Research Article

Ahmed Mohammed, Vicenţiu D. Rădulescu* and Antonio Vitolo

Blow-up solutions for fully nonlinear equations: Existence, asymptotic estimates and uniqueness

https://doi.org/10.1515/anona-2018-0134
Received June 13, 2018; accepted June 13, 2018

Abstract: The primary objective of the paper is to study the existence, asymptotic boundary estimates and uniqueness of large solutions to fully nonlinear equations $H(x, u, Du, D^2u) = f(u) + h(x)$ in bounded C^2 domains $\Omega \subseteq \mathbb{R}^n$. Here H is a fully nonlinear uniformly elliptic differential operator, f is a non-decreasing function that satisfies appropriate growth conditions at infinity, and h is a continuous function on Ω that could be unbounded either from above or from below. The results contained herein provide substantial generalizations and improvements of results known in the literature.

Keywords: Large solutions, existence and uniqueness, fully nonlinear elliptic equations

MSC 2010: 35J60, 35J70

1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^2 boundary $\partial\Omega$. We consider the infinite boundary value problem

$$\begin{cases} H[u] = f(u) + h(x) & \text{in } \Omega \\ \quad u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $H[u] := H(x, u, Du, D^2u)$ is a fully nonlinear uniformly elliptic operator. For $u \in C^2(\Omega)$, as usual, Du stands for the gradient of u while D^2u denotes the Hessian matrix of u.

Let $\mathcal{S}_n$ be the set of $n \times n$ real symmetric matrices. Throughout this paper, we fix constants $0 < \lambda \leq \Lambda$ and we set $\mathcal{A}_{\lambda,\Lambda} := \{A \in \mathcal{S}_n : \lambda I_n \leq A \leq \Lambda I_n\}$.

To specify our assumptions on H, we first recall the so-called Pucci extremal operators $\mathcal{P}^-_{\lambda,\Lambda} : \mathcal{S}_n \to \mathbb{R}$ and $\mathcal{P}^+_{\lambda,\Lambda} : \mathcal{S}_n \to \mathbb{R}$ (see [8]) defined by

$$\mathcal{P}^+_{\lambda,\Lambda}(X) := \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AX) \quad \text{and} \quad \mathcal{P}^-_{\lambda,\Lambda}(X) := \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AX) \quad \text{for all } X \in \mathcal{S}_n.$$

Here $\operatorname{tr}(X)$ stands for the trace of $X \in \mathcal{S}_n$.

Ahmed Mohammed, Department of Mathematical Sciences, Ball State University, Muncie, IN 47306, USA,
e-mail: amohammed@bsu.edu
***Corresponding author: Vicenţiu D. Rădulescu,** Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia; and Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland; and Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania, e-mail: vicentiu.radulescu@imfm.si. https://orcid.org/0000-0003-4615-5537
Antonio Vitolo, Department of Civil Engineering, Universitá di Salerno, 84084 Fisciano (Salerno), Italy, e-mail: vitolo@unisa.it

Given non-negative functions $\gamma, \chi \in C(\Omega)$ let us set

$$\mathcal{M}^+(x,t,p,X) := \mathcal{P}^+_{\lambda,\Lambda}(X) + \gamma(x)|p| + \chi(x)t^-, \tag{1.2}$$

$$\mathcal{M}^-(x,t,p,X) := \mathcal{P}^-_{\lambda,\Lambda}(X) - \gamma(x)|p| - \chi(x)t^+ \tag{1.3}$$

for $(x,t,p,X) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n$, where $t^\pm = \max(\pm t, 0)$.

The class of functions $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n \to \mathbb{R}$ considered in this work will include

$$H(x,t,p,X) := \mathcal{P}^\pm_{\lambda,\Lambda}(X) + \mathcal{K}(x,p) - \chi(x)t,$$

where $\mathcal{K} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that

$$|\mathcal{K}(x,q) - \mathcal{K}(x,p)| \le \gamma(x)|q-p|$$

for some non-negative $\gamma, \chi \in C(\Omega)$.

In this paper, we will consider mappings $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n \to \mathbb{R}$ such that for all $x, y \in \Omega$, $s, t \in \mathbb{R}$, $p, q \in \mathbb{R}^n$ and $X, Y \in \mathcal{S}_n$ the following hold:

(H-1) H is continuous, $H(x,0,0,O) = 0$ and

$$\mathcal{M}^-(x, t-s; q-p; Y-X) \le H(x,t,q,Y) - H(x,s,p,X) \le \mathcal{M}^+(x, t-s; q-p; Y-X).$$

(H-2) $|H(x,t,p,X) - H(y,t,p,X)| \le K\|X\||x-y| + \omega((1+|p|)|x-y|)$, where $K \ge 0$ is a constant and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(0^+) = 0$.

Here and throughout, $\mathbb{R}_+$ stands for the set of positive real numbers.

We now turn to the nonlinearity f in (1.1). Throughout this paper we will assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies both of the following conditions:

(f-1) f is non-decreasing, positive in $\mathbb{R}_+$ and $f(0) = 0$.

(f-2) f satisfies the Keller–Osserman condition; namely

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)\,ds,\ t \ge 0.$$

Further conditions on f, as well as on $h \in C(\Omega)$, that will be needed in this work will be explained later.

The study of large solutions has a long history. Perhaps a systematic study of large solutions started with the works of Keller [28] and Osserman [40]. Since then a huge amount of work has emerged focusing on existence and uniqueness of large solutions. An exhaustive list on large solutions is impossible and we only list [1, 5, 10, 11, 17, 18, 21, 22, 24, 30–33, 41, 42, 47] and refer the interested reader to the references therein. We wish to single out the papers of García-Melián [22], López-Gómez and Luis Maire [30], and Marcus and Véron [32] on uniqueness of large solutions of $\Delta u = f(u)$ on smooth bounded domains under some general conditions on f. However, in order to put the problems we wish to consider in this paper in perspective, let us recall some works that are directly related to problem (1.1) with $h(x) \not\equiv 0$ on Ω. In [45], Verón studied the existence and uniqueness of solutions to problem (1.1) when $h \in C(\Omega)$ is non-positive, $H[u] = \operatorname{tr}(A(x)D^2u)$ is uniformly elliptic, and $f(t) = |t|^{\kappa-1}t$ for some $\kappa > 1$. Likewise, in [15], Diaz and Letelier investigated larges solutions of $\operatorname{div}(|Du|^{p-2}Du) = f(u) + h(x)$, $p > 1$, in bounded C^2 domains when f is a non-decreasing function that satisfies a condition of Keller–Osserman type suited for the p-Laplace operator and $h \in C(\Omega)$ is non-positive. In [2], Alarcón and Quaas study existence, asymptotic boundary behavior and uniqueness of solutions to problem (1.1). In the paper [2], the authors consider the case when $H[u]$ depends on D^2u only, f satisfies the usual Keller–Osserman condition and $h \in C(\Omega)$ is non-positive. In a related work [46], one of the authors, Amendola and Galise show that $H(Du, D^2u) = c|u|^{p-1}u + |u|^{q-1} + h(x)$ has at most one positive large solution on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with “local graph property" introduced by Marcus and Verón [31]. Here $c \in \mathbb{R}$, $0 < p < q$, $h \le 0$ on Ω and H is a uniformly elliptic operator that is “homogeneous” of degree $k \in [p, q]$ which satisfies appropriate structural conditions. We refer to [46] for more details on the results and conditions imposed.

In a recent paper, García-Melián studied existence and uniqueness of large solutions to $\Delta u = |u|^{p-1}u + h(x)$ in bounded C^2 domains, where $h \in C(\Omega)$ is allowed to change sign. See also [47]. To the best of our knowledge, this more challenging case of a sign-changing inhomogeneous term h is investigated for the first time in the paper [23]. In [23], the author obtains existence of large solutions to the aforementioned equation for a large class of unbounded $h \in C(\Omega)$ and uniqueness result is proven under the restriction that h is bounded on Ω from above. Motivated by the work of [23], one of the authors and Porru [37] extended the work of [23] to large solutions of $Lu = f(u) + h$, where L is a linear uniformly elliptic equations in non-divergence form with possibly unbounded lower-order terms. In [37], existence and uniqueness results are obtained when the inhomogeneous term $h \in C(\Omega)$ is allowed to be unbounded from above but with some restrictions, and with bounded coefficients for the first-order and zero-order terms.

The main objective of the present paper is to extend many of the above results to solutions of problem (1.1) by relaxing the conditions used in most of the aforementioned papers. In fact, the results contained herein are new for solutions of (1.1) when h is unbounded from above, even when $H[u] = \Delta u$. Another feature of the current work is that we obtain existence of solutions not only when h is unbounded on Ω, but also when the coefficients of H are unbounded on Ω.

The paper is organized as follows. In Section 2, we state the main results of the paper. These results discuss existence, asymptotic boundary estimates and uniqueness of solutions to problem (1.1). Section 3 presents some basic facts that are consequences of the assumptions made in the Introduction. We also recall several useful results from the literature that will be used in our work. The ABP maximum principle will play a recurring role in our work. In Section 4 we will develop several existence results. Depending on the rate of growth of h near the boundary, we will either relax the conditions needed on f or require more restriction.

Asymptotic boundary estimates of solutions to problem (1.1) will be developed in Section 5. In the investigation of such estimates, a condition on f introduced by Martin Dindoš in [16] will have a prominent role.

Uniqueness of solution to problem (1.1) will be investigated in Section 6. In its most general form, the uniqueness result will use a condition on h that manifests through the growth of a solution ψ of $\mathcal{M}^-[\psi] = -h^+$. In particular, our uniqueness result allows $h \in L^n(\Omega) \cap C(\Omega)$ in problem (1.1).

Finally, we have included an Appendix where some useful results on existence to boundary value problems involving H with unbounded coefficients are studied. These results are used in the main body of the paper and are of independent interest.

2 Main results

In this section we state the main results of the paper. To avoid use of technicalities, we have chosen to present these results in less general terms than given in the main body of the paper.

We begin by considering the non-increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\int_{\phi(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = t, \qquad t > 0. \tag{2.1}$$

Our first existence result, as well as many others, will depend on the sizes of

$$\Theta_f^*(h^\pm) := \limsup_{d(x)\to 0} \frac{h^\pm(x)}{f(\phi(d(x)))}, \tag{2.2}$$

where $d(x)$ is the distance of $x \in \Omega$ to the boundary $\partial\Omega$ of Ω. We will denote (2.2) simply as $\Theta^*(h)$ when there is no ambiguity concerning f.

Another important feature of our work is that we allow unbounded coefficients $\gamma, \chi \in C(\Omega)$ subject to the conditions:

(C-γ) $\lim_{d(x)\to 0} \gamma(x)d(x) = 0$,

(C-χ) $\lim_{d(x)\to 0} \chi(x)d^2(x) = 0$.

To state our first existence result, we recall the following condition introduced by Dindoš in [16]. There is $\theta > 1$ such that

(f-θ) $\ell := \liminf_{t\to\infty} \dfrac{f(\theta t)}{\theta f(t)} > 1.$

This condition, or a strengthened form thereof, will also appear in the study of boundary asymptotic estimates as well as in our uniqueness result. We remark that (f-θ), together with (f-1) implies (f-2). See Remark 3.6.

To obtain existence of solutions to problem (1.1) with the coefficients γ and χ allowed to be unbounded on Ω according to (C-γ) and (C-χ), we need control on the rate of growth of f at infinity and the following condition provides such control:

(f-3) $\alpha := \liminf_{t\to\infty} \dfrac{F(t)}{tf(t)} > 0.$

Perhaps a word on notational use is in order here. If we wish to use any condition (f-x) on a function $g : \mathbb{R} \to \mathbb{R}$, we will simply quote it as condition (g-x).

Referring to Section 3 for condition (f-4) we now state our first existence result.

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain. Assume that* (H-1), (H-2), (C-γ), (C-χ), (f-1), (f-2), (f-4) *hold. Suppose that there is $g : \mathbb{R} \to \mathbb{R}$ that satisfies* (g-1), (g-3) *and* (g-θ) *such that $f \le g$ at infinity. Then there exists a constant $\Theta > 0$ such that problem* (1.1) *admits a maximal solution whenever $h \in C(\Omega)$ satisfies $h(x) = O(f(\phi(d(x))))$ as $d(x) \to 0$ with $\Theta^*(h^+) < \Theta$. Here $\Theta = \Theta(\lambda, \alpha, \theta, \ell)$, where θ, ℓ, α are the parameters in condition* (g-3) *and* (g-θ).

We note that when f satisfies (f-3) and (f-θ), we may take f as the function g in Theorem 2.1, and if f satisfies (f-3), the choice $g(t) = t^p f(t)$, where $p > 0$ will do in Theorem 2.1.

A complementary existence result can be obtained by prescribing an indirect control on the size of h^+. This control is imposed on the growth, near the boundary, of a non-negative solution ψ of a PDE related to the Pucci maximal operator as follows:

(D-h) The equation

$$\mathcal{M}^+[\psi] \le -h^+ \tag{2.3}$$

admits a non-negative solution $\psi \in C(\Omega)$.

We refer to Remark 4.8 for a discussion on this condition.

Based on a result of Ancona [4], see also [34], we can relax condition (C-γ), while at the same time we need to strengthen condition (C-χ) to the following conditions, respectively:

(B-γ) $\sup_{x\in\Omega} \gamma(x)d(x) < \infty,$

(C-χ_η) $\sup_{x\in\Omega} \dfrac{d^2(x)}{\eta(d(x))}\chi(x) < \infty,$

where $\eta : (0, R] \to \mathbb{R}_+$ is a non-decreasing function for some $R \ge \operatorname{diam}(\Omega)$ and satisfies the Dini condition

$$\int_0^R \frac{\eta(t)}{t}\,dt < \infty.$$

It will be convenient to refer to such a function as a Dini continuous function. Assuming that

$$h^+(x) = O(d(x)^{-2}\eta(d(x))) \quad \text{as } d(x) \to 0,$$

one can use Lemma 4.3 to establish the existence of a maximal solution of (1.1). Moreover, this maximal solution is positive provided that the following is sufficiently small:

$$h_\eta := \sup_\Omega \frac{h^+(x)}{d^{-2}(x)\eta(d(x))} < \infty. \tag{2.4}$$

This allows for instance when $h^+(x) = O(d^{-2+\delta}(x))$ as $d(x) \to 0$ for some $\delta > 0$. We wish to emphasize here that the only conditions on f needed are (f-1) and (f-2).

