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## Random walk of solitary and shock waves in nonlocal disordered media

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**Abstract.** We study the random walk of solitons and characteristic lines of shock fronts in the presence of disorder for the one-dimensional nonlinear Schrödinger equation in Kerr-like media. We analyze the interplay of nonlocality and randomness, and the way their competition affects strongly coherent nonlinear waves is theoretically and numerically investigated.

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**1. Introduction**

Nonlinear phenomena, as solitons [1, 2] and dissipative and dispersive shock waves (SWs) [3–8], are largely affected by the presence of nonlocality [9–11] and disorder [12–14]. A nonlocal response is present when the light beam induces a refractive index change on a spatial region greater than the beam waist; this effect is typically found in physical systems displaying long-range correlation [15–20]. However, considering a nonlocal response also implies accounting for disorder. Indeed, a large spatial region that interacts with the propagating electromagnetic field in a nonlocal medium commonly also involves material fluctuations. These are due to electronic and thermal effects and their interplay with nonlinearity produces a complex scenario to be investigated. In fact, the way nonlinear coherent phenomena, as SWs and solitons, are affected by randomness is one of the most active research areas in nonlinear optics [21–24]. Disorder frustrates the coherent interaction between light and matter and delays the formation of solitons or SWs. In these respects, the amount of disorder and nonlinearity can be retained as control parameters in ‘phase-diagrams’ [14, 25, 26], identifying regimes in which various phenomena can be observed.

In optics, one-dimensional (1D) propagation in a nonlinear medium is described by the nonlinear Schrödinger equation (NLS) [1]

$$2ik\partial_z A + \partial_x^2 A + 2k^2 \frac{\Delta n}{n_0} A = 0, \quad (1)$$

where  $A$  is the beam envelope normalized such that  $I = |A|^2$  is the optical intensity,  $\Delta n(r) = n_2|A|^2$  is the intensity-dependent component of the refractive index,  $n = n_0 + n_2|A|^2$ .  $z$  is the propagation distance,  $x$  the transverse coordinate,  $k = 2\pi n_0/\lambda$  is the wave vector and  $\lambda$  is the wavelength.

In this paper, we introduce a disorder term in (1) by taking

$$\Delta n(x, z) = n_2|A|^2 + \Delta n_R, \quad (2)$$

where  $\Delta n_R = \Delta n_R^0 V(x, z)$  and  $V(x, z)$  is a Gaussian potential such that  $\langle \Delta n_R(x) \Delta n_R(x') \rangle = (\Delta n_R^0)^2 \delta(x - x')$ . The refractive index perturbation  $\Delta n_R$  is associated to a length scale competing with the scale of the nonlinear interaction between light and matter, as detailed below.

Equation (1) with the disorder term (2) can be normalized by introducing spatial scales weighting the relative roles of disorder, nonlinearity and diffraction. Nonlocality [9, 10, 14, 23] also introduces a spatial scale. The nonlocal response of the medium can be written as a convolution integral of the wave field and a response function  $\chi(x - x')$ :

$$\Delta n(x, z, I) = \int_{-\infty}^{+\infty} dx' \chi(x - x') [n_2 I(x', z) + \Delta n_R(x', z)]. \quad (3)$$

The response function typically depends on a length scale  $L_{\text{nlloc}}$  which gives its spatial extension, being the local limit recovered by  $\chi(x - x') = \delta(x - x')$ . The typical response function has an exponential form  $\chi(x) = e^{-|x|/L_{\text{nlloc}}}/(2L_{\text{nlloc}})$ . In the highly nonlocal limit, the response function can be approximated by a constant,  $\chi(x) = \chi_0$ , with  $\chi_0 = 1/2L_{\text{nlloc}}$  in the exponential case.

### 1.1. The three length scales

When we add disorder and nonlocality in (1), the resulting equation is

$$2ik\partial_z A + \partial_x^2 A + 2k^2 \frac{n_2}{n_0} A \chi * |A|^2 + 2k^2 \frac{\Delta n_R^0}{n_0} A \chi * V = 0, \quad (4)$$

where we have introduced the convolution integral  $D * B = \int dx' D(x - x') B(x')$ .

The *diffraction (or Rayleigh) length*  $L_d$  for a Gaussian beam, propagating along the  $z$ -axis and with a beam waist  $w_0$ , is defined as

$$L_d = kw_0^2. \quad (5)$$

The weight of nonlinearity is measured by the *nonlinear length*  $L_{\text{nl}}$ , which defines the length over which the nonlinear phase shift  $\Delta\phi_{\text{nl}} = 4\pi n_2 I_0 z/\lambda$  is equal to 1:

$$L_{\text{nl}} = \frac{n_0}{2kn_2 I_0} \quad (6)$$

with  $I_0$  the peak intensity.

We define the *randomness length*  $L_R$  as the length over which the phase shift of a beam propagating in an homogeneous medium with refractive index  $\Delta n_R^0$ ,  $\Delta\phi_R = 4\pi \Delta n_R^0 z/\lambda$  is equal to 1:

$$L_R = \frac{n_0}{2k\Delta n_R^0} \quad (7)$$

and it is determined by the strength of the random refractive index perturbation  $\Delta n_R^0$ .

$\Delta n_R^0$  is related to the localization length  $l$ , which scales as  $l \propto 1/(\Delta n_R^0)^{2/3}$  [12, 27], and measures the spatial extension of the disorder induced transversal confinement.

