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Best proximity point results for modified α-proximal C-contraction mappings

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Abstract
First we introduce new concepts of contraction mappings, then we establish certain best proximity point theorems for such kind of mappings in metric spaces. Finally, as consequences of these results, we deduce best proximity point theorems in metric spaces endowed with a graph and in partially ordered metric spaces. Moreover, we present an example and some fixed point results to illustrate the usability of the obtained theorems.

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1 Introduction

A wide variety of problems arising in different areas of pure and applied mathematics, such as difference and differential equations, discrete and continuous dynamic systems, and variational analysis, can be modeled as fixed point equations of the form $x = Tx$. Therefore, fixed point theory plays a crucial role for solving equations of above kind, whose solutions are the fixed points of the mapping $T : X \to X$, where X is a nonempty set. Areas of potential applications of this theory include physics, economics, and engineering in dealing with the study of equilibrium points (which are fixed points of certain mappings). On the other hand, if T is a nonself-mapping, the above fixed point equation could have no solutions and, in this case, it is of a certain interest to determine an approximate solution x that is optimal in the sense that the distance between x and Tx is minimum. In this context, best proximity point theory is an useful tool in studying such kind of element. We recall the following concept.

Definition 1.1 Let A, B be two nonempty subsets of a metric space (X, d) and $T : A \to B$ be a nonself-mapping. An element $x \in A$ such that $d(x, Tx) = d(A, B)$ is a best proximity point of the nonself-mapping T.

Clearly, if T is a self-mapping, a best proximity point is a fixed point, that is, $x = Tx$.

From the beginning, best proximity point theory of nonself-mappings has been studied by many authors; see the pioneering papers of Fan [1] and Kirk *et al.* [2]. The investigation of several variants of conditions for the existence of a best proximity point can be found in [3–12]. In particular, some significant best proximity point results for multivalued mappings are presented in [13]; see also the references therein.

Inspired and motivated by the above facts, in this paper, we introduce new concepts of contraction mappings. Then we establish certain best proximity point theorems for such kind of mappings in metric spaces. As consequences of these results, we deduce best proximity point theorems in metric spaces endowed with a graph and in partially ordered metric spaces. Moreover, we present an example and some fixed point results to illustrate the usability of the obtained theorems.

2 Preliminaries

In this section, we collect some useful definitions and results from fixed point theory.

Samet *et al.* [14] defined the notion of α-admissible mapping as follows.

Definition 2.1 ([14]) Let $\alpha : X \times X \to [0,+\infty)$ be a function. We say that a self-mapping $T : X \to X$ is α-admissible if

$$x,y \in X, \quad \alpha(x,y) \ge 1 \quad \Longrightarrow \quad \alpha(Tx,Ty) \ge 1.$$

By using this concept, they proved some fixed point results.

Theorem 2.1 ([14]) *Let (X,d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Assume that the following conditions hold*:

(i) *for all $x,y \in X$ we have*

$$\alpha(x,y)d(Tx,Ty) \le \psi\big(d(x,y)\big), \tag{1}$$

where $\psi : [0,+\infty) \to [0,+\infty)$ is a nondecreasing function such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$,

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$,*

(iii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

Later on, working on these ideas a wide variety of papers appeared in the literature; see for instance [15–17]. Finally, we recall that Karapinar *et al.* [18] introduced the notion of triangular α-admissible mapping as follows.

Definition 2.2 ([18]) Let $\alpha : X \times X \to (-\infty,+\infty)$ be a function. We say that a self-mapping $T : X \to X$ is triangular α-admissible if

$$\text{(i)} \quad x,y \in X, \quad \alpha(x,y) \ge 1 \quad \Longrightarrow \quad \alpha(Tx,Ty) \ge 1,$$

$$\text{(ii)} \quad x,y,z \in X, \quad \begin{cases} \alpha(x,z) \ge 1, \\ \alpha(z,y) \ge 1 \end{cases} \quad \Longrightarrow \quad \alpha(x,y) \ge 1.$$

For more details and applications of this line of research, we refer the reader to some related papers of the authors and others [19–25].