Theorem 2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded smooth domain. Assume* (H-1), (H-2), (B-γ), (C-χ_η), (f-1), (f-2). *If $h \in C(\Omega)$ is such that $h_\eta < \infty$ for some Dini continuous function η, then problem* (1.1) *admits a maximal solution. Moreover, there exists a constant $c > 0$ such that the solution is positive whenever $h_\eta < c$.*

The proof relies on the existence of a positive solution ψ to problem (2.3). This approach based on condition (D-h) also proves to be useful in dealing with uniqueness for unbounded h, at least when the coefficients γ and χ are non-negative constants. The analysis on uniqueness will be carried out through boundary asymptotic estimates of solutions of (1.1).

To obtain boundary asymptotic estimates, we need Dindoš' condition as well as control from below on h. In fact, we need to assume $h^-(x) = O(f(\phi(d(x)))$ as $d(x) \to 0$, or equivalently $\Theta^*_f(h^-) < \infty$. However, we need the coefficients γ and χ be bounded, which without loss of generality, we take to be non-negative constants.

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded smooth domain. Assume* (H-1) *and* (H-2) *with γ and χ non-negative constants,* (f-1) *and* (f-θ) *for some $\theta > 1$. Let $h \in C(\Omega)$ be such that $\Theta^*(h^-) < \infty$ and $h_\eta < \infty$ for some Dini continuous function η. There exist constants $0 < A_* \leq A^* < \infty$ such that*

$$A_* \leq \liminf_{d(x)\to 0} \frac{u(x)}{\phi(d(x))} \leq \limsup_{d(x)\to 0} \frac{u(x)}{\phi(d(x))} \leq A^*$$

for all solutions u of (1.1).

We need further assumptions on H and on f in order to get uniqueness. These are the sub-homogeneity property: For all $(\sigma, x, t, p, X) \in (1, \infty) \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n$

(H-3) $H(x, \sigma t, \sigma p, \sigma X) \leq \sigma H(x, t, p, X)$

and the monotonicity condition

(f-m) $\frac{f(t)}{t}$ is non-decreasing at infinity.

The following uniqueness result holds.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain and assume* (H-1)–(H-3) *with γ, χ non-negative constants. Assume that f satisfies conditions* (f-1), (f-3), (f-θ) *for all $\theta > 1$ and* (f-m). *Suppose also that $h \in C(\Omega)$ satisfies $\Theta^*(h^-) < \infty$ and $h^+(x) = O(d^{-2}(x)\eta(d(x)))$ as $d(x) \to 0$ for some Dini continuous function η. Then problem* (1.1) *admits at most one solution.*

We should point out that the above asymptotic estimate and uniqueness results, which are stated here with the condition $h^+(x) = O(d^{-2}(x)\eta(d(x)))$ as $d(x) \to 0$ for some Dini continuous function η, have been established in this paper in a more general framework through control of the growth of the functions ψ given in condition (D-h). To the best of our knowledge, this approach appears here for the first time (see also [35]). The optimal growth on h^+ that this method leads to remains an open problem.

3 Preliminaries

Throughout the entire paper we suppose that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with C^2 boundary. In this work it will be convenient to use the following notations. Given $\delta > 0$,

$$\Omega_\delta := \{x \in \Omega : d(x) < \delta\}$$

and

$$\Omega^\delta := \{x \in \Omega : d(x) > \delta\},$$

where $d(x)$ denotes the distance of $x \in \Omega$ to the boundary $\partial\Omega$. Since Ω is a bounded C^2 domain, we note that there is $\mu > 0$ such that $d \in C^2(\overline{\Omega}_\mu)$ and $|\nabla d(x)| = 1$ on Ω_μ. See [25, Lemma 14.16] for a proof. In fact, by modifying the distance function d appropriately, we can suppose that d is a positive C^2 function on Ω. For instance one can use $(1 - \varphi)d + \varphi$ instead of d, where $\varphi \in C^2_c(\Omega)$ is a cut-off function with $0 \leq \varphi \leq 1$ on Ω, $\varphi \equiv 0$ on Ω_{μ_0} for some $0 < \mu_0 < \mu$ and $\varphi \equiv 1$ on Ω^μ. Therefore hereafter, we will always suppose that d is this modified distance function and that d is in $C^2(\overline{\Omega})$ with $|Dd| \equiv 1$ on Ω_{μ_0}.

It is helpful to keep in mind the following alternative description of the Pucci extremal operators:

$$\mathcal{P}^+_{\lambda,\Lambda}(X) = \Lambda \operatorname{tr}(X^+) - \lambda \operatorname{tr}(X^-) = \Lambda \sum_{e_i(X)>0} e_i(X) + \lambda \sum_{e_i(X)<0} e_i(X),$$
$$\mathcal{P}^-_{\lambda,\Lambda}(X) = \lambda \operatorname{tr}(X^+) - \Lambda \operatorname{tr}(X^-) = \lambda \sum_{e_i(X)>0} e_i(X) + \Lambda \sum_{e_i(X)<0} e_i(X),$$

where X^+ and X^- are the positive and negative parts of X, respectively, and $e_i(X)$, $i = 1, \dots, n$, are the eigenvalues of X, counted according multiplicity, in non-decreasing order.

The positive homogeneity, duality, sub-additive and super-additive properties of the Pucci extremal operators (see [8]) lead to the following useful properties of $\mathcal{M}^\pm$:

$$\mathcal{M}^\pm(x, c(t, p, X)) = c\mathcal{M}^\pm(x, t, p, X), \tag{3.1}$$
$$\mathcal{M}^-(x, t, p, X) = -\mathcal{M}^+(x, -t, -p, -X), \tag{3.2}$$
$$\mathcal{M}^+(x, t, p, X) + \mathcal{M}^-(x, s, q, Y) \le \mathcal{M}^+(x, t+s, p+q, X+Y) \le \mathcal{M}^+(x, t, p, X) + \mathcal{M}^+(x, s, q, Y), \tag{3.3}$$
$$\mathcal{M}^-(x, t, p, X) + \mathcal{M}^-(x, s, q, Y) \le \mathcal{M}^-(x, t+s, p+q, X+Y) \le \mathcal{M}^+(x, t, p, X) + \mathcal{M}^-(x, s, q, Y) \tag{3.4}$$

for all $c \ge 0$ and $(x, t, p, X), (x, s, q, Y) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n$.

Remark 3.1. From (H-1) it follows that H is uniformly elliptic, that is,

$$\lambda \operatorname{tr}(Y - X) \le H(x, t, p, Y) - H(x, t, p, X) \le \Lambda \operatorname{tr}(Y - X) \quad \text{whenever } X \le Y.$$

Moreover, (H-1) implies that H is non-increasing in t:

$$H(x, t, p, X) - H(x, s, p, X) \le 0$$

for $s \le t$.

Given $k \in C(\Omega \times \mathbb{R})$, a function $u \in C^2(\Omega)$ is said to be a classical solution of equation $H[u] = k(x, u)$ in Ω if and only if

$$H(x, u(x), Du(x), D^2u(x)) = k(x, u(x)) \quad \text{for all } x \in \Omega. \tag{3.5}$$

However, in this paper we consider functions $u \in C(\Omega)$ which are solutions in the viscosity sense, according to the following definition.

Let $u \in \mathrm{USC}(\Omega)$ (upper semicontinuous in Ω), resp. $u \in \mathrm{LSC}(\Omega)$ (lower semicontinuous in Ω). Then u is said to be a viscosity subsolution (resp., supersolution) in Ω of (3.5) if and only if for each $x \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum (resp. minimum) at x we have

$$H(x, u(x), D\varphi(x), D^2\varphi(x)) \ge k(x, u(x)) \quad (\text{resp., } H(x, u(x), D\varphi(x), D^2\varphi(x)) \le k(x, u(x))).$$

A function $u \in C(\Omega)$ that is both a viscosity subsolution and viscosity supersolution in Ω of (3.5) is called a viscosity solution in Ω.

Remark 3.2. It is well known that a function $u \in C^2(\Omega)$ is a classical subsolution (supersolution) of (3.5) if and only if u is a viscosity subsolution (supersolution) of (3.5). The forward implication follows directly from the definition. For the reverse, we refer to [8, Corollary 2.6].

We note the following consequence of condition (H-1):

$$\mathcal{M}^-[u] \le H[u] \le \mathcal{M}^+[u] \tag{3.6}$$

for any function $u \in C^2(\Omega)$, where $\mathcal{M}^\pm[u] := \mathcal{M}^\pm(x, u, Du, D^2u)$.

In the sequel we will make an extensive use of a fundamental tool for pointwise estimates of viscosity solutions of elliptic equations, known as the Alexandroff–Bakelman–Pucci maximum principle (see, for instance, [3, 6, 9, 43]). For the convenience of the reader we recall below the version needed here.

For this, we first remark that if $k \in C(\Omega)$ and $H[w] \ge k(x)$ for some $w \in C(\Omega)$, then by (3.6) it follows that $\mathcal{M}^+[w] \ge k(x)$. Note also that the latter implies that $w^+(x) = \max(w(x), 0)$ satisfies $\mathcal{M}^+[w^+] \ge -k^-(x)$. Therefore, setting

$$\mathcal{M}^+_\gamma[w^+] := \mathcal{P}^+_{\lambda,\Lambda}(D^2w^+) + \gamma|Dw^+|,$$

we also have

$$\mathcal{M}^+_\gamma[w^+] \geq -k^-(x).$$

Consequently, the standard Alexandroff–Bakelman–Pucci maximum principle (see [6, Proposition 2.12]) leads to the following.

Proposition 3.3 (ABP estimate). *Let $\mathcal{O} \subseteq \mathbb{R}^n$ be a bounded domain with diameter R. Suppose that H satisfies condition* (H-1)*, assuming $\gamma = \|\gamma^+\|_{L^\infty(\mathcal{O})} < \infty$. For $k \in C(\mathcal{O}) \cap L^n(\mathcal{O})$, let $w \in C(\overline{\mathcal{O}})$ be a viscosity subsolution of equation $H[w] = k(x)$ in $\mathcal{O}$. There is a non-negative constant C, depending only on n, λ, Λ, and γR, such that*

$$\sup_{\mathcal{O}} w \leq \sup_{\partial\mathcal{O}} w^+ + CR\|k^-\|_{L^n(\mathcal{O})}.$$

In particular, under the assumptions of Proposition 3.3, the following sign propagation property holds:

$$H[w] \geq 0 \text{ in } \mathcal{O}, \ \ w \leq 0 \text{ on } \partial\mathcal{O} \implies w \leq 0 \text{ in } \mathcal{O}.$$

One then obtains a useful comparison principle by combining Proposition 3.3 and the following result which is based on [14, Proposition 2.1]. A justification for the reformulation presented below is sketched in [33, Lemma 2.5].

Lemma 3.4. *Let $\mathcal{O} \subseteq \mathbb{R}^n$ be a bounded domain, and let $a(t)$, $b(t)$ be continuous functions on $\mathbb{R}$. Suppose that H satisfies* (H-1) *and* (H-2)*. If $H[u] \geq a(u)$ and $H[v] \leq b(v)$ for some $u, v \in C(\overline{\mathcal{O}})$, then*

$$\mathcal{M}^+[u - v] \geq a(u) - b(v) \quad \text{in } \widetilde{\mathcal{O}} := \{x \in \mathcal{O} : u(x) > v(x)\}.$$

As mentioned in the Introduction, we will assume throughout the paper that $f : \mathbb{R} \to \mathbb{R}$ satisfies conditions (f-1) and (f-2). We recall some useful consequences of these assumptions.

Remark 3.5. It is well known that if f satisfies (f-1) and (f-2), then both the following limits hold:

$$\lim_{t\to\infty} \frac{\sqrt{F(t)}}{f(t)} = 0, \quad \lim_{t\to\infty} \frac{t}{f(t)} = 0, \quad t > 0.$$

The reader is referred to [24, 26] for a proof.

The non-increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined in (2.1) satisfies

$$\lim_{t\to 0} \phi(t) = \infty,$$

and

$$\phi'(t) = -\sqrt{2F(\phi(t))}, \quad \phi''(t) = f(\phi(t)).$$

Here we mention some easy, but useful consequences of the Dindoš' condition (f-θ).

Remark 3.6. First we point out that assuming (f-1), condition (f-θ) with $\theta > 1$ implies (f-2). More precisely, we have

$$\liminf_{t\to\infty} \frac{f(t)}{t^q} > 0$$

for some $q > 1$. For a proof we refer to [35, Lemma 2.2].

Remark 3.7. Note that, by iterating (f-θ), we also have, for all $j \in \mathbb{N}$,

$$\liminf_{t\to\infty} \frac{f(\theta^j t)}{\theta^j f(t)} \geq \ell^j \tag{3.7}$$

as well as

$$\limsup_{t\to\infty} \frac{f(\theta^{-j} t)}{\theta^{-j} f(t)} \leq \ell^{-j}.$$

Remark 3.8. We remark that if f satisfies (f-1) and (f-3), then $0 \leq \alpha \leq \frac{1}{2}$, where α is the infimum in condition (f-3). We refer to [36, Lemma 6.1] for a proof.

We also recall the following two lemmas from [36], and [39], respectively.

Lemma 3.9. *Suppose that f satisfies* (f-1), (f-2) *and* (f-3). *Then:*

$$\limsup_{t\to\infty}\frac{\sqrt{F(t)}}{f(t)\int_t^\infty F(s)^{-\frac{1}{2}}\,ds}<\infty, \tag{i}$$

$$\limsup_{t\to\infty}\frac{t}{f(t)(\int_t^\infty F(s)^{-\frac{1}{2}}\,ds)^2}<\infty. \tag{ii}$$

Remark 3.10. Lemma 3.9 (ii) leads to the following observation when f satisfies conditions (f-1), (f-2) and (f-3). Suppose $h\in C(\Omega)$ such that $h^+(x)=O(d^{-2}(x)\phi(d(x)))$ as $d(x)\to 0$. Then $\Theta^*(h^+)<\infty$.