### 1.2. Rescaled nonlinear Schrödinger equation

Scaling the coordinates as  $x \rightarrow w_0 x$ ,  $z \rightarrow 2Lz$  with  $L \equiv \sqrt{L_d L_{\text{nl}}}$ , and by introducing the complex field  $\psi = A/\sqrt{I_0}$ , we obtain from (4)

$$i\varepsilon_1 \partial_z \psi + \varepsilon_1^2 \partial_x^2 \psi \pm \psi K * |\psi|^2 + \varepsilon_2^2 \psi K * V = 0, \quad (8)$$

where

$$\varepsilon_1 = \sqrt{\frac{L_{\text{nl}}}{L_d}} \quad (9)$$

and

$$\varepsilon_2 = \sqrt{\frac{L_{\text{nl}}}{L_{\text{R}}}} \quad (10)$$

are the only relevant quantities, evidencing the relative weight of nonlinearity, randomness and diffraction. The rescaled response function in the exponential case takes the form  $K(x) = 1/(2\sigma) e^{-|x|/\sigma}$ , with  $\sigma = L_{\text{nlloc}}/w_0$ .

In the next section, we show how different regimes can be obtained when varying the length scales  $L_{\text{d}}$ ,  $L_{\text{nl}}$  and  $L_{\text{R}}$ , and correspondingly,  $\epsilon_1$  and  $\epsilon_2$ .

## 2. Ordered case

In an homogeneous sample, the weight of the fluctuations of refractive index  $\Delta n_{\text{R}}^0$  is negligible, hence the randomness length scale is infinite  $L_{\text{R}} \gg L_{\text{nl}}$ ,  $L_{\text{d}}$  or, equivalently,  $\epsilon_2 = 0$ . The rescaled NLS equation becomes

$$i\varepsilon_1 \partial_z \psi + \varepsilon_1^2 \partial_x^2 \psi \pm |\psi|^2 \psi = 0, \quad (11)$$

where we consider the local case  $K(x) = \delta(x)$ .

### 2.1. Solitonic regime

The generation of solitary waves results from the balance between the diffraction and nonlinearity, in this case  $L_{\text{d}} \simeq L_{\text{nl}}$ ,  $\epsilon_1 \simeq 1$ .

Limiting ourselves to the focusing case, equation (8)

$$i\partial_z \psi + \partial_x^2 \psi + |\psi|^2 \psi = 0 \quad (12)$$

admits a localized solution, taking the form  $\psi(x, z) = u(x) e^{i\beta z}$  with

$$u(x) = u_0 \operatorname{sech}(x/W_0), \quad (13)$$

$W_0 = \sqrt{2}/u_0$  and  $\beta = u_0^2/2$ . The condition on the length scales  $L_{\text{d}} = L_{\text{nl}}$  gives ‘the existence curve’,  $u_0 W_0 = \text{const}$  that relates amplitude and waist of soliton. In the 1D case, the localized solution is always stable and no catastrophic collapse events occur. In the nonlocal case,  $u(x)$  is found numerically [14].

### 2.2. Shock waves regime

The SWs generation is a strongly nonlinear phenomenon and has also been studied in the presence of nonlocality [9–11]. To investigate such a regime, we deal with the hydrodynamical approximation [9, 28], when considering a weakly diffractive regime,  $L_{\text{d}} \gg L_{\text{nl}}$  and  $\epsilon_1 \ll 1$ . The evolution of a Gaussian beam  $\psi(x) = u_0(x) e^{i\beta z}$  with  $u_0(x) = e^{-x^2/2}$  is obtained by the Wentzel, Kramers and Brillouin (WKB) transformation

$$\psi(x, z) = \sqrt{\rho(x, z)} \exp[i\phi(x, z)/\epsilon_1], \quad (14)$$

where  $\rho(x, z)$  is the slowly varying beam intensity and  $\phi(x, z)$  its phase. By inserting equation (14) in (11), the following Euler-like equations are obtained:

$$\begin{aligned} O(\epsilon) : \rho_z + \partial_x(\rho v) &= 0, \\ O(1) : \phi_z + v^2 \mp \rho &= 0, \end{aligned} \quad (15)$$

where we have defined the velocity of the fluid  $v \equiv \phi_x$  as the phase chirp and we neglect the ‘quantum’ pressure term,  $\frac{1}{2}(\frac{1}{\sqrt{\rho}}\partial_x^2\sqrt{\rho})$ . We derive the  $O(1)$  equation with respect to  $x$ , and obtain

$$\left(\frac{\partial v}{\partial z} + 2v\frac{\partial v}{\partial x}\right) = \pm\frac{\partial\rho}{\partial x}. \quad (16)$$

At the beginning of the evolution, the strong nonlinearity lets the field intensity  $\rho$  to act as a pressure distribution on the velocity of the ‘optical fluid’. In the focusing case, the pressure profile induces the fluid (the flat phase distribution) to accumulate in the center causing the shock to occur in correspondence of the peak intensity. In the defocusing case, this results in an accumulation of the energy density on the beam sides with the formation of two symmetric shocks, as described below.

### 3. Weakly disordered system

When the disorder is a perturbation,  $L_R \gg L_{nl}, L_d$  and  $\varepsilon_2 \ll \varepsilon_1$ , it is interesting to analyze the way it affects the evolution of solitary and SWs.

#### 3.1. Solitonic regime

We consider a nonlocal nonlinear medium in the presence of disorder. We assume that  $L_R \gg L_{nl}, L_d$  and the diffraction and nonlinearity satisfy the condition for soliton existence:  $L_{nl} \simeq L_d$ . As a consequence, equation (8) becomes

$$i\partial_z\psi + \partial_x^2\psi \pm \psi K * |\psi|^2 + \psi K * \eta = 0, \quad (17)$$

where we introduced the variable  $\eta = \varepsilon_2^2 V(x, z) = (\Delta n_R^0 / |n_2| I_0) V(x, z)$  as a small perturbation. We use a perturbational approach and show that an increasing degree of nonlocality hampers the Brownian motion of self-trapped wave-packets [14]. In fact, the nonlocality acts as a spatial filter on the wave propagation, canceling fluctuations.

