \frac{\Delta_{12}}{D_1 D_2} + \\ & + \ldots + \frac{\Delta_n}{D_n} + \ldots + \frac{\Delta_1}{D_1} + Q \,, \end{aligned} \tag{12}$$

where Q is a potential irreducible quotient which would give no contibution upon integration. The residues Δ's are polynomials in the (components of the) loop momenta. Therefore, by integrating both sides, one obtaines the decomposition of the original integral $\mathcal{I}_{12\ldots n}$

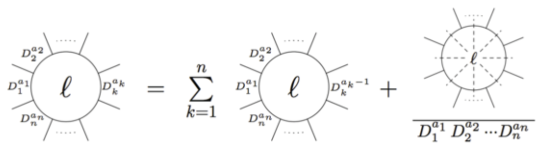

Figure 2: Multiloop integrand decomposition formula. A generic ℓ-loop integrand with a certain number of denominators, each raised to an arbitrary power, is expressed as combination of integrands where the power of a given denomiator is reduced by one, plus a term corresponding to the residue, depicted by a cut diagram.

in terms of *indepentent integrals*, corresponding to the monomials appearing in the residues which give a non-vanishing contribution upon the loop integration. The integrand decomposition (12) implyes that, exactly as it happened for the tree-level amplitudes, also the integrands of multi-loop amplitudes can be decomposed in terms of independent building blocks simply by *partial fractioning*.

While in the one-loop case the independent integrals are analyically known, in the multi-loop case, their classification and evaluation is an open problem.

It is important to observe that the set of integrals arising after the integrand decomposition is *not* a minimal set. Within the continuous dimensional regularization scheme, the invariance of Feynman integrals under the redefinition of loop-momenta is responsible for the existence of relations known as *integration-by-parts identities* (IBPs) [42]. Such relations can be exploited in order to identify a minimal set of independent integrals, dubbed *master integrals* (MIs), that can be used as a basis of functions for the virtual contributions to scattering amplitudes. Loop momenutm shift invariance is a symmetry that cannot be captured by the polynomial decomposition of the integrand. Therefore, IBPs constitute additional relations that can further reduce the number of integrals appearing in the integrand decomposition. It is highly desirable to develop a method for multiloop amplitudes which combines the benefits of integrand decomposition and IBPs to achieve the decomposition of high-multiplicity amplitudes in terms of a minimal set of MIs, which currently constitutes one of the two obstructions preventing the automation of evaluationing multi-loop integrals for generic scattering reactions. The other being the evaluation of MIs, once they are identified.

4. Differential Equations and Feynman Integrals

The *method of differential equations* [43–45], reviewed in [46–48], is one of the most effective techniques for computing dimensionally regulated multi-loop integrals.

The MIs are functions of the kinematic invariants constructed from the external momenta and of the masses of the (internal and external) particles. Remarkably, the aforementioned IBP relations imply that the MIs obey linear systems of first-order differential equations (DE's) in the kinematic invariants, which can be used for the determination of their actual expression. In fact, any ℓ-loop integral $\mathcal{I}$ is a homogeneous function of external momenta p_i and masses m_i, whose degree $\gamma = \gamma(d, \ell)$ depends on the space-time dimensions d, the number of loop ℓ, and on the powers of denominators. Accordingly, one can write the Euler relation,

$$\Big(\sum_i p_i \cdot \partial_{p_i} + \sum_j m_j\, \partial_{m_j}\Big)\mathcal{I} = \gamma(d,\ell)\mathcal{I} \ , \qquad (13)$$

where $\partial_x \equiv \partial/\partial x$. On the other hand, the derivatives in the l.h.s. can be calculated explicitly, and, by means of IBPs, they turn into linear combinations involving $\mathcal{I}$ and the other MIs. Therefore, Euler equation for homogeneous functions becomes a tool for deriving the system of differential equations obeyd by MIs. *The* solution of the system, namely the MIs, is finally determined by imposing the *boundary conditions* at special values of the kinematic variables, properly chosen either in correspondence of configurations that reduce the MIs to simpler integrals or in correspondence of pseudo-thresholds. In this latter case, the boundary conditions are obtained by imposing the *regularity* of the MIs around unphysical singularities, ruling out divergent behavior of the general solution of the systems.

