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Research Article

On Power Sums Involving Lucas Functions Sequences

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We present some general formulas related to sum of powers, also with alternating sign, involving Lucas functions sequences. In particular, our formulas give a synthesis of various identities involving sum of powers of well-known polynomial sequences such as Fibonacci, Lucas, Pell, Jacobsthal, and Chebyshev polynomials. Finally, we point out some interesting divisibility properties between polynomials arising from our results.

1. Introduction

The spirit of this paper is to develop the interesting ideas of Horadam presented in [1, 2], finding some general formulas for special sum of powers (also with alternating signs) related to a wide class of functions and in particular to some important classes of well-known polynomials. We will use a slight different notation with respect to Horadam.

Definition 1. One defines the Lucas functions sequence $(W_n^{(d)}(x))_{n=0}^{+\infty}$ as the sequence of functions satisfying the recurrence relation

$$\begin{aligned} W_0^{(d)}(x) &= \Delta^{d-2}\left(1+(-1)^d\right), \\ W_1^{(d)}(x) &= \Delta^{d-2}\left(a(x)+(-1)^d b(x)\right), \\ W_{n+1}^{(d)}(x) &= p(x)W_n^{(d)}(x)+q(x)W_{n-1}^{(d)}(x) \qquad n \geq 1, \end{aligned} \tag{1}$$

where d is an integer and $p(x)$, $q(x)$ are polynomials such that

$$\begin{aligned} a(x)+b(x) &= p(x), \\ a(x)b(x) &= -q(x), \\ \Delta &= \sqrt{p^2(x)+4q(x)} = a(x)-b(x). \end{aligned} \tag{2}$$

One supposes that all the functions in (1) and (2) are well defined for suitable values of the variable x.

Clearly from the previous definition, we have

$$\begin{aligned} W_n^{(d)}(x) &= \Delta^{d-2}\left(a^n(x)+(-1)^d b^n(x)\right), \\ a(x) &= \frac{p(x)+\Delta}{2}, \quad b(x) = \frac{p(x)-\Delta}{2}, \end{aligned} \tag{3}$$

and, in particular, when d is a positive integer, the sequence (1) is a polynomial sequence. It is worth noting that very important families of polynomials satisfy the recurrence (1), such as Chebyshev, Fibonacci, Lucas, and Jacobsthal polynomials as Horadam pointed out in [1]. For example, considering $p(x) = 2x, q(x) = -1$, we retrieve for $d = 1$ the sequence $(U_{n-1}(x))_{n=1}^{+\infty}$ and for $d = 2$ the sequence $(2T_n(x))_{n=0}^{+\infty}$ involving Chebyshev polynomials of second and first kind, respectively.

In the next section, we present our results on power sums of the kind $\sum_{k=1}^{n}(-1)^{\delta k}(W_{2kl+r}^{(d)}(x)/q^{lk}(x))^s$ with $\delta \in \{0,1\}$, l, r integers, $l \neq 0$, and n, s positive integers. Finally we discuss interesting consequences of our formulas, related to some divisibility properties for polynomials obtained generalizing the so-called Melham convolutions. These convolutions were introduced in [3] for Fibonacci and Lucas numbers and studied in many recent papers, for example, [4–9], also with their extensions to Fibonacci, Lucas, and Chebyshev polynomials. For the sake of simplicity from now on we omit the dependence on x.

2. Power Sums

First of all, we give some straightforward calculation rules related to the functions $W_n^{(d)}$, which we will use along our proofs.

Proposition 2. *For the functions defined by the recurrence (1), the following rules hold:*

$$W_{-n}^{(d)} = (-1)^{d+n} q^{-n} W_n^{(d)}, \tag{4}$$

$$W_n^{(d)} = \Delta^{d-e} W_n^{(e)}, \quad e \equiv d \bmod 2, \tag{5}$$

$$\Delta^2 W_m^{(h)} W_n^{(k)} = W_{m+n}^{(h+k)} + (-1)^{k+n} q^n W_{m-n}^{(h+k)}, \tag{6}$$

where d, e, h, k, m, n are integers.

Proof. These rules are direct consequences of relations (3) and easy calculations. Some of them are also listed in the paper of Horadam [2]. □

We start with two useful lemmas which will enable us to obtain our general formulas.

