Research Article

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Harnack inequality for non-divergence structure semi-linear elliptic equations

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Abstract: In this paper we establish a Harnack inequality for non-negative solutions of $Lu = f(u)$ where L is a non-divergence structure uniformly elliptic operator and f is a non-decreasing function that satisfies an appropriate growth conditions at infinity.

Keywords: Non-divergence structure elliptic operator, Harnack inequality, large solution, Alexandroff–Bakelman–Pucci maximum principle

1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. We consider a second order differential operator in non-divergence form given by

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u, \tag{1.1}$$

where the coefficients a_{ij}, b_j and c are assumed to be measurable on Ω. We also suppose that the coefficient matrix $A(x) := (a_{ij}(x))$ is an $n \times n$ real symmetric matrix and that L is uniformly elliptic in Ω in the sense that

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R}^n,$$

for some $0 < \lambda \leq \Lambda$. We will use the notation L_0 for the principal part of L, that is,

$$L_0 u = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Our purpose in this work is to establish a Harnack inequality for non-negative solutions of

$$Lu = f(u) \quad \text{in } \Omega, \tag{1.2}$$

where the non-linearity f is an increasing function on $\mathbb{R}^+ = (0, \infty)$ that satisfies appropriate growth conditions at infinity.

The Harnack inequality is an important tool in the investigation of qualitative properties of solutions to second order elliptic as well as parabolic PDEs. The most recent and remarkable application was demonstrated in Perelman's use of a version of the Harnack inequality for the Ricci flow to settle the century old Poincaré's conjecture which states that any closed 3-manifold with trivial fundamental group is diffeomorphic to S^3 (see [10]). We refer the reader to the paper [7] for a comprehensive discussion of the Harnack inequality and its history.

Let us now review some works on the Harnack inequality on semi-linear elliptic equations that are closely related to the content of this paper. In [4], Finn and McOwen establish a Harnack inequality for non-negative

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solutions of $\Delta u = u^q$ for $q > 1$. In [2], Dindoš extends the results of Finn and McOwen by replacing the nonlinearity $f(t) = t^q$, $q > 1$, by a strictly convex function f that satisfies conditions (f1) and (f2) stated in Section 2 below. The condition (f2), which was introduced by Dindoš in the context of Harnack inequalities, appears to play a particularly crucial role in the study of the Harnack inequality for non-negative solutions of the semilinear equation $\Delta u = f(u)$. Our main objective in this paper is to extend Dindoš' Harnack inequality results to non-negative solutions of (1.2) in Ω, where L is as in (1.1) and f is a continuous function which satisfies conditions that are weaker than some of those used in [2]. Finally, we provide an example to show that Dindoš' condition (f2) cannot be relaxed for the stated Harnack inequality to hold.

The paper is organized as follows. In Section 2 we recall some useful facts and fix notations. An L^∞-bound on non-negative solutions of $Lu = f(u)$ in Ω is established in Section 3. In Section 4 we state and prove our main result: the Harnack inequality for non-negative solutions of $Lu = f(u)$ in Ω. We also demonstrate that Dindoš' condition (f2) cannot be relaxed for this Harnack inequality to hold. Finally, we include an Appendix where we state and prove some useful technical results that are used in the proof of the Harnack inequality.

2 Preliminaries

One of the most celebrated results in the area of second order elliptic and parabolic equations in non-divergence form is the Harnack inequality of Krylov and Safonov for non-negative solutions of $Lu = g$ in Ω, which we recall below, in the form stated in [1, Theorem 4.1]. The reader is referred to [6] for a proof.

Theorem 2.1 (Krylov and Safonov). *Given $z \in \Omega$ and $R > 0$ with $B(z, 2R) \subseteq \Omega$, suppose $g \in L^n(B(z, 2R))$ and that there are constants α, β such that $\|b(x)\| \le \beta$ and $|c(x)| \le \alpha$ for all $x \in B(z, 2R)$. Let $u \in W^{2,n}(B(z, 2R))$ satisfy $u \ge 0$ in $B(z, 2R)$ and $Lu = g$ in $B(z, 2R)$. Then*

$$\sup_{B(z,R)} u \le C\Big(\inf_{B(z,R)} u + R\|g\|_{L^n(B(z,2R))}\Big),$$

where $C := C(n, \Lambda/\lambda, \beta R, \alpha R^2)$.

For the remainder of this paper, we will suppose that there are some positive constants Θ^*, Θ_* such that

$$\|b(x)\| := \Big(\sum_{j=1}^n b_j^2(x)\Big)^{1/2} \le \Theta^*, \quad |c(x)| \le \Theta_* \ (x \in \Omega).$$

Going back to (1.2) we will assume the following on f, defined on $\mathbb{R}^+ := (0, \infty)$:

(f1) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing and continuous such that $f(t) > 0$ for $t > 0$.