3 Main results in metric spaces

Let A, B be two nonempty subsets of a metric space (X,d). Following the usual notation, we put

$$A_0 := \{x \in A : d(x,y) = d(A,B), \text{ for some } y \in B\},$$
$$B_0 := \{y \in B : d(x,y) = d(A,B), \text{ for some } x \in A\}.$$

If $A \cap B \neq \emptyset$, then A_0 and B_0 are nonempty. Further, it is interesting to notice that A_0 and B_0 are contained in the boundaries of A and B, respectively, provided A and B are closed subsets of a normed linear space such that $d(A,B) > 0$ (see [26]). Also, we will use the following definition; see [27] for more details.

Definition 3.1 Let A, B be two nonempty subsets of a metric space (X,d). The pair (A,B) is said to have the V-property if, for every sequence $\{y_n\}$ of B that satisfies the condition $d(x,y_n) \to d(x,B)$ for some $x \in A$, there is $y \in B$ such that $d(x,y) = d(x,B)$.

From now on, denote with Ψ the family of all continuous and nondecreasing functions $\psi : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ such that $\psi(x,y) = 0$ if and only if $x = y = 0$.

Definition 3.2 Let A, B be two nonempty subsets of a metric space (X,d) and $\alpha : A \times A \to [0,+\infty)$ be a function. We say that a nonself-mapping $T : A \to B$ is triangular α-proximal admissible if, for all $x, y, z, x_1, x_2, u_1, u_2 \in A$,

$$\text{(T1)} \quad \begin{cases} \alpha(x_1,x_2) \geq 1, \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B) \end{cases} \implies \alpha(u_1,u_2) \geq 1,$$

$$\text{(T2)} \quad \begin{cases} \alpha(x,z) \geq 1, \\ \alpha(z,y) \geq 1 \end{cases} \implies \alpha(x,y) \geq 1.$$

Definition 3.3 Let A, B be two nonempty subsets of a metric space (X,d) and $\alpha : A \times A \to [0,+\infty)$ be a function. We say that a nonself-mapping $T : A \to B$ is

(i) a modified α-proximal C-contraction if, for all $u, v, x, y \in A$,

$$\begin{cases} \alpha(x,y) \geq 1, \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B) \end{cases}$$
$$\implies d(u,v) \leq \frac{1}{2}\big(d(x,v) + d(y,u)\big) - \psi\big(d(x,v), d(y,u)\big), \tag{2}$$

(ii) an α-proximal C-contraction of type (I) if, for all $u, v, x, y \in A$,

$$\begin{cases} d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B) \end{cases}$$
$$\implies \alpha(x,y)d(u,v) \leq \frac{1}{2}\big(d(x,v) + d(y,u)\big) - \psi\big(d(x,v), d(y,u)\big),$$

where $0 \leq \alpha(x,y) \leq 1$ for all $x, y \in A$,

(iii) an α-proximal C-contraction of type (II) if, for all $u, v, x, y \in A$,

$$\begin{cases} d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{cases}$$
$$\implies \quad (\alpha(x, y) + \ell)^{d(u,v)} \leq (\ell + 1)^{\frac{1}{2}(d(x,v)+d(y,u)) - \psi(d(x,v), d(y,u))},$$

where $\ell > 0$.

Remark 3.1 Every α-proximal C-contraction of type (I) and α-proximal C-contraction of type (II) mappings are modified α-proximal C-contraction mappings.

Now we give our main result.

Theorem 3.1 *Let A, B be two nonempty subsets of a metric space (X, d) such that A is complete and A_0 is nonempty. Assume that $T : A \to B$ is a continuous modified α-proximal C-contraction such that the following conditions hold*:

(i) *T is a triangular α-proximal admissible mapping and $T(A_0) \subseteq B_0$,*
(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1.$$

Then T has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x, Tx) = d(A, B) = d(y, Ty)$, we have $\alpha(x, y) \geq 1$.

