

## Research Article

# Legendre Wavelets Method for Solving Fractional Population Growth Model in a Closed System

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A new operational matrix of fractional order integration for Legendre wavelets is derived. Block pulse functions and collocation method are employed to derive a general procedure for forming this matrix. Moreover, a computational method based on wavelet expansion together with this operational matrix is proposed to obtain approximate solution of the fractional population growth model of a species within a closed system. The main characteristic of the new approach is to convert the problem under study to a nonlinear algebraic equation.

## 1. Introduction

In recent years, fractional calculus and differential equations have found enormous applications in mathematics, physics, chemistry, and engineering because of the fact that a realistic modeling of a physical phenomenon having dependence not only at the time instant but also on the previous time history can be successfully achieved by using fractional calculus. The applications of the fractional calculus have been demonstrated by many authors. For examples, it has been applied to model the nonlinear oscillation of earthquakes, fluid-dynamic traffic, frequency dependent damping behavior of many viscoelastic materials, continuum and statistical mechanics, colored noise, solid mechanics, economics, signal processing, and control theory [1–5]. However, during the last decade fractional calculus has attracted much more attention of physicists and mathematicians. Due to the increasing applications, some schemes have been proposed to solve fractional differential equations. The most frequently used methods are Adomian decomposition method (ADM) [6, 7], homotopy perturbation method [8], homotopy analysis method [9], variational iteration method (VIM) [10], fractional differential transform method (FDTM) [11, 12], fractional difference method (FDM) [13], power series method [14], generalized block pulse operational matrix method [15],

and Laplace transform method [16]. Also, recently the Haar wavelets [17], Legendre wavelets [18, 19], and the Chebyshev wavelets of first kind [20–23] and second kind [24] have been developed to solve the fractional differential equations. It is worth noting that wavelets are localized functions, which are the basis for energy-bounded functions and in particular for  $L^2(R)$ , so that localized pulse problems can be easily approached and analyzed [25–28].

Approximation by orthogonal family of basis functions has found wide applications in science and engineering. The most commonly used orthogonal families of functions in recent years are sine-cosine functions, block pulse functions, Legendre, Chebyshev, and Laguerre polynomials and also orthogonal wavelets, for example Haar, Legendre, Chebyshev, and CAS wavelets. The main advantages of using an orthogonal basis is that the problem under consideration reduces to a system of linear or nonlinear algebraic system equations [18]; thus this act not only simplifies the problem enormously but also speeds up the computation work during the implementation. This work can be done by truncating the series expansion in orthogonal basis function for the unknown solution of the problem and using the operational matrices [29]. There are two main approaches for numerical solution of fractional differential equations.

One approach is based on using the operational matrix of fractional derivative to reduce the problem under consideration into a system of algebraic equations and solving this system to obtain the numerical solution of the problem. Another useful approach is based on converting the underlying fractional differential equations into fractional integral equations, and using the operational matrix of fractional integration, to eliminate the integral operations and reducing the problem into solving a system of algebraic equations. The operational matrix of fractional Riemann-Liouville integration is given by

$$I^\alpha \Psi(x) \approx P^\alpha \Psi(x), \quad (1)$$

where  $\Psi(x) = [\psi_1(x), \psi_2(x), \dots, \psi_{\widehat{m}}]^T$ , in which  $\psi_i(x)$  ( $i = 1, 2, \dots, \widehat{m}$ ) are orthogonal basis functions which are orthogonal with respect to a specific weight function on a certain interval  $[a, b]$  and  $P^\alpha$  is the operational matrix of fractional integration of  $\Psi(x)$ . Notice that  $P^\alpha$  is a constant  $\widehat{m} \times \widehat{m}$  matrix and  $\alpha$  is an arbitrary positive constant.