Lemma 3. *The following equality holds:*

$$\left(\frac{W_{2\lambda+r}^{(d)}}{q^\lambda}\right)^s = \frac{\Delta^{s(d-2)+2-\alpha}}{2} \sum_{h=0}^{s} \binom{s}{h} \left((-1)^{d+r} q^r\right)^h \frac{W_{(2\lambda+r)(s-2h)}^{(\alpha)}}{q^{\lambda(s-2h)}}, \tag{7}$$

where d, λ, r are integers, s is positive integer, and $\alpha \in \{1,2\}$ with $\alpha \equiv ds \bmod 2$.

Proof. Using the binomial theorem, we have from relations (3)

$$\left(W_{2\lambda+r}^{(d)}\right)^s = \Delta^{s(d-2)} \sum_{h=0}^{s} \binom{s}{h} (-1)^{d(s-h)} a^{(2\lambda+r)h} b^{(2\lambda+r)(s-h)} \tag{8}$$

or equivalently

$$\left(W_{2\lambda+r}^{(d)}\right)^s = \Delta^{s(d-2)} \sum_{h=0}^{s} \binom{s}{h} (-1)^{dh} a^{(2\lambda+r)(s-h)} b^{(2\lambda+r)h} \tag{9}$$

and if we sum (8) and (9) picking up $(ab)^{(2\lambda+r)h} = (-q)^{(2\lambda+r)h}$ we have

$$\left(\frac{W_{2\lambda+r}^{(d)}}{q^\lambda}\right)^s = \frac{\Delta^{s(d-2)}}{2} \sum_{h=0}^{s} \binom{s}{h} (-1)^{(d+r)h} \cdot q^{-\lambda(s-2h)+rh} \left(a^{(2\lambda+r)(s-2h)} + (-1)^{ds} b^{(2\lambda+r)(s-2h)}\right) \tag{10}$$

from which we easily find (7). □

Now, from identity (7), it is straightforward to obtain expressions for the powers $(W_{2\lambda+r}^{(d)}/q^\lambda)^s$ in the cases s odd or s even. We only need to apply some little calculations and rule (4) to change sign in the subscripts. After a suitable rearrangement of the summations involved, if $s = 2m+1$ we get

$$\Delta^{2m(d-2)+d-\alpha} \sum_{h=0}^{m} \binom{2m+1}{m-h} \left((-1)^{d+r} q^r\right)^{m-h} \cdot \frac{W_{(2\lambda+r)(2h+1)}^{(\alpha)}}{q^{\lambda(2h+1)}}; \tag{11}$$

otherwise, if $s = 2m$, we find

$$\Delta^{2m(d-2)} \sum_{h=1}^{m} \binom{2m}{m-h} \left((-1)^{d+r} q^r\right)^{m-h} \frac{W_{2(2\lambda+r)h}^{(2)}}{q^{2\lambda h}} + \binom{2m}{m} \left((-1)^{d+r} q^r \Delta^{2(d-2)}\right)^m. \tag{12}$$

Lemma 4. *Let one consider $\delta \in \{0,1\}$, $\alpha \in \{1,2\}$, integers l, r with $l \neq 0$, and positive integers N, n. Then one has*

$$\sum_{k=1}^{n} (-1)^{\delta k} \frac{W_{(2lk+r)N}^{(\alpha)}}{q^{klN}} = \Delta^{-2} (-1)^{\delta n} \frac{W_{(l(2n+1)+r)N}^{(\alpha+\beta)} + (-1)^{n\delta+1} q^{nlN} W_{(l+r)N}^{(\alpha+\beta)}}{q^{nlN} W_{lN}^{(\beta)}}, \tag{13}$$

where $\beta \in \{1,2\}$ and $\beta \equiv \delta + lN + 1 \bmod 2$.

Proof. From rule (6), we have

$$W_{(2lk+r)N}^{(\alpha)} W_{lN}^{(\delta+lN+1)} = \Delta^{-2} \left(W_{(l(2k+1)+r)N}^{(\delta+\alpha+lN+1)} + (-1)^{\delta+1} q^{lN} W_{(l(2k-1)+r)N}^{(\delta+\alpha+lN+1)}\right) \tag{14}$$