Proof By (ii) there exist $x_0, x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1.$$

On the other hand, $T(A_0) \subseteq B_0$, then there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

Now, since T is triangular α-proximal admissible, we have $\alpha(x_1, x_2) \geq 1$. Thus

$$d(x_2, Tx_1) = d(A, B) \quad \text{and} \quad \alpha(x_1, x_2) \geq 1.$$

Since $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A, B).$$

Then we have

$$d(x_2, Tx_1) = d(A, B), \qquad d(x_3, Tx_2) = d(A, B), \qquad \alpha(x_1, x_2) \geq 1.$$

Again, since T is triangular α-proximal admissible, we obtain $\alpha(x_2, x_3) \geq 1$ and hence

$$d(x_3, Tx_2) = d(A, B), \qquad \alpha(x_2, x_3) \geq 1.$$

By continuing this process, we construct a sequence $\{x_n\}$ such that

$$\begin{cases} \alpha(x_{n-1}, x_n) \geq 1, \\ d(x_n, Tx_{n-1}) = d(A, B), \\ d(x_{n+1}, Tx_n) = d(A, B), \end{cases} \tag{3}$$

for all $n \in \mathbb{N}$. Now, from (2) with $u = x_n$, $v = x_{n+1}$, $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{1}{2}\bigl(d(x_{n-1}, x_{n+1}) + d(x_n, x_n)\bigr) - \psi\bigl(d(x_{n-1}, x_{n+1}), d(x_n, x_n)\bigr) \\ &= \frac{1}{2}d(x_{n-1}, x_{n+1}) - \psi\bigl(d(x_{n-1}, x_{n+1}), 0\bigr) \\ &\leq \frac{1}{2}d(x_{n-1}, x_{n+1}) \\ &\leq \frac{1}{2}\bigl(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\bigr), \end{aligned} \tag{4}$$

which implies $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$. It follows that the sequence $\{d_n\}$, where $d_n := d(x_n, x_{n+1})$, is decreasing and so there exists $d \geq 0$ such that $d_n \to d$ as $n \to +\infty$. Then, taking the limit as $n \to +\infty$ in (4), we obtain

$$d \leq \frac{1}{2} \lim_{n \to +\infty} d(x_{n-1}, x_{n+1}) \leq \frac{1}{2}(d + d) = d,$$

that is,

$$\lim_{n \to +\infty} d(x_{n-1}, x_{n+1}) = 2d. \tag{5}$$

Again taking the limit as $n \to +\infty$ in (4), by (5) and the continuity of ψ, we get

$$d \leq d - \psi(2d, 0),$$

and so $\psi(2d, 0) = 0$. Therefore, by the property of ψ, we get $d = 0$, that is,

$$\lim_{n \to +\infty} d(x_{n+1}, x_n) = 0. \tag{6}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there are $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k

$$n(k) > m(k) > k, \quad d(x_{n(k)}, x_{m(k)}) \geq \varepsilon, \qquad d(x_{n(k)-1}, x_{m(k)}) < \varepsilon.$$

This implies that, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon &\leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\ &< d(x_{n(k)}, x_{n(k)-1}) + \varepsilon. \end{aligned}$$

Taking the limit as $k \to +\infty$ in the above inequality and using (6), we get

$$\lim_{k\to+\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{7}$$

Again, from

$$d(x_{n(k)}, x_{m(k)}) \le d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \le d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)+1}, x_{n(k)}),$$

taking the limit as $k \to +\infty$, by (6) and (7) we deduce

$$\lim_{k\to+\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \tag{8}$$

Similarly, we deduce

$$\lim_{k\to+\infty} d(x_{n(k)}, x_{m(k)+1}) = \varepsilon \tag{9}$$

and

$$\lim_{k\to+\infty} d(x_{m(k)}, x_{n(k)+1}) = \varepsilon. \tag{10}$$

We shall show that

$$\alpha(x_{m(k)}, x_{n(k)}) \ge 1, \quad \text{where } n(k) > m(k) > k. \tag{11}$$

Since T is a triangular α-proximal admissible mapping and

$$\begin{cases} \alpha(x_{m(k)}, x_{m(k)+1}) \ge 1, \\ \alpha(x_{m(k)+1}, x_{m(k)+2}) \ge 1, \end{cases}$$

by (T2) of Definition 3.2, we have

$$\alpha(x_{m(k)}, x_{m(k)+2}) \ge 1.$$

Again, since T is a triangular α-proximal admissible mapping and

$$\begin{cases} \alpha(x_{m(k)}, x_{m(k)+2}) \ge 1, \\ \alpha(x_{m(k)+2}, x_{m(k)+3}) \ge 1, \end{cases}$$

by (T2) of Definition 3.2 we have

$$\alpha(x_{m(k)}, x_{m(k)+3}) \ge 1.$$

Thus, by continuing this process, we get (11).