Letting  $\psi = \phi \exp(i\beta z)$ , equation (17) is written as

$$i\partial_z\phi + \partial_x^2\phi - \beta\phi + \phi K * |\phi|^2 = is(x, \psi, \psi_x, \psi_{xx}, \dots, z), \quad (18)$$

where  $s$  is taken as a perturbation term, depending on  $\psi$  and its transverse derivative at any order, and  $\beta$  is the nonlinear wave-vector. For  $s = 0$ , the unperturbed solitary wave (SW) is written as

$$\phi_0 = u(x - X + 2\Omega z, \beta) \exp(i\theta - i\Omega x - i\Omega^2 z), \quad (19)$$

where  $X$  is the center of the self-trapped wave,  $\theta$  is the phase and  $2\Omega$  is the velocity.

In the following, we derive dynamical equations for the first order perturbation of these parameters. By letting  $\phi = \phi_0 + \phi_1$ , the linearized evolution equation is

$$\partial_z\phi_1 = \mathcal{L}(\phi_1) + s \quad (20)$$

with

$$\mathcal{L}(\phi_1) = -i\beta\phi_1 + i\phi_{1,xx} + i\phi_0 K * (\phi_0\phi_1^* + \phi_0^*\phi_1) + i\phi_1 K * |\phi_0|^2. \quad (21)$$

The analysis can be limited to SW with  $\Omega = 0$ . The first order perturbation  $\phi_1$  is represented by a small variation of  $X$ ,  $\Omega$ ,  $\theta$  and  $\beta$ :

$$\phi_1 = f_X\delta X + f_\theta\delta\theta + f_\beta\delta\beta + (f_\Omega - Xf_\theta)\delta\Omega, \quad (22)$$

where the auxiliary functions are defined as  $f_\theta = i\phi_0$ ,  $f_\beta = \partial_\beta \phi_0$ ,  $f_X = \partial_X \phi_0$  and  $f_\Omega = -i(x - X)\phi_0$ , and being  $\delta X(z)$ ,  $\delta\theta(z)$ ,  $\delta\beta(z)$  and  $\delta\Omega(z)$  the  $z$ -dependent perturbations to SW parameters. We introduce the adjoint functions  $\hat{f}$  given by  $\hat{f}_\theta = if_\beta$ ,  $\hat{f}_\beta = -if_\theta$ ,  $\hat{f}_\Omega = -if_X$ ,  $\hat{f}_X = if_\Omega$  and such that  $(\hat{f}_a, f_b) = \mathcal{N}_a \delta_{a,b}$  with  $a$  and  $b$  two SW parameters ( $X$ ,  $\Omega$ ,  $\theta$  or  $\beta$ ) and introducing the scalar product  $(\hat{f}, f) = \Re \int (\hat{f})^* f dx$ . It is  $\mathcal{N}_\theta = \mathcal{N}_\beta = (1/2)(dP/d\beta) = P'/2$  and  $\mathcal{N}_X = \mathcal{N}_\Omega = (1/2)P$ , with  $P = (\phi_0, \phi_0)$  the SW power.

Being  $\hat{\mathcal{L}}(if) = -i\mathcal{L}(f)$ , the projection over the adjoint functions of equation (20) gives

$$\begin{aligned} \delta\dot{\theta} - X\delta\dot{\Omega} &= \delta\beta + \frac{2S_\theta}{P'}, & \delta\dot{\beta} &= \frac{2S_\beta}{P'}, \\ \delta\dot{X} &= -2\delta\Omega + \frac{2S_X}{P}, & \delta\dot{\Omega} &= \frac{2S_\Omega}{P}, \end{aligned} \quad (23)$$

where  $S_\alpha = (\hat{f}_\alpha, s)$ , and the dot is the derivative with respect to  $z$ .

Equations (23) hold for any  $s$ ; for the random perturbation in (17), we have

$$s = i\eta u e^{i\theta}, \quad (24)$$

where  $\langle \eta(x)\eta(x') \rangle = \eta_N^2 \delta(x - x')$  and  $\eta_N = \langle (\Delta n_R^0)^2 \rangle^{1/2} / |n_2| I_0$ . Equations (23) become

$$\begin{aligned} \delta\dot{\theta} &= X\delta\Omega + \delta\beta + \frac{1}{P'} \frac{d}{d\beta} \int u^2(x - X)(K * f) dx, \\ \delta\dot{\beta} &= 0, & \delta\dot{X} &= -2\Omega, \end{aligned} \quad (25)$$

$$\delta\dot{\Omega} = -\frac{2}{P} \int u(x - X)u_x(x - X)(K * f) dx,$$

from which

$$\delta\Omega(z) = -\frac{2}{P} \int_0^t \int \int u(x - X)u_x(x - X)K(x - x')f(x', z') dx' dx dz' \quad (26)$$

and averaging over disorder,

$$\langle \delta\Omega(z)\delta\Omega(z') \rangle = \frac{4\langle f \rangle}{P^2} C \min(z, z'), \quad (27)$$

where

$$C = \int \int \int u(x_1 - X)u(x_2 - X)u(x_3 - X)u_x(x_3 - X)K(x_1 - x_2)K(x_3 - x_2) dx_1 dx_2 dx_3. \quad (28)$$