The next result, a consequence of Lemma 3.9, will prove useful in establishing the existence of solutions to problem (1.1).

Corollary 3.11. *Suppose that condition* (f-1), (f-2), (f-3) *are satisfied. Assuming, in addition,* (C-γ), *we have*

$$\lim_{d\to 0}\frac{\sqrt{F(\phi(d))}}{f(\phi(d))}\gamma(x)=0. \tag{3.8}$$

Assuming, in addition, (C-χ), *we have*

$$\lim_{d\to 0}\frac{\phi(d)}{f(\phi(d))}\chi(x)=0.$$

Proof. To show (3.8), observe that

$$\frac{\sqrt{F(\phi(d(x)))}}{f(\phi(d(x)))}|\gamma(x)|=\frac{\sqrt{F(\phi(d(x)))}}{d(x)f(\phi(d(x)))}|\gamma(x)|\,d(x)=\frac{\sqrt{F(\phi(d(x)))}}{f(\phi(d(x)))\int_{\phi(d(x))}^\infty\frac{ds}{\sqrt{2F(s)}}}|\gamma(x)|\,d(x).$$

Therefore, in light of Lemma 3.9 (i) and condition (C-γ), recalling that $\phi(\delta)\to\infty$ as $\delta\to 0$, the right-hand side tends to zero as $d(x)\to 0$. In a similar way, using Lemma 3.9 (ii) and condition (C-χ), we get

$$\frac{\phi(d(x))}{f(\phi(d(x)))}\chi(x)=\frac{\phi(d(x))}{f(\phi(d(x)))d^2(x)}\chi(x)\,d^2(x)=\frac{\phi(d(x))}{f(\phi(d(x)))(\int_{\phi(d(x))}^\infty\frac{ds}{\sqrt{2F(s)}})^2}\chi(x)\,d^2(x)\to 0\quad\text{as } d(x)\to 0. \qquad\square$$

The next lemma will be useful in the proof of Theorem 6.2, and hence Theorem 2.4.

Lemma 3.12. *Let f satisfy conditions* (f-1) *and* (f-3). *Then:*
(i) *Given any $\kappa>0$, there are positive constants t_κ and c_κ such that $f(\kappa t)\geq c_\kappa f(t)$ for all $t>t_\kappa$.*
(ii) *If, in addition,* (f-θ) *holds, then given $\varrho>1$, there are constants $\delta_\varrho>0$ and $c_\varrho>0$ such that $\phi(\varrho t)\geq c_\varrho\phi(t)$ for all $0<t<\delta_\varrho$.*

We should point out that the constants c_κ and δ_κ in Lemma 3.12 (i) depend on the parameter α in condition(f-3), while the constants c_ϱ and δ_ϱ Lemma 3.12 (ii) also depend on θ and ℓ_θ in condition (f-θ). See [35, Lemmas 2.12, 2.13 and 2.15].

The following condition which holds for any odd function f that satisfies (f-1) will be needed in one of our existence results:

(f-4) $\lim_{t\to-\infty}f(t)=-\infty$.

4 Existence

We start this section with a result that shows the existence of supersolutions of (1.1) in balls $B\Subset\Omega$ of suitably small radii.

Lemma 4.1. *Suppose that assumptions* (H-1), (f-1), (f-2) *are satisfied. Suppose also $h\in C(\Omega)$. If $B\Subset\Omega$ is a ball of sufficiently small radius, then there exists a supersolution $v\in C(B)$:*

$$\begin{cases} H[v]\leq f(v)+h(x) & \textit{in } B,\\ v=\infty & \textit{on } \partial B.\end{cases} \tag{4.1}$$

Proof. We may suppose that B is centered at the origin, that is, $B := \{x \in \mathbb{R}^n : |x| < R\}$ with $R < \operatorname{dist}(0, \partial\Omega)$.

Let us start with the case $h \equiv 0$. We look for a solution of the form $w = \phi(\varrho) > 0$ with $\varrho(x) = R^2 - |x|^2$. By (H-1), we have

$$\begin{aligned} H[w] &\le \mathcal{M}^+[w] \\ &\le \mathcal{P}^+_{\lambda,\Lambda}(D^2 w) + \gamma(x)|Dw| \\ &\le f(w)\Big[4\Lambda|x|^2 + \frac{\sqrt{2F(\phi(\varrho))}}{f(\phi(\varrho))}(2n\Lambda + 2\gamma(x)|x|)\Big] \\ &\le f(w)\Big[4\Lambda R^2 + \frac{\sqrt{2F(\phi(\varrho))}}{f(\phi(\varrho))}(2n\Lambda + 2\gamma_0 R)\Big], \end{aligned}$$

where $\gamma_0 = \max_{\overline{B}} \gamma(x)$. Taking a smaller concentric ball B, and using Remark 3.5, we take R sufficiently small so that $4\Lambda R^2 < \frac{1}{2}$ and then

$$\frac{\sqrt{2F(\phi(\varrho))}}{f(\phi(\varrho))}(2n\Lambda + 2\gamma_0 R) < \frac{1}{2}.$$

Thus w solves problem (4.1) with $h \equiv 0$.

Next, we consider an arbitrary $h \in C(\Omega)$. Suppose $h(x) \ge -h_0$ and $\gamma(x) \le \gamma_0$ in B, with $h_0, \gamma_0 \in \mathbb{R}_+$. Set $v(x) := w(x) + C(R^2 - |x|^2)$, with $w(x)$ as considered above and $C > 0$ to be suitably chosen. Then

$$H[v] \le \mathcal{M}^+[w] + C\mathcal{M}^+[R^2 - |x|^2] \le f(w) - 2C(\lambda n - \gamma_0 R).$$

We now shrink R further, if necessary, to have $\lambda n - \gamma_0 R \ge \frac{\lambda n}{2}$ and then we take $C > 0$ large enough in order that $C\lambda n > h_0$. Since $f(w) \le f(v)$, we see that v solves (4.1). □

Next, we show the existence of solutions of problem (1.1) when Ω is replaced by any $\mathcal{O} \Subset \Omega$. To accomplish this, we can apply, on noting that $\gamma, \chi, h \in C(\overline{\mathcal{O}})$, the existence theorem [13, Theorem 1.1].

Theorem 4.2. *Assume* (H-1), (H-2), (f-1), (f-2). *If* $\mathcal{O} \Subset \Omega$, *then problem* (1.1) *with* $\mathcal{O}$ *instead of* Ω *has a solution.*

Proof. For $j \in \mathbb{N}$, let $u_j \in C(\mathcal{O})$ be a solution of

$$\begin{cases} H[u] = f(u) + h(x) & \text{in } \mathcal{O}, \\ u = j & \text{on } \partial\mathcal{O}. \end{cases} \tag{4.2}$$

(See [13, Theorem 1.1].) By Lemma 3.4, the difference $w := u_j - u_{j+1}$ satisfies the differential inequality

$$\mathcal{M}^+[w] \ge f(u_j) - f(u_{j+1}) \quad \text{in } \widetilde{\mathcal{O}} := \mathcal{O} \cap \{u_j > u_{j+1}\}.$$

Therefore, assuming that $\widetilde{\mathcal{O}}$ is non-empty, we would have

$$\mathcal{M}^+[w] \ge 0 \quad \text{in } \widetilde{\mathcal{O}}.$$

By the maximum principle, Proposition 3.3, we have $u_j \le u_{j+1}$ in $\widetilde{\mathcal{O}}$ which is a contradiction. It follows that $\{u_j\}$ is a non-decreasing sequence.

Let $B(z, R) \Subset \mathcal{O}$ be a ball of sufficiently small radius R, and let v be the corresponding solution of (4.1) provided by Lemma 4.1. A further application of the maximum principle shows that $u_j \le v$ in $B(z, R)$. Therefore

$$C_1 \equiv \min_{B(z,\frac{R}{2})} u_1 \le u_j(x) \le \max_{B(z,\frac{R}{2})} v \equiv C_2 \quad \text{in } B(z, \tfrac{R}{2}).$$

Hence $\{u_j\}$ is a sequence of locally uniformly bounded solutions of (4.2), and so via the Harnack inequality also locally equi-Hölder continuous in $B(z, \frac{R}{2})$ (see [7, 44]). By Ascoli–Arzelà and stability results on viscosity solutions (see [12]), and taking into account the monotonicity of the sequence $\{u_j\}$, we have that

$$u(x) := \lim_{j\to\infty} u_j(x)$$

is a continuous viscosity solution of problem (1.1). See, for instance, [20, 46]. □

The following lemma shows that a maximal solution can be found for problem (1.1) if a subsolution exists.

Lemma 4.3. *Assume* (H-1), (H-2), (f-1), (f-2) *and* $h \in C(\Omega)$. *If the problem*

$$\begin{cases} H[w] \geq f(w) + h(x) & \text{in } \Omega, \\ w = \infty & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

has a solution w, *then problem* (1.1) *has a maximal solution* u *such that* $u \geq w$ *in* Ω. *If* $h^+ \equiv 0$, *then the solution is non-negative in* Ω.

Proof. Let $\{\Omega_j\}$ be an exhaustion of Ω by smooth domains so that $\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \Omega$ and $\bigcup_{j\in\mathbb{N}} \Omega_j = \Omega$. For each $j \in \mathbb{N}$ we take the solution $u = u_j$ of problem (1.1) with Ω_j instead of Ω (h is bounded in Ω_j), provided by Theorem 4.2. On a fixed Ω_j the solutions u_k, with $k > j$, are bounded and by the maximum principle $u_k \leq u_j$ on Ω_j. Moreover, $\{u_k\}_{k\geq j}$ is a non-increasing sequence on Ω_j. Let us set

$$u(x) := \lim_{j\to\infty} u_j(x) \quad \text{in } \Omega.$$

Using the subsolution w in Ω of the hypothesis and supersolutions on balls of sufficiently small radius as provided by Lemma 4.1, we can show that the sequence $\{u_j\}$ is uniformly bounded on each domain $\mathcal{O} \Subset \Omega$. Consequently, the sequence $\{u_j\}$ is equi-Hölder continuous. Therefore $u(x)$ is a continuous viscosity solution of equation $H[u] = f(u) + h(x)$ in Ω (see the proof of Theorem 4.2). It is clear from the maximum principle that $u_j \geq w$ in Ω_j for each j. Therefore $u \geq w$ in Ω and

$$\liminf_{d(x)\to 0} u(x) \geq \liminf_{d(x)\to 0} w(x) = \infty.$$

To prove the second assertion in the lemma, suppose $h^+ \equiv 0$ in Ω. Then by conditions (H-1) and (f-1) we note that w^+ is also a solution of (4.3). Therefore, comparison with each u_j as in the above shows that maximal solution u constructed above satisfies $u \geq w^+ \geq 0$ in Ω, which was to be shown. □

We are ready to prove our first existence theorem for problem (1.1), where an auxiliary function g satisfying Dindoš' condition will be employed. As mentioned in the Introduction, we will refer to conditions (f-1)–(f-4), (f-θ), respectively, as conditions (g-1)–(g-4), (g-θ) when we use g instead of f. Similarly, we denote by ϕ_g, θ_g, ℓ_g and $\Theta_g^*(h^\pm)$, respectively, the function ϕ, the numbers θ, $\ell > 1$ in (f-θ) and Remark 3.7, and the quantities $\Theta_g^*(h^\pm)$ in (2.2) when we consider g instead of f.

In the statement of the theorem it will be convenient to use the following notation for any positive constant $\theta_* > 0$:

$$\ell_*(g;\theta_*) := \liminf_{t\to\infty} \frac{g(\theta_* t)}{\theta_* g(t)}. \tag{4.4}$$

We recall that for g that satisfies (g-1) and Dindoš' condition there is $\theta_* > 1$ such that $\ell_*(g;\theta_*) > 1$.

Theorem 4.4. *Assume* (H-1), (H-2), (f-1), (f-2), (f-4), (C-γ), (C-χ). *Suppose there exists a function* $g : \mathbb{R} \to \mathbb{R}$ *satisfying conditions* (g-1), (g-3), *and that there is* $\theta_* > 0$ *such that* $\ell_* := \ell_*(g;\theta_*) > 1$ *and*

$$\rho := \limsup_{t\to\infty} \frac{f(t)}{g(t)} < \lambda\ell_*. \tag{4.5}$$

Then there exists a positive constant $\Theta = \Theta(\lambda, \rho, \theta_*, \ell_*)$ *such that problem* (1.1) *admits a maximal solution provided* $h \in C(\Omega)$ *satisfies* $h^+(x) = O(g(\phi_g(d(x))))$ *as* $d(x) \to 0$ *with* $\Theta_g^*(h^+) < \Theta$. *In fact, we may choose* $\Theta = \theta_*^{-1}(\lambda - \rho\ell_*^{-1})$.

Remark 4.5. If g satisfies Dindoš' condition with $\theta_g > 1$ and $\ell_g > 1$, then (4.5) is equivalent to

$$\rho := \limsup_{t\to\infty} \frac{f(t)}{g(t)} < \infty. \tag{4.6}$$

In fact, suppose that (4.6) holds. In Remark 3.7 we have seen that the function g satisfies (3.7) with $\theta = \theta_g$ and $\ell = \ell_g$ for all $j \in \mathbb{N}$. We choose $j \in \mathbb{N}$ large enough to have $\rho < \lambda\ell_g^j$, obtaining (4.5) with $\ell_* = \ell_g^j$. We also point out that if $\lambda \geq 1$ and $f \leq g$ holds at infinity, then we can take $\theta_* = \ell_* = 1$ in the theorem.

Proof. Let us first observe that (f-2) and (4.5) show that g satisfies the Keller–Osserman condition (g-2). For notational simplicity, let us denote ϕ_g and $\Theta_g^*(h^+)$ by ϕ and Θ^*, respectively. According to Lemma 4.3 it is enough to show that problem (4.3) admits a solution. To this end, we search for a solution in the form

$$w(x) = \theta_*^{-1}\phi_g(d(x)) - A$$

with the constant $A > 0$ to be suitably chosen.