On the other hand, we know that

$$\begin{cases} d(x_{m(k)+1}, Tx_{m(k)}) = d(A,B), \\ d(x_{n(k)+1}, Tx_{n(k)}) = d(A,B). \end{cases}$$

Therefore, from (2) we have

$$\begin{aligned} d(x_{m(k)+1}, x_{n(k)+1}) \leq{} & \frac{1}{2}\bigl(d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})\bigr) \\ & - \psi\bigl(d(x_{m(k)}, x_{n(k)+1}), d(x_{n(k)}, x_{m(k)+1})\bigr). \end{aligned}$$

Taking the limit as $k \to +\infty$ in the above inequality and using (8), (9), (10) and the continuity of ψ, we get

$$\varepsilon \leq \frac{1}{2}(\varepsilon + \varepsilon) - \psi(\varepsilon, \varepsilon)$$

and hence $\psi(\varepsilon, \varepsilon) = 0$, which leads to the contradiction $\varepsilon = 0$. Thus, $\{x_n\}$ is a Cauchy sequence. Since A is complete, then there is $z \in A$ such that $x_n \to z$. Now, from

$$d(x_{n+1}, Tx_n) = d(A,B), \quad \text{for all } n \in \mathbb{N} \cup \{0\},$$

taking the limit as $n \to +\infty$, we deduce $d(z, Tz) = d(A,B)$, because of the continuity of T.

Finally we prove the uniqueness of the point $x \in A$ such that $d(x, Tx) = d(A,B)$. Indeed, suppose that there exist $x, y \in A$ which are best proximity points, that is, $d(x, Tx) = d(A,B) = d(y, Ty)$. Since $\alpha(x,y) \geq 1$, we have

$$\begin{aligned} d(x,y) &\leq \frac{1}{2}\bigl(d(x,y) + d(y,x)\bigr) - \psi\bigl(d(x,y), d(y,x)\bigr) \\ &= d(x,y) - \psi\bigl(d(x,y), d(x,y)\bigr), \end{aligned}$$

which implies $d(x,y) = 0$, that is, $x = y$. □

Corollary 3.1 *Let A, B be two nonempty subsets of a metric space (X,d) such that A is complete and A_0 is nonempty. Assume that $T : A \to B$ is a continuous α-proximal C-contraction mapping of type (I) or a continuous α-proximal C-contraction mapping of type (II) such that the following conditions hold*:

(i) *T is a triangular α-proximal admissible mapping and $T(A_0) \subseteq B_0$*,

(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1.$$

Then T has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x, Tx) = d(A,B) = d(y, Ty)$, we have $\alpha(x,y) \geq 1$.

In analogy to the main result but omitting the continuity hypothesis of T, we can state the following theorem.

Theorem 3.2 *Let A, B be two nonempty subsets of a metric space (X,d) such that A is complete, the pair (A,B) has the V-property and A_0 is nonempty. Assume that $T:A\to B$ is a modified α-proximal C-contraction such that the following conditions hold*:

(i) *T is a triangular α-proximal admissible mapping and $T(A_0)\subseteq B_0$*,

(ii) *there exist $x_0,x_1\in A_0$ such that*

$$d(x_1,Tx_0)=d(A,B)\quad\text{and}\quad\alpha(x_0,x_1)\geq 1,$$

(iii) *if $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1})\geq 1$ and $x_n\to x\in A$ as $n\to+\infty$, then $\alpha(x_n,x)\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$.*

Then T has a best proximity point. Further, the best proximity point is unique if, for every $x,y\in A$ such that $d(x,Tx)=d(A,B)=d(y,Ty)$, we have $\alpha(x,y)\geq 1$.