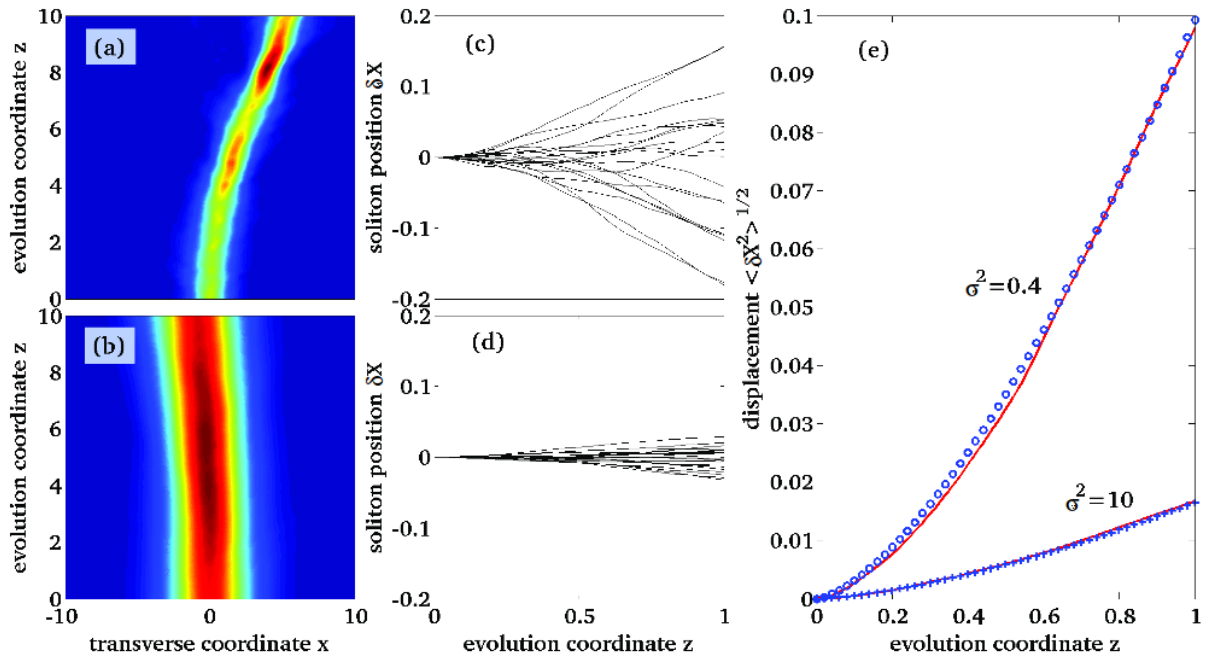
From (23), we obtain the mean position

$$\delta X(z) = -2 \int_0^z \delta\Omega(z') dz' \quad (29)$$

from which  $\langle \delta X \rangle = 0$  and

$$\langle \delta X(z)^2 \rangle = 4 \int_0^z \int_0^z \langle \Omega(z_1)\Omega(z_2) \rangle dz_1 dz_2 = \frac{16\eta_N^2 C}{3P^2} z^3. \quad (30)$$

Equation (30) is the nonlocal expression of the Gordon–Haus effect, that describes the Brownian motion of coherently amplified solitons [13]. In equation (30), the quantity  $C$  measures the effect



**Figure 1.** (a) Typical dynamics of a solitary wave from the numerical solutions of (17) in the presence of randomness and exponential low nonlocality ( $P = 6$ ,  $\sigma^2 = 0.4$ ,  $\eta_N = 0.01$ ) and (b) exponential high nonlocality ( $P = 6$ ,  $\sigma^2 = 10$ ,  $\eta_N = 0.01$ ); center of mass trajectories of the same bound-state ( $P = 6$ ) for 20 disorder realizations ( $\eta_N = 0.01$ ) for (c) small nonlocality ( $\sigma^2 = 0.4$ ) and, (d), high nonlocality ( $\sigma^2 = 10$ ). (e) Comparison of the numerical (continuous line,  $P = 6$ ,  $\eta_N = 0.01$ ) and theoretical results (circles for  $\sigma^2 = 0.4$  and crosses for  $\sigma^2 = 10$  standard deviation of the SW position versus  $z$  for 20 disorder realizations).

on locality on the soliton fluctuations. In particular, in the highly nonlocal limit  $K(x) \rightarrow K_0$ , for a bell-shaped soliton profile [ $u(x) = u(-x)$ ], we have

$$C = K_0^2 \left[ \int u(x_1 - X) u_x(x_1 - X) dx_1 \right]^2 = 0. \quad (31)$$

In the highly nonlocal regime, the random fluctuations of the fundamental soliton vanish.

Physically, when we increase the degree of nonlocality, the spectral content of the nonlocal response shrinks, and, as a consequence, averages out the randomness [15]. This result is valid for every kind of nonlocality.

To verify this theoretical analysis, we have numerically integrated the stochastic partial differential equation (17) for a 1D exponential nonlocality, using a pseudo-spectral stochastic Runge–Kutta algorithm [29, 30]. Panels (a) and (b) of figure 1 show a typical evolution starting from a bound state and displaying the random deviation of the SW for two different degrees of nonlocality  $\sigma$ . We stress that the high nonlocality degree for the case in panel (b) considerably dampens the random walk of the soliton with respect to the quasi-local case in panel (a). In panels (c) and (d) of figure 1, we report various trajectories for a fixed SW power for  $\sigma^2 = 0.4$  (c) and  $\sigma^2 = 10$  (d). Panel (e) shows the calculated standard deviation for the two degrees of



nonlocality above: the analytical prediction of equation (30) is well reproduced by the numerical simulations.