In view of successful application of wavelet operational matrices in numerical solution of integral and differential equations, together with the characteristics of wavelet functions, we believe that they can be applicable in solving fractional population growth model. In this paper, the operational matrix of fractional order integrations for Legendre wavelets is derived, and a general procedure based on collocation method and block Pulse functions (BPFs) for forming this matrix is presented. Then, by using this matrix a computational method for solving fractional population growth model in a closed system is proposed. This paper is organized as follows. In Section 2, some necessary definitions of the fractional calculus are reviewed. In Section 3, the Legendre wavelets with some of their properties are presented. In Section 4, the proposed method for solving fractional population growth model in a closed system is described. Finally a conclusion is drawn in Section 5.

## 2. Preliminaries

In this section, we present some notations, definitions, and preliminary facts that will be used further in this paper.

The Riemann-Liouville fractional integral operator  $I^\alpha$  of order  $\alpha \geq 0$  on the usual Lebesgue space  $L^1[0, b]$  is given by [30]

$$(I^\alpha u)(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s) ds, & \alpha > 0, \\ u(x), & \alpha = 0. \end{cases} \quad (2)$$

The Riemann-Liouville fractional derivative of order  $\alpha > 0$  is normally defined as

$$D^\alpha u(x) = \left( \frac{d}{dx} \right)^m I^{m-\alpha} u(x), \quad (m-1 < \alpha \leq m), \quad (3)$$

where  $m$  is an integer.

The fractional derivative of order  $\alpha > 0$  in the Caputo sense is given by [30]

$$D_*^\alpha u(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} u^{(m)}(s) ds, \quad (4)$$

$$(m-1 < \alpha \leq m),$$

where  $m$  is an integer,  $x > 0$ , and  $u^{(m)} \in L^1[0, b]$ .

The useful relation between the Riemann-Liouville operator and Caputo operator is given by the following expression:

$$I^\alpha D_*^\alpha u(x) = u(x) - \sum_{k=0}^{m-1} u^{(k)}(0^+) \frac{x^k}{k!}, \quad (m-1 < \alpha \leq m), \quad (5)$$

where  $m$  is an integer,  $x > 0$ , and  $u^{(m)} \in L^1[0, b]$ .

## 3. The Legendre Wavelets

In this section, we briefly present some properties of Legendre wavelets.

*3.1. Constructing the Legendre Wavelets.* Here we introduce a process to construct the Legendre wavelets on the unit interval  $[0, 1]$ , using recursive wavelet construction which has been proposed in [31, 32] for piecewise polynomials on  $[0, 1]$ . For this purpose, we first introduce some notations. Throughout this work,  $\mathbb{N}$  denotes the set of all natural numbers,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  and  $\mathbb{Z}_\mu = \{0, 1, \dots, \mu-1\}$ , for a positive integer  $\mu$ .

For an integer  $\mu > 1$ , we consider the following contractive mappings on the interval  $I = [0, 1]$ :

$$\psi_\epsilon(t) = \frac{t + \epsilon}{\mu}, \quad t \in [0, 1], \quad \epsilon \in \mathbb{Z}_\mu. \quad (6)$$

It is obvious that the mappings  $\{\psi_\epsilon\}$  satisfy the following properties:

$$\psi_\epsilon(I) \subset I, \quad \forall \epsilon \in \mathbb{Z}_\mu, \quad (7)$$

$$\bigcup_{\epsilon \in \mathbb{Z}_\mu} \psi_\epsilon(I) = I.$$

Now, let  $F_0$  denote the finite dimensional linear space on  $[0, 1]$  that is spanned by the Legendre polynomials  $P_0(2x-1)$ ,  $P_1(2x-1)$ ,  $\dots$ , and  $P_{M-1}(2x-1)$ , where  $M \in \mathbb{N}$  and  $P_m$  are the Legendre polynomials of degree  $m$ , namely,

$$F_0 = \text{span} \{P_m(2x-1) \mid x \in [0, 1], m \in \mathbb{Z}_\mu\}. \quad (8)$$

It is well known that the Legendre polynomials  $P_m$  are orthogonal with respect to the weight function  $w(x) = 1$  on the interval  $[-1, 1]$ .