For any given scattering process the set of MIs is not unique, and, in practice, their choice is rather arbitrary. Usually MIs are identified after applying the Laporta reduction algorithm [49]. Afterward, convenient manipulations of the basis of MIs may be performed. Proper choices of MIs can simplify the form of the systems of differential equations, hence, of their solution, although general criteria for determining such optimal sets are not available. The systems of DE's for MIs can be efficiently solved by means of algebraic methods, observing that, with a *good* choice of MIs, the system can be cast in a form - which we define *canonical* - where the dependence on the dimensional parameter $\epsilon = (4-d)/2$ is factorized from the kinematic [50]. The integration of canonical systems is simple, and the analytic properties of its general solution are manifestly inherited from the associated matrix. In fact, the latter becomes the kernel of the representation of the solutions in terms of repeated integrations.

We have recently suggested a convenient form for the initial system of MIs, and proposed an algorithm to find the transformation matrix yielding to a canonical system [51]. We reached out to the problem of the Schrödinger Equation in the interaction picture in Quantum Mechanics (QM), in presence of an Hamiltonian with a linear perturbation. After establishing an analogy between the perturbation parameter of QM, and the space-time dimensions of regulated Feynman integrals, we suggest that a convenient set of MIs should obey a systems of DEs whose associated matrix is linear in ϵ. In this case, by generalizing what happens in QM, which involves commutative integral matrices, to the non-commutative case, we found that the transformation absorbing the constant term and yielding to a new system of DE's where the ϵ-dependence is factorized can be obtained by using Magnus exponential matrix [52–54]. Moreover, the integration of the canonical system, can be written as Magnus' (or equivalently Dyson's) series expansion in ϵ. Magnus exponential is not unitary, as it happens in the quantum mechanical case, but the proposed method can be considered also inspired by unitarity.

5. Magnus series expansion

Consider a generic linear matrix differential equation [54]

$$\partial_x Y(x) = A(x)Y(x) \ , \quad Y(x_0) = Y_0 \ . \qquad (14)$$

If $A(x)$ commutes with its integral $\int_{x_0}^x d\tau A(\tau)$, *e.g.* in the scalar case, the solution can be written as $Y(x) = e^{\int_{x_0}^x d\tau A(\tau)}\, Y_0$. In the general non-commutative case, one can use the *Magnus theorem* [52] to write the solution as,

$$Y(x) = e^{\Omega(x,x_0)}\, Y(x_0) \equiv e^{\Omega(x)}\, Y_0 \ , \qquad (15)$$

where $\Omega(x)$ is written as a series expansion, known as *Magnus expansion*, $\Omega(x) \equiv \sum_{n=1}^{\infty} \Omega_n(x)$. The first three terms of the expansion read as,

$$\begin{aligned}
\Omega_1(x) &= \int_{x_0}^{x} d\tau_1 A(\tau_1) \ , \\
\Omega_2(x) &= \frac{1}{2}\int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2\, [A(\tau_1), A(\tau_2)] \ , \\
\Omega_3(x) &= \frac{1}{6}\int_{x_0}^{t} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 \ \times \\
&\qquad \Big([A(\tau_1),[A(\tau_2),A(\tau_3)]] + \\
&\qquad +[A(\tau_3),[A(\tau_2),A(\tau_1)]]\Big) \ . \qquad (16)
\end{aligned}$$

Magnus expansion can be considered as the continuous analogue of the Baker-Campbell-Hausdorff (BCH) formula. We remark that if A and its integral commute, the Magnus series is truncated at the first order, $\Omega = \Omega_1$. In what follows, we will use the symbol $\Omega[A](x)$ to denote the Magnus expansion obtained using A as kernel.