### 3.2. Shock waves regime

Due to their strongly nonlinear and coherent origin, the SWs will be heavily affected by disorder during focusing and defocusing propagation. By the hydrodynamical approach, we investigate the competition between strong nonlinearity and disorder and we find that an increasing amount of disorder delays the shock point up to the inhibition of the wave breaking.

To enter the shock regime, we consider a weak diffraction regime with  $L_d \gg L_R \gg L_{nl}$ , and the local case,  $K = \delta(x)$ :

$$i\epsilon_1 \partial_z \psi + \epsilon_1^2 \partial_x^2 \psi \pm |\psi|^2 \psi + \eta \psi = 0, \quad (32)$$

letting  $\psi(x, z) = \sqrt{\rho(x, z)} e^{i\phi(x, z)/\epsilon_1}$ , and retaining only leading orders in  $\epsilon_1$ , we have

$$\begin{aligned} O(\epsilon_1) : \rho_z + \nabla_x(\rho v) &= 0, \\ O(1) : \phi_z + v^2 \mp \rho &= \eta, \end{aligned} \quad (33)$$

where  $v \equiv \phi_x$  is the phase chirp. We derive the first order equation with respect to  $x$ :

$$\frac{\partial v}{\partial z} + 2v \frac{\partial v}{\partial x} = -\frac{dU}{dx}, \quad (34)$$

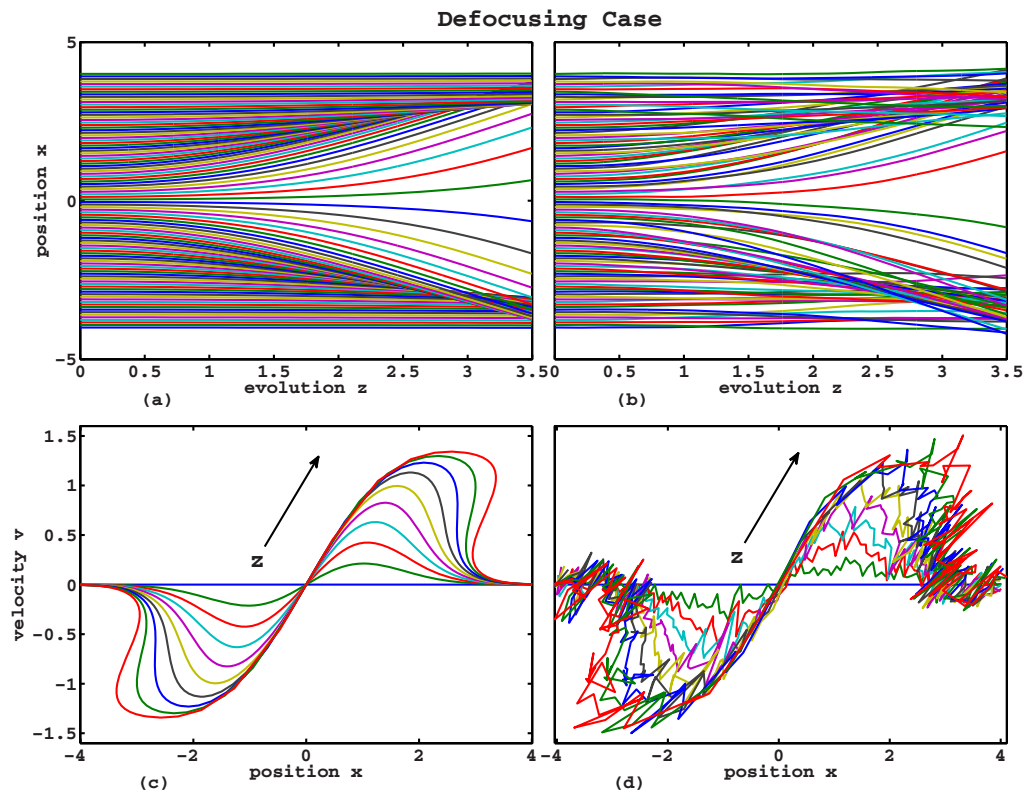
where the potential function  $U$  is  $U_{\text{foc}} = \eta - \rho$  in the focusing case and  $U_{\text{def}} = \eta + \rho$  in the defocusing case. We assume a Gaussian profile for the input beam  $\rho = \exp(-x^2/2)$ , and we apply the method of characteristics [31]. Equation (34) is reduced to

$$\begin{aligned} \frac{dx}{dz} &= 2v, \\ \frac{dv}{dz} &= -\frac{dU}{dx}, \end{aligned} \quad (35)$$

and by deriving the first equation with respect to  $z$  (hereafter, we make the ansatz  $z \rightarrow z/\sqrt{2}$  to simplify the notation), we have

$$\frac{d^2x}{dz^2} = -\frac{dU}{dx} = -\frac{dU_{nl}}{dx} - \frac{d\eta}{dx}, \quad (36)$$

which is equation of motion of a unitary mass particle in the potential  $U = U_{nl} + \eta$  with  $U_{nl} = e^{-x^2/2}$  the deterministic contribution deriving from nonlinearity. The stochastic term  $f = -d\eta/dx$  is taken as a Langevin force with Gaussian distribution,  $\langle f(z) f(z') \rangle = f_N^2 \delta(z - z')$  with  $f_N \approx \eta_N$ . Numerical simulations, obtained by a stochastic Runge–Kutta algorithm [32], provide solutions of equation (36) for several initial conditions  $x(0)$ ,  $v(0)$  and noise realizations  $\eta(x)$ . For each pair of initial conditions, we change the noise configuration in order to account for the dependence of  $\eta_N$  on  $x$ . The shock position corresponds to the singularity of the phase chirp  $|dv/dx| \rightarrow \infty$  and can be directly measured as the point of large steepness of the beam intensity [10]. In this unitary mass particle approach and in the absence of disorder, the shock point can be extrapolated by the coalescence of multiple trajectories, as shown in figure 2(a) (defocusing case) and in figure 3(a) (focusing case). In fact, the characteristic lines give the direction of energy flow and the shock formation corresponds to the accumulation of light density in specific regions of the transverse dimension  $x$ : at the edges of the laser beam for