(f2) There exists $\theta > 1$ such that

$$\liminf_{t\to\infty} \frac{f(\theta t)}{\theta f(t)} > 1.$$

Condition (f2) was introduced by Dindoš, in [2], in his investigation of the Harnack inequality for non-negative solutions of $\Delta u = f(u)$, where f, in addition to satisfying (f1) and (f2), is assumed to be strictly convex.

Let us record two lemmas that will be useful for us. We provide proofs in the Appendix.

Lemma 2.2. *Suppose f satisfies* (f1) *and* (f2). *There exist constants $\sigma > 0$, $t^* > 0$ and $q > 1$ such that*

$$f(t) \ge \sigma t^q \quad \text{for all } t \ge t^*. \tag{2.1}$$

Let us note the following immediate consequence of Lemma 2.2.

Remark 2.3. Suppose f satisfies (f1) and (f2). Then f satisfies the Keller–Osserman condition, namely

$$\int_1^\infty \frac{ds}{\sqrt{F(s)}} < \infty, \quad \text{where} \quad F(t) := \int_0^t f(s)\, ds. \tag{2.2}$$

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Suppose f satisfies the Keller–Osserman condition (2.2). Then the following function is well defined on $\mathbb{R}^+ = (0, \infty)$:

$$\Psi(t) := \int_t^\infty \frac{ds}{\sqrt{F(s) - F(t)}}. \tag{2.3}$$

The function Ψ is continuous on $(0, \infty)$, and the change of variable $\xi = F(s) - F(t)$ shows that it is also decreasing. Moreover, $\Psi(t) \to 0$ as $t \to \infty$ (see estimate (A.8) in the Appendix where the proof of Lemma 2.4 is given). We will use Φ to denote the inverse of Ψ so that

$$\int_{\Phi(t)}^\infty \frac{ds}{\sqrt{F(s) - F(\Phi(t))}} = t \quad \text{for all } t > 0.$$

The following lemma provides a useful ingredient in the proof of the Harnack inequality for non-negative solutions of (1.2).

Lemma 2.4. *If f satisfies* (f1) *and* (f2), *then*

$$\limsup_{t \to 0^+} \frac{t^2 f(\Phi(t))}{\Phi(t)} < \infty.$$

Let us recall some more results due to Keller [8] and Osserman [11] that will prove to be useful for us later. Suppose f satisfies the Keller–Osserman condition (2.2). Given $R > 0$ and $z \in \mathbb{R}^n$, let $B := B(z, R)$ be the ball of radius R in $\mathbb{R}^n$ centered at z. If κ is a positive constant, then the boundary value problem

$$\begin{cases} \Delta w = \kappa f(w) & \text{in } B, \\ w = \infty & \text{on } \partial B \end{cases} \tag{2.4}$$

admits a positive C^2 radial solution $w(x) = \varphi(|x - z|)$ (see [5, 8, 11]). In fact, for $r = |x - z|$, φ satisfies the ODE

$$(r^{n-1}\varphi')' = r^{n-1}\kappa f(\varphi), \quad 0 \le r < R, \quad \varphi'(0) = 0, \quad \varphi(R) = \infty,$$

and the following inequalities:

$$0 \le \varphi'(r) \le \frac{r}{n}\kappa f(\varphi(r)), \quad 0 \le r < R, \tag{2.5}$$

$$\frac{\kappa}{n} f(\varphi(r)) \le \varphi''(r) \le \kappa f(\varphi(r)), \quad 0 \le r < R. \tag{2.6}$$

From (2.5) it follows that

$$|\nabla w(x)| \le \frac{R}{n}\kappa f(w(x)) \quad x \in B.$$

Moreover, as a consequence of (2.6), we have the following:

$$\sqrt{\frac{\kappa}{n}} R \le \int_{\varphi(0)}^\infty \frac{ds}{\sqrt{2(F(s) - F(\varphi(0)))}} \le \sqrt{\kappa} R.$$

We should point out that φ is a strictly convex function on $[0, R)$, and hence the Hessian $D^2 w(x)$ is positive definite in $B(z, R)$.

For an easy reference we summarize the above results in the following lemma.

Lemma 2.5 (Keller–Osserman). *Suppose f satisfies* (f1) *and the Keller–Osserman condition* (2.2). *Given $\kappa > 0$, $R > 0$ and $z \in \mathbb{R}^n$, there exists a strictly convex C^2 radial function w that satisfies* (2.4) *in $B(z, R)$. Moreover,*

$$\Phi(\sqrt{2\kappa} R) \le w(z) \le \Phi\Big(\sqrt{\frac{2\kappa}{n}} R\Big).$$

Furthermore, since φ is non-decreasing, we have

$$w(x) \ge \Phi(\sqrt{2\kappa} R) \quad \text{for all } x \in B(z, R). \tag{2.7}$$

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3 An L^∞-bound for solutions

We begin this section by recalling an inequality from the theory of matrices. See, for instance, [3] .