Denoting by G the antiderivative of g vanishing at the origin, direct computation in Ω_μ, where $|Dd| = 1$, yields

$$\begin{aligned} H[w] &\geq \mathcal{M}^-[w] \\ &\geq \mathcal{P}^-_{\lambda,\Lambda}(D^2 w) - \gamma(x)|Dw| - \chi(x)w^+ \\ &\geq \theta_*^{-1}\{\mathcal{P}^-_{\lambda,\Lambda}(\phi''(d)Dd\otimes Dd) + \mathcal{P}^-_{\lambda,\Lambda}(\phi'(d)D^2 d) - \gamma(x)\phi'(d)|Dd| - \chi(x)\phi(d)\} \\ &\geq \theta_*^{-1} g(\phi(d))\Big\{\lambda - \frac{\sqrt{2G(\phi(d))}}{g(\phi(d))}(\mathcal{P}^+_{\lambda,\Lambda}(D^2 d) + \gamma(x)) - \frac{\phi(d)}{g(\phi(d))}\chi(x)\Big\}. \end{aligned} \tag{4.7}$$

In (4.7), given $\varepsilon > 0$, we may pass to a smaller $\mu = \mu(\varepsilon) > 0$, if necessary, to ensure that

$$\frac{\sqrt{2G(\phi(d))}}{g(\phi(d))}|\mathcal{P}^+_{\lambda,\Lambda}(D^2 d)| < \frac{\lambda\varepsilon}{3}, \tag{4.8}$$

$$\frac{\sqrt{2G(\phi(d))}}{g(\phi(d))}\gamma(x) < \frac{\lambda\varepsilon}{3}, \tag{4.9}$$

$$\frac{\phi(d)}{g(\phi(d))}\chi(x) < \frac{\lambda\varepsilon}{3}. \tag{4.10}$$

In (4.8) we used Remark 3.5 and the fact that $d \in C^2(\overline{\Omega})$. In (4.9) and (4.10) we employed Corollary 3.11.

For $\mu > 0$ small enough, the following chain of inequalities holds in Ω_μ:

$$\begin{aligned} H[w] &\geq \lambda\theta_*^{-1} g(\phi(d))(1-\varepsilon) \quad \text{from (4.7), (4.8), (4.9), and (4.10)} \\ &= (f(\theta_*^{-1}\phi(d)) + h^+(x))\frac{\lambda(1-\varepsilon)}{\frac{f(\theta_*^{-1}\phi(d))}{\theta_*^{-1}g(\phi(d))} + \frac{h^+(x)}{\theta_*^{-1}g(\phi(d(x)))}} \\ &= (f(\theta_*^{-1}\phi(d)) + h^+(x))\frac{\lambda(1-\varepsilon)}{\frac{f(\theta_*^{-1}\phi(d))}{g(\theta_*^{-1}\phi(d))}\frac{g(\theta_*^{-1}\phi(d))}{\theta_*^{-1}g(\phi(d))} + \frac{h^+(x)}{\theta_*^{-1}g(\phi(d(x)))}} \\ &\geq (f(\theta_*^{-1}\phi(d)) + h^+(x))\frac{\lambda(1-\varepsilon)}{((\rho+\varepsilon)\ell_*^{-1} + \theta_*(\Theta^*+\varepsilon))(1+\varepsilon)}. \end{aligned} \tag{4.11}$$

To get (4.11), we used (2.2), (4.4), and (4.5). Now, if $\Theta^* < \theta_*^{-1}(\lambda - \rho\ell_*^{-1})$, taking $\varepsilon \in (0,1)$ small enough, by (4.5) we have

$$H[w] \geq f(\theta_*^{-1}\phi(d)) + h(x), \quad x \in \Omega_\mu.$$

Since $f(\theta_*^{-1}\phi(d)) \geq f(\theta_*^{-1}\phi(d) - A) = f(w)$, it follows that w is a subsolution in Ω_μ.

To finish the proof, we will choose $A > 0$ large enough so that w is a subsolution also in $\Omega^{\frac{\mu}{2}}$. For this, let

$$m = \min_{\Omega^{\frac{\mu}{2}}} H[\theta_*^{-1}\phi(d(x))], \quad M_h = \max_{\Omega^{\frac{\mu}{2}}} h(x).$$

By (f-4) we can choose A such that

$$f(\theta_*^{-1}\phi(d) - A) < m - M_h$$

so that in $\Omega^{\frac{\mu}{2}}$, once again by (H-1),

$$H[w] \geq m \geq f(\theta_*^{-1}\phi(d) - A) + M_h \geq f(w) + h(x).$$

This concludes the proof. □

Theorem 2.1 now follows as a direct consequence of Theorem 4.4 as we show below.

Proof of Theorem 2.1. By assumption we note that $\rho \le 1$, where ρ is the constant in (4.5). Since g satisfies condition (g-θ), we use Remark 3.7 to choose $j \in \mathbb{N}$ large enough such that $\lambda \ell_g^j > \rho$. Therefore with $\ell_* = \ell_g^j$ we observe that (4.5) holds. Now we take Θ as in Theorem 4.4 to obtain the conclusion of Theorem 2.1. □

We also make note of the following special case.

Corollary 4.6. *Assume* (H-1), (H-2), (f-1), (f-2), (f-4), (C-γ), (C-χ). *If $f(t) = o(t^p)$ at infinity for some $p > 1$, then for any $h \in C(\Omega)$ such that $h^+(x) = O(d^{-2p/(p-1)})$ as $d(x) \to 0$, problem* (1.1) *admits a maximal solution. In particular, if $f(t) = o(t^p)$ at infinity for any $p > 1$, and $h^+(x) = O(d^{-Q})$ for some constant $Q > 0$ as $d(x) \to 0$, then problem* (1.1) *has a maximal solution.*

Proof. Let $g(t) = |t|^{p-1}t$. Note that in this case (4.4) holds for any $\theta_* > 0$. In particular, the hypothesis on h allows us to choose $\theta_* > 0$ small enough such that $\theta_*^{-1}\lambda > \Theta^*(h^+)$. Since in (4.5) we have $\rho = 0$, and $\Theta_g^*(h^+) < \Theta = \theta_*^{-1}\lambda$, Theorem 4.4 shows that problem (1.1) admits a maximal solution. To prove the second assertion, it suffices to observe that given a constant $Q > 0$ we pick $p > 1$ such that $\frac{2p}{p-1} > Q$. □

Corollary 4.6 shows, for instance, that the following problem admits a solution for any $h \in C(\Omega)$ such that $d(x)^q h^+(x)$ is bounded in Ω for some $q > 0$:

$$\begin{cases} H[u] = u\log^3(|u| + 1) + h(x) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega. \end{cases}$$

Remark 4.7. If f satisfies (f-3) and (4.4) such that $\lambda\ell_* > 1$, then one can use f for g in (4.5).

An alternative existence result can be obtained by imposing indirect control on h through the solvability of (2.3). We readily note that $\psi \ge 0$ in Ω satisfies (2.3) if and only if the following holds:

$$\mathcal{M}_\gamma^+[\psi] := \mathcal{P}_{\lambda,\Lambda}^+(D^2\psi) + \gamma(x)|D\psi| = \mathcal{M}^+[\psi] \le -h^+(x) \quad \text{in } \Omega. \tag{4.12}$$

Remark 4.8. Condition (D-h) is satisfied, for instance, if

$$\mathcal{M}_\gamma^+[v] = -h^+(x), \quad x \in \Omega,$$

admits a solution $v \in C(\Omega)$ that is bounded in Ω from below. In fact, setting $c = \inf_\Omega v$ in this situation, then $\psi := v - c$ is a non-negative solution of (4.12). If, in addition, $\gamma \in L^q(\Omega)$ for some $q > n$ and $h^+ \in L^p(\Omega)$ for some $p > p_0$, then the equation $\mathcal{M}_\gamma^+[v] = -h^+$ has a solution $v \in C(\overline{\Omega})$, in which case condition (D-h) holds. Here, $p_0 = p_0(n, \lambda, \Lambda) \in (\frac{n}{2}, n)$ is the Escauriaza exponent [19] (see also Crandall and Święch, [27]) such that for any $h \in L^p(\Omega)$ with $p > p_0$, solutions of $\mathcal{P}_{\lambda,\Lambda}^+(D^2u) \ge h(x)$ satisfy the maximum principle. We refer the reader to [29, Theorem 7.1] for details.

In the present approach for existence of solutions to (1.1), we can relax condition (C-γ) by requiring the weaker condition (B-γ), while we need to strengthen condition (C-χ) to condition (C-χ_η) as described in the Introduction.

We now have the following existence result.

Theorem 4.9. *Assume* (H-1), (H-2), (B-γ), (C-χ_η), (f-1), (f-2). *If $h \in C(\Omega)$ and* (D-h) *holds with ψ bounded above, then problem* (1.1) *admits a maximal solution.*

Proof. Let v be a solution of

$$\begin{cases} \mathcal{M}^-[v] = f(v) & \text{in } \Omega, \\ v = \infty & \text{on } \partial\Omega. \end{cases}$$

We direct the reader to the Appendix, Lemma A.4, for justification of the existence of such a solution.

We now use condition (D-h) to find a solution $\psi \ge 0$ of (2.3) which is bounded above on Ω. Then $w := v - \psi$ satisfies

$$\mathcal{M}^-[w] \ge \mathcal{M}^-[v] - \mathcal{M}^+[\psi] = f(v) + h^+ \ge f(w) + h \quad \text{in } \Omega.$$

Consequently,

$$H[w] \geq f(w) + h \quad \text{in } \Omega.$$

Since $\psi(x)$ is bounded above and $v(x) \to \infty$, we note that $w(x) \to \infty$ as $x \to \partial\Omega$. Therefore w is a large solution of (4.3). We invoke Lemma 4.3 to conclude the proof. □

The following gives a generalization of the existence results of Alarcón and Quaas [2].

Corollary 4.10. *Assume* (H-1), (H-2), (B-γ), (C-χ_η), (f-1), (f-2). *If $h \in C(\Omega) \cap L^p(\Omega)$ for some $p > p_0$, where p_0 is the Escauriaza exponent, then problem* (1.1) *admits a maximal solution.*

Proof. Note that by Remark 4.8, condition (D-h) holds. Therefore Theorem 4.9 shows that problem (1.1) admits a maximal solution. □

As pointed out in the Introduction condition (2.4) allows us to show existence of solutions to (1.1), and the problem admits a positive solution provided h_η is sufficiently small. This is the content of Theorem 2.2, which we now prove.

Proof of Theorem 2.2. Let η be the Dini continuous function as provided in the hypothesis. Then according to [4], there is a positive function $\psi_1 \in C^\infty(\Omega) \cap C(\overline{\Omega})$ such that

$$\mathcal{M}^+[\psi_1] \leq -\frac{\eta(d(x))}{d^2(x)}, \quad x \in \Omega. \tag{4.13}$$

We refer to the Appendix for how this assertion follows from the work of Ancona in [4]. Therefore (2.4) together with (4.13) shows that condition (D-h) holds. Thus Theorem 4.9 shows that problem (1.1) admits a solution.

Let $v \geq 0$ be a large solution of $\mathcal{M}^-[v] = f(v)$ in Ω. We refer to the Appendix for the existence of such a large solution. By the Harnack inequality[1] (see [38]) we note that actually $v > 0$ in Ω. Let $w := v - c\psi_1$, where $c > 0$ is chosen such that $c \max_\Omega \psi_1 < \min_\Omega v$. It follows that $w > 0$ in Ω and if $h_\eta < c$, then (4.13) implies that

$$\mathcal{M}^+[c\psi_1] \leq -\frac{\eta(d(x))}{d^2(x)} c \leq -\frac{\eta(d(x))}{d^2(x)} h_\eta \leq -h^+(x), \quad x \in \Omega.$$

Consequently, we have

$$H[w] = H[v - c\psi_1] \geq \mathcal{M}^-[v - c\psi_1] \geq \mathcal{M}^-[v] - c\mathcal{M}^+[\psi_1] \geq f(v) + h^+ \geq f(w) + h.$$

We now invoke Lemma 4.3 to conclude that (1.1) has a solution u such that $w \leq u$ in Ω, and thus completing the proof of the theorem. □

5 Boundary asymptotic estimates

In this and the subsequent section, except in Theorem 5.3 and in the Appendix, we will assume that γ and χ in (1.2) and (1.3) are non-negative constants.

Boundary asymptotic estimates of solutions to (1.1) can be derived provided condition (D-h) holds with $\psi(x) = O(\phi(d(x))$ as $d(x) \to 0$. The size of the following quantity will play a critical role in this derivation:

$$\Xi^*(\psi) := \limsup_{d(x)\to 0} \frac{\psi(x)}{\phi(d(x))}.$$

We have the following theorem on asymptotic boundary estimates of solutions to (1.1).

1 Let $\rho > 0$ such that $v > 0$ in Ω_ρ. For a fixed $p > 1$, let $g(t) = f(t)$ for $t \leq \max_{\Omega^\rho} v$ and $g(t) = f(\max_{\Omega^\rho} v)t^p/(\max_{\Omega^\rho v})^p$ for $t > \max_{\Omega^\rho} v$. Then v is a solution of $\mathcal{M}^-[v] = g(v)$ in Ω^ρ. Since g satisfies all the assumptions of [38, Theorem 2.8], we see that $v > 0$ in Ω^ρ by the Harnack inequality.

Theorem 5.1. *Assume* (H-1), (H-2) *with* γ, χ *non-negative constants,* (f-1), (f-2).