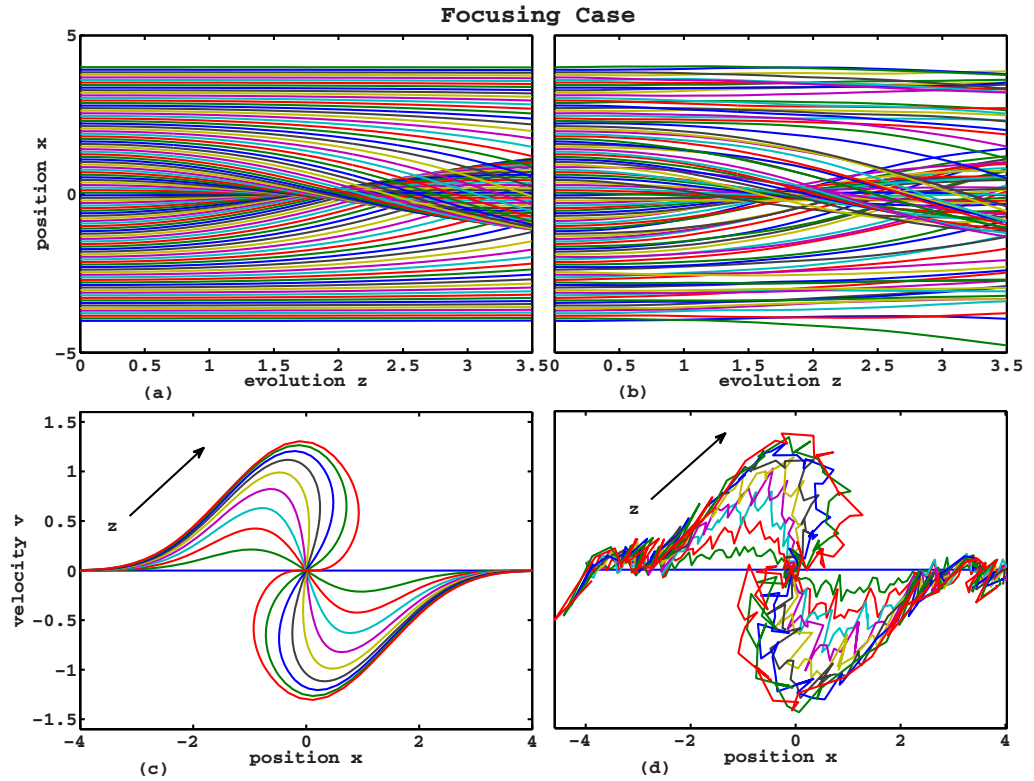


**Figure 2.** Trajectories from initial uniformly distributed conditions versus  $z$ . The collisions among particles give the shock position: in the defocusing case, the shock event occurs at the edges of the beam profile, (a) without disorder ( $\eta_N = 0$ ) and (b), for  $\eta_N = 0.1$ .

defocusing nonlinearity, and at the center of the laser beam for focusing nonlinearity. When we analyze the phase space of  $x$  and  $v$  (figure 2(c) for defocusing nonlinearity and figure 3(c) for focusing nonlinearity), the shock can be observed as a folding of particle velocities into a multivalued function. Physically, being that the light field is a single-value function, it happens that diffraction regularizes the wavefront, breaking the wave and causing the so-called *undular bore* [8, 16, 33]. This phenomenon corresponds to coherent fast oscillations both of the field intensity and the phase chirp, which smoothly join the singularity region with the rest of the wave front.

In the presence of disorder, the light scattering lowers the field intensity and moves the shock threshold away along the propagation distance. In the particle analogy, the characteristic lines are randomized (figures 2(b) and 3(b)) and the particles collide along greater distances, undergoing a Brownian motion and delaying the trajectory accumulation that determines the shock event.

In figures 4 and 5, we show, respectively, the numerical histograms of the particle positions in the defocusing and focusing cases. Panel (a) reports the accumulation point of the caustics in the ordered systems; it has a fixed position, corresponding to the singularity of the phase chirp,  $dv/dx \rightarrow \infty$ . In the disordered cases, panel (b), the trajectories are randomized and intersect at different points. We choose as the shock position  $z_s(\eta_N)$ , the mean value among the positions in which the histograms in figures 4 and 5 (panels (b)) are above the 90% of its maximum. It is an