Given any $n \times n$ symmetric real matrix A and an $n \times n$ positive semi-definite matrix B the following inequality holds:

$$\lambda_{\min}(A)\,\mathrm{tr}(B) \le \mathrm{tr}(AB) \le \lambda_{\max}(A)\,\mathrm{tr}(B).$$

Here $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are the minimum and maximum eigenvalues of A. In the following lemma we will use the right-hand side inequality to obtain a super-solution for equation (1.2).

Lemma 3.1. *Suppose f satisfies* (f1) *and* (f2). *There exists $R_0 > 0$ such that for any given $z \in \Omega$ and $0 < R < R_0$ with $B := B(z, R) \subseteq \Omega$, the equation*

$$Lw = f(w) \quad \text{in } B \qquad \text{and} \qquad w = \infty \quad \text{on } \partial B$$

admits a super-solution. Moreover, the following estimate holds:

$$\Phi\Big(\frac{R}{\sqrt{2\Lambda}}\Big) \le w(z) \le \Phi\Big(\frac{R}{\sqrt{2n\Lambda}}\Big). \tag{3.1}$$

Proof. We use Lemma 2.2 to choose $t_0 > 0$ such that

$$f(t) \ge 4\Theta_* t \quad \text{for all } t \ge t_0.$$

Let $R_0 > 0$ such that $\Phi(R_0/\sqrt{2\Lambda}) \ge t_0$. Given $0 < R < R_0$ and $z \in \Omega$ such that $B(z, R) \subseteq \Omega$, let w be the strict convex solution of (2.4) in $B(z, R)$, given by Lemma 2.5 with $\kappa := (4\Lambda)^{-1}$. On recalling (2.7), we have $w(x) \ge t_0$ in $B(z, R)$. Then

$$\begin{aligned}
Lw - \frac{1}{2}f(w) &= L_0 w - \frac{1}{4}f(w) + b\cdot\nabla w + c(x)w - \frac{1}{4}f(w)\\
&= \mathrm{tr}\big(A(x)D^2 w(x)\big) - \frac{1}{4}f(w) + b\cdot\nabla w + c(x)w - \frac{1}{4}f(w)\\
&\le \lambda_{\max}(A(x))\,\mathrm{tr}(D^2 w(x)) - \frac{1}{4}f(w) + \Theta^*|\nabla w| + \Theta_* w - \frac{1}{4}f(w)\\
&\le \Lambda\Delta w(x) - \frac{1}{4}f(w) + \frac{\Theta^* R\kappa}{n}f(w) + w\Big(\Theta_* - \frac{f(w)}{4w}\Big)\\
&= \Big(\kappa\Lambda - \frac{1}{4}\Big)f(w) + \frac{\Theta^* R\kappa}{n}f(w) + w\Big(\Theta_* - \frac{f(w)}{4w}\Big).
\end{aligned}$$

Thus,

$$Lw - \frac{1}{2}f(w) \le \frac{\Theta^* R\kappa}{n}f(w) \quad \text{in } B(z, R),\ 0 < R < R_0.$$

Recalling that $\kappa = (4\Lambda)^{-1}$, we further restrict R_0 so that

$$0 < R_0 \le \frac{2n\Lambda}{\Theta^*}.$$

This completes the proof of the lemma. □

One last condition needed on f is the following:
(f3) The function $t \mapsto f(t)/t$ is non-decreasing on $(0, \infty)$.

Remark 3.2. We note that (f3) implies that f is sub-linear near 0. In particular, we have $f(0+) = 0$. Also, any convex function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(0+) = 0$ satisfies (f3).

Given $x \in \Omega$, we will use the notation $d(x)$ for $\mathrm{dist}(x, \partial\Omega)$. The following estimate on solutions of $Lu = f(u)$ on Ω is another crucial result used for deriving the Harnack inequality.

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Theorem 3.3. *Suppose f satisfies* (f1), (f2) *and* (f3). *There exists a non-increasing function* $\eta: \mathbb{R}^+ \to \mathbb{R}^+$ *such that for any non-negative solution u of* $Lu = f(u)$ *in* Ω, *we have*

$$u(x) \le \eta(d(x)), \quad x \in \Omega. \tag{3.2}$$

Proof. Let us fix t_0 such that $f(t) \ge 4\Theta_* t$ for $t \ge t_0$. Moreover, we take $R_0 > 0$ as in the proof of Lemma 3.1 so that $\Phi(R_0/\sqrt{2\Lambda}) \ge t_0$. Let

$$\Omega_{R_0} := \{x \in \Omega : d(x) < R_0\}.$$

To construct a non-increasing function η such that (3.2) holds for any non-negative solution u of $Lu = f(u)$ in Ω, we consider the sets Ω_{R_0} and $\Omega \setminus \Omega_{R_0}$ separately. To this end, let u be any non-negative solution of $Lu = f(u)$ in Ω.