and we observe that

$$W_{lN}^{(\delta+lN+1)} \sum_{k=1}^{n} (-1)^{\delta k} \frac{W_{(2lk+r)N}^{(\alpha)}}{q^{lkN}} = \Delta^{-2} \sum_{k=1}^{n} \left((-1)^{\delta k} \frac{W_{(l(2k+1)+r)N}^{(\delta+\alpha+lN+1)}}{q^{klN}} - (-1)^{\delta(k-1)} \frac{W_{(l(2k-1)+r)N}^{(\delta+\alpha+lN+1)}}{q^{(k-1)lN}}\right) = \Delta^{-2} \left((-1)^{\delta n} \cdot \frac{W_{(l(2n+1)+r)N}^{(\delta+\alpha+lN+1)}}{q^{nlN}} - W_{(l+r)N}^{(\delta+\alpha+lN+1)}\right) \tag{15}$$

since the sum telescopes. Thus we easily obtain (13) using rule (5). □

Thanks to rule (6), we point out that the right member of (13) could also be expressed in the following equivalent form:

$$(-1)^{\delta n}\Delta^{-2}\left(\frac{\Delta^2 W^{(\alpha)}_{(l(n+1)+r)N}W^{(\beta)}_{nlN}}{q^{nlN}W^{(\beta)}_{lN}} + \left((-1)^{\delta n+1}-(-1)^{\beta+nlN}\right)\frac{W^{(\alpha+\beta)}_{(l+r)N}}{W^{(\beta)}_{lN}}\right), \tag{16}$$

which becomes

$$(-1)^{\delta n}\frac{W^{(\alpha)}_{(l(n+1)+r)N}W^{(1)}_{nlN}}{q^{nlN}W^{(1)}_{lN}} \tag{17}$$

when $\delta \equiv lN \bmod 2$, since in this case $\beta = 1$, or

$$(-1)^{\delta n}\frac{W^{(\alpha)}_{nlN}W^{(\beta)}_{nlN}}{q^{nlN}W^{(\beta)}_{lN}} \tag{18}$$

when $W^{(\alpha+\beta)}_{(l+r)N} = 0$, in other words, by definition (1), when $r = -l$ and $\alpha+\beta = 3$.

Now we have all what we need to find out our general formulas.

Theorem 5. *For all integers d, l, r, m, n, δ, with $l \neq 0$, $m \geq 0$, $n \geq 1$, and $\delta \in \{0,1\}$, the following identities hold:*

$$\sum_{k=1}^{n}(-1)^{\delta k}\left(\frac{W^{(d)}_{2lk+r}}{q^{kl}}\right)^{2m+1} = (-1)^{\delta n} \cdot \Delta^{(2m+1)(d-2)-\alpha}\sum_{h=0}^{m}\binom{2m+1}{m-h}\left((-1)^{d+r}q^r\right)^{m-h} \cdot \mathscr{W}_{2h+1}, \tag{19}$$

where

$$\mathscr{W}_{2h+1} = \frac{W^{(\alpha+\beta)}_{(l(2n+1)+r)(2h+1)} + (-1)^{n\delta+1}q^{nl(2h+1)}W^{(\alpha+\beta)}_{(l+r)(2h+1)}}{q^{nl(2h+1)}W^{(\beta)}_{l(2h+1)}}, \tag{20}$$

$\alpha, \beta \in \{1,2\}$, $\alpha \equiv d \bmod 2$, $\beta \equiv \delta + l + 1 \bmod 2$, and

$$\sum_{k=1}^{n}(-1)^{\delta k}\left(\frac{W^{(d)}_{2lk+r}}{q^{kl}}\right)^{2m} = (-1)^{\delta n}\Delta^{2m(d-2)}\sum_{h=1}^{m}\binom{2m}{m-h}\left((-1)^{d+r}q^r\right)^{m-h}\mathscr{W}_{2h} + \binom{2m}{m}\left((-1)^{d+r}q^r\Delta^{2(d-2)}\right)^m\mathscr{U}(\delta,n), \tag{21}$$

where

$$\mathscr{W}_{2h} = \frac{W^{(\delta+1)}_{2(l(2n+1)+r)h} + (-1)^{n\delta+1}q^{2nlh}W^{(\delta+1)}_{2(l+r)h}}{q^{2nlh}W^{(\delta+1)}_{2lh}}, \tag{22}$$

$$\mathscr{U}(\delta,n) = \begin{cases}\mathscr{U}(0,n) = n \\ \mathscr{U}(1,n) = 0 & \text{if } n \text{ is even} \\ \mathscr{U}(1,n) = -1 & \text{if } n \text{ is odd.}\end{cases} \tag{23}$$