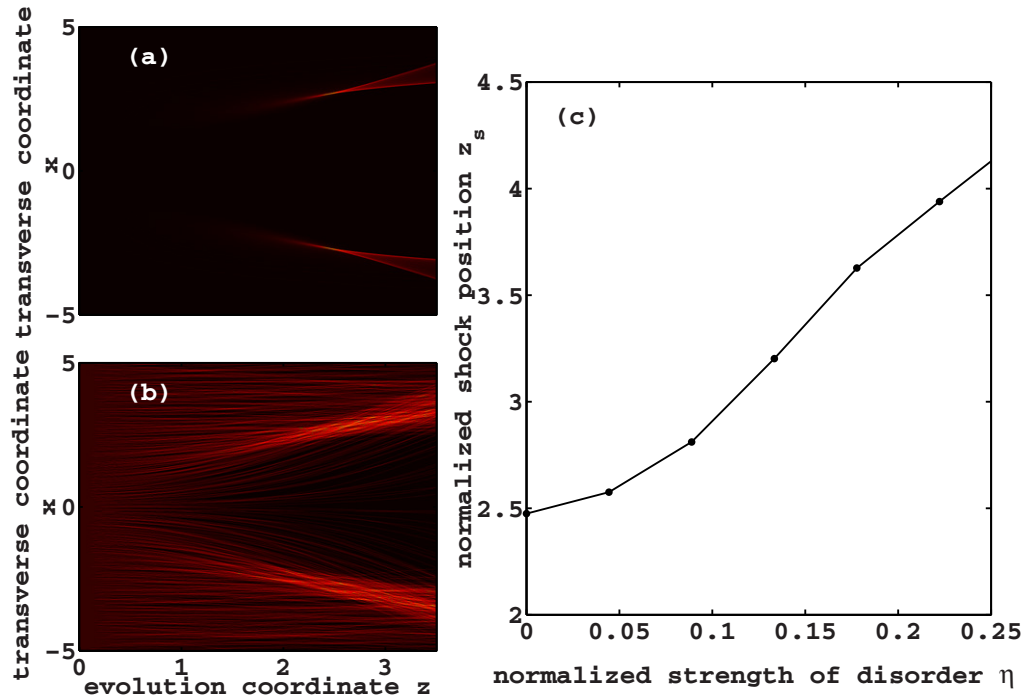
**Figure 3.** Trajectories from initial uniformly distributed conditions versus  $z$ . The collision points between particles signal the shock position: in the focusing case, the shock event happens in the center of the beam profile, (a) without disorder ( $\eta_N = 0$ ) and (b), for  $\eta_N = 0.1$ .

arbitrary criterion, which allows us to assess the probability that the shock event can be found in a specific spatial region.

In panels (c) of figures 4 and 5, the shock position is reported as a function of normalized strength of disorder  $\eta_N$ . As  $\eta_N$  increases, the shock process is delayed with respect to the ordered case. When the amount of disorder exceeds a critical value, the shock generation is inhibited. This threshold can be analytically determined as that corresponding to a random index perturbation  $\Delta n_R$  comparable with the nonlinear term:  $\langle (\Delta n_R^0)^2 \rangle^{1/2} \cong |n_2| I_0$ , which gives  $\eta_N \cong 1$ . In equation (32), the stochastic term related to  $\eta$  becomes greater than the nonlinear contribution. When this happens, the hydrodynamical limit in (36) is no longer valid. In fact, the stochastic fluctuations of the material become so large as to mask the nonlinear effect.

In the hydrodynamical regime, the shock position  $z_s$  scales as  $1/\sqrt{P}$ . When disorder is added, we found that the shock position becomes like  $z_s(\eta_N)/\sqrt{P}$  with  $z_s(\eta_N)$  given in figures 4(c) and 5(c). Hence, nonlinearity and disorder act as opposite effects: on the one hand, the shock formation is favored ( $z_s$  decreases) when nonlinearity grows, but on the other hand, by introducing disorder, light is scattered and the phenomenon is hampered ( $z_s$  increases).

**3.2.1. Nonlocal case.** Here we consider the way nonlocality affects the formation of SWs in the presence of disorder. Similarly to disorder, nonlocality delays the shock generation.



**Figure 4.** Histograms of particle positions in the defocusing case (a) for  $\eta_N = 0$  and (b)  $\eta_N = 0.1$ ; (c) normalized shock position versus the degree of disorder  $\eta_N$ . The shock position is delayed by the presence of disorder.

Furthermore, nonlocality also allows the shock to form in focusing media whereas filamentation or modulation instability prevail on the shock formation in local media [9].

We start by considering the nonlocal version of equation (32)

$$\begin{aligned} i\epsilon_1 \partial_z \psi + \epsilon_1^2 \partial_x^2 \psi \pm \theta \psi &= 0, \\ -\sigma^2 \partial_x^2 \theta + \theta &= |\psi|^2 + \eta, \end{aligned} \quad (37)$$