Proof Following the proof of Theorem 3.1, there exist a Cauchy sequence $\{x_n\}\subseteq A$ and $z\in A$ such that (3) holds and $x_n\to z$ as $n\to+\infty$. On the other hand, for all $n\in\mathbb{N}$, we can write

$$\begin{aligned}d(z,B)&\leq d(z,Tx_n)\\&\leq d(z,x_{n+1})+d(x_{n+1},Tx_n)\\&=d(z,x_{n+1})+d(A,B).\end{aligned}$$

Taking the limit as $n\to+\infty$ in the above inequality, we get

$$\lim_{n\to+\infty}d(z,Tx_n)=d(z,B)=d(A,B).\tag{12}$$

Since the pair (A,B) has the V-property, then there exists $w\in B$ such that $d(z,w)=d(A,B)$ and hence $z\in A_0$. Moreover, since $T(A_0)\subseteq B_0$, then there exists $v\in A$ such that

$$d(v,Tz)=d(A,B).$$

Now, by (iii) and (3), we have $\alpha(x_n,z)\geq 1$ and $d(x_{n+1},Tx_n)=d(A,B)$ for all $n\in\mathbb{N}\cup\{0\}$. Also, since T is a modified α-proximal C-contraction, we get

$$d(x_{n+1},v)\leq\frac{1}{2}\bigl(d(x_n,v)+d(z,x_{n+1})\bigr)-\psi\bigl(d(x_n,v),d(z,x_{n+1})\bigr).$$

Taking the limit as $n\to+\infty$ in the above inequality, we have

$$d(z,v)\leq\frac{1}{2}d(z,v)-\psi\bigl(d(z,v),0\bigr)$$

which implies, $d(z,v)=0$, that is, $v=z$. Hence z is a best proximity point of T. The uniqueness of the best proximity point follows easily proceeding as in Theorem 3.1. □

Next, we use an example to illustrate the efficiency of the new theorem.

Example 3.1 Let $X = \mathbb{R}$ be endowed with the usual metric $d(x,y) = |x-y|$, for all $x, y \in X$. Consider $A = (-\infty, -1]$, $B = [1, +\infty)$ and define $T : A \to B$ by

$$Tx = \begin{cases} -x+1, & \text{if } x \in (-\infty, -14), \\ x^2+1, & \text{if } x \in [-14, -12), \\ 4x^4+5, & \text{if } x \in [-12, -10), \\ -x^3+2, & \text{if } x \in [-10, -8), \\ 10, & \text{if } x \in [-8, -6), \\ \ln(|x|+1), & \text{if } x \in [-6, -4), \\ -x+|x+3||x+4|e^{-x}, & \text{if } x \in [-4, -2), \\ 1, & \text{if } x \in [-2, -1]. \end{cases}$$

Also, define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 4, & \text{if } x, y \in [-2, -1], \\ \frac{1}{2}, & \text{otherwise}, \end{cases}$$

and $\psi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ by

$$\psi(s,t) = \frac{1}{2}(s+t), \quad \text{for all } s, t \in X.$$

Clearly, the pair (A, B) has the V-property and $d(A,B) = 2$. Now, we have

$$A_0 = \bigl\{x \in A : d(x,y) = d(A,B) = 2, \text{ for some } y \in B\bigr\} = \{-1\},$$
$$B_0 = \bigl\{y \in B : d(x,y) = d(A,B) = 2, \text{ for some } x \in A\bigr\} = \{1\}.$$

It is immediate to see that $T(A_0) \subseteq B_0$, $d(-1, T(-1)) = d(A,B) = 2$ and $\alpha(-1,-1) \geq 1$.

