

ICM11

## The consequence of different loading rates in elasto/viscoplasticity

Fabio De Angelis<sup>a\*</sup>, Donato Cancellara<sup>a</sup>, Mariano Modano<sup>a</sup>, Mario Pasquino<sup>a</sup>

<sup>a</sup>*Department of Structural Engineering,  
University of Naples Federico II, via Claudio 21, 80125 Naples, Italy*

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### Abstract

In the present paper computational applications are illustrated with reference to elasto/viscoplastic problems. The influence of different loading programs on the inelastic behaviour of rate-sensitive elasto/viscoplastic materials is illustrated with specific numerical examples. An associated formulation of the evolutive laws is adopted. Different loading procedures are taken into account by considering different values of the loading rates and of the intrinsic properties of the material. A suitable integration scheme is applied and a numerical example is considered by analysing different loading programs. Numerical computations and results are reported which illustrate the rate-dependency of the constitutive model in use. Consequently the significance of the loading program is emphasized with reference to the non-linear response of rate-dependent elasto/viscoplastic materials.

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Selection and peer-review under responsibility of ICM11

*Keywords:* Elasto/viscoplasticity; Finite element method; Numerical procedures.

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### 1. Introduction

In elastoplasticity, the system of variational inequalities is associated with a class of return mapping algorithms based on the generalized mid-point rule. Application of this operator split methodology is based on an elastic prediction and a plastic correction phase, see e.g. Wilkins [1], Krieg and Key [2], Krieg and Krieg [3], Nagtegaal [4], Ortiz and Popov [5], Simo and Taylor [6]. Accuracy analysis and use of generalized midpoint rule algorithms have been analysed in detail by Ortiz and Popov [5].

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\* Corresponding author. Tel.: +39.081.7683735; fax: +39.081.7683332.  
E-mail address: [fabio.deangelis@unina.it](mailto:fabio.deangelis@unina.it)

The need to take into account the numerical integration procedure into the evaluation of the tangent operator was first indicated by Nagtegaal [4] and then developed by Simo and Taylor [6] by the application of a consistent tangent operator which restores the quadratic rate of convergence typical of iterative solution schemes based on Newton's method.

In viscoplasticity, the derivation of a class of return mapping algorithms associated with the system of variational inequalities is still a considerable task relative to the rate-independent behaviour, see e.g. Crisfield [7] and Zienkiewicz and Taylor [8]. In particular Zienkiewicz and Corneau [9] discussed integration procedures and considered time step restrictions for the Euler forward difference method in quasi-static elasto/viscoplasticity. Hughes and Taylor [10] reconsidered the application of implicit methods, by the use of an algorithmic procedure which requires the inversion of a compliance matrix. Integration algorithms for viscoplastic models involving non-smooth yield surfaces are reported by Simo et al. [11], while stability properties of algorithms are investigated in Simo and Govindjee [12]. Integration procedures for viscoplastic models are also found in Ju [13] and in Peric [14], who proposed a perturbation method for the solution of stiff equations arising in low-rate-sensitive materials. A summary of the various generalized mid-point rule algorithms may be found in Croizet et al. [15]. Models including non-linear kinematic hardening behaviour and integration procedures applied for complex material models are analysed by Chaboche and Cailletaud [16]. A comprehensive account can be found e.g. in Simo and Hughes [17].

In the present paper the consequences of the loading rate on the non-linear response of elasto/viscoplastic materials is illustrated and a specific numerical example is detailed. A solution procedure is pursued which may be applied to general cases, see e.g. De Angelis [18]. In order to take into account different loading rates a non-dimensional loading program parameter is introduced which accounts for the velocity of the imposed displacement, the intrinsic properties of the material and the geometry of the problem.

Numerical computations and results for both rate-independent and rate-dependent material models are finally reported in order to illustrate the significance of different loading rates on the non-linear behaviour of elasto/viscoplastic materials.

## 1. Constitutive problem of evolution in elasto/viscoplasticity

Let us consider a body  $B$ , whose reference configuration  $\Omega$  in  $\mathbb{R}_e^n$ ,  $1 \leq n \leq 3$ , defines a bounded region with particles labelled  $\mathbf{x} \in \Omega$ . Let  $T \in \mathbb{R}_+$  be the time interval of interest, while  $\mathbf{V}$  indicates the space of displacements,  $\mathbf{D}$  the strain space and  $\mathbf{S}$  the dual stress space. We denote by  $\mathbf{u}: \Omega \times T \rightarrow \mathbf{V}$  the displacement of particle and by  $\boldsymbol{\sigma}: \Omega \times T \rightarrow \mathbf{S}$  the stress tensor. The strain tensor is defined as  $\boldsymbol{\varepsilon} = \Delta_s(\mathbf{u}): \Omega \times T \rightarrow \mathbf{D}$ , where  $\Delta_s$  is the symmetric part of the gradient. The assumption of an infinitesimal theory with quasi-static deformations is adopted. Consequently the total strain  $\boldsymbol{\varepsilon}$  is additively decomposed into an elastic part  $\boldsymbol{\varepsilon}^e$  and a part  $\boldsymbol{\varepsilon}^{vp}$  where combined plastic and viscous effects are represented, so that  $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^{vp}$ .