In order to construct an orthonormal basis for  $L^2[0, 1]$ , for each  $\epsilon \in \mathbb{Z}_\mu$  we define an isometry  $T_\epsilon$  on  $L^2[0, 1]$  as follows:

$$(T_\epsilon f)(x) = \begin{cases} \sqrt{\mu} f(\psi_\epsilon^{-1}(x)), & x \in \psi_\epsilon(I), \\ 0, & x \notin \psi_\epsilon(I). \end{cases} \quad (9)$$

Starting from the space  $F_0$ , we define a sequence of spaces  $\{F_k \mid k \in \mathbb{N}_0\}$  using the recurrence formula

$$F_{k+1} = \bigoplus_{\epsilon \in \mathbb{Z}_\mu} T_\epsilon F_k, \quad k \in \mathbb{N}_0, \quad (10)$$

where  $\oplus$  denotes the direct sum; that is, if  $A$  and  $B$  are two subspaces of  $L^2[0, 1]$  with  $A \cap B = \{0\}$ , then

$$A \oplus B = \{f + g : f \in A, g \in B\}. \quad (11)$$

The sequence of spaces  $\{F_k \mid k \in \mathbb{N}_0\}$  is nested, that is, [32]:

$$F_0 \subset F_1 \subset \dots \subset F_k \subset F_{k+1} \subset \dots, \quad (12)$$

$$\dim F_k = M\mu^k, \quad k \in \mathbb{N}_0.$$

Moreover, similar to Theorem 2.4 in [33], it can be proved that

$$\overline{\bigcup_{k=0}^{\infty} F_k} = L^2[0, 1]. \quad (13)$$

Now, we construct an orthonormal basis for each of the spaces  $F_k$ . We first notice that

$$G_0 = \left\{ \sqrt{2m+1} P_m(2x-1) \mid x \in [0, 1], m \in \mathbb{Z}_\mu \right\} \quad (14)$$

is an orthonormal basis for  $F_0$ , and moreover for  $f(x) \in L^2[0, 1]$  with compact support and for  $\epsilon \neq \epsilon'$  we have

$$\text{supp} \{T_\epsilon f\} \cap \text{supp} \{T_{\epsilon'} f\} = \emptyset, \quad \epsilon \neq \epsilon', \quad (15)$$

where  $\text{supp}(f)$  denotes the support of the function  $f$ . It can be simply seen that [31]

$$G_k = \left\{ T_{\epsilon_0} \circ \dots \circ T_{\epsilon_{k-1}} \left( \sqrt{2m+1} P_m(2x-1) \right) \mid m \in \mathbb{Z}_M, \epsilon_\ell \in \mathbb{Z}_\mu, \ell \in \mathbb{Z}_k \right\} \quad (16)$$

is an orthonormal basis for  $F_k$ , where “ $\circ$ ” denotes composition of functions. In other words, if for  $n = 1, 2, \dots, \mu^k$ ,  $k \in \mathbb{N}$ , we set

$$\begin{aligned} \psi_{nm}(x) &= \psi(k, m, n, x) \\ &= \begin{cases} \sqrt{2m+1} \mu^{k/2} P_m(2\mu^k x - 2n+1), & x \in \left[ \frac{n-1}{\mu^k}, \frac{n}{\mu^k} \right), \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (17)$$

then  $\{\psi_{nm}(x) \mid n = 1, 2, \dots, \mu^k, m \in \mathbb{Z}_M\}$  forms an orthonormal basis for  $F_k$ .