Case 1: Suppose $z \in \Omega_{R_0}$. We consider the ball $B := B(z, R)$ for $0 < R < d(z)$. Let w be the super-solution of $Lu = f(u)$ in $B(z, R)$ given in Lemma 3.1. From (3.1), and recalling that $w(x) \ge w(z)$ for all $x \in B(z, R)$, we see that

$$w(x) \ge \Phi(R/\sqrt{2\Lambda}) \ge \Phi(R_0/\sqrt{2\Lambda}) \ge t_0.$$

Consequently, we see that

$$c(x) - \frac{f(w)}{w} \le 0, \quad x \in B(z, R). \tag{3.3}$$

Let us consider the following differential operator:

$$\widehat{L}v := \sum_{i\,j=1}^{n} a_{ij}(x)v_{x_i x_j} + \sum_{j=1}^{n} b_j(x)v_{x_j} + \Big(c(x) - \frac{f(w)}{w}\Big)v.$$

We note that $\widehat{L}w = Lw - f(w) \le 0$ in $B(z, R)$. Suppose $\mathcal{O} := \{x \in B : u(x) > w(x)\}$ is non-empty. We note $\mathcal{O} \subset\subset B$. As a consequence of (f3) we see that

$$\widehat{L}u := u\Big(\frac{f(u)}{u} - \frac{f(w)}{w}\Big) \ge 0 \quad \text{in } \mathcal{O}.$$

Since $u \le w$ on $\partial\mathcal{O}$, we conclude by the (Alexandroff–Bakelman–Pucci) maximum principle that $u \le w$ on $\mathcal{O}$, which is a contradiction. Therefore, $u \le w$ in $B(z, R)$. In particular,

$$u(z) \le w(z) \le \Phi\Big(\frac{R}{\sqrt{2n\Lambda}}\Big).$$

Letting $R \to d(z)$, we conclude that

$$u(z) \le \Phi\Big(\frac{d(z)}{\sqrt{2n\Lambda}}\Big) \quad \text{for all } z \in \Omega_{R_0}. \tag{3.4}$$

Case 2: Now consider the set $\Omega_{0\epsilon} = \{x \in \Omega : d(x) > R_0 - \epsilon\}$, so that $\Omega \setminus \Omega_{R_0} \subseteq \Omega_{0\epsilon}$. Let

$$v_\epsilon(x) := \Phi\Big(\frac{R_0 - \epsilon}{\sqrt{2n\Lambda}}\Big) \quad x \in \Omega_{0\epsilon}.$$

Since $\partial\Omega_{0\epsilon} \subseteq \mathcal{O}$, estimate (3.4) shows that $u \le v_\epsilon$ on $\partial\Omega_{0\epsilon}$. Since $v_\epsilon \ge t_0$ on $\Omega_{0\epsilon}$, we also note that

$$c(x)v_\epsilon \le 4\Theta_* v_\epsilon \le f(v_\epsilon) \quad \text{in } \Omega_{0\epsilon}.$$

Therefore, estimate (3.3) holds in $\Omega_{0\epsilon}$ with w replaced by v_ϵ. Now if we set

$$\widehat{L}_\epsilon v := \sum_{i\,j=1}^{n} a_{ij}(x)v_{x_i x_j} + \sum_{j=1}^{n} b_j(x)v_{x_j} + \Big(c(x) - \frac{f(v_\epsilon)}{v_\epsilon}\Big)v,$$

then, using $\widehat{L}_\epsilon$ in place of $\widehat{L}$, we can argue as in Case 1 (with v_ϵ in place of w) to show that

$$u(x) \le v_\epsilon(x) = \Phi\Big(\frac{R_0 - \epsilon}{\sqrt{2n\Lambda}}\Big) \quad \text{in } \Omega_{0\epsilon}.$$

We then let $\epsilon \to 0^+$ and conclude $u \le \Phi(R_0/\sqrt{2\Lambda})$ in $\Omega \setminus \Omega_{R_0}$.

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Putting the two cases together, we conclude that $u(x) \le \eta(d(x))$ in Ω, where

$$\eta(t) := \begin{cases} \Phi\Big(\dfrac{t}{\sqrt{2n\Lambda}}\Big) & \text{if } 0 < t < R_0, \\ \Phi\Big(\dfrac{R_0}{\sqrt{2n\Lambda}}\Big) & \text{if } t \ge R_0. \end{cases} \tag{3.5}$$

This completes the proof. □

Remark 3.4. We wish to emphasize that only the bound $c(x) \le \Theta_*$ in Ω from above is used in the proofs of Lemma 3.1 and Theorem 3.3.