Within the framework of the generalized standard material (Halphen and Nguyen [19], Lemaitre and Chaboche [20]) we introduce a dual pair of internal variables, a kinematic one  $\boldsymbol{\alpha} \in \mathbb{R} \times \mathbf{X}$  and the corresponding static one  $\boldsymbol{\chi} \in \mathbb{R} \times \mathbf{X}'$  defined as  $\boldsymbol{\alpha} = (\alpha_{iso}, \boldsymbol{\alpha}_{kin})$  and  $\boldsymbol{\chi} = (\chi_{iso}, \boldsymbol{\chi}_{kin})$  where  $\alpha_{iso} \in \mathbb{R}$  and  $\chi_{iso} \in \mathbb{R}$  model isotropic hardening while  $\boldsymbol{\alpha}_{kin} \in \mathbf{X}$  and  $\boldsymbol{\chi}_{kin} \in \mathbf{X}'$  model kinematic hardening, being  $\mathbf{X}$  and  $\mathbf{X}'$  dual spaces. In case of a linear hardening behaviour static and kinematic internal variables are connected by the hardening relation  $\boldsymbol{\chi} = \mathbf{H} \boldsymbol{\alpha}$ , where  $\mathbf{H}$  denotes the hardening matrix  $\mathbf{H} = \text{diag}[H_{iso}, \mathbf{H}_{kin}]$ .

The closed convex elastic domain in the generalized stress space is expressed as  $C = \{(\boldsymbol{\sigma}, \boldsymbol{\chi}) \in \mathbf{S} \times \mathbb{R} \times \mathbf{X}' : f(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leq 0\}$ , where the zero level set of the convex function  $f: \mathbf{S} \times \mathbb{R} \times \mathbf{X}' \rightarrow \mathbb{R}$  defines the generalized yield criterion of the material by providing the boundary of the elastic domain  $\partial C = \{(\boldsymbol{\sigma}, \boldsymbol{\chi}) \in \mathbf{S} \times \mathbb{R} \times \mathbf{X}' :$

$f(\boldsymbol{\sigma}, \boldsymbol{\chi})=0$ . Constitutive equations in viscoplasticity derive from the optimality conditions of a properly regularized functional representing viscoplastic dissipation. In fact the viscoplastic model problem may be viewed as a regularization with penalization (Yosida [21]) of the plastic problem. As a consequence different specializations of the viscoplastic constitutive relations are obtained by properly specializing the penalty function (see e.g. De Angelis [18]). For instance adopting the Perzyna [22] viscoplastic constitutive model the evolutive equations are expressed as

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}^{vp} &= (1/\eta) \langle \Phi(f(\boldsymbol{\sigma}, \boldsymbol{\chi})) \rangle \partial_{\boldsymbol{\sigma}} f(\boldsymbol{\sigma}, \boldsymbol{\chi}) \\ -\dot{\boldsymbol{\alpha}} &= (1/\eta) \langle \Phi(f(\boldsymbol{\sigma}, \boldsymbol{\chi})) \rangle \partial_{\boldsymbol{\chi}} f(\boldsymbol{\sigma}, \boldsymbol{\chi}) \end{aligned} \tag{1}$$

where  $\eta > 0$  has the meaning of a viscosity coefficient and the MacAuley bracket  $\langle \cdot \rangle$  is defined as  $\langle x \rangle = (x + |x|)/2$ . A standard choice of the flow function for linear viscous effects is  $\Phi(f(\boldsymbol{\sigma}, \boldsymbol{\chi})) = f(\boldsymbol{\sigma}, \boldsymbol{\chi})$ . Other proposed expressions of the flow function for nonlinear viscous effects are reported e.g. in Skrzypek and Hetnarski [23].