**3.2. Function Approximation.** A function  $f(x)$  defined over  $[0, 1)$  may be expanded by the Legendre wavelets as

$$u(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x), \quad (18)$$

where  $c_{nm} = (u(x), \psi_{nm}(x))$ , and  $(\cdot, \cdot)$  denotes the inner product. If the infinite series in (18) is truncated, then it can be written as

$$u(x) \approx \sum_{n=1}^{\mu^k} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) = C^T \Psi(x), \quad (19)$$

where  $T$  indicates transposition,  $C$  and  $\Psi(x)$  are  $\widehat{m} = \mu^k M$  column vectors which are given by

$$C = \left[ c_{10}, \dots, c_{1M-1} \mid c_{20}, \dots, c_{2M-1} \mid \dots \mid c_{\mu^k 0}, \dots, c_{\mu^k M-1} \right]^T,$$

$$\Psi(x) = \left[ \psi_{10}(x), \dots, \psi_{1M-1}(x) \mid \psi_{20}(x), \dots, \right. \\ \left. \psi_{2M-1}(x) \mid \dots \mid \psi_{\mu^k 0}(x), \dots, \psi_{\mu^k M-1}(x) \right]^T. \quad (20)$$

Taking the collocation points

$$t_i = \frac{(2i-1)}{2\widehat{m}}, \quad i = 1, 2, \dots, \widehat{m}, \quad (21)$$

we define the wavelet matrix  $\Phi_{\widehat{m} \times \widehat{m}}$  as

$$\Phi_{\widehat{m} \times \widehat{m}} = \left[ \Psi\left(\frac{1}{2\widehat{m}}\right), \Psi\left(\frac{3}{2\widehat{m}}\right), \dots, \Psi\left(\frac{2\widehat{m}-1}{2\widehat{m}}\right) \right]. \quad (22)$$

Indeed  $\Phi_{\widehat{m} \times \widehat{m}}$  has the following form:

$$\Phi_{\widehat{m} \times \widehat{m}} = \begin{pmatrix} A & 0 & 0 & \dots & 0 \\ 0 & A & 0 & \dots & 0 \\ 0 & 0 & A & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & A \end{pmatrix}, \quad (23)$$

where  $A$  is an  $M \times M$  matrix given by

$$A = \begin{pmatrix} \psi_{10}\left(\frac{1}{2\widehat{m}}\right) & \psi_{10}\left(\frac{3}{2\widehat{m}}\right) & \cdots & \psi_{10}\left(\frac{2\widehat{m}-1}{2\widehat{m}}\right) \\ \psi_{11}\left(\frac{1}{2\widehat{m}}\right) & \psi_{11}\left(\frac{3}{2\widehat{m}}\right) & \cdots & \psi_{11}\left(\frac{2\widehat{m}-1}{2\widehat{m}}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \psi_{\mu^k M-1}\left(\frac{1}{2\widehat{m}}\right) & \psi_{\mu^k M-1}\left(\frac{3}{2\widehat{m}}\right) & \cdots & \psi_{\mu^k M-1}\left(\frac{2\widehat{m}-1}{2\widehat{m}}\right) \end{pmatrix}. \tag{24}$$

For example, for  $\mu = 3, k = 1, M = 2$ , the Legendre matrix can be expressed as:

$$\Phi_{6 \times 6} = \begin{pmatrix} 1.7321 & 1.7321 & 0.0 & 0.0 & 0.0 & 0.0 \\ -1.5000 & 1.5000 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.7321 & 1.7321 & 0.0 & 0.0 \\ 0.0 & 0.0 & -1.5000 & 1.5000 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.7321 & 1.7321 \\ 0.0 & 0.0 & 0.0 & 0.0 & -1.5000 & 1.5000 \end{pmatrix}. \tag{25}$$

**3.3. Operational Matrix of Fractional Order Integration.** The fractional integration of order  $\alpha$  of the vector function  $\Psi(x)$  can be expressed as

$$(I^\alpha \Psi)(x) \simeq P^\alpha \Psi(x), \tag{26}$$

where  $P^\alpha$  is the  $\widehat{m} \times \widehat{m}$  operational matrix of fractional integration of order  $\alpha$ . In the following we obtain an explicit form of the matrix  $P$ . For this purpose, we need to introduce a new family of basis functions, namely, block pulse functions (BPFs).