Now, let $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1$. Therefore, $x, y, z \in [-2, -1]$, that is, $\alpha(x,z) \geq 1$. Also suppose

$$\begin{cases} \alpha(x,y) \geq 1, \\ d(u, Tx) = d(A,B) = 2, \\ d(v, Ty) = d(A,B) = 2, \end{cases}$$

then

$$\begin{cases} x, y \in [-2, -1], \\ d(u, Tx) = 2, \\ d(v, Ty) = 2. \end{cases}$$

Hence, $u = v = -1$, that is, $\alpha(u,v) \geq 1$. Further,

$$d(u,v) = 0 \leq \frac{1}{2}\bigl(d(x,v) + d(y,u)\bigr) - \psi\bigl(d(x,v), d(y,u)\bigr),$$

that is, T is a triangular α-proximal admissible and modified α-proximal C-contraction mapping. Moreover, if $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subseteq [-2, -1]$ and hence $x \in [-2, -1]$. Consequently, $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore all the conditions of Theorem 3.2 hold for this example and T has a best proximity point. Here $z = -1$ is the best proximity point of T.

We conclude this section with another corollary.

Corollary 3.2 *Let A, B be two nonempty subsets of a metric space (X, d) such that A is complete, the pair (A, B) has the V-property and A_0 is nonempty. Assume that $T : A \to B$ is a continuous α-proximal C-contraction mapping of type (I) or a continuous α-proximal C-contraction mapping of type (II) such that the following conditions hold*:

(i) *T is a triangular α-proximal admissible mapping and $T(A_0) \subseteq B_0$,*

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1,$$

(iii) *if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in A$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x, Tx) = d(A, B) = d(y, Ty)$, we have $\alpha(x, y) \geq 1$.

4 Some results in metric spaces endowed with a graph

Consistent with Jachymski [28], let (X, d) be a metric space and Δ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [29], p.309) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^N$ of $N + 1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected (see for details [28, 30]).

Recently, some results have appeared providing sufficient conditions for a mapping to be a Picard operator if (X, d) is endowed with a graph. The first result in this direction was given by Jachymski [28].

Definition 4.1 ([28]) Let (X, d) be a metric space endowed with a graph G. We say that a self-mapping $T : X \to X$ is a Banach G-contraction or simply a G-contraction if T preserves the edges of G, that is,

$$\text{for all } x, y \in X, \quad (x, y) \in E(G) \implies (Tx, Ty) \in E(G)$$

and T decreases weights of the edges of G in the following way:

$$\exists \alpha \in (0, 1), \quad \text{for all } x, y \in X, (x, y) \in E(G) \implies d(Tx, Ty) \leq \alpha d(x, y).$$

Definition 4.2 Let A, B be two nonempty closed subsets of a metric space (X, d) endowed with a graph G. We say that a nonself-mapping $T : A \to B$ is a G-proximal C-contraction if, for all $u, v, x, y \in A$,

$$\begin{cases} (x,y) \in E(G), \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B) \end{cases}$$
$$\implies \quad d(u,v) \le \frac{1}{2}\big(d(x,v) + d(y,u)\big) - \psi\big(d(x,v), d(y,u)\big)$$

and

$$\begin{cases} (x,y) \in E(G), \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B) \end{cases} \implies \quad (u,v) \in E(G).$$

Theorem 4.1 *Let A, B be two nonempty closed subsets of a metric space (X, d) endowed with a graph G. Assume that A is complete, A_0 is nonempty and $T : A \to B$ is a continuous G-proximal C-contraction mapping such that the following conditions hold*:

(i) $T(A_0) \subseteq B_0$,

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad (x_0, x_1) \in E(G),$$

(iii) *for all $(x,y) \in E(G)$ and $(y,z) \in E(G)$, we have $(x,z) \in E(G)$.*

Then T has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x, Tx) = d(A,B) = d(y, Ty)$, we have $(x,y) \in E(G)$.