5.1. Magnus and Dyson series

Magnus series is related to Dyson series [54], and their connection can be obtained starting from the Dyson expansion of the solution of the system (14),

$$Y(x) \quad = \quad Y_0 + \sum_{n=1}^{\infty} Y_n(x) \,, \tag{17}$$

in terms of the *time-ordered* integrals Y_n,

$$Y_n(x) \equiv \int_{x_0}^{x} d\tau_1 \cdots \int_{x_0}^{\tau_{n-1}} d\tau_n \, A(\tau_1)A(\tau_2)\cdots A(\tau_n) \,. \tag{18}$$

Comparing Eq. (15) and (17) we have

$$\sum_{j=1}^{\infty} \Omega_j(x) = \log\left(Y_0 + \sum_{n=1}^{\infty} Y_n(x) \right) . \tag{19}$$

6. Master Integrals from Differential Equations

We consider a linear system of first order differential equations

$$\partial_x f(\epsilon, x) = A(\epsilon, x)\, f(\epsilon, x) \,, \tag{20}$$

where f is a vector of MIs, while x is a variable depending on kinematic invariants and masses. We suppose that A depends linearly on ϵ,

$$A(\epsilon, x) = A_0(x) + \epsilon A_1(x) \,, \tag{21}$$

and we change the basis of MIs via the Magnus exponential obtained by using A_0 as kernel,

$$f(\epsilon, x) = B_0(x)\, g(\epsilon, x) \,, \quad B_0(x) \equiv e^{\Omega[A_0](x,x_0)} \,. \tag{22}$$

Because of Magnus Theorem, B_0 obeys the equation,

$$\partial_x B_0(x) = A_0(x) B_0(x) \,, \tag{23}$$

which implies that the new basis g of MIs fulfills a system of differential equations in the *canonical* factorized form,

$$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x) \,. \tag{24}$$

The matrix $\hat{A}_1$ is related to A_1 through,

$$\hat{A}_1(x) = B_0^{-1}(x) A_1(x) B_0(x) \,, \tag{25}$$

and does not depend on ϵ. The solution of Eq. (24) can be found by using Magnus exponential with $\epsilon\hat{A}_1$ as kernel

$$g(\epsilon, x) = B_1(\epsilon, x) g_0(\epsilon) \,, \quad B_1(\epsilon, x) = e^{\Omega[\epsilon\hat{A}_1](x,x_0)} \,, \tag{26}$$

where the vector g_0 corresponds to the boundary values of the MIs. Therefore, the solution of the original system Eq. (20) finally reads,

$$f(\epsilon, x) = B_0(x) B_1(\epsilon, x) g_0(\epsilon) \,. \tag{27}$$

It is worth to notice that $\Omega[\epsilon\hat{A}_1]$ in Eq. (26) depends on ϵ, while $\Omega[A_0]$ in Eq. (22) does not.
We found that the convergence of Magnus exponential can be accelerated by splitting A_0 into a diagonal term, and a matrix with vanishing diagonal entries.

6.1. Iterated integrals

The solutions of differential equations can be naturally cast in terms of iterated integrals, with rational kernels, known as Multiple Polylogarithms (MPLs) [55–58], defined as,

$$G(\vec{w}_n; x) \equiv G(w_1, \vec{w}_{n-1}; x) \equiv \int_0^x dt \frac{G(\vec{w}_{n-1}; t)}{t - w_1} \,, \tag{28}$$

$$G(\vec{0}_n; x) \equiv \frac{1}{n!} \log^n(x) \,, \tag{29}$$

with $\vec{w}_n$ being a vector of n arguments. The number n is referred to as the *weight* of $G(\vec{w}_n; x)$ and amounts to the number of iterated integrations needed to define it. G-polylogarithms fulfill shuffle algebra relations of the type