We define a  $\widehat{m}$ -set of BPFs as [34, 35]

$$b_i(x) = \begin{cases} 1, & \frac{i}{\widehat{m}} \leq x < \frac{(i+1)}{\widehat{m}}, \\ 0, & \text{otherwise,} \end{cases} \tag{27}$$

where  $i = 0, 1, 2, \dots, (\widehat{m} - 1)$ .

The functions  $b_i(x)$  are disjoint and orthogonal.

The Legendre wavelets may be expanded into a  $\widehat{m}$ -set of BPFs as

$$\Psi(x) \simeq \Phi_{\widehat{m} \times \widehat{m}} B_{\widehat{m}}(x), \tag{28}$$

where  $B_{\widehat{m}}(x) = [b_0(x), b_1(x), \dots, b_i(x), \dots, b_{\widehat{m}-1}(x)]^T$ .

In [34], Kilicman et al. have given the block pulse operational matrix of fractional integration  $P_B^\alpha$  as

$$(I^\alpha B_{\widehat{m}})(x) \simeq P_B^\alpha B_{\widehat{m}}(x), \tag{29}$$

where

$$P_B^\alpha = \frac{1}{\widehat{m}^\alpha} \frac{1}{\Gamma(\alpha + 2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{\widehat{m}-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{\widehat{m}-2} \\ 0 & 0 & 1 & \cdots & \xi_{\widehat{m}-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{30}$$

and  $\xi_i = (i + 1)^{\alpha+1} - 2i^{\alpha+1} + (i - 1)^{\alpha+1}$ .

Next, we derive the Legendre wavelets operational matrix of fractional integration. By considering (26) and using (28), and (29) we have

$$\begin{aligned} (I^\alpha \Psi)(x) &\simeq (I^\alpha \Phi_{\widehat{m} \times \widehat{m}} B_{\widehat{m}})(x) = \Phi_{\widehat{m} \times \widehat{m}} (I^\alpha B_{\widehat{m}})(x) \\ &\simeq \Phi_{\widehat{m} \times \widehat{m}} P_B^\alpha B_{\widehat{m}}(x). \end{aligned} \tag{31}$$

Thus, by considering (28) and (31), we obtain the Legendre wavelets operational matrix of fractional integration as

$$(I^\alpha \Psi)(x) \simeq \Phi_{\widehat{m} \times \widehat{m}} P_B^\alpha \Phi_{\widehat{m} \times \widehat{m}}^{-1}. \tag{32}$$

To illustrate the calculation procedure we choose  $\mu = 3, k = 1, M = 2$ , and  $\alpha = 1/2$ ; thus we have:

$$P^{(1/2)} = \begin{pmatrix} 0.43433 & 0.14689 & 0.35988 & -0.069510 & 0.23430 & -0.017626 \\ -0.11016 & 0.17991 & 0.052129 & -0.028562 & 0.013219 & -0.0032248 \\ 0.0 & 0.0 & 0.43433 & 0.14689 & 0.35988 & -0.069510 \\ 0.0 & 0.0 & -0.11016 & 0.17991 & 0.052129 & -0.028562 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.43433 & 0.14689 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.11016 & 0.17991 \end{pmatrix}. \tag{33}$$

### 4. Application for Fractional Population Growth Model

As we have already mentioned, the fractional order models are more accurate than integer order models; that is, there are more degrees of freedom in the fractional order models. In this section, we will apply Legendre wavelets for solving a fractional population growth model. The model is characterized by the nonlinear fractional Volterra integrodifferential equation [36] as follows:

$$D_*^\alpha p(\bar{t}) - \bar{a}p(\bar{t}) + \bar{b}[p(\bar{t})]^2 + \bar{c}p(\bar{t}) \int_0^{\bar{t}} p(\tau) d\tau = 0, \tag{34}$$