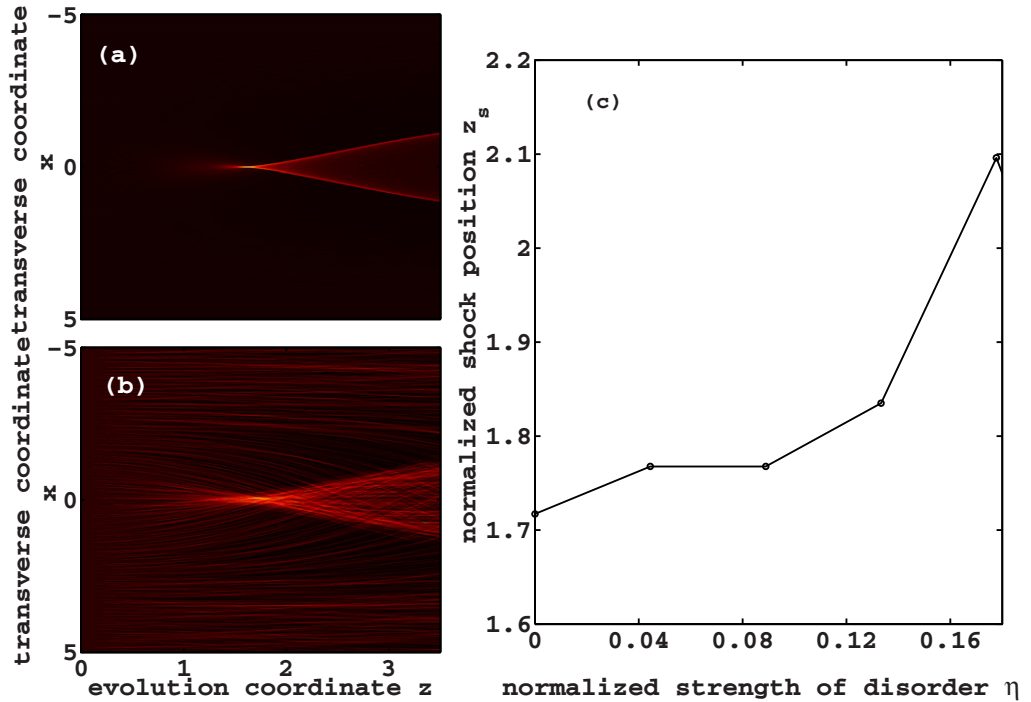
where  $\theta = 2kL_{nl} \Delta n / n_0 = K * (\pm |\psi|^2 + \eta)$  and  $\Delta n$  is the nonlocal nonlinear disordered refractive index perturbation (3) with a degree of nonlocality given by  $\sigma$ .

We make a WKB transformation of the field  $\psi(x, z) = \sqrt{\rho(x, z)} e^{i\phi(x, z)\epsilon_1}$  and substituting in (37), we obtain for the leading orders in  $\epsilon_1$ :

$$\begin{aligned} O(\epsilon_1) : \rho_z + \nabla_x(\rho v) &= 0, \\ O(1) : v_z + 2v_x v \mp \partial_x \theta &= 0, \\ O(1) : -\sigma^2 \partial_x^2 \theta + \theta &= \rho + \eta, \end{aligned} \quad (38)$$

where  $v = \partial_x \phi$  is the phase chirp. The intensity  $\rho$  adiabatically changes in  $z$  with respect to the phase chirp and the refractive index modulation  $\theta$ . Hence, we take  $\rho$  constant versus  $z$ ,  $\rho(x) = e^{-x^2/2}$ . As a consequence, the exact solution  $\theta(x)$  can be obtained from the third equation in (38). By using the method of characteristics, the second equation of (38) can be reduced to

$$\begin{aligned} \frac{dx}{dz} &= 2v, \\ \frac{dv}{dz} &= -\frac{dU}{dx}, \end{aligned} \quad (39)$$



**Figure 5.** Histograms of particle positions in the focusing case for (a)  $\eta_N = 0$  and (b)  $\eta_N = 0.1$ ; (c) normalized shock position versus the degree of disorder  $\eta_N$ .

where the potential  $U$  is  $U_{\text{foc}} = K * (\eta - \rho)$  in the focusing case and  $U_{\text{def}} = K * (\eta + \rho)$  in the defocusing case. The nonlocal nonlinear response  $K * \rho$  is the counterpart of the deterministic potential represented by the field intensity  $\rho$  in the local case (see (36)).

In the nonlocal case, the potential well (focusing case) and the potential barrier (defocusing case) are much more flat with respect to the local case. As a consequence, the intersection of characteristic lines leading to the shock generation is slowed down and the shock position is delayed along  $z$ . The potential profiles are randomized by the presence of  $K * \eta$  in (39) that further act as a SW damper, corrugating the surface over which the effective particles evolve.

Being the SWs generated in correspondence of the chirp singularity  $|\text{d}\phi/\text{d}x| \rightarrow \infty$  and, approximating the phase as  $\phi \approx \theta$ , we have

$$\begin{aligned} \frac{\text{d}\phi}{\text{d}x} &\approx \int \frac{\text{d}K(x-x')}{\text{d}x} (|\psi(x')|^2 + \eta(x')) \text{d}x' \\ &= \int K(x-x') \frac{\text{d}[|\psi(x')|^2 + \eta(x')]}{\text{d}x'} \text{d}x'. \end{aligned} \quad (40)$$

In the highly nonlocal limit  $K(x-x') = K_0$ , and the integral (40) vanishes. As a result, by increasing the nonlocality degree, the shock position is gradually delayed, up to its inhibition in the highly nonlocal limit.

Hence, both nonlocality and disorder tend to destroy the shock formation. The phenomena dampen the effective pressure gradient that causes the thickening of the ‘optical fluid’ resulting in the shock generation. On the one hand, nonlocality flattens the pressure profile over the spatial region of the nonlocal response (given by  $\sigma$ ). On the other hand, disorder scatters the light field, reducing the field intensity accumulation on the shock points.

#### 4. Conclusions

We have investigated the interplay between nonlinearity and disorder in the nonlocal NLS. Three main length scales, namely diffraction, randomness and nonlinear lengths, control the excitation of solitary and SWs in weakly disordered media. When solitons are considered, the nonlocality limits the scattering action of disorder on the soliton trajectory, averaging out the randomness and stabilizing the solitary waves. In the SW regime, a nonlocal nonlinearity delays shock formation with respect to the local case. The presence of disorder further delays the shock generation while nonlocality on the random potential does not significantly change the scenario. Given that disorder is inextricably linked to nonlocality, we believe that these results can be relevant when considering thermal and thermodiffusive media [7, 9, 34], photorefractive systems [17], Bose–Einstein condensate [18, 19], liquid crystals [20] or plasma [35, 36].

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