Proof Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Firstly we prove that T is a triangular α-proximal admissible mapping. To this aim, assume

$$\begin{cases} \alpha(x,y) \ge 1, \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B). \end{cases}$$

Therefore, we have

$$\begin{cases} (x,y) \in E(G), \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B). \end{cases}$$

Since T is a G-proximal C-contraction mapping, we get $(u,v) \in E(G)$, that is, $\alpha(u,v) \geq 1$ and

$$d(u,v) \leq \frac{1}{2}\big(d(x,v) + d(y,u)\big) - \psi\big(d(x,v), d(y,u)\big).$$

Also, let $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$, then $(x,z) \in E(G)$ and $(z,y) \in E(G)$. Consequently, from (iii), we deduce that $(x,y) \in E(G)$, that is, $\alpha(x,y) \geq 1$. Thus T is a triangular α-proximal admissible mapping with $T(A_0) \subseteq B_0$. Moreover, T is a continuous modified α-proximal C-contraction. From (ii) there exist $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A,B)$ and $(x_0,x_1) \in E(G)$, that is, $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$. Hence, all the conditions of Theorem 3.1 are satisfied and T has a unique fixed point. □

Similarly, by using Theorem 3.2, we can prove the following theorem.

Theorem 4.2 *Let A, B be two nonempty closed subsets of a metric space (X,d) endowed with a graph G. Assume that A is complete, the pair (A,B) has the V-property and A_0 is nonempty. Also suppose that $T : A \to B$ is a G-proximal C-contraction mapping such that the following conditions hold*:

(i) $T(A_0) \subseteq B_0$,

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad (x_0,x_1) \in E(G),$$

(iii) *for all $(x,y) \in E(G)$ and $(y,z) \in E(G)$, we have $(x,z) \in E(G)$*,

(iv) *if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$*.

Then T has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x,Tx) = d(A,B) = d(y,Ty)$, we have $(x,y) \in E(G)$.

5 Some results in partially ordered metric spaces

In recent years, Ran and Reurings [31] initiated the study of weaker contraction conditions by considering self-mappings in partially ordered metric space. Further these results were generalized by many authors; see for instance [32, 33]. Here we consider some recent results of Mongkolkeha *et al.* [34] and Sadiq Basha *et al.* [35].

Definition 5.1 ([35]) Let $(X,d,\preceq)$ be a partially ordered metric space. We say that a nonself-mapping $T : A \to B$ is proximally ordered-preserving if and only if, for all $x_1, x_2, u_1, u_2 \in A$,

$$\begin{cases} x_1 \preceq x_2, \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B) \end{cases} \implies \quad u_1 \preceq u_2.$$

Theorem 5.1 (Theorem 2.2 of [34]) *Let A, B be two nonempty closed subsets of a partially ordered complete metric space $(X,d,\preceq)$ such that A_0 is nonempty. Assume that $T : A \to B$ satisfies the following conditions*:

(i) *T is continuous and proximally ordered-preserving such that $T(A_0) \subseteq B_0$*,

(ii) *there exist elements* $x_0, x_1 \in A_0$ *such that*

$$d(x_1, Tx_0) = d(A,B) \quad \text{and} \quad x_0 \preceq x_1,$$

(iii) *for all* $x, y, u, v \in A$,

$$\begin{cases} x \preceq y, \\ d(u, Tx) = d(A,B), \\ d(y, Ty) = d(A,B) \end{cases} \implies d(u,v) \le \frac{1}{2}\big(d(x,v) + d(y,u)\big) - \psi\big(d(x,v), d(y,u)\big).$$

Then T *has a best proximity point.*

Proof Define $\alpha : A \times A \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

Firstly we prove that T is a triangular α-proximal admissible mapping. To this aim, assume

$$\begin{cases} \alpha(x,y) \ge 1, \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B). \end{cases}$$

Therefore, we have

$$\begin{cases} x \preceq y, \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B). \end{cases}$$

Now, since T is proximally ordered-preserving, then $u \preceq v$, that is, $\alpha(u,v) \ge 1$. Consequently, condition (T1) of Definition 3.2 holds. Also, assume

$$\begin{cases} \alpha(x,z) \ge 1, \\ \alpha(z,y) \ge 1, \end{cases}$$

so that $\begin{cases} x \preceq z, \\ z \preceq y, \end{cases}$ and consequently $x \preceq y$, that is, $\alpha(x,y) \ge 1$. Hence, condition (T2) of Definition 3.2 holds. Further, by (ii) we have

$$d(x_1, Tx_0) = d(A,B) \quad \text{and} \quad \alpha(x_0, x_1) \ge 1.$$

Moreover, from (iii) we get

$$\begin{cases} \alpha(x,y) \ge 1, \\ d(u, Tx) = d(A,B), \\ d(y, Ty) = d(A,B) \end{cases} \implies d(u,v) \le \frac{1}{2}\big(d(x,v) + d(y,u)\big) - \psi\big(d(x,v), d(y,u)\big).$$

Thus all the conditions of Theorem 3.1 hold and T has a best proximity point. □

Similarly, omitting the continuity hypothesis of T, we can give the following result.