$$p(0) = p_0, \quad 0 < \alpha \leq 1,$$

where  $\alpha$  is a constant parameter describing the order of the time fractional derivative,  $\bar{a} > 0$  is the birth rate coefficient,  $\bar{b} > 0$  is the crowding coefficient,  $\bar{c} > 0$  is the toxicity coefficient,  $p_0$  is the initial population, and  $p(\bar{t})$  is the population of identical individuals at time  $\bar{t}$  which exhibits crowding and sensitivity to the amount of toxins produced [37]. The coefficient  $\bar{c}$  indicates the essential behavior of the population evolution before its level falls to zero in the long run. It is worth mentioning that when the toxicity coefficient is zero, (34) reduces to the well-known logistic equation [37, 38]. The last term contains the integral which indicates the total metabolism or total amount of toxins produced since time zero. The individual death rate is proportional to this integral, and also the population death rate due to toxicity must include a factor  $p$ . Due to the fact that the system is closed, the presence of the toxic term always causes the population level falling to zero in the long run, as it will be seen later. The relative size of the sensitivity to toxins,  $\bar{c}$ , determines the manner in which the population evolves before its extinction. It is worth noting that in case  $\alpha = 1$ , the fractional equation reduces to a classical logistic growth model, so the proposed method can be also applied in this situation. Here we apply the scale time and population by introducing the non-dimensional variables  $t = \bar{c}\bar{t}/\bar{b}$  and  $u = \bar{b}p/\bar{a}$ , to obtain the following non-dimensional problem:

$$\kappa D_*^\alpha u(t) - u(t) + [u(t)]^2 + u(t) \int_0^t u(\tau) d\tau = 0, \tag{35}$$

$$u(0) = u_0, \quad 0 < \alpha \leq 1,$$

where  $u(t)$  is the scaled population of identical individuals at time  $t$  and  $\kappa = \bar{c}/\bar{a}\bar{b}$  is a prescribed non-dimensional

parameter. The only equilibrium solution of (35) is the trivial solution  $u(t) = 0$ , and the analytical solution for  $\alpha = 1$  is [39]

$$u(t) = u_0 \exp\left(\frac{1}{\kappa} \int_0^t \left(1 - u(\tau) - \int_0^\tau u(s) ds\right) d\tau\right). \tag{36}$$

In recent years, several numerical methods have been proposed to solve the classical and fractional population growth model, for instance, the reader is advised to see [36–43] and references therein. Here we use the operational matrix of fractional integration for solving nonlinear fractional integrodifferential population model (35). For this purpose, we first approximate  $D_*^\alpha u(t)$  as

$$D_*^\alpha u(t) \approx U^T \Psi(t), \tag{37}$$

where  $U$  is an unknown vector which should be found and  $\Psi(t)$  is the vector which is defined in (20).

By using initial condition and (5), we have

$$u(t) \approx U^T P^\alpha \Psi(t) + u_0. \tag{38}$$

Since  $\Psi(t) \approx \Phi_{\bar{m} \times \bar{m}} B_{\bar{m}}(t)$ , from (38), we have:

$$u(t) \approx U^T P^\alpha \Phi_{\bar{m} \times \bar{m}} B_{\bar{m}}(t) + u_0 [1, 1, \dots, 1] B_{\bar{m}}(t). \tag{39}$$

Define

$$A^T = [a_1, a_2, \dots, a_{\bar{m}}] = U^T P^\alpha \Phi_{\bar{m} \times \bar{m}} + u_0 [1, 1, \dots, 1]. \tag{40}$$

By using (38) and (39), we have  $u(t) \approx A^T B_{\bar{m}}(t)$ . From (27), we have

$$[u(t)]^2 \approx [a_1^2, a_2^2, \dots, a_{\bar{m}}^2] B_{\bar{m}}(t) = \bar{A}^T B_{\bar{m}}(t). \tag{41}$$