Proof. The identities (19) and (21) are direct consequences of (11) and (12), combined with (13) of Lemma 4. In the case $N = 2h+1$, we have $\beta \equiv \delta + l + 1 \bmod 2$; on the other hand, if $N = 2h$, we have $\beta = \delta + 1$. Clearly in the latter, we also have to take in account that $\alpha = 2$, so, applying rule (5), we obtain

$$(-1)^{\delta n}\mathscr{W}_{2h} = \frac{W^{(\delta+1)}_{2(l(2n+1)+r)h} + (-1)^{n\delta+1}q^{2nlh}W^{(\delta+1)}_{2(l+r)h}}{q^{2nlh}W^{(\delta+1)}_{2lh}}. \tag{24}$$

Finally, when we consider the sum related to the term $\binom{2m}{m}\left((-1)^{d+r}q^r\Delta^{2(d-2)}\right)^m$ in (12), we observe that $\sum_{k=1}^{n}(-1)^{\delta k} = \mathscr{U}(\delta,n)$. □

In order to highlight in the next section some interesting divisibility properties between polynomials, we end with some results which made us able to rewrite equalities (19) and (21). For our purposes, we recall two important formulas due to Girard and Waring.

Lemma 6 (Girard–Waring formulas). *For all nonnegative integers n and real numbers X, Y, the following identities hold:*

$$X^n + Y^n = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{n}{n-k}\binom{n-k}{k}(X+Y)^{n-2k}(XY)^k, \tag{25}$$

$$\frac{X^{n+1}-Y^{n+1}}{X-Y} = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}(X+Y)^{n-2k}(XY)^k. \tag{26}$$

Clearly these two formulas have a long history and should be widely known, so we only refer the reader to the original books of Girard [10] and Waring [11]. We also mention the paper of Gould [12], in which the reader will find some interesting remarks about the history and the use of these formulas and their generalizations. We observe that formula (26) also holds in the case $X = Y$; indeed

$$\lim_{Y\to X}\frac{X^{n+1}-Y^{n+1}}{X-Y} = X^n(n+1), \tag{27}$$

and $X^n(n+1)$ corresponds to the right member of (26) via the identity

$$n+1 = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}2^{n-2k}. \tag{28}$$

Applying these formulas, we can find alternative expressions for $W_{(2A+B)N}^{(D)} + (-1)^{\gamma} q^{AN} W_{BN}^{(D)}$ in the cases N odd or N even, under the convention that ratios of the kind $W_{BN}^{(D)}/W_{B}^{(D)}$ when $B = 0$ and D odd will take the value N, according to the result of the limit $\lim_{B\to 0}(a^{BN} - b^{BN})/(a^{B} - b^{B})$.

Proposition 7. *Let one consider integers D, γ, A, B, C, and nonnegative integer h. Then the following equality holds:*

$$\begin{aligned} &W_{(2A+B)(2h+1)}^{(D)} + (-1)^{\gamma} q^{A(2h+1)} W_{B(2h+1)}^{(D)} \\ &\quad = \left(W_{2A+B}^{(D)} + (-1)^{\gamma} q^{A} W_{B}^{(D)}\right)(2h+1) q^{2Ah} \frac{W_{B(2h+1)}^{(D+1)}}{W_{B}^{(D+1)}} \\ &\quad + \left(W_{2A+B}^{(D)} + (-1)^{\gamma} q^{A} W_{B}^{(D)}\right)^{2} \\ &\quad \cdot \mathscr{R}_{h}\left(W_{2A+B}^{(D)} + (-1)^{\gamma} q^{A} W_{B}^{(D)}, (-1)^{\gamma+1} q^{A} W_{B}^{(D)}\right), \end{aligned} \tag{29}$$

where

$$\begin{aligned} \mathscr{R}_{h}(u, v) &= \sum_{t=0}^{h-1} C_{t}(h, q, \Delta) \\ &\cdot \sum_{j=1}^{2h-2t} \binom{2h+1-2t}{j+1} u^{j-1} v^{2h-2t-j}, \end{aligned} \tag{30}$$

$$\begin{aligned} C_{t}(h, q, \Delta) &= \frac{2h+1}{2h+1-t} \\ &\cdot \binom{2h+1-t}{t} \left((-1)^{D+B+1} q^{2A+B}\right)^{t} \Delta^{-(D-2)(2h-2t)}. \end{aligned} \tag{31}$$