(i) *Suppose that* $h \in C(\Omega)$ *satisfies* $\Theta^*(h^-) < \infty$. *If there exists a positive constant* $A^* \geq 1$ *such that*

$$\frac{\Theta^*(h^-)}{A^*} < \liminf_{t\to\infty} \frac{f(A^* t)}{A^* f(t)} - \Lambda, \tag{5.1}$$

then for any continuous subsolution u *of* (1.1) *we have*

$$\limsup_{d(x)\to 0} \frac{u(x)}{\phi(d(x))} \leq A^*. \tag{5.2}$$

(ii) *Suppose that* $h \in C(\Omega)$ *satisfies* (D-h). *If there exists a positive constant* $A_* \leq 1$ *such that*

$$\limsup_{t\to\infty} \frac{f(A_* t)}{A_* f(t)} < \lambda \quad \textit{and} \quad \Xi^*(\psi) < A_*, \tag{5.3}$$

then for any supersolution u *of* (1.1) *we have*

$$0 < A'_* := A_* - \Xi^*(\psi) \leq \liminf_{d(x)\to 0} \frac{u(x)}{\phi(d(x))}.$$

Proof. In the proof that follows it will be convenient to write Θ^* for $\Theta^*(h^-)$. For any $0 < \rho < \mu$ let us consider the following subsets of Ω:

$$\Omega_\rho^- := \{x \in \Omega : \rho < d(x) < \mu\}, \quad \Omega_\rho^+ := \{x \in \Omega : 0 < d(x) < \mu - \rho\}.$$

We start with the proof of (i), which will be carried out by showing that

$$w^*(x) := A^* \phi(d(x) - \rho), \quad x \in \Omega_\rho^-,$$

is a supersolution of the PDE in (1.1) on Ω_ρ^- for all $0 < \rho < \mu$ and sufficiently small $\mu > 0$. Let $u \in C(\Omega)$ be a supersolution of (1.1) which we may suppose $u > 0$ in Ω_ρ^-. Given $\varepsilon > 0$, we use Remark 3.5 to obtain, for a sufficiently small $\mu > 0$,

$$\frac{\sqrt{2F(\phi(d-\rho))}}{f(\phi(d-\rho))}(|\mathcal{P}^-_{\lambda,\Lambda}(D^2 d)| + \gamma) \leq \Lambda\varepsilon.$$

Then, recalling (3.6) and the expression of ϕ', ϕ'', computation shows that

$$\begin{aligned} \mathcal{M}^+[w^*] &\leq \mathcal{P}^-_{\lambda,\Lambda}(D^2 w^*) + \gamma|Dw^*| \\ &\leq A^* f(\phi(d-\rho))\Big[\Lambda + \frac{\sqrt{2F(\phi(d-\rho))}}{f(\phi(d-\rho))}(|\mathcal{P}^-_{\lambda,\Lambda}(D^2 d)| + \gamma)\Big] \\ &\leq A^* f(\phi(d-\rho))\Lambda(1+\varepsilon). \end{aligned} \tag{5.4}$$

On using the assumption that $\Theta^*(h^-) < \infty$, we obtain the following estimates. Let us note that, according to (5.1), we can take μ sufficiently small that

$$\frac{f(A^*\phi(d-\rho))}{A^* f(\phi(d-\rho))} - \frac{h^-(x)}{A^* f(\phi(d-\rho))} > 0 \quad \text{in } \Omega_\mu.$$

By shrinking $\mu > 0$ further, if necessary, the following hold in Ω_μ:

$$\begin{aligned} A^* f(\phi(d-\rho)) &= \frac{f(A^*\phi(d-\rho)) - h^-(x)}{\frac{f(A^*\phi(d-\rho))}{A^* f(\phi(d-\rho))} - \frac{h^-(x)}{A^* f(\phi(d-\rho))}} \\ &\leq \frac{f(A^*\phi(d-\rho)) - h^-(x)}{\frac{f(A^*\phi(d-\rho))}{A^* f(\phi(d-\rho))} - \frac{h^-(x)}{A^* f(\phi(d(x)))}} \\ &\leq \frac{f(A^*\phi(d-\rho)) - h^-(x)}{\frac{f(A^*\phi(d-\rho))}{A^* f(\phi(d-\rho))} - (1+\varepsilon)\frac{\Theta^* + \varepsilon}{A^*}}. \end{aligned} \tag{5.5}$$

By (5.1), we can choose $\mu > 0$ sufficiently small so that

$$\frac{f(A^*\phi(d-\rho))}{A^* f(\phi(d-\rho))} \geq (1+\varepsilon)\Big(\Lambda + \frac{\Theta^* + \varepsilon}{A^*}\Big). \tag{5.6}$$

Using (5.6) in (5.5), we find

$$A^* f(\phi(d-\rho)) \leq \frac{f(A^*\phi(d-\rho)) - h^-(x)}{\Lambda(1+\varepsilon)}. \tag{5.7}$$

Estimating (5.4) with (5.7), we get

$$\mathcal{M}^+[w^*] \leq f(A^*\phi(d-\rho)) - h^-(x) \leq f(A^*\phi(d-\rho)) + h(x), \tag{5.8}$$

which, on recalling (3.6), shows that w^* is a supersolution in Ω_ρ^- of the PDE in (1.1).

Next, let $u \in C(\Omega)$ be a subsolution of (1.1) and set $B^* := \max\{u(x) : d(x) \geq \mu\}$. Then $u \leq w^* + B^*$ on $\partial\Omega_\rho^-$ and we note that the following inequalities hold on Ω_ρ^-:

$$\begin{aligned} H[w^* + B^*] &\leq \mathcal{M}^+[w^*] \\ &\leq f(w^*) + h \quad \text{from (5.8)} \\ &\leq f(w^* + B^*) + h. \end{aligned}$$

By the comparison principle we conclude that $u \leq w^* + B^*$ in Ω_ρ^-. Therefore

$$\frac{u(x)}{\phi(d(x)-\rho)} - \frac{B^*}{\phi(d(x)-\rho)} \leq A^* \quad \text{for } x \in \Omega_\rho^-.$$

On letting $\rho \to 0^+$, we see that the following holds on Ω_μ:

$$\frac{u(x)}{\phi(d(x))} - \frac{B^*}{\phi(d(x))} \leq A^*,$$

from which we get (5.2) upon letting $d(x) \to 0$.

Now we turn to the proof of (ii). For this consider the function

$$w_*(x) := A_*\phi(d(x)+\rho), \quad x \in \Omega_\rho^+$$

and we wish to show that w_* is a subsolution of equation $H[w_*] = f(w_*)$ in Ω_ρ^+ provided μ is sufficiently small. For $\varepsilon \in (0, \frac{1}{2})$ to be chosen small enough, using Remark 3.5, we take a sufficiently small $\mu > 0$ in order that

$$\frac{\sqrt{2F(\phi(d+\rho))}}{f(\phi(d+\rho))}(|\mathcal{P}^+_{\lambda,\Lambda}(D^2 d)| + \gamma) + \frac{\phi(d+\rho)}{f(\phi(d+\rho))} \leq \lambda\varepsilon.$$

Then, on recalling (3.6) and the expression for ϕ', ϕ'', direct computation in Ω_ρ^+ shows that

$$\begin{aligned} H[w_*] &\geq \mathcal{M}^-[w_*] \geq \mathcal{P}^-_{\lambda,\Lambda}(D^2 w_*) - \gamma|Dw_*| - \chi w_* \\ &\geq A_* f(\phi(d+\rho))\Big[\lambda - \frac{\sqrt{2F(\phi(d+\rho))}}{f(\phi(d+\rho))}(|\mathcal{P}^+_{\lambda,\Lambda}(D^2 d)| + \gamma) - \frac{\phi(d+\rho)}{f(\phi(d+\rho))}\chi\Big] \\ &\geq A_*\lambda(1-\varepsilon) f(\phi(d+\rho)). \end{aligned} \tag{5.9}$$

In the above, provided $\mu > 0$ is small enough, by (5.3) we can make

$$\frac{f(A_*\phi(d+\rho))}{A_* f(\phi(d+\rho))} \leq \lambda(1-\varepsilon).$$

Inserting this in (5.9), we get

$$H[w_*] \geq f(A_*\phi(d+\rho)) \geq f(w_*),$$

and this shows that w_* is a subsolution in Ω_ρ^+ as claimed.

Set $B_* := A_*\phi(\mu) > 0$. By the structure conditions,

$$H[w_* - B_*] \geq H[w_*] - \mathcal{M}^+[B_*] \geq f(w_*) \geq f(w_* - B_*)$$

and $w_* - B_* \leq u$ on $\partial\Omega_\rho^+$. On the other hand, considering the function $\psi(x)$ provided by condition (D-d), we also have in Ω_ρ^+

$$H[u + \psi] \leq H[u] + \mathcal{M}^+[\psi] \leq f(u) + h(x) - h^+(x) \leq f(u).$$

Moreover, since $\psi \geq 0$ we have $w_* - B_* \leq u + \psi$ on $\partial\Omega_\rho^+$. Therefore, by the comparison principle (see Proposition 3.3 and Lemma 3.4) we find

$$w_* - B_* \leq u + \psi \quad \text{in } \Omega_\rho^+,$$

and therefore we have

$$A_* \leq \frac{u(x)}{\phi(d(x) + \rho)} + \frac{\psi(x) + B_*}{\phi(d(x) + \rho)} \quad \text{for } x \in \Omega_\rho^+.$$

On letting $\rho \to 0^+$, we see that the following holds on Ω_μ:

$$A_* \leq \frac{u(x)}{\phi(d(x))} + \frac{\psi(x) + B_*}{\phi(d(x))}.$$

On recalling that $\phi(d(x)) \to \infty$, as $d(x) \to 0$, and using condition (5.3), we get

$$A_* \leq \liminf_{d(x)\to 0} \frac{u(x)}{\phi(d(x))} + \Xi^*(\psi),$$

and this concludes the proof of the second part of the theorem with $A'_* := A_* - \Xi^*(\psi)$. □

If f satisfies Dindoš' condition (f-θ) for some $\theta > 1$, then we can easily choose constants $0 < A_* \leq 1 \leq A^*$ such that (5.1) and (5.3) both hold. This leads to the following corollary.

Corollary 5.2. *Assume* (H-1), (H-2) *with γ, χ non-negative constants,* (f-1), (f-θ) *and assume that $h \in C(\Omega)$ satisfies $\Theta^*(h^-) < \infty$. Then there exist constants $0 < A_* \leq A^* < \infty$ such that if* (D-h) *holds with $\psi \in C(\Omega)$ and $\Xi^*(\psi) < A_*$, then*

$$A_* \leq \liminf_{d(x)\to 0} \frac{u(x)}{\phi(d(x))} \leq \limsup_{d(x)\to 0} \frac{u(x)}{\phi(d(x))} \leq A^* \tag{5.10}$$

for all solutions u of (1.1).

Proof. By hypothesis, f satisfies condition (f-θ) for some $\theta > 1$. We now take the smallest $j \in \mathbb{N}$, depending on ℓ, λ and Λ, such that

$$\Theta^*(h^-) < \theta^j(\ell^j - \Lambda), \quad \theta^{-j} \leq 1 \quad \text{and} \quad \ell^{-j} < \lambda.$$

Then, recalling Remark 3.7, we see that inequality (5.1) holds with the choice $A^* = \theta^j$. If we now also require $\Xi^*(\psi) < \theta^{-j}$, then both inequalities in (5.3) hold with the choice $A^* = \theta^{-j}$. We invoke Theorem 5.1 (i) and (ii) to complete the proof. □

In addition to requiring condition (D-h) in the statement of Theorem 4.9, we also needed the solution ψ of (2.3) to be bounded on Ω from above. Thanks to Theorem 5.1 (i), we can now relax this restriction as we now show.

Theorem 5.3. *Assume* (H-1), (H-2) *with γ, χ satisfying* (C-γ) *and* (C-χ_η), *respectively,* (f-1) *and* (f-θ). *There exists a positive constant $\Xi = \Xi(\theta, \ell, \lambda)$ such that if condition* (D-h) *holds with a solution ψ such that $\Xi^*(\psi) < \Xi$, then problem* (1.1) *admits a maximal solution.*

Proof. By Lemma A.4 of the Appendix let v be a large solution of $\mathcal{M}^-[v] = f(v)$ in Ω. Consider the function $w = v - \psi$ with ψ such that $\Xi^*(\psi) < \Xi$. Here $\Xi > 0$ is a constant to be suitably chosen soon. As in the proof of Theorem 4.9, we can show that $\mathcal{M}^-[w] \geq f(w) + h$.

It remains to prove that $w(x) \to \infty$ as $d(x) \to 0$, and then invoke Lemma 4.1 in order to complete the proof. To this end, we use (f-θ) to find $j \in \mathbb{N}$ large enough (see Remark 3.7) in order that $\ell^{-j} < \lambda$, set $A_* = \theta^{-j}$ and $\Xi = \frac{\theta^{-j}}{4}$. If $\Xi^*(\psi) < \Xi$, then the assumption of Theorem 5.1 are satisfied and

$$A'_* = A_* - \Xi^*(\psi) \geq A_* - \Xi = \frac{3}{4}\theta^{-j},$$

so that

$$\liminf_{d(x)\to 0} \frac{v(x)}{\phi(d(x))} \geq A'_* > 0.$$

As a consequence, we have

$$\begin{aligned}\liminf_{d(x)\to 0} \frac{w(x)}{\phi(d(x))} &\geq \liminf_{d(x)\to 0} \frac{v(x)}{\phi(d(x))} - \limsup_{d(x)\to 0} \frac{\psi(x)}{\phi(d(x))}\\ &\geq A'_* - \Xi^*(\psi) \geq \frac{3}{4}\theta^{-j} - \Xi = \frac{\theta^{-j}}{2} > 0,\end{aligned}$$

and $w(x) \to \infty$ when $d(x) \to 0$, as we wanted to show. □

The following is an immediate consequence of Theorem 5.3.

Corollary 5.4. *Assume that H, f and γ, χ satisfy the hypotheses of Theorem 5.3. If, in addition, $\gamma \in L^q(\Omega)$ for some $q > n$ and $h^+ \in L^p(\Omega)$ for some $p > p_0$, then problem* (1.1) *has a maximal solution.*

Proof. It suffices to note, by Remark 4.8, that condition (D-h) holds with non-negative $\psi \in C(\overline{\Omega})$, and hence $\Xi^*(\psi) = 0$. Therefore the conclusion follows from Theorem 5.3. □

We conclude this section with the proof of Theorem 2.3.