Also, we have

$$\int_0^t u(\tau) d\tau \approx A^T P_B B_{\bar{m}}(t) = C^T B_{\bar{m}}(t), \tag{42}$$

where  $C^T = A^T P_B$ . Now using (27), (39), and (42), we have

$$u(t) \int_0^t u(\tau) d\tau \approx \bar{U}^T B_{\bar{m}}(t), \tag{43}$$

where

$$\bar{U}^T = [a_1 c_1, a_2 c_2, \dots, a_{\bar{m}} c_{\bar{m}}]. \tag{44}$$

Now by substituting (37), (39), (41) and (43), into (35), we obtain

$$(\bar{\kappa} U^T \Phi_{\bar{m} \times \bar{m}} - A^T + \bar{A}^T + \bar{U}^T) B_{\bar{m}}(t) \approx 0, \tag{45}$$

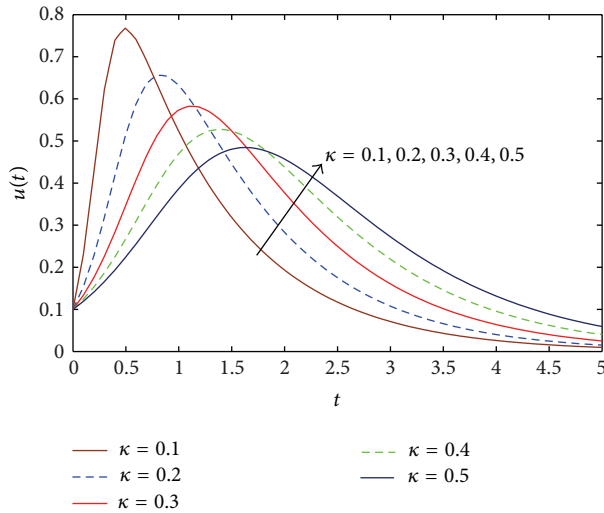


FIGURE 1: Numerical solutions of the classical population growth model for different values of  $\kappa$ .

and by replacing  $\approx$  by  $=$ , we obtain the following system of nonlinear algebraic equations:

$$\kappa U^T \Phi_{\widehat{m} \times \widehat{m}} - A^T + \widetilde{A}^T + \widehat{U}^T = 0. \quad (46)$$

Finally by solving this system and determining  $A$ , we obtain the approximate solution of the problem as  $u(t) = A^T \Psi(t)$ .

As a numerical example, we consider the nonlinear fractional integrodifferential equation (35) with the initial condition  $u(0) = 0.1$ , which is investigated in several papers, for instance see [36–43]. Here our purpose is to study the mathematical behavior of the solution of this fractional population growth model as the order of the fractional derivative changes. In particular, we seek to study the rapid growth along the logistic curve that will reach a peak then slow exponential decayed for different values of  $\alpha$ . To see the behavior solution of this problem for different values of  $\alpha$ , we will take advantage of the proposed method and consider the following two special cases.

*Case 1.* We investigate the classical population growth model ( $\alpha = 1$ ) for some different small values  $\kappa$ . The behavior of the numerical solutions for  $\widehat{m} = 162$  ( $\mu = 3$ ,  $k = 3$ , and  $M = 6$ ) is shown in Figure 1. From Figure 1 it can be seen that as  $\kappa$  increases, the amplitude of  $u(t)$  decreases, whereas the exponential decay increases.

*Case 2.* In this case we investigate the fractional population growth model (35) for different values of  $\alpha$  and  $\kappa$ .

From Figures 2, 3, and 4 it can be simply seen that as the order of the fractional derivative decreases, the amplitude of  $u(t)$  decreases, whereas the exponential decay increases and also it can be concluded that as  $\kappa$  increases, the maximum of  $u(t^*)$  of  $u(t)$  decreases. This tendency is similar to the case  $\alpha = 1$ , which we have already mentioned.