Proof. Thanks to the Girard–Waring formula (25) of Lemma 6, we can rewrite

$$\begin{aligned} &W_{(2A+B)(2h+1)}^{(D)} \\ &\quad = \Delta^{D-2}\left(\left(a^{2A+B}\right)^{2h+1} + \left((-1)^{D} b^{2A+B}\right)^{2h+1}\right), \\ &q^{A(2h+1)} W_{B(2h+1)}^{(D)} \\ &\quad = \Delta^{D-2}\left(\left(q^{A} a^{B}\right)^{2h+1} + \left((-1)^{D} q^{A} b^{B}\right)^{2h+1}\right) \end{aligned} \tag{32}$$

obtaining

$$\begin{aligned} &W_{(2A+B)(2h+1)}^{(D)} + (-1)^{\gamma} q^{A(2h+1)} W_{B(2h+1)}^{(D)} \\ &\quad = \sum_{t=0}^{h} C_{t}(h, q, \Delta) \\ &\quad \cdot \left(\left(W_{2A+B}^{(D)}\right)^{2h+1-2t} + (-1)^{\gamma}\left(q^{A} W_{B}^{(D)}\right)^{2h+1-2t}\right) \\ &\quad = C_{h}(h, q, \Delta)\left(W_{2A+B}^{(D)} + (-1)^{\gamma} q^{A} W_{B}^{(D)}\right) \\ &\quad + \sum_{t=0}^{h-1} C_{t}(h, q, \Delta) \\ &\quad \cdot \left(\left(W_{2A+B}^{(D)}\right)^{2h+1-2t} + (-1)^{\gamma}\left(q^{A} W_{B}^{(D)}\right)^{2h+1-2t}\right). \end{aligned} \tag{33}$$

Now, since

$$\begin{aligned} &\left(W_{2A+B}^{(D)}\right)^{2h+1-2t} + (-1)^{\gamma}\left(q^{A} W_{B}^{(D)}\right)^{2h+1-2t} \\ &\quad = \sum_{j=1}^{2h+1-2t} \binom{2h+1-2t}{j} \left(W_{2A+B}^{(D)} + (-1)^{\gamma} q^{A} W_{B}^{(D)}\right)^{j} \\ &\quad \cdot \left((-1)^{\gamma+1} q^{A} W_{B}^{(D)}\right)^{2h+1-2t-j} = \left(W_{2A+B}^{(D)} + (-1)^{\gamma}\right. \\ &\quad \left.\cdot q^{A} W_{B}^{(D)}\right)(2h+1-2t)\left(q^{A} W_{B}^{(D)}\right)^{2h-2t} \\ &\quad + \sum_{j=1}^{2h-2t} \binom{2h+1-2t}{j+1} \left(W_{2A+B}^{(D)} + (-1)^{\gamma} q^{A} W_{B}^{(D)}\right)^{j+1} \\ &\quad \cdot \left((-1)^{\gamma+1} q^{A} W_{B}^{(D)}\right)^{2h-2t-j} \end{aligned} \tag{34}$$

the last member of (33) is equal to

$$\begin{aligned} &\left(W_{2A+B}^{(D)} + (-1)^{\gamma} q^{A} W_{B}^{(D)}\right)\Bigg(C_{h}(h, q, \Delta) \\ &\quad + \sum_{t=0}^{h-1} C_{t}(h, q, \Delta)(2h+1-2t)\left(q^{A} W_{B}^{(D)}\right)^{2h-2t}\Bigg) \\ &\quad + \left(W_{2A+B}^{(D)} + (-1)^{\gamma} q^{A} W_{B}^{(D)}\right)^{2} \mathscr{R}_{h}\left(W_{2A+B}^{(D)}\right. \\ &\quad \left. + (-1)^{\gamma} q^{A} W_{B}^{(D)}, (-1)^{\gamma+1} q^{A} W_{B}^{(D)}\right), \end{aligned} \tag{35}$$

where, by definition (31) of $C_{t}(h, q, \Delta)$ and by the Girard–Waring formula (26) of Lemma 6, we have with simple calculations

$$\begin{aligned} &C_{h}(h, q, \Delta) \\ &\quad + \sum_{t=0}^{h-1} C_{t}(h, q, \Delta)(2h+1-2t)\left(q^{A} W_{B}^{(D)}\right)^{2h-2t} \\ &\quad = (2h+1) \sum_{t=0}^{h} (-1)^{t} \binom{2h-t}{t} \left((-1)^{D+B} q^{2A+B}\right)^{t} \\ &\quad \cdot \Delta^{-(D-2)(2h-2t)} \left(q^{A} W_{B}^{(D)}\right)^{2h-2t} = (2h+1) q^{2Ah} \\ &\quad \cdot \frac{W_{B(2h+1)}^{(D+1)}}{W_{B}^{(D+1)}}. \end{aligned} \tag{36}$$