Proof of Theorem 2.3. We wish to show that $\Xi^*(\psi)$, where ψ is as in condition (D-h), is sufficiently small as required for the conclusion of Theorem 5.3 to hold. Since all other assumptions of Theorem 5.3 hold, this would establish the desired result. Nevertheless, the assumption concerning the smallness of $\Xi^*(\psi)$ is satisfied since for some Dini continuous function η we have $h^+(x) = O(\eta(d(x))d^{-2}(x))$ as $d(x) \to 0$. Due to the existence result of Ancona already used in Theorem 2.2 and discussed in (A.2) of the Appendix, there is a non-negative $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\mathcal{M}^+(\psi) \leq -\frac{\eta(d(x))}{d^2(x)}.$$

Therefore, on noting that $\Xi^*(\psi) = 0$, the proof is complete. □

6 Uniqueness

In this section we discuss uniqueness of solutions of (1.1) under some additional conditions on H and f. The asymptotic boundary behavior of solutions results of Section 5 will be crucial in developing the uniqueness result. We will assume H satisfies the sub-homogeneity condition (H-3) and we will also suppose that f satisfies the Dindoš' condition (f-θ) for all $\theta > 1$.

Remark 6.1. Suppose that (f-θ) holds for all $\theta > 1$. Then $\theta \mapsto \ell_\theta$ is non-decreasing on $(1, \infty)$. In particular, given $\bar{\theta} > 1$ and $1 < \tau < \ell_{\bar{\theta}}$, there is $\bar{t} := t(\bar{\theta}, \tau)$ such that

$$f(\theta t) \geq \tau f(t) \quad \text{for all } (\theta, t) \in (\bar{\theta}, \infty) \times (\bar{t}, \infty).$$

The next result shows that any two solutions of (1.1) have the same rate of growth near the boundary. This result is based on condition (D-h) and will be proved under the technical assumption that the limit supremum $\Xi^*(\psi)$ is actually a limit which will be denoted by $\Xi(\psi)$:

$$\Xi^*(\psi) := \lim_{d(x)\to 0} \frac{\psi(x)}{\phi(d(x))} \equiv \Xi(\psi).$$

Theorem 6.2. *Let* (H-1)–(H-3) *hold with γ, χ non-negative constants. Assume that f satisfies conditions* (f-1), (f-3) *and* (f-θ) *for all $\theta > 1$. Suppose that $h \in C(\Omega)$ is such that $\Theta^*(h^-) < \infty$ and that* (D-h) *holds for some $\psi \in C(\Omega)$. There is a constant $\Xi > 0$ such that if $\Xi^*(\psi) = \Xi(\psi) < \Xi$, then we have*

$$\lim_{d(x)\to 0} \frac{u(x)}{v(x)} = 1 \tag{6.1}$$

for any solutions u and v of (1.1).

Proof. Let u and v be two solutions of (1.1) in Ω, and let $\Xi > 0$ be the constant in Corollary 5.2. If $\Xi(\psi) < \Xi$, then the estimates (5.10) lead to

$$\theta := \limsup_{d(x)\to 0} \frac{u(x)}{v(x)} < \infty.$$

To prove the theorem, we show that $\theta \le 1$, for then reversing the roles of u and v we will have

$$1 \le \liminf_{d(x)\to 0} \frac{v(x)}{u(x)} \le \limsup_{d(x)\to 0} \frac{v(x)}{u(x)} = \theta \le 1.$$

By way of contradiction, let us suppose $\theta > 1$. Let $\Psi(x) := \psi(x) - \Xi(\psi)\phi(d(x))$. Note that

$$\theta := \limsup_{d(x)\to 0} \frac{u(x)}{v(x) + \Psi(x)}.$$

We fix $\varepsilon_0 > 0$ small enough such that $\theta - \varepsilon_0 > 1$. Let $w := v + \Psi$. Given $\varepsilon \in (0, \varepsilon_0)$, there is $\delta := \delta(\varepsilon) \in (0, \mu)$ such that

$$\begin{cases} \dfrac{u(x)}{w(x)} \le \theta + \varepsilon & \text{if } d(x) \le \delta, \\ \dfrac{u(x_\varepsilon)}{w(x_\varepsilon)} > \theta - \varepsilon & \text{for some } x_\varepsilon \text{ with } d(x_\varepsilon) < \dfrac{2}{3}\delta. \end{cases} \tag{6.2}$$

According to Corollary 5.2, we observe that

$$\liminf_{d(x)\to 0} \frac{w(x)}{\phi(d(x))} \ge A_*$$

for some positive constant A_*. Therefore $w(x) \to \infty$ as $d(x) \to 0$. For the remainder of the proof we will suppose $\mu > 0$ is sufficiently small such that all of the following hold for $x \in \Omega_\mu$:

(i) Since (f-θ) holds for all $\theta > 1$ and recalling Remark 6.1 for $\varepsilon \in (0, \varepsilon_0)$, we have

$$f((\theta - \varepsilon)v(x)) - (\theta - \varepsilon)f(v(x)) \ge mf(v(x)) > 0, \tag{6.3}$$

with $m := \ell_{\theta-\varepsilon_0} - 1 > 0$ (set $\overline{\theta} = \theta - \varepsilon_0$ and apply Remark 6.1 with $\tau := \frac{1}{2}(1 + \ell_{\overline{\theta}})$).

(ii) Use Corollary 5.2 to obtain

$$\frac{1}{2}A_*\phi(d(x)) \le w(x), \quad v(x) \le 2A^*\phi(d(x)). \tag{6.4}$$

(iii) By Lemma 3.12, (ii) there is a positive constant c_ϕ such that

$$\phi(3r) \ge c_\phi \phi(r) \quad \text{for } 0 < r < \mu. \tag{6.5}$$

Let $0 < \varepsilon < \varepsilon_0$ be fixed, but arbitrary and let $\delta = \delta(\varepsilon)$ be the corresponding positive number such that (6.2) holds.

Let us now consider

$$\mathcal{O} := \{x \in \Omega : u(x) > (\theta - \varepsilon)w(x)\} \cap B(x_\varepsilon, r), \quad \text{where } r := \frac{1}{2}d(x_\varepsilon) < \frac{1}{3}\delta,$$

so that

$$r \le d(x) \le 3r \quad \text{for all } x \in \mathcal{O} \subseteq \Omega_\delta. \tag{6.6}$$

Let us note that, for sufficiently small $\mu > 0$, see (4.7),

$$\mathcal{M}^-[\phi(d(x))] > 0, \quad x \in \Omega_\mu.$$

By (H-1) and (H-3) we have the following on Ω_μ:

$$\begin{aligned} H[(\theta - \varepsilon)w] &\le (\theta - \varepsilon)H[v] + (\theta - \varepsilon)\mathcal{M}^+[\Psi] \\ &\le (\theta - \varepsilon)H[v] + (\theta - \varepsilon)\mathcal{M}^+[\psi] - (\theta - \varepsilon)\Xi(\psi)\mathcal{M}^-[\phi(d(x))] \\ &= (\theta - \varepsilon)f(v) - (\theta - \varepsilon)h^-. \end{aligned} \tag{6.7}$$

By appealing to Lemma 3.12 (i) and Lemma 3.9, respectively, and by shrinking μ if necessary, we can suppose that both of the following hold for $0 < r < \mu$, and some positive constant c_f, not necessarily the same:

$$f\Big(\frac{1}{2}A_* c_\phi \phi(r)\Big) \geq c_f f(\phi(r)) \tag{6.8}$$

and

$$f(\phi(r)) = \frac{1}{2r^2}\Bigg(\int_{\phi(r)}^{\infty} \frac{ds}{\sqrt{F(s)}}\Bigg)^2 f(\phi(r)) \geq c_f \frac{\phi(r)}{2r^2}. \tag{6.9}$$

From the monotonicity of the functions f, and ϕ (non-increasing), we find that for all $x \in \mathcal{O} \subseteq \Omega_\mu$,

$$\begin{aligned} f(v(x)) &\geq f\Big(\frac{1}{2}A_*\phi(d(x))\Big) && \text{(by (6.4))} \\ &\geq f\Big(\frac{1}{2}A_*\phi(3r)\Big) \geq f\Big(\frac{1}{2}A_* c_\phi \phi(r)\Big) && \text{(by (6.5))} \\ &\geq c_f f(\phi(r)) && \text{(by (6.8)).} \end{aligned} \tag{6.10}$$

On noting (6.7) and the fact that $H[u] \geq f(u) + h$, we invoke Lemma 3.4 to find that the following hold on $\mathcal{O}$:

$$\begin{aligned} \mathcal{M}_y^+[u - (\theta - \varepsilon)w] &= \mathcal{M}^+[u - (\theta - \varepsilon)w] \\ &\geq (f(u) - (\theta - \varepsilon)f(v)) + h^+ + (\theta - \varepsilon - 1)h^- \\ &\geq f((\theta - \varepsilon)v) - (\theta - \varepsilon)f(v) \\ &\geq mf(v) \quad \text{(by (6.3)).} \end{aligned} \tag{6.11}$$

Using (6.10) in (6.11) and recalling (6.6), we find that

$$\mathcal{M}_y^+[u - (\theta - \varepsilon)w] \geq mc_f f(\phi(r)), \quad x \in \mathcal{O}. \tag{6.12}$$

We now observe that for all $y \in \overline{B}(x_\varepsilon, r)$ we have

$$\begin{aligned} f(\phi(r)) &\geq \frac{c_f}{2}\frac{\phi(r)}{r^2} && \text{(by (6.9))} \\ &\geq \frac{c_f}{2}\frac{\phi(d(y))}{r^2} && \text{(by (6.6))} \\ &\geq \frac{c_f}{4A^*}\frac{w(y)}{r^2} && \text{(by (6.4)).} \end{aligned} \tag{6.13}$$

Therefore, from (6.12) and (6.13) we conclude

$$\mathcal{M}_y^+[u - (\theta - \varepsilon)w] \geq \frac{C}{r^2} w(y) \quad \text{on } \mathcal{O} \text{ and for all } y \in \overline{B}(x_\varepsilon, r). \tag{6.14}$$

In (6.14) we have set $C := \frac{1}{4A^*} m c_f^2$ which, we should note, is independent of ε.

For arbitrary, but fixed $y \in \overline{B}(x_\varepsilon, r)$ we consider the following auxiliary function:

$$z(x) := aCw(y)\Big(1 - \frac{|x - x_\varepsilon|^2}{r^2}\Big).$$

Since $z(x)$ is concave and smooth, choosing $0 < a \leq (2(\Lambda n + b\mu))^{-1}$ we have

$$\begin{aligned} \mathcal{M}_y^-[z] &= -\Lambda \Delta z - b|Dz| \\ &\geq -\Lambda \|\Delta z\|_{L^\infty(\overline{\Omega})} - b\|Dz\|_{L^\infty(\overline{\Omega})} \\ &\geq -2aC[\Lambda n + b\mu]\frac{w(y)}{r^2} \\ &\geq -\frac{C}{r^2} w(y). \end{aligned} \tag{6.15}$$

From (6.14), (6.15) and (3.3) the following inequality holds on $\mathcal{O}$:

$$\mathcal{M}_y^+[u - (\theta - \varepsilon)w + z] \geq M_y^+[u - (\theta - \varepsilon)w] + M_y^-[z] \geq 0 \quad \text{for all } y \in \overline{B}(x_\varepsilon, r).$$

By the Alexandroff–Bakelman–Pucci maximum principle (see Proposition 3.3), we find that there is $y_\varepsilon \in \partial\mathcal{O}$ such that

$$u(x_\varepsilon) - (\theta - \varepsilon)w(x_\varepsilon) + z(x_\varepsilon) \le [u(y_\varepsilon) - (\theta - \varepsilon)w(y_\varepsilon) + z(y_\varepsilon)]^+. \tag{6.16}$$

We infer that $y_\varepsilon \in \partial B(x_\varepsilon, r)$. In fact, supposing the contrary $y_\varepsilon \in B(x_\varepsilon, r)$, then $u(y_\varepsilon) = (\theta - \varepsilon)w(y_\varepsilon)$, and hence we would have a contradiction:

$$z(x_\varepsilon) < u(x_\varepsilon) - (\theta - \varepsilon)w(x_\varepsilon) + z(x_\varepsilon) \le z(y_\varepsilon).$$

Therefore indeed $y_\varepsilon \in \partial B(x_\varepsilon, r)$. Consequently, from (6.16) we obtain

$$aCw(y) = z(x_\varepsilon) \le u(y_\varepsilon) - (\theta - \varepsilon)w(y_\varepsilon) \quad \text{for all } y \in \overline{B}(x_\varepsilon, r). \tag{6.17}$$

Therefore, since $d(y_\varepsilon) < \delta$, we use (6.2) to estimate (6.17) as

$$aCw(y) \le (\theta + \varepsilon)w(y_\varepsilon) - (\theta - \varepsilon)w(y_\varepsilon) = 2\varepsilon w(y_\varepsilon). \tag{6.18}$$

Setting $y = y_\varepsilon$ in (6.18) and rearranging, we find

$$aCw(y_\varepsilon) \le 2\varepsilon w(y_\varepsilon).$$

Taking, $\varepsilon = \min\{\frac{1}{2}\varepsilon_0, \frac{1}{4}aC\}$ leads to a contradiction, thus completing the proof of the theorem. □

Our next result is a uniqueness theorem, and this requires the monotonicity condition (f-m) on f stated in the Introduction.

Theorem 6.3. *Suppose that the assumptions of Theorem 6.2 on H and f are satisfied and that* (f-m) *holds as well. Suppose also that $h \in C(\Omega)$ is such that $\Theta^*(h^-) < \infty$ and* (D-h) *holds with ψ. There is a constant $\Xi > 0$ such that if $\Xi^*(\psi) = \Xi(\psi) < \Xi$, then problem* (1.1) *admits at most one solution.*

Proof. Let $\psi \in C(\Omega)$ be as in (D-h). We suppose that $\Xi(\psi) < \Xi$, where Ξ is the positive constant in Theorem 6.2. Note that Theorem 6.2 applies. Let u be a positive solution of (1.1). According to Theorem 5.3, and by taking Ξ smaller if needed, problem (1.1) admits a maximal solution v and therefore $u \le v$ in Ω. For the purpose of obtaining a contradiction, we suppose that $u(x_0) < v(x_0)$ for some $x_0 \in \Omega$. Let $\varepsilon > 0$ be arbitrary. Consider $w_\varepsilon := (1 + \varepsilon)u + \varepsilon\psi$. Then the set

$$\mathcal{O}_\varepsilon := \{x \in \Omega : w_\varepsilon(x) < v(x)\}$$

contains x_0 for all sufficiently small $\varepsilon > 0$. Given $z \in \Omega$, it is clear that there is ε such that $z \in \mathcal{O}_\varepsilon$. If this is not the case, then $(1 + \varepsilon)u(z) + \varepsilon\psi(z) \ge v(z)$ for all sufficiently small $\varepsilon > 0$. But then this implies $u(z) = v(z)$. On recalling that $u \le v$ in Ω and hence by Lemma 3.4, we have $\mathcal{M}_\gamma^+[v - u] \ge f(v) - f(u) \ge 0$, the Strong Maximum Principle implies that $u = v$ in Ω, contradicting $u(x_0) < v(x_0)$. As a consequence of (6.1) we see that $\mathcal{O}_\varepsilon \Subset \Omega$ for each $\varepsilon > 0$.