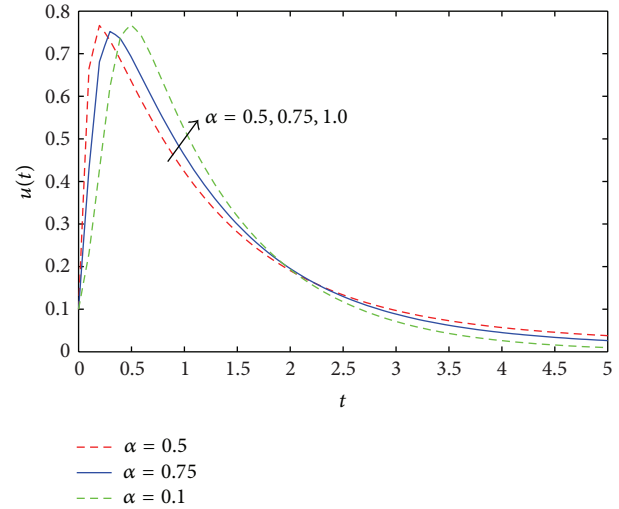


FIGURE 2: Numerical solutions of the fractional population growth model for  $\kappa = 0.1$ .

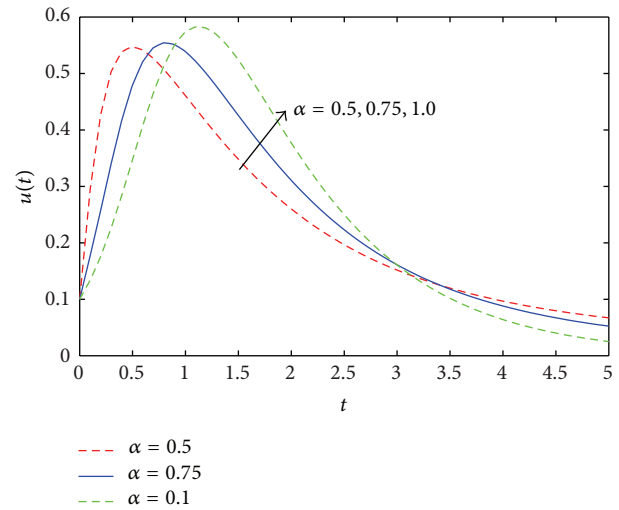


FIGURE 3: Numerical solutions of the fractional population growth model for  $\kappa = 0.3$ .

## 5. Conclusion

In this paper, the operational matrix of fractional order integration for Legendre wavelets was derived. Block pulse functions and collocation method were employed to derive a general procedure for forming this matrix. Moreover, a wavelet expansion together with this operational matrix was used to obtain approximate solution of the fractional population growth model of a species within a closed system. The main characteristic of the new approach is to convert the problem under study to a system of nonlinear algebraic equations by introducing the operational matrix of fractional integration for these basis functions. Analysis of the behavior of the model showed that it increases rapidly along the logistic curve followed by a slow exponential decay after reaching a maximum point, and also when the order of

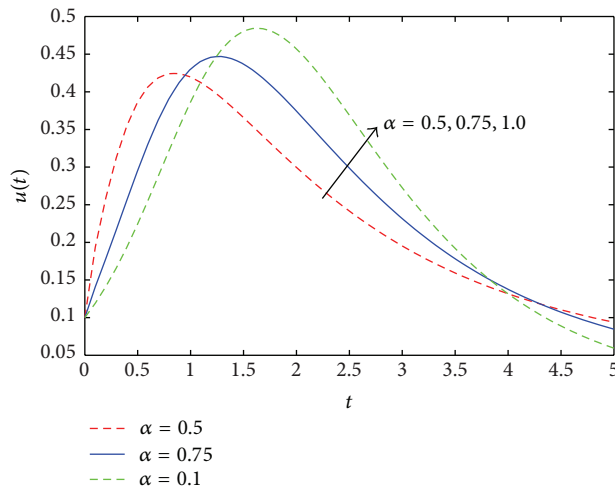


FIGURE 4: Numerical solutions of the fractional population growth model for  $\kappa = 0.5$ .

the fractional derivative  $\alpha$  decreases, the amplitude of the solution decreases, whereas the exponential decay increases.

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