4 Harnack inequality for semi-linear equations

Given $x \in \Omega$, we use the notation

$$\delta_j(x) = \frac{j}{3} d(x), \quad j = 1, 2.$$

Theorem 4.1 (Harnack inequality). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and suppose that f satisfies* (f1), (f2) *and* (f3). *Given $z \in \Omega$, there exists a positive constant C, depending only on n, Λ/λ, $\Theta^* d(z)$ and $\Theta_* d^2(z)$ and independent of any non-negative solution u of $Lu = f(u)$ in Ω, such that*

$$\sup_{B(z,d(z)/3)} u \le C \inf_{B(z,d(z)/3)} u. \tag{4.1}$$

Proof. Let $u \ge 0$ be any solution of $Lu = f(u)$ in Ω. Fix $\epsilon > 0$. Then $\widetilde{L}(u + \epsilon) = \epsilon c(x)$ in Ω, where

$$\widetilde{L}w := a_{ij}(x)w_{x_i x_j} + b_i(x)w_{x_i} + (c(x) - V(x))w \quad \text{and} \quad V(x) := \frac{f(u(x))}{u(x) + \epsilon}.$$

Note that $\tilde{c}(x) := c(x) - V(x) \le \Theta_*$ in Ω. Therefore, recalling Remark 3.4, Theorem 3.3 shows that

$$u(x) \le \eta(d(x)), \quad x \in \Omega, \tag{4.2}$$

where $\eta\colon (0, \infty) \to (0, \infty)$ is the non-increasing function defined in (3.5).

Given $z \in \Omega$, we invoke the Harnack inequality of Krylov and Safonov, Theorem 2.1, to obtain

$$\sup_{B(z,\delta_1(z))} (u + \epsilon) \le C\Big(\inf_{B(z,\delta_1(z))} (u + \epsilon) + \epsilon\Theta_* |\Omega|^{1/n} \operatorname{diam}\Omega\Big), \tag{HI$_\epsilon$}$$

where the constant C depends on n, $\frac{\Lambda}{\lambda}$, $\Theta^*\, d(z)$ and $d^2(z)(\max_{x \in B(z,\delta_2(z))} |\tilde{c}(x)|)$.

Since $|\tilde{c}| \le |c| + V$, to show that the constant C in (HI$_\epsilon$) depends only on n, Λ/λ, $\Theta^* d(z)$ and $\Theta_* d^2(z)$, it suffices to show that $d^2(z)M(z)$ is uniformly bounded on Ω, independently of ϵ, where

$$M(z) := \max_{x \in B(z,\delta_2(z))} V(x).$$

To this end, let us begin by noting that condition (f3) and (4.2) show that

$$V(x) = \frac{f(u(x))}{u(x) + \epsilon} \le \frac{f(\eta(d(x)))}{\eta(d(x)) + \epsilon} \le \frac{f(\eta(d(x)))}{\eta(d(x))}, \quad x \in \Omega. \tag{4.3}$$

If $x \in B(z, \delta_2(z))$, then $d(x) \ge d(z)/3 = \delta_1(z)$. Therefore, for any $x \in B(z, \delta_2(z))$, recalling that η is non-increasing and using (f3) once again, we estimate

$$V(x) \le \frac{f(\eta(d(x)))}{\eta(d(x))} \le \frac{f(\eta(\delta_1(z)))}{\eta(\delta_1(z))}. \tag{4.4}$$

Hence, by Lemma 2.4, for any $z \in \Omega$ we have

$$d^2(z)M(z) \le d^2(z)\Big(\max_{x \in B(z,\delta_2(z))} V(x)\Big) \le \frac{d^2(z) f(\eta(\delta_1(z)))}{\eta(\delta_1(z))} = 9\frac{\delta_1^2(z) f(\eta(\delta_1(z)))}{\eta(\delta_1(z))} \le C.$$

Therefore, we have shown that the constant C in (HI$_\epsilon$) is independent of $z \in \Omega$, $\epsilon > 0$ and the solution u. Now we let $\epsilon \to 0$ in (HI$_\epsilon$) to get the desired Harnack inequality. □

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Remark 4.2. Note that $f(0) = 0$ is a necessary condition for the Harnack inequality (4.1) to hold for non-negative solutions of $\Delta u = f(u)$ in Ω. To see this suppose $f(0) > 0$. Then

$$\int_0^\infty \frac{ds}{\sqrt{F(s)}} := R^* < \infty.$$

Thus, $\Psi(0) = R^*$, where Ψ is the function defined in (2.3). Therefore, $\Phi(R^*) = 0$. Now let $z \in \Omega$, and choose $R \geq \sqrt{n}R^*$ such that $\Omega \subseteq B(z, R)$. Let w be the solution of problem (2.4), with $\kappa = 1$, in $B(z, R)$ given in Lemma 2.5. According to Lemma 2.5, we have

$$0 \leq w(z) \leq \Phi\Big(\frac{R}{\sqrt{n}}\Big) \leq \Phi(R^*) = 0.$$

But then

$$\sup_{B(z,d(z)/3)} w \leq C \inf_{B(z,d(z)/3)} w \leq w(z) = 0.$$

Thus, $w \equiv 0$ on $B(z, d(z)/3)$. Of course this is not possible if $f(0) > 0$.