By (f-m), we let $t_0 > 0$ such that $\frac{f(t)}{t}$ is non-decreasing for $t \ge t_0$. Let $0 < \varrho < \mu$ such that $u(x) \ge t_0$ for all $x \in \Omega_\varrho$. We note that there is $\varepsilon_0 > 0$ such that

$$\mathcal{O}_{\varepsilon,\varrho} := \mathcal{O}_\varepsilon \cap \Omega_\varrho, \quad 0 < \varepsilon < \varepsilon_0,$$

is non-empty. Otherwise there is a sequence $\{\varepsilon_j\}$ that converges to zero and

$$\mathcal{O}_{\varepsilon_j} \cap \Omega_\varrho = \emptyset.$$

Then $(1 + \varepsilon_j)u(x) + \varepsilon_j\psi \ge v(x)$ on Ω_ϱ, for all j. This implies $u \ge v$ on Ω_ϱ, and hence $u = v$ on Ω_ϱ. But then $\mathcal{M}_\gamma^+[v - u] \ge f(v) - f(u) \ge 0$ in Ω, and $v = u$ on $\partial\Omega^\varrho$ implies, by the Alexandroff–Bakelman–Pucci maximum principle, that $u \ge v$ on Ω^ϱ and therefore $u = v$ in Ω, which contradicts the assumption that $u(x_0) < v(x_0)$.

By (H-1) we see that

$$H[w_\varepsilon] \le H[(1 + \varepsilon)u] + \varepsilon\mathcal{M}^+[\psi]. \tag{6.19}$$

Therefore, by (H-3) and (6.19) we have the following in $\mathcal{O}_{\varepsilon,\varrho}$:

$$\begin{aligned} H[w_\varepsilon] &\le (1 + \varepsilon)H[u] + \varepsilon\mathcal{M}^+[\psi] \le (1 + \varepsilon)f(u) + (1 + \varepsilon)h - \varepsilon h^+ \le f((1 + \varepsilon)u) + h - \varepsilon h^- \\ &\le f(w_\varepsilon) + h \quad \text{(by (f-1))}. \end{aligned}$$

On $\mathcal{O}_{\varepsilon,\varrho}$, $0 < \varepsilon < \varepsilon_0$, we see that

$$\begin{aligned}\mathcal{M}_\gamma^+[v - w_\varepsilon] &\geq f(v) - f(w_\varepsilon) \quad \text{(by Lemma 3.4)}\\ &\geq 0.\end{aligned}$$

Recalling $\mathcal{O}_{\varepsilon,\varrho} \Subset \Omega$, we see that $v - w_\varepsilon \in C(\overline{\mathcal{O}}_{\varepsilon,\varrho})$, and hence by the ABP maximum principle, Proposition 3.3, we have

$$v - w_\varepsilon \leq \max_{\partial\mathcal{O}_{\varepsilon,\varrho}}(v - w_\varepsilon). \tag{6.20}$$

Let us note that $\partial\mathcal{O}_{\varepsilon,\varrho} = (\partial\mathcal{O}_\varepsilon \cap \Omega_\varrho) \cup (\mathcal{O}_\varepsilon \cap \partial\Omega_\varrho)$. We also observe that the maximum on the right of (6.20) cannot occur on $\partial\mathcal{O}_\varepsilon \cap \Omega_\rho$. Therefore for $0 < \varepsilon \leq \varepsilon_0$ the maximum on the right of (6.20) is achieved on $\mathcal{O}_\varepsilon \cap \partial\Omega_\varrho$. Since $\mathcal{O}_\varepsilon \cap \partial\Omega = \emptyset$, this means

$$v - w_\varepsilon \leq \max_{\mathcal{O}_\varepsilon \cap \{x : d(x) = \varrho\}}(v - w_\varepsilon).$$

We let $\varepsilon \to 0^+$ to obtain

$$v - u \leq \max_{d(x)=\varrho}(v - u) := \kappa \quad \text{in } \Omega_\varrho.$$

On noting that

$$\begin{cases}\mathcal{M}_\gamma^+[v - (u + \kappa)] \geq f(v) - f(u) \geq 0 & \text{in } \Omega,\\ v \leq u + \kappa & \text{on } \partial\Omega^\rho,\end{cases}$$

we conclude, by the maximum principle, that $v \leq u + \kappa$ on Ω^ρ. Consequently, $v \leq u + \kappa$ on Ω. Let us also note that $\kappa \geq v(x_0) - u(x_0) > 0$. Since $\mathcal{M}_\gamma^+[v - (u + \kappa)] \geq 0$ in Ω, by the Strong Maximum Principle (see [3] for instance) we conclude $v = u + \kappa$ in Ω. Now we find that

$$\begin{aligned}f(u) + h(x) = H(x, u, Du, D^2u) &= H(x, u, D(u + \kappa), D^2(u + \kappa))\\ &\geq H(x, u + \kappa, D(u + \kappa), D^2(u + \kappa)) \qquad \text{(by (H-1), see Remark 3.1)}\\ &= H(x, v, Dv, D^2v) = f(v) + h(x).\end{aligned}$$

But $u \leq v$ in Ω implies that $f(u) \leq f(v)$. Therefore we have $f(u) = f(v) = f(u + \kappa)$ in Ω. If $x^* \in \Omega_\varrho$, then according to (f-4) we have

$$\frac{f(u(x^*))}{u(x^*)} \leq \frac{f(u(x^*) + \kappa)}{u(x^*) + \kappa}.$$

Therefore, since $f(u(x^*)) = f(u(x^*) + \kappa)$, we have $u(x^*) \geq u(x^*) + \kappa$. Of course, this is not possible. This proves the uniqueness theorem. □

Remark 6.4. Suppose that the assumptions of Theorem 6.3 on H and f hold. Assume that $h^+ \in C(\Omega) \cap L^p(\Omega)$ for some $p > p_0$, where p_0 is the Escauriaza exponent. Then problem (1.1) admits at most one solution. Thus Theorem 6.3 improves the uniqueness result of [2].

Finally, we give the proof of Theorem 2.4.

Proof of Theorem 2.4. Note that all the assumptions of Theorem 6.3, except for the existence of ψ that satisfies (D-h) with $\Xi^*(\psi) = \Xi(\psi)$ small enough, are stated explicitly in the theorem to be proved. As already observed in the proof of Theorem 2.3, this missing assumption is satisfied with $\Xi^*(\psi) = \Xi(\psi) = 0$ by the hypothesis made on h^+. Thus the proof is completed upon invoking Theorem 6.3. □

A Appendix

Given a bounded C^2 domain $\Omega \subseteq \mathbb{R}^n$, in this appendix, we wish to study the existence of non-negative solution to

$$\begin{cases}H[u] = f(u) & \text{in } \Omega,\\ u = \infty & \text{on } \partial\Omega,\end{cases}$$

under the assumption that H satisfies (H-1), (H-2) and f satisfies (f-1), (f-2). The coefficients $\gamma, \chi \in C(\Omega)$ are allowed to be unbounded with their growth near the boundary controlled in accordance with conditions (C-γ) and (C-χ_η).

Our analysis is based on a result due to Ancona [4, Proposition 11, Remark 6.1]. Let η be a Dini continuous function. According to the result in [4] cited above, there is a positive function $\psi \in C^\infty(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} L\psi \leq -\dfrac{\eta(d(x))}{d^2(x)} & \text{in } \Omega, \\ \psi = 0 & \text{on } \Omega, \end{cases} \tag{A.1}$$

for any uniformly elliptic differential operator $Lw := \operatorname{tr}(A(x)D^2w) + b(x) \cdot Dw$ with fixed ellipticity constants $0 < \lambda \leq \Lambda$. Here $A(x) := [a_{ij}(x)]$ with $A(x) \in \mathcal{A}_{\lambda,\Lambda}$, and $|b(x)|$ is continuous on Ω such that $|b(x)|d(x)$ sufficiently small near the boundary $\partial\Omega$ (see [4, Remark 6.1]).

Since $\mathcal{P}^+_{\lambda,\Lambda}(D^2w) := \sup\{\operatorname{tr}(AD^2w) : \lambda I_n \leq A \leq \Lambda I_n\}$, it follows that (A.1) is valid with $\mathcal{M}^+_\gamma$ taking the place of L. Therefore we have

$$\begin{cases} H[\psi] \leq \mathcal{M}^+[\psi] = \mathcal{M}^+_\gamma[\psi] \leq -\dfrac{\eta(d(x))}{d^2(x)} & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.2}$$

As a consequence of (A.2) we may proceed as in [4, Theorem 4] to prove the following existence result. While we only need a special case of the next lemma for our purpose, we present it in the generality stated as it may be of independent interest. Due to the nonlinearity of the underlying operator the proof requires careful consideration.

Lemma A.1. *Let $\vartheta \in C(\Omega)$ with $d^2(x)|\vartheta(x)| \leq \eta(d(x))$ near the boundary $\partial\Omega$ for some Dini continuous function η. For any constant $\kappa \geq 0$ the Dirichlet problem*

$$\begin{cases} H[w] = \vartheta(x) & \text{in } \Omega, \\ w = \kappa & \text{on } \partial\Omega, \end{cases} \tag{A.3}$$

has a viscosity solution $w \in C(\overline{\Omega})$.

Proof. We present the proof in three steps. Throughout the proof we fix an exhaustion $\{\mathcal{O}_j\}$ of Ω by smooth subdomains $\mathcal{O}_j \Subset \Omega$.

Step 1. We first show the existence of a solution $w \in C(\overline{\Omega})$ to

$$\begin{cases} H[w] = -\vartheta^- & \text{in } \Omega, \\ w = \kappa & \text{on } \partial\Omega. \end{cases} \tag{A.4}$$

By hypothesis, there is a positive constant $c = c(\vartheta)$ such that

$$\vartheta^-(x) \leq \frac{c\eta(d(x))}{d^2(x)} \quad \text{in } \Omega.$$

Then, taking (A.2) into consideration, we see that $\mathcal{M}^+[c\psi] \leq -\vartheta^-$ in Ω. Let $z := c\psi + \kappa$. Then $\mathcal{M}^+[z] \leq -\vartheta^-$ in Ω and $z = \kappa$ on $\partial\Omega$. For each j, let $w_j \in C(\overline{\mathcal{O}}_j)$ be the solution of

$$\begin{cases} H[w_j] = -\vartheta^- & \text{in } \mathcal{O}_j, \\ w_j = \kappa & \text{on } \partial\mathcal{O}_j. \end{cases} \tag{A.5}$$

For the existence of a solution to problem (A.5) we refer, for instance, to [13, Theorem 1.1]. Note that $\mathcal{M}^+[\kappa - w_j] \geq \vartheta^-$, and therefore by Proposition 3.3 we see that $w_j \geq \kappa$ in $\mathcal{O}_j$. Similarly, we have $w_j \leq w_{j+1}$ in $\mathcal{O}_j$ for all j. Furthermore, we have

$$\mathcal{M}^+[w_j - z] \geq 0 \quad \text{in } \mathcal{O}_j.$$

Since $z \geq \kappa$ in Ω, again by Proposition 3.3, we have $w_j \leq z$ in $\mathcal{O}_j$. Let

$$w := \lim_{j\to\infty} w_j \quad \text{in } \Omega.$$

Then we note that w is a viscosity solution of $H[w] = -\vartheta^-$ in Ω, and since $\kappa \leq w \leq z$, we conclude that w is a solution of (A.4) as desired.

Step 2. Here we show the existence of a solution $v \in C(\overline{\Omega})$ to

$$\begin{cases} H[v] = \vartheta^+ & \text{in } \Omega, \\ v = \kappa & \text{on } \partial\Omega. \end{cases} \tag{A.6}$$

On recalling the assumptions on ϑ and χ, we note that there is a positive constant $C := C(\vartheta, \kappa, \chi)$ such that

$$\vartheta^+ + \kappa\chi \le \frac{C\eta(d(x))}{d^2(x)} \quad \text{in } \Omega.$$

By (A.2) we conclude that $\mathcal{M}^+[C\psi] \le -\vartheta^+ - \kappa\chi$. Let $z := -C\psi + \kappa$. Then $\mathcal{M}^-[z] \ge \mathcal{M}^-[-Cw] - \kappa\chi = \vartheta^+$ in Ω and $z = \kappa$ on $\partial\Omega$. Thus $H[z] \ge \vartheta^+$ in Ω and $z = \kappa$ on $\partial\Omega$. For each positive integer j, let $v_j \in C(\overline{O}_j)$ such that

$$\begin{cases} H[v_j] = \vartheta^+ & \text{in } \mathcal{O}_j, \\ v_j = \kappa & \text{on } \partial\mathcal{O}_j. \end{cases}$$

Then $\mathcal{M}^+[z - v_j] \ge 0$ in $\mathcal{O}_j$ and since $z \le \kappa$ in Ω, we see from the maximum principle that hence $z \le v_j$ in $\mathcal{O}_j$. Similarly, by the maximum principle $v_j \le \kappa$ in $\mathcal{O}_j$. As before, $v_j \le v_{j+1}$ in $\mathcal{O}_j$. Consequently, the limit v solves (A.6) and $v \ge \kappa$ in Ω.