□

Proposition 8. *Let one consider integers D, γ, A, B, C, and nonnegative integer h. Then, if D is odd, one has*

$$\begin{aligned}
&W^{(D)}_{2(2A+B)h} + (-1)^{\gamma} q^{2Ah} W^{(D)}_{2Bh} \\
&\quad = q^{2A(h-1)} \left(W^{(D)}_{2(2A+B)} + (-1)^{\gamma} q^{2A} W^{(D)}_{2B}\right) \frac{W^{(D)}_{2Bh}}{W^{(D)}_{2B}} \\
&\quad + W^{(D)}_{2(2A+B)} \left(W^{(2)}_{2(2A+B)} - q^{2A} W^{(2)}_{2B}\right) \\
&\quad \cdot \mathcal{S}_h \left(W^{(2)}_{2(2A+B)} - q^{2A} W^{(2)}_{2B}, q^{2A} W^{(2)}_{2B}\right);
\end{aligned} \tag{37}$$

else, if D is even, one gets

$$\begin{aligned}
&W^{(D)}_{2(2A+B)h} + (-1)^{\gamma} q^{2Ah} W^{(D)}_{2Bh} = \left(W^{(D)}_{2(2A+B)} - q^{2A} W^{(D)}_{2B}\right) \\
&\quad \cdot \mathcal{T}_h \left(W^{(2)}_{2(2A+B)} - q^{2A} W^{(2)}_{2B}, q^{2A} W^{(2)}_{2B}\right) \\
&\quad + \left(1 + (-1)^{\gamma}\right) q^{2Ah} W^{(D)}_{2Bh},
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
&\mathcal{S}_h (u, v) \\
&\quad = \sum_{t=0}^{\lfloor (h-1)/2 \rfloor} K_t (q) \sum_{j=0}^{h-2-2t} \binom{h-1-2t}{j+1} u^j v^{h-2-2t-j}, \\
&\qquad K_t (q) = \binom{h-1-t}{t} \left(-q^{4A+2B}\right)^t, \\
&\mathcal{T}_h (u, v) = \sum_{t=0}^{\lfloor h/2 \rfloor} H_t (q) \sum_{j=0}^{h-1-2t} \binom{h-2t}{j+1} u^j v^{h-1-2t-j}, \\
&\qquad H_t (q) = \frac{h}{h-t} \binom{h-t}{t} \left(-q^{4A+2B}\right)^t.
\end{aligned} \tag{39}$$

Proof. In order to prove (37) if D is odd, we observe that

$$\begin{aligned}
W^{(D)}_{2(2A+B)h} &= \Delta^{D-2} \left(\left(a^{2(2A+B)}\right)^h - \left(b^{2(2A+B)}\right)^h\right), \\
q^{2Ah} W^{(D)}_{2Bh} &= \Delta^{D-2} \left(\left(q^{2A} a^{2B}\right)^h - \left(q^{2A} b^{2B}\right)^h\right).
\end{aligned} \tag{40}$$

Thus, applying Girard–Waring formula (26), we have

$$\begin{aligned}
&W^{(D)}_{2(2A+B)h} + (-1)^{\gamma} q^{2Ah} W^{(D)}_{2Bh} = \sum_{t=0}^{\lfloor (h-1)/2 \rfloor} K_t (q) \\
&\quad \cdot \left(\left(W^{(2)}_{2(2A+B)}\right)^{h-1-2t} W^{(D)}_{2(2A+B)} \right. \\
&\quad \left. + (-1)^{\gamma} \left(q^{2A} W^{(2)}_{2B}\right)^{h-1-2t} q^{2A} W^{(D)}_{2B}\right).
\end{aligned} \tag{41}$$