However, we can relax the hypothesis (f3) on f and still obtain the Harnack inequality for solutions of $Lu = f(u)$ in Ω that are uniformly bounded away from zero with the constant in the inequality that now depends on the uniform lower bound. To be precise, we consider the following condition on f:
(f3)* There exists $\tau > 0$ such that $t \mapsto f(t)/t$ is non-decreasing on (τ, ∞).
Given $\delta > 0$, let

$$\mathcal{H}_\delta := \{u \in W^{2,n}(\Omega) : u \text{ is a solution of } Lu = f(u) \text{ in } \Omega \text{ such that } u \geq \delta \text{ on } \Omega\}.$$

Theorem 4.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and suppose that f satisfies* (f1), (f2) *and* (f3)*. *Given $\delta > 0$, there exists a positive constant C, independent of any $u \in \mathcal{H}_\delta$ but depending on $f(\tau)\delta^{-1}$, such that* (4.1) *holds for all $u \in \mathcal{H}_\delta$.*

Proof. Let us first note that condition (f3) has been used only in the proofs of Theorem 3.3 and Theorem 4.1. We thus need to show how these proofs need to be modified when condition (f3)* replaces condition (f3). In the proof of Theorem 3.3, we only need to choose t_0 large enough such that $t_0 > \tau$. The same proof then shows that Theorem 3.3 holds for all solutions of $Lu = f(u)$ in Ω. In the proof of Theorem 4.1, condition (f3)* can be used to estimate $V(x)$ as follows. Let $u \in \mathcal{H}_\delta$ be arbitrary. First we note that estimate (4.3), and hence (4.4) hold whenever $u(x) \geq \tau$. If $u(x) < \tau$, then

$$V(x) \leq \frac{f(u(x))}{u(x) + \epsilon} \leq \frac{f(\tau)}{\delta}.$$

Therefore, for any $z \in \Omega$ we have

$$d^2(z)M(z) \leq d^2(z)\Big(\max_{x \in B(z,\delta_2(z))} V(x)\Big) \leq \max\Big(\frac{d^2(z)f(\eta(\delta_1(z)))}{\eta(\delta_1(z))}, \frac{d^2(z)f(\tau)}{\delta}\Big),$$

which is uniformly bounded in Ω by a constant independent of u and ϵ. As a consequence, the constant C in (HI_ϵ) is independent of u and ϵ as well. Letting $\epsilon \to 0$ in (HI_ϵ) completes the proof. □

Remark 4.4. One may wonder if condition (f2) can be relaxed in Theorem 4.1 and ask whether conditions (f1), (2.2) and (f3) are sufficient for the Harnack inequality of Theorem 4.1 to hold. To address this it will be convenient to first state Theorem 4.1 for the special case when the non-divergence structure operator L reduces to the Laplacian.

Corollary 4.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and suppose f satisfies conditions* (f1), (f2) *and* (f3). *There exists a constant C, depending on dimension only, such that for any non-negative solution of $\Delta u = f(u)$ in Ω, any $z \in \Omega$ and $\rho(z) = \frac{1}{3}dist(z, \partial\Omega)$, we have*

$$\sup_{B(z,\rho(z))} u \leq C \inf_{B(z,\rho(z))} u. \tag{4.5}$$

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We now show that this form of the Harnack inequality fails if Dindoš' condition (f2) does not hold. In other words, we exhibit an example of f that satisfies conditions (f1), (2.2) and (f3) but not (f2) such that inequality (4.5) cannot hold with a constant C, depending on dimension only. For this, we consider the following nonlinearity:

$$f(t) = (t+1)(\log(t+1))^4 + 2(t+1)(\log(t+1))^3, \quad t \geq 0.$$

It is easy to see that f satisfies conditions (f1), (2.2) and (f3), but not (f2). Below we produce a positive solution of $\Delta u = f(u)$ in $B(0, R)$ such that

$$\frac{\sup\{u(x) : x \in B(z, \frac{1}{3}d(z))\}}{\inf\{u(x) : x \in B(z, \frac{1}{3}d(z))\}} \tag{4.6}$$

is unbounded over $z \in B(0, R)$, thus showing that inequality (4.5) cannot hold for a constant C independent of $z \in B(0, R)$.