Step 3. We now show the existence of a solution to (A.3). Let w and v be solutions of (A.4) and (A.6), respectively. We recall that $v \le \kappa \le w$ in Ω. Let $u_j \in C(\overline{\mathcal{O}}_j)$ be a solution of

$$\begin{cases} H[u_j] = \vartheta & \text{in } \mathcal{O}_j, \\ u_j = v & \text{on } \partial\mathcal{O}_j. \end{cases}$$

Then

$$\mathcal{M}^+[v - u_j] \ge \vartheta^- \ge 0 \quad \text{in } \mathcal{O}_j.$$

By the maximum principle we have $v \le u_j$ in Ω. Similarly, $\mathcal{M}^+[u_j - w] \ge \vartheta^+ \ge 0$ in $\mathcal{O}_j$. Moreover, $u_j = v \le w$ on $\partial\mathcal{O}_j$. Therefore $u_j \le w$ in $\mathcal{O}_j$. We also observe that $u_j \le u_{j+1}$ in $\mathcal{O}_j$. Therefore the limit $u := \lim_{j\to\infty} u_j$ satisfies $v \le u \le w$, and hence u satisfies (A.3) as was to be shown. □

One can use Lemma A.1 to establish solvability of the Dirichlet problem where the right-hand side depends on the unknown. To be specific, we consider the following Dirichlet problem:

$$\begin{cases} H[u] = f(u) & \text{in } \Omega, \\ u = \kappa & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{D}_H$}$$

for any non-negative constant κ.

For this we need the following comparison lemma, which is a direct consequence of Proposition 3.3 and Lemma 3.4. For completeness we include the short proof.

Lemma A.2. *Suppose $w, v \in C(\overline{\Omega})$ such that $H[v] \ge f(v)$ and $H[w] \le f(w)$ in Ω. If $v \le w$ on $\partial\Omega$, then $v \le w$ on Ω.*

Proof. Given $\varepsilon > 0$, we note that $v < w + \varepsilon$ in Ω_δ for any sufficiently small $\delta > 0$. Moreover, it follows from condition (H-1) that

$$H[w + \varepsilon] \le H(x, w + \varepsilon, Dw, D^2 w) \le H[w] \le f(w) \le f(w + \varepsilon).$$

Suppose that $\mathcal{O} := \{w + \varepsilon > v\} \cap \Omega^\delta$ is non-empty. By Lemma 3.4, we have $\mathcal{M}^+[w + \varepsilon - v] \ge f(w + \varepsilon) - f(v) \ge 0$ in $\mathcal{O} := \{w + \varepsilon > v\} \cap \Omega^\delta$. Therefore, by ABP, $v \le w + \varepsilon$ in $\mathcal{O}$, which is a contradiction. It follows that $\mathcal{O}$ is empty and $v \le w + \varepsilon$. Since ε and δ are arbitrary, we obtain the desired conclusion. □

Now we can state the following result on the solvability of problem ($\mathcal{D}_H$).

Lemma A.3. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies* (f-1). *Given a constant $\kappa \ge 0$, problem* ($\mathcal{D}_H$) *has a solution $u \in C(\overline{\Omega})$.*

Proof. We recall from (3.6) that $H[\kappa] \leq \mathcal{M}^+[\kappa] = 0$. Let $u_0 \in C(\overline{\Omega})$ such that

$$\begin{cases} H[u_0] = f(\kappa) & \text{in } \Omega, \\ u_0 = \kappa & \text{on } \partial\Omega. \end{cases}$$

The existence of such solutions follows from Lemma A.1. Since $\mathcal{M}^+[u_0 - \kappa] \geq 0$ in Ω, by Lemma A.2 we find that $u_0 \leq \kappa$ in Ω.

For $j \geq 1$, let $u_j \in C(\overline{\mathcal{O}}_j)$ be a solution of

$$\begin{cases} H[u_j] = f(u_j) & \text{in } \mathcal{O}_j, \\ u_j = u_0 & \text{on } \partial\mathcal{O}_j. \end{cases} \tag{A.7}$$

Existence of a solution to problem (A.7) follows, for instance, from [13, Theorem 1.1]. Since $\mathcal{M}^+[u_j - \kappa] \geq 0$ in Ω, we proceed as in the above to conclude that $u_j \leq \kappa$ in Ω. Consequently,

$$\mathcal{M}^+[u_0 - u_j] \geq f(\kappa) - f(u_j) \geq 0 \quad \text{in } \mathcal{O}_j.$$

Therefore, again by the comparison principle we find that $u_0 \leq u_j$ in $\mathcal{O}_j$ for all j. Similarly, it follows that $u_j \leq u_{j+1}$ in $\mathcal{O}_j$. Thus in summary, we have shown that

$$u_0 \leq u_j \leq u_{j+1} \leq \kappa \quad \text{in } \mathcal{O}_j \text{ for all } j = 1, 2, \ldots.$$

By Ascoli–Arzelá and the stability of viscosity solutions we conclude that $u := \lim u_j$ is a solution of $(\mathcal{D}_H)$. □

Lemma A.4. *Let f satisfy conditions* (f-1) *and* (f-2). *The following problem has a solution:*

$$\begin{cases} H[u] = f(u) & \textit{in } \Omega, \\ u = \infty & \textit{on } \partial\Omega. \end{cases} \tag{A.8}$$

Proof. Now that we know problem $(\mathcal{D}_H)$ is solvable for any constant $\kappa \geq 0$, we can proceed as in the proof of Theorem 4.2 to show that problem (A.8) has a solution $u \in C(\Omega)$. Since $f(0) = 0$, we note that $u \geq 0$ in Ω. □

Funding: Vicenţiu D. Rădulescu was supported by the Slovenian Research Agency grants P1-0292, J1-8131, J1-7025, N1-0064, and N1-0083. He also acknowledges the support through a grant of the Romanian Ministry of Research and Innovation, CNCS-UEFISCDI, project number PN-III-P4-ID-PCE-2016-0130, within PNCDI III.

References

[1] N. Abatangelo, Very large solutions for the fractional Laplacian: Towards a fractional Keller–Osserman condition, *Adv. Nonlinear Anal.* **6** (2017), no. 4, 383–405.

[2] S. Alarcón and A. Quaas, Large viscosity solutions for some fully nonlinear equations, *NoDEA Nonlinear Differential Equations Appl.* **20** (2013), no. 4, 1453–1472.

[3] M. E. Amendola, L. Rossi and A. Vitolo, Harnack inequalities and ABP estimates for nonlinear second-order elliptic equations in unbounded domains, *Abstr. Appl. Anal.* **2008** (2008), Article ID 178534.

[4] A. Ancona, On strong barriers and an inequality of Hardy for domains in $\mathbf{R}^n$, *J. Lond. Math. Soc. (2)* **34** (1986), no. 2, 274–290.

[5] C. Bandle and M. Marcus, "Large" solutions of semilinear elliptic equations: Existence, uniqueness and asymptotic behaviour, *J. Anal. Math.* **58** (1992), 9–24.

[6] L. Caffarelli, M. G. Crandall, M. Kocan and A. Święch, On viscosity solutions of fully nonlinear equations with measurable ingredients, *Comm. Pure Appl. Math.* **49** (1996), no. 4, 365–397.

[7] L. A. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, *Ann. of Math. (2)* **130** (1989), no. 1, 189–213.

[8] L. A. Caffarelli and X. Cabré, *Fully Nonlinear Elliptic Equations*, Amer. Math. Soc. Colloq. Publ. 43, American Mathematical Society, Providence, 1995.

[9] I. Capuzzo-Dolcetta, F. Leoni and A. Vitolo, The Alexandrov–Bakelman–Pucci weak maximum principle for fully nonlinear equations in unbounded domains, *Comm. Partial Differential Equations* **30** (2005), no. 10–12, 1863–1881.

[10] O. Costin and L. Dupaigne, Boundary blow-up solutions in the unit ball: Asymptotics, uniqueness and symmetry, *J. Differential Equations* **249** (2010), no. 4, 931–964.

[11] O. Costin, L. Dupaigne and O. Goubet, Uniqueness of large solutions, *J. Math. Anal. Appl.* **395** (2012), no. 2, 806–812.

[12] M. G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), no. 1, 1–67.

[13] M. G. Crandall, M. Kocan, P. L. Lions and A. Święch, Existence results for boundary problems for uniformly elliptic and parabolic fully nonlinear equations, *Electron. J. Differential Equations* **1999** (1999), Paper No. 24.

[14] F. Da Lio and B. Sirakov, Symmetry results for viscosity solutions of fully nonlinear uniformly elliptic equations, *J. Eur. Math. Soc. (JEMS)* **9** (2007), no. 2, 317–330.

[15] G. Díaz and R. Letelier, Explosive solutions of quasilinear elliptic equations: Existence and uniqueness, *Nonlinear Anal.* **20** (1993), no. 2, 97–125.

[16] M. Dindoš, Large solutions for Yamabe and similar problems on domains in Riemannian manifolds, *Trans. Amer. Math. Soc.* **363** (2011), no. 10, 5131–5178.

[17] S. Dumont, L. Dupaigne, O. Goubet and V. Rădulescu, Back to the Keller–Osserman condition for boundary blow-up solutions, *Adv. Nonlinear Stud.* **7** (2007), no. 2, 271–298.

[18] L. Dupaigne, M. Ghergu and V. Rădulescu, Lane–Emden–Fowler equations with convection and singular potential, *J. Math. Pures Appl. (9)* **87** (2007), no. 6, 563–581.

[19] L. Escauriaza, $W^{2,n}$ a priori estimates for solutions to fully nonlinear equations, *Indiana Univ. Math. J.* **42** (1993), no. 2, 413–423.

[20] M. J. Esteban, P. L. Felmer and A. Quaas, Superlinear elliptic equation for fully nonlinear operators without growth restrictions for the data, *Proc. Edinb. Math. Soc. (2)* **53** (2010), no. 1, 125–141.

[21] G. Galise and A. Vitolo, Viscosity solutions of uniformly elliptic equations without boundary and growth conditions at infinity, *Int. J. Differ. Equ.* **2011** (2011), Article ID 453727.

[22] J. García-Melián, Uniqueness of positive solutions for a boundary blow-up problem, *J. Math. Anal. Appl.* **360** (2009), no. 2, 530–536.

[23] J. García-Melián, Large solutions for an elliptic equation with a nonhomogeneous term, *J. Math. Anal. Appl.* **434** (2016), no. 1, 872–881.

[24] M. Ghergu and V. D. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*, Oxford Lecture Ser. Math. Appl. 37, Clarendon Press, Oxford, 2008.

[25] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Grundlehren Math. Wiss. 224, Springer, Berlin, 1977.

[26] F. Gladiali and G. Porru, Estimates for explosive solutions to p-Laplace equations, in: *Progress in Partial Differential Equations. Vol. 1* (Pont-à-Mousson 1997), Pitman Res. Notes Math. Ser. 383, Longman, Harlow (1998), 117–127.

[27] J. A. Goldstein, An appreciation of my teacher, M. M. Rao, in: *Stochastic Processes and Functional Analysis*, Lecture Notes Pure Appl. Math. 238, Dekker, New York (2004), 3–5.

[28] J. B. Keller, On solutions of $\Delta u = f(u)$, *Comm. Pure Appl. Math.* **10** (1957), 503–510.

[29] S. Koike and A. Święch, Weak Harnack inequality for fully nonlinear uniformly elliptic PDE with unbounded ingredients, *J. Math. Soc. Japan* **61** (2009), no. 3, 723–755.

[30] J. López-Gómez and L. Maire, Uniqueness of large positive solutions, *Z. Angew. Math. Phys.* **68** (2017), no. 4, Article ID 86.

[31] M. Marcus and L. Véron, Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997), no. 2, 237–274.

[32] M. Marcus and L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: The subcritical case, *Arch. Ration. Mech. Anal.* **144** (1998), no. 3, 201–231.

[33] M. Marcus and L. Véron, The boundary trace and generalized boundary value problem for semilinear elliptic equations with coercive absorption, *Comm. Pure Appl. Math.* **56** (2003), no. 6, 689–731.

[34] J. H. Michael, A general theory for linear elliptic partial differential equations, *J. Differential Equations* **23** (1977), no. 1, 1–29.

[35] A. Mohammed and G. Porru, Harnack inequality for non-divergence structure semilinear elliptic equations, *Adv. Nonlinear Anal.* (2016), DOI 10.1515/anona-2016-0050.

[36] A. Mohammed and G. Porru, Large solutions for non-divergence structure equations with singular lower order terms, *Nonlinear Anal. Real World Appl.* **35** (2017), 470–482.

[37] A. Mohammed and G. Porru, Large solutions to non-divergence structure semilinear elliptic equations with inhomogeneous term, *Adv. Nonlinear Anal.* (2017), DOI 10.1515/anona-2017-0065.

[38] A. Mohammed, G. Porru and A. Vitolo, Harnack inequality for nonlinear elliptic equations with strong absorption, *J. Differential Equations* **263** (2017), no. 10, 6821–6843.

[39] A. Mohammed and A. Vitolo, Large solutions of fully nonlinear equations: Existence and uniqueness, preprint.

[40] R. Osserman, On the inequality $\Delta u \geq f(u)$, *Pacific J. Math.* **7** (1957), 1641–1647.

[41] V. D. Rădulescu, Singular phenomena in nonlinear elliptic problems: From blow-up boundary solutions to equations with singular nonlinearities, Handbook of Differential Equations: Stationary Partial Differential Equations. Vol. IV, Handb. Differ. Equ., Elsevier, Amsterdam (2007), 485–593.
[42] D. Repovš, Asymptotics for singular solutions of quasilinear elliptic equations with an absorption term, *J. Math. Anal. Appl.* **395** (2012), no. 1, 78–85.
[43] A. Święch, $W^{1,p}$-interior estimates for solutions of fully nonlinear, uniformly elliptic equations, *Adv. Differential Equations* **2** (1997), no. 6, 1005–1027.
[44] N. S. Trudinger, Comparison principles and pointwise estimates for viscosity solutions of nonlinear elliptic equations, *Rev. Mat. Iberoam.* **4** (1988), no. 3–4, 453–468.
[45] L. Véron, Semilinear elliptic equations with uniform blow-up on the boundary, *J. Anal. Math.* **59** (1992), 231–250.
[46] A. Vitolo, M. E. Amendola and G. Galise, On the uniqueness of blow-up solutions of fully nonlinear elliptic equations, *Discrete Contin. Dyn. Syst.* **2013** (2013), 771–780.
[47] Z. Zhang, A boundary blow-up elliptic problem with an inhomogeneous term, *Nonlinear Anal.* **68** (2008), no. 11, 3428–3438.