$$G(\vec{m}; x)\, G(\vec{n}; x) = \sum_{\vec{p} = \vec{m} \sqcup\!\sqcup \vec{n}} G(\vec{p}; x) \,, \tag{30}$$

where shuffle product $\vec{m} \sqcup\!\sqcup \vec{n}$ denotes all possible merges of $\vec{m}$ and $\vec{n}$ preserving their respective orderings. Because of shuffle relations, MPLs obey functional relations which can be exploited to simplify the analytic expressions of scattering amplitudes. These identities yield the identification of minimal sets of functions, and allow for analytic continuation of the results in different kinematic regimes. As a consequence, new tools, found among number theoretic concepts, such as *Hopf algebras*, *symbol calculus*, and *coproduct*, have been developed in order to simplify complex results and to express them in terms of a few independent functions (see [59] for a recent review).

MPLs do not exhaust the classes of functions that could appear along the evaluation of Feynman integrals. It is known that processes involving massive particles in the loops can be source of elliptic integrals [60, 61]. Algebraic properties of elliptic integrals represent a current bottleneck for the development of new techniques for automating the evaluation of MIs beyond one-loop.

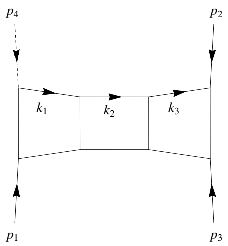

Figure 3: The three-loop ladder box diagram, with one off-shell leg: the solid lines stand for massless particles; the dashed line represents a massive particle. Momentum conservation is $\sum_{i=1}^{4} p_i = 0$, with $p_i^2 = 0$ $(i = 1,\ 2,\ 3)$ and $p_4^2 = m_H^2$.

6.2. Applications

We made use of Magnus theorem for the determination of non-trivial integrals, like the two-loop vertex diagrams for the electron form factors in QED and the two-loop box integrals for the $2 \to 2$ massless scattering [51], the two-loop corrections to the $pp \to Hj$, as well as for evaluating the three-loop ladder diagrams for $pp \to Hj$ (in the infinite top-mass approximation) [62]. The latter is a formidable calcualtion involving the solution of a system of eitghty-five MIs. In this case, after identifying a set of MIs obeying a linear system of differential equations in $x = -s/m_H^2$ and $y = -t/m_H^2$, by means of a Magnus tranform, the system can be brought in canonical form, reading as,

$$df(x,y) = \epsilon\, A(x,y)\, f(x,y) \tag{31}$$

where f is the vector of MIs, and $df = \partial_x f dx + \partial_y f dy$. The matrix A is purely logarithmic,

$$\begin{aligned} A(x,y) \;=\;& a_1 \ln(x) + a_2 \ln(1-x) \\ & + a_3 \ln(y) + a_4 \ln(1-y) \\ & + a_5 \ln(x+y) + a_6 \ln(1-(x+y))\,, \end{aligned} \tag{32}$$

where the a_i $(i = 1,\ldots,6)$ are 85×85 matrices whose entries are just rational numbers. The logarithmic form of A trivializes the solution, which can be written as a Dyson series in ϵ, where the coefficient of the series are combinationa of MPLs with uniform weight (where the weight increases as the order in ϵ does). Boundary conditions have been fixed by imposing the regularity of the solutions in special kinematic configurations. Suprisingly, to fix the boundary values of all 85 MIs, only 2 simple integrals had to be independently provided.