Since $f(0) = 0$ and f satisfies (f1) as well as the Keller–Osserman condition (2.2), for a given $R > 0$ there exists a radially symmetric non-negative solution u of

$$\Delta u = f(u) \quad \text{in } B(0, R) \subseteq \mathbb{R}^n, \qquad u(x) \to \infty \quad \text{as } |x| \to R.$$

By the strong maximum principle of Vasquez [12], we note that $u(x) > 0$ for $|x| < R$. Let $v(r) = u(|x|)$ for $r = |x|$. Then

$$v'' + \frac{n-1}{r} v' = f(v), \quad v'(0) = 0, \quad v(R) = \infty.$$

If we multiply by v' and integrate over $(0, r)$, we find

$$(v')^2 + 2(n-1)\int_0^r \frac{(v')^2}{s}\, ds = (v+1)^2(\log(v+1))^4 - (v(0)+1)^2(\log(v(0)+1))^4.$$

Since $v'(r) \to \infty$ as $r \to R$ and v' is increasing, we have (see [9, Lemma 2.1])

$$\lim_{r\to R} \frac{\int_0^r (v')^2/s\, ds}{(v')^2} = 0.$$

Therefore, for r near R we have

$$\frac{4}{9}(v+1)^2(\log(v+1))^4 < (v')^2 < (v+1)^2(\log(v+1))^4$$

and

$$\frac{2}{3} < \frac{v'}{(v+1)(\log(v+1))^2} < 1.$$

Integration over (r, R) yields

$$e^{1/(R-r)} - 1 < v(r) < e^{3/[2(R-r)]}.$$

Now, if $r_1 < r_2 < R$ with r_2 close to R, then we have

$$\sup_{(r_1,r_2)} v(r) > \frac{1}{2}e^{1/(R-r_2)}, \quad \inf_{(r_1,r_2)} v(r) \leq e^{3/[2(R-r_1)]}.$$

If $R - r_2 = (R - r_1)/2$, then

$$\sup_{(r_1,r_2)} v(r) \geq \frac{1}{2}e^{2/(R-r_1)}, \quad \inf_{(r_1,r_2)} v(r) \leq e^{3/[2(R-r_1)]}. \tag{4.7}$$

Let $z = ((r_1 + r_2)/2, 0, \dots, 0)$, and consider the ball $B(z, \rho)$ of radius $\rho := (r_2 - r_1)/2$. Notice that

$$\rho = \frac{1}{3}\operatorname{dist}(z, \partial B).$$

Then, from (4.7), we find

$$\frac{\sup_{B(z,\rho)} v(r)}{\inf_{B(z,\rho)} v(r)} \geq \frac{1}{2}\exp\Big(\frac{1}{2(R-r_1)}\Big) = \frac{1}{2}\exp\Big(\frac{1}{8\rho}\Big) = \frac{1}{2}\exp\Big(\frac{3}{8\operatorname{dist}(z, \partial B)}\Big).$$

Therefore, we see that (4.6) becomes arbitrarily large as $d(z) \to 0$.

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A Appendix

Proof of Lemma 2.2. By condition (f2) we pick ϱ such that

$$1 < \varrho < \liminf_{t\to\infty} \frac{f(\theta t)}{\theta f(t)}. \tag{A.1}$$

Then there exists $M_\varrho > 0$ such that

$$f(\theta t) \geq (\varrho\theta) f(t) \quad \text{for all } t \geq M_\varrho. \tag{A.2}$$

By iterating (A.2), we see that

$$f(\theta^k t) \geq (\varrho\theta)^k f(t) \quad t \geq M_\varrho \tag{A.3}$$

for any positive integer k. Thus, for any positive integer k with $t = M_\varrho$, inequality (A.3) becomes

$$f(\theta^k M_\varrho) \geq (\varrho\theta)^k f(M_\varrho).$$

Let $t \geq M_\varrho$, and let $k \in \mathbb{N} \cup \{0\}$ with

$$\theta^k M_\varrho \leq t < \theta^{k+1} M_\varrho. \tag{A.4}$$

Let also

$$q := \frac{\ln(\theta\varrho)}{\ln\theta} = 1 + \frac{\ln\varrho}{\ln\theta}.$$

We note that $q > 1$, and $\varrho\theta = \theta^q$. Using (A.4), we obtain the following chain of inequalities:

$$f(t) \geq f(\theta^k M_\varrho) \geq (\varrho\theta)^k f(M_\varrho) = C(\varrho\theta)^{k+1} = C(\theta^q)^{k+1} = C(\theta^{k+1})^q = C(\theta^{k+1}M_\varrho)^q M_\varrho^{-q} > CM_\varrho^{-q} t^q, \tag{A.5}$$

where $C := \frac{f(M_\varrho)}{\varrho\theta}$. Since $q > 1$, we conclude that f satisfies condition (2.1), thus completing the proof of Lemma 2.2. □

Remark A.1. Suppose f satisfies (f1) and (f2). Given $\tau > 0$, there exists $s_\tau > 0$ such that

$$f(s) \geq \tau s \quad \text{for all } s \geq s_\tau.$$

This follows directly from (A.5).