A von Mises yield criterion with linear hardening is considered in the form  $f(\boldsymbol{\sigma}, \boldsymbol{\chi}_{kin}, \chi_{iso}) = \|\text{dev } \boldsymbol{\sigma} - \boldsymbol{\chi}_{kin}\| - \kappa(\chi_{iso}) = \|\boldsymbol{\eta}\| - (\sqrt{2/3})(\sigma_{yo} + \chi_{iso}) \leq 0$ , where  $\text{dev } \boldsymbol{\sigma}$  is the stress deviator,  $\boldsymbol{\eta} = \text{dev } \boldsymbol{\sigma} - \boldsymbol{\chi}_{kin}$  is the relative stress,  $\kappa(\chi_{iso}) = (\sqrt{2/3})(\sigma_{yo} + \chi_{iso})$  represents the current radius of the yield surface in the deviatoric plane and  $\sigma_{yo}$  denotes the uniaxial yield stress of the vergin material. In the assumption of linear hardening behaviour, the static internal variable related to isotropic hardening is specified as  $\chi_{iso} = H_{iso} \alpha_{iso}$ , and the dual kinematic internal variable  $\alpha_{iso}$  is represented by the equivalent viscoplastic strain  $e^{vp} = [(\sqrt{2/3}) \|\dot{\boldsymbol{\varepsilon}}^{vp}\| dt$ .

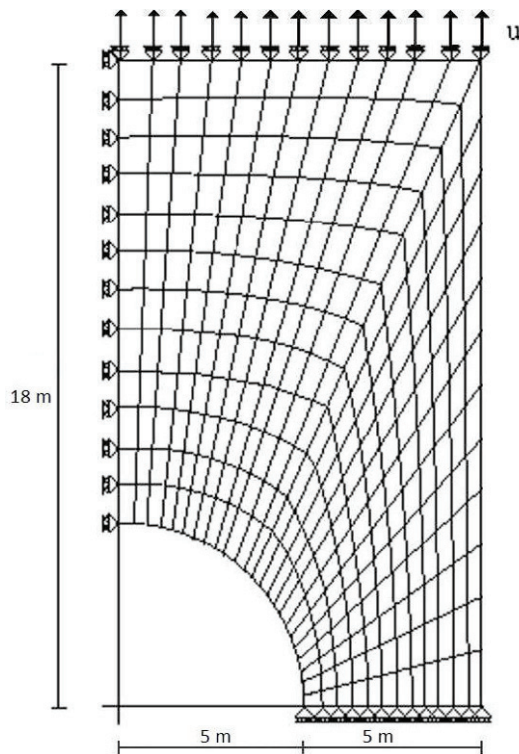


Fig. 1. Perforated strip: geometry of the problem and finite element mesh;

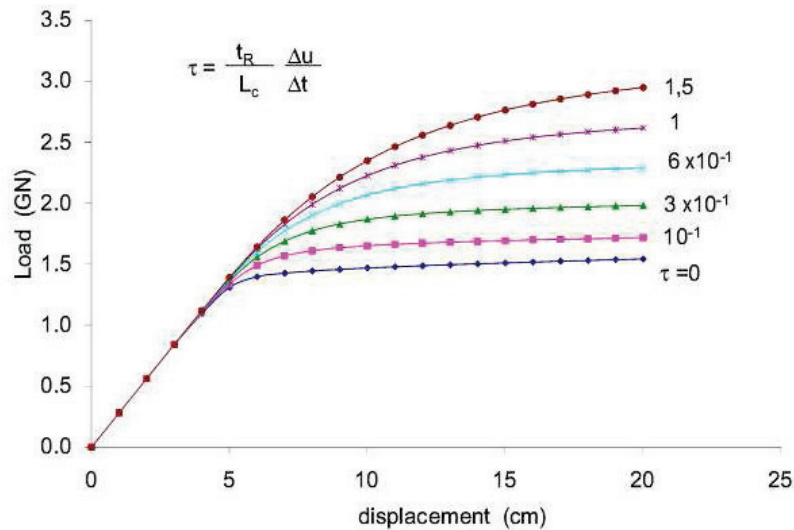


Fig. 2. Perforated strip: load vs. displacement curves;

## 2. Effects of different loading rates on the material behavior: computational applications

The effect of different loading rates on the mechanical response of elasto/viscoplastic material behavior is investigated by considering the boundary value problem of an infinitely long rectangular strip with a circular hole in its axial direction, subjected to increasing extension in a direction perpendicular to the axis of the strip and parallel to one of its sides. For symmetry reasons only one quarter of the section is analysed. Loading is performed by controlling the vertical displacement of the top and bottom boundaries of the strip. The geometry of the problem and the loading conditions are illustrated in fig. 1.