Further investigation is required for clarifying whether the possibility of choosing a set of master integrals obeying a system that is linear in ϵ is a general feature or just accidental, and, eventually, how to find it in a systematic way. In this respect, we think that it might be worth to consider systems of differential equations for master integrals that are not necessarily all defined at the same value of space-time dimensions, because regularity in the $\epsilon \to 0$ limit may be a property of shifted-dimension master integrals, and the shifting amount could depend on the topology of the diagrams. Moreover, even in the case a set of MIs obeying a system of differential equations linear in ϵ is found, the convergence of the Magnus series, needed for finding a finite matrix that implements the transformation to the canonical form, is not guaranteed a priori. Therefore, we can turn the arrow of the implications around, and conjecture that the existence of a canonical set of MIs that can be expressed in terms of MPLs of uniform weight and obeying a canonical system of differential equations implies the existence of a (finite) Magnus exponential matrix that rules the transformation of the canonical basis to a basis obeying a system that is linear in ϵ. Nevertheless, it is also known that polylogarithms do not exhaust the set of functions appearing in the evaluation of Feynman integrals, where also elliptic functions do arise. What does happen in these cases? Does a ϵ-linear system exist? Can one find a converging Magnus exponential matrix? Can the convergence of the Magnus exponential capture the elliptic or the polylogarithmic character of the MIs? Answering to these questions requires definitely further studies and more applications to cases of increasing complexity, which we plan for the near future.

7. Conclusions

In this contribution, I have analyzed the algebraic patterns underlying the structure of scattering amplitudes in gauge theory. I focused on three domains, such as the decomposition of amplitudes in terms of independent functions, the construction of the systems of differential equations the latter obey, and the functions needed to solve them. We have seen the central role played by *unitarity* in the context of evaluating scattering amplitudes. It not only inspired a method to perform the amplitudes decomposition, by means of unitarity-cuts, but it also suggested a technique for the evaluation of master integrals, by means of matrix exponentials, similar to the unitary time-evolution in quantum mechanics.

In particular, I discussed a unique framewok for decomposing amplitudes at all orders in perturbation theory, simply based on integrand partial fractioning. This idea captures the *essence* of complex integration, yet avoiding it: *i)* Feynman integrals are multivariate integrals of rational functions; *ii)* cut-integration with com-

plex variables amounts to apply (generalized) Residue Theorem; *iii)* applying residue theorem to rational functions amounts to partial fractioning.
Later, I presented how master integrals can be evaluated within the differential equations method, by means of the Magnus exponential series.

The presented techniques can be applied to scattering amplitudes in gauge theory, which can be applied in supersymmetric contexts as well as in absence of supersymmetry.

Let me conclude recalling that during the last decade the theoretical physics programme devoted to improving our abilities in computing high-multiplicity one-loop amplitudes has received a strong boost. If we gave an answer, or better, more than one answer, to the problem of automating seminumerical calculations of one-loop amplitudes, providing automatic analytic calculation of one-loop amplitudes is also within reach. The efforts to extend this progress to higher orders has begun, with the development of techniques not only dealing with virtual contributions, but also with the integration over the phase-space and the development of subtraction methods - the other, dark side, of radiative corrections. First satisfactory results have been accomplished, but the road ahead looks still like a minefield, and much work is still required.

By combining methods and techniques from different fields of Theoretical Physics and of Mathematics, such as Collider Phenomenology, Quantum Field Theory, String Theory, General Relativity, Algebraic Geometry and Number Theory, the new interdisciplinary field of *Amplitudes* offers a potential in developing new theories for the description of quantum interactions at a more fundamental level.

Acknowledgements

I wish to thank the organizers of the Silafae X, for the enthusiastic environment, and for their kind hospitality. Also, I thank H. van Deurzen, S. Di Vita, R. Fazio, N. Greiner, G. Heinrich, G. Luisoni, E. Mirabella, G. Ossola, T. Peraro, U. Schubert, F. Tramontano, W. Torres-Bobadilla, V. Yundin, as well as all the members of the GoSam team, for their vivid collaboration on the discussed topics.

The research contained in this work is supported by the Alexander von Humboldt Foundation, in the framework of the Sofja Kovalevskaja Award 2010, endowed by the German Federal Ministry of Education and Research.

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