Proof of Lemma 2.4. We start by making some preparatory observations. Note that

$$F(2t) = \int_0^{2t} f(s)\,ds = 2\int_0^{t} f(2s)\,ds \geq 2\int_0^{t} f(s)\,ds = 2F(t).$$

For $s \geq 2t$ we see that

$$\frac{F(t)}{F(s)} \leq \frac{F(t)}{F(2t)} \leq \frac{1}{2}.$$

Thus, for $s \geq 2t$ we have

$$\sqrt{1 - \frac{F(t)}{F(s)}} \geq \frac{1}{\sqrt{2}}.$$

Using this, we find that

$$\int_{2t}^{\infty} \frac{ds}{\sqrt{F(s) - F(t)}} = \int_{2t}^{\infty} \frac{ds}{\sqrt{F(s)}\sqrt{1 - F(t)/F(s)}} \leq \sqrt{2}\int_{2t}^{\infty} \frac{ds}{\sqrt{F(s)}}. \tag{A.6}$$

By the mean-value theorem, for some c with $t \leq c \leq s$, we have

$$F(s) - F(t) = f(c)(s - t) \geq f(t)(s - t).$$

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Therefore, for $s \geq t > 0$ the following holds:

$$F(s) - F(t) \geq f(t)(s - t). \tag{A.7}$$

Thus, for $t > 0$,

$$\begin{aligned}
\Psi(t) &= \int_t^\infty \frac{ds}{\sqrt{F(s) - F(t)}} \\
&= \int_t^{2t} \frac{ds}{\sqrt{F(s) - F(t)}} + \int_{2t}^\infty \frac{ds}{\sqrt{F(s) - F(t)}} \\
&\leq \int_t^{2t} \frac{ds}{\sqrt{f(t)(s-t)}} + \sqrt{2} \int_{2t}^\infty \frac{ds}{\sqrt{F(s)}} \quad \text{(by (A.6) and (A.7))} \\
&= \frac{1}{\sqrt{f(t)}} \int_0^t \frac{ds}{\sqrt{s}} + \sqrt{2} \int_{2t}^\infty \frac{ds}{\sqrt{F(s)}} \\
&= 2\sqrt{\frac{t}{f(t)}} + \sqrt{2} \int_{2t}^\infty \frac{ds}{\sqrt{F(s)}}.
\end{aligned} \tag{A.8}$$

Next we estimate the last integral in (A.8). With $\varrho > 1$ as chosen in (A.1) and the corresponding positive constant M_ϱ such that (A.2) holds, we recall that (A.3) holds for any positive integer k and all $t \geq M_\varrho$. Now observe that for any $t \geq M_\varrho$,

$$\int_{2t}^\infty \frac{ds}{\sqrt{F(s)}} = 2\int_t^\infty \frac{ds}{\sqrt{F(2s)}} \leq 2\int_t^\infty \frac{ds}{\sqrt{sf(s)}} = 2\sum_{k=0}^\infty \int_{\theta^k t}^{\theta^{k+1} t} \frac{ds}{\sqrt{sf(s)}} = 2\sum_{k=0}^\infty \int_{\theta^k t}^{\theta^{k+1} t} \frac{ds}{\sqrt{s^2(f(s)/s)}}. \tag{A.9}$$

For $\theta^k t \leq s \leq \theta^{k+1} t$, from (A.3) we see that

$$\frac{f(s)}{s} \geq \frac{(\varrho\theta)^k f(t)}{\theta^{k+1} t} = \frac{\varrho^k f(t)}{\theta t}.$$

Using this last inequality in (A.9), we find that

$$\int_{2t}^\infty \frac{ds}{\sqrt{F(s)}} \leq 2\sum_{k=0}^\infty \sqrt{\frac{\theta t}{\varrho^k f(t)}} \int_{\theta^k t}^{\theta^{k+1} t} \frac{ds}{s} = 2\sqrt{\theta}\ln(\theta)\sqrt{\frac{t}{f(t)}} \sum_{k=0}^\infty \left(\frac{1}{\sqrt{\varrho}}\right)^k. \tag{A.10}$$

Combining (A.8) and (A.10), we have shown that there exists a positive constant C, that depends on θ and ϱ only, such that for all $t \geq M_\varrho$,

$$\int_t^\infty \frac{ds}{\sqrt{F(s) - F(t)}}\, ds \leq C\sqrt{\frac{t}{f(t))}} \quad \text{for all } t \geq M_\varrho.$$

Consequently, we have

$$\frac{f(t)}{t}\Psi^2(t) \leq C \quad \text{for all } t \geq M_\varrho. \tag{A.11}$$

Let $t_0 > 0$ be such that $\Phi(t) \geq M_\varrho$ for all $0 < t < t_0$. Then from (A.11), we conclude that there exists $C > 0$ such that for all $0 < t < t_0$,

$$\frac{t^2 f(\Phi(t))}{\Phi(t)} \leq C.$$

This proves Lemma 2.4. □

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