Theorem 5.2 (see Theorem 2.6 of [34]) *Let A, B be two nonempty closed subsets of a partially ordered complete metric space $(X, d, \preceq)$ such that A_0 is nonempty and the pair (A, B) has the V-property. Assume that $T : A \to B$ satisfies the following conditions*:

(i) *T is proximally ordered-preserving such that $T(A_0) \subseteq B_0$*,

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad x_0 \preceq x_1,$$

(iii) *for all $x, y, u, v \in A$*,

$$\begin{cases} x \preceq y, \\ d(u, Tx) = d(A, B), \\ d(y, Ty) = d(A, B) \end{cases} \implies d(u, v) \le \frac{1}{2}\big(d(x, v) + d(y, u)\big) - \psi\big(d(x, v), d(y, u)\big),$$

(iv) *if $\{x_n\}$ is an increasing sequence in A converging to $x \in A$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then T has a best proximity point.

6 Application to fixed point theorems

In this section we briefly collect some fixed point results which are consequences of the results presented in the main section. Stated precisely, from Theorem 3.1, we obtain the following theorems.

Theorem 6.1 *Let (X, d) be a complete metric space. Assume that $T : X \to X$ is a continuous self-mapping satisfying the following conditions*:

(i) *T is triangular α-admissible*,

(ii) *there exists x_0 in X such that $\alpha(x_0, Tx_0) \ge 1$*,

(iii) *for all $x, y \in X$*,

$$\alpha(x, y) d(Tx, Ty) \le \frac{1}{2}\big(d(x, Ty) + d(y, Tx)\big) - \psi\big(d(x, Ty), d(y, Tx)\big).$$

Then T has a fixed point.

Theorem 6.2 *Let (X, d) be a complete metric space. Assume that $T : X \to X$ is a continuous self-mapping satisfying the following conditions*:

(i) *T is triangular α-admissible*,

(ii) *there exists x_0 in X such that $\alpha(x_0, Tx_0) \ge 1$*,

(iii) *for all $x, y \in X$*,

$$\big(\alpha(x, y) + \ell\big)^{d(Tx, Ty)} \le (\ell + 1)^{\frac{1}{2}(d(x, Ty) + d(y, Tx)) - \psi(d(x, Ty), d(y, Tx))},$$

where $\ell > 0$.

Then T has a fixed point.

Analogously, from Theorem 3.2, we obtain the following theorems, which do not require the continuity of T.

Theorem 6.3 *Let (X,d) be a complete metric space. Assume that $T : X \to X$ is a self-mapping satisfying the following conditions*:

(i) *T is triangular α-admissible*,

(ii) *there exists x_0 in X such that $\alpha(x_0, Tx_0) \geq 1$*,

(iii) *for all $x, y \in X$*,

$$\alpha(x,y)d(Tx,Ty) \leq \frac{1}{2}\big(d(x,Ty) + d(y,Tx)\big) - \psi\big(d(x,Ty), d(y,Tx)\big),$$

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$*.

Then T has a fixed point.

Theorem 6.4 *Let (X,d) be a complete metric space. Assume that $T : X \to X$ is a self-mapping satisfying the following conditions*:

(i) *T is triangular α-admissible*,

(ii) *there exists x_0 in X such that $\alpha(x_0, Tx_0) \geq 1$*,

(iii) *for all $x, y \in X$*,

$$\big(\alpha(x,y) + 1\big)^{d(Tx,Ty)} \leq 2^{[\frac{1}{2}(d(x,Ty)+d(y,Tx)) - \psi(d(x,Ty),d(y,Tx))]},$$

(iv) *if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in A$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$*.

Then T has a fixed point.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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