Since

$$\begin{aligned}
&\left(W^{(2)}_{2(2A+B)}\right)^{h-1-2t} W^{(D)}_{2(2A+B)} + (-1)^{\gamma} \left(q^{2A} W^{(2)}_{2B}\right)^{h-1-2t} \\
&\quad \cdot q^{2A} W^{(D)}_{2B} = \left(q^{2A} W^{(2)}_{2B}\right)^{h-1-2t} \\
&\quad \cdot \left(W^{(D)}_{2(2A+B)} + (-1)^{\gamma} q^{2A} W^{(D)}_{2B}\right) \\
&\quad + W^{(D)}_{2(2A+B)} \sum_{j=1}^{h-1-2t} \binom{h-1-2t}{j} \\
&\quad \cdot \left(W^{(2)}_{2(2A+B)} - q^{2A} W^{(2)}_{2B}\right)^j \left(q^{2A} W^{(2)}_{2B}\right)^{h-1-2t-j},
\end{aligned} \tag{42}$$

the right member of (41) becomes

$$\begin{aligned}
&\left(W^{(D)}_{2(2A+B)} + (-1)^{\gamma} q^{2A} W^{(D)}_{2B}\right) \\
&\quad \cdot \sum_{t=0}^{\lfloor (h-1)/2 \rfloor} K_t (q) \left(q^{2A} W^{(2)}_{2B}\right)^{h-1-2t} \\
&\quad + W^{(D)}_{2(2A+B)} \left(W^{(2)}_{2(2A+B)} - q^{2A} W^{(2)}_{2B}\right) \\
&\quad \cdot \mathcal{S}_h \left(W^{(2)}_{2(2A+B)} - q^{2A} W^{(2)}_{2B}, q^{2A} W^{(2)}_{2B}\right),
\end{aligned} \tag{43}$$

where, again by Girard–Waring formula (26), we obtain

$$\sum_{t=0}^{\lfloor (h-1)/2 \rfloor} K_t (q) \left(q^{2A} W^{(2)}_{2B}\right)^{h-1-2t} = q^{2A(h-1)} \frac{W^{(D)}_{2hB}}{W^{(D)}_{2B}}. \tag{44}$$

On the other hand, by means of Girard–Waring formula (25), when D is even, we have

$$\begin{aligned}
W^{(D)}_{2(2A+B)h} &= \Delta^{D-2} \left(\left(a^{2(2A+B)}\right)^h + \left(b^{2(2A+B)}\right)^h\right) \\
&= \Delta^{D-2} \sum_{t=0}^{\lfloor h/2 \rfloor} H_t (q) \left(W^{(2)}_{2(2A+B)}\right)^{h-2t} \\
q^{2Ah} W^{(D)}_{2Bh} &= \Delta^{D-2} \left(\left(q^{2A} a^{2B}\right)^h + \left(q^{2A} b^{2B}\right)^h\right) \\
&= \Delta^{D-2} \sum_{t=0}^{\lfloor h/2 \rfloor} H_t (q) \left(q^{2A} W^{(2)}_{2B}\right)^{h-2t};
\end{aligned} \tag{45}$$

thus

$$\begin{aligned}
&W^{(D)}_{2(2A+B)h} + (-1)^{\gamma} q^{2Ah} W^{(D)}_{2Bh} \\
&\quad = W^{(D)}_{2(2A+B)h} - q^{2Ah} W^{(D)}_{2Bh} + \left(1 + (-1)^{\gamma}\right) q^{2Ah} W^{(D)}_{2Bh} \\
&\quad = \Delta^{D-2} \sum_{t=0}^{\lfloor h/2 \rfloor} H_t (q) \left(\left(W^{(2)}_{2(2A+B)}\right)^{h-2t} - \left(q^{2A} W^{(2)}_{2B}\right)^{h-2t}\right) \\
&\quad + \left(1 + (-1)^{\gamma}\right) q^{2Ah} W^{(D)}_{2Bh}.
\end{aligned} \tag{46}$$

Now substituting the identity

$$\begin{aligned}\left(W_{2(2A+B)}^{(2)}\right)^{h-2t} &= \sum_{j=0}^{h-2t}\binom{h-2t}{j} \\ &\quad \cdot \left(W_{2(2A+B)}^{(2)} - q^{2A}W_{2B}^{(2)}\right)^{j} \\ &\quad \cdot \left(q^{2A}W_{2B}^{(2)}\right)^{h-2t-j}\end{aligned} \tag{47}$$

in the last member of (46), with some simple calculations we can finally find (38). □

As straightforward consequences of identities (29), (37), and (38) applied in equalities (19) and (21), we state the following two corollaries of Theorem 5.