The adopted mesh consists of 325 nodes and 288 elements, in particular a 4-node bilinear isoparametric quadrilateral element has been used. The mechanical properties of the material are: elastic modulus  $E=70 \cdot 10^3$  MPa, Poisson's ratio  $\nu=0.2$ , yield limit  $\sigma_{y0}=243$  MPa, hardening moduli  $H_{iso}=H_{kin}=1.5 \cdot 10^2$  MPa. The imposed displacement is given in single steps  $\Delta u$ , up to a final displacement  $u_{max}$ . In the computational analysis a constitutive model of the Perzyna type is assumed, with linear viscous effects. Furthermore, in order to take into account the rate-dependence of the material behaviour, a non-dimensional loading program parameter  $\tau = (t_R/L_c)(\Delta u/\Delta t)$  is introduced, which accounts for the velocity  $\Delta u/\Delta t$  of the imposed displacement  $\Delta u$ , the intrinsic properties of the material by means of the relaxation time  $t_R = \eta/2G$  and the geometry through a characteristic length of the structural model  $L_c=L/c$ , where  $L$  is the length of the strip,  $c$  is a suitably chosen constant assumed equal to 2900 and  $G$  is the shear modulus. In fig. 2 load versus displacement curves are plotted for different loading rates, where the load is considered to be the sum of the nodal reactions on the bounded upper edge. The rate-independent plastic behaviour is correctly recovered for  $\tau=0$ . Non-null increasing values of the loading program parameter correspond to increasing values of the rate of the prescribed displacement on the bounded upper edge.

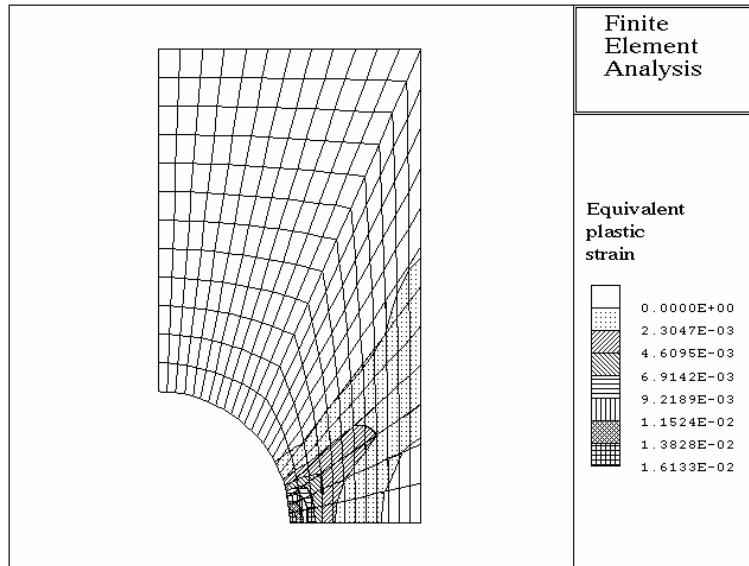


Fig. 3. Contour plot of equivalent plastic strain for  $u = 6$  cm and  $\tau = 0$  ;

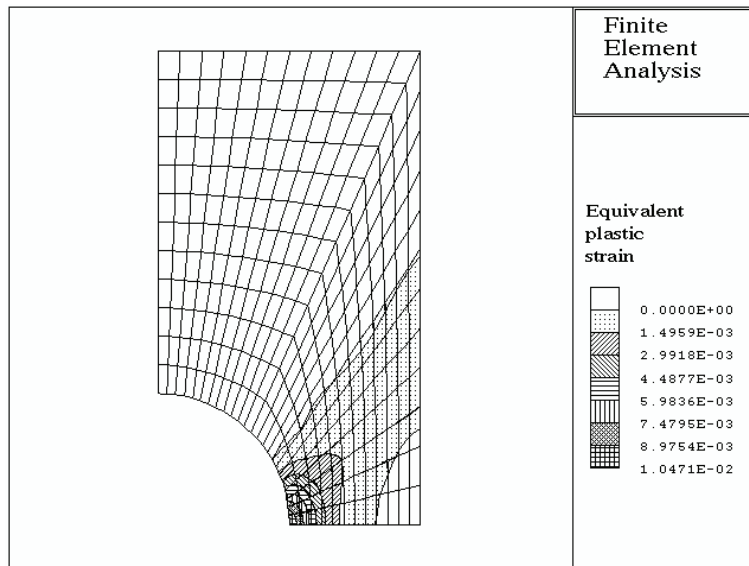


Fig. 4. Contour plot of equivalent plastic strain for  $u = 6$  cm and  $\tau = 0.1$  ;

The evolution of the plastic process shows that the plastic strain evolves from the inner boundary of the circular hole to the right edge of the external boundary (fig. 3). A straightforward evaluation of the rate-dependent material behaviour is readily accomplished by comparing the contour plots of the equivalent plastic strain obtained for a prescribed displacement  $u=6$  cm with different loading programs,

respectively  $\tau=0$  which corresponds to a rate-independent material behavior (see fig. 3), and  $\tau=0.1$  which corresponds to a rate-dependent material behavior (see fig. 4).

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