Corollary 9. *Under the same hypotheses of Theorem 5, one has*

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{\delta k}\left(\frac{W_{2lk+r}^{(d)}}{q^{kl}}\right)^{2m+1} &= \frac{W_{l(2n+1)+r}^{(\alpha+\beta)} + (-1)^{n\delta+1}q^{nl}W_{(l+r)}^{(\alpha+\beta)}}{q^{nl}W_{l+r}^{(\alpha+\beta+1)}}\Sigma_{1,2m+1} \\ &\quad + \left(W_{l(2n+1)+r}^{(\alpha+\beta)} + (-1)^{n\delta+1}q^{nl}W_{l+r}^{(\alpha+\beta)}\right)^{2}\Sigma_{2,2m+1},\end{aligned} \tag{48}$$

where

$$\begin{aligned}\Sigma_{1,2m+1} &= (-1)^{\delta n}\Delta^{(2m+1)(d-2)-\alpha}\sum_{h=0}^{m}\binom{2m+1}{m-h} \\ &\quad \cdot \left((-1)^{(d+r)}q^{r}\right)^{m-h}(2h+1)\frac{W_{(l+r)(2h+1)}^{(\alpha+\beta+1)}}{W_{l(2h+1)}^{(\beta)}}, \\ \Sigma_{2,2m+1} &= (-1)^{\delta n}\Delta^{(2m+1)(d-2)-\alpha}\sum_{h=1}^{m}\binom{2m+1}{m-h} \\ &\quad \cdot \left((-1)^{(d+r)}q^{r}\right)^{m-h}\frac{\mathscr{R}_h}{q^{nl(2h+1)}W_{l(2h+1)}^{(\beta)}}, \\ \mathscr{R}_h &= \mathscr{R}_h\left(W_{l(2n+1)+r}^{(\alpha+\beta)}\right. \\ &\quad \left.+(-1)^{\delta n+1}q^{nl}W_{l+r}^{(\alpha+\beta)}, (-1)^{\delta n+1}q^{nl}W_{l+r}^{(\alpha+\beta)}\right);\end{aligned} \tag{49}$$

moreover we have

$$\Delta^{2} \mid W_{l(2n+1)+r}^{(\alpha+\beta)} + (-1)^{n\delta+1}q^{nl}W_{l+r}^{(\alpha+\beta)}. \tag{50}$$

Proof. We only need to consider the numerator of $\mathscr{W}_{2h+1}$ in (20) of Theorem 5 and use equality (29) of Proposition 7 with $A = nl$, $B = l + r$, $\gamma = \delta n + 1$, and $D = \alpha + \beta$. Finally equality (50) is obvious when $\alpha + \beta = 3, 4$, thanks to rule (5), and if $\alpha + \beta = 2$, that is, $\alpha = \beta = 1$, we have from rule (6)

$$\begin{aligned}\Delta^{2}W_{l(n+1)+r}^{(1)}W_{ln}^{(1)} &= W_{l(2n+1)+r}^{(2)} + (-1)^{nl+1}q^{nl}W_{l+r}^{(2)} \\ &= W_{l(2n+1)+r}^{(2)} + (-1)^{\delta n+1}q^{nl}W_{l+r}^{(2)}\end{aligned} \tag{51}$$

since $\beta = 1$ corresponds to $\delta \equiv l \bmod 2$. □

Corollary 10. *Under the same hypotheses of Theorem 5, if $\delta = 0$, for all positive integers n, one has*

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{W_{2lk+r}^{(d)}}{q^{kl}}\right)^{2m} &= \frac{W_{2(l(2n+1)+r)}^{(1)} - q^{2nl}W_{2(l+r)}^{(1)}}{q^{2nl}W_{2(l+r)}^{(1)}}\Sigma_{1,2m} \\ &\quad + W_{2(l(2n+1)+r)}^{(1)}\left(W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)}\right)\Sigma_{2,2m} \\ &\quad + n\binom{2m}{m}\left((-1)^{d+r}q^{r}\Delta^{2(d-2)}\right)^{m}\end{aligned} \tag{52}$$

with

$$\Sigma_{1,2m} = \Delta^{2m(d-2)}\sum_{h=1}^{m}\binom{2m}{m-h}\left((-1)^{d+r}q^{r}\right)^{m-h}\frac{W_{2(l+r)h}^{(1)}}{W_{2lh}^{(1)}}, \tag{53}$$

$$\Sigma_{2,2m} = \Delta^{2m(d-2)}\sum_{h=1}^{m}\binom{2m}{m-h}\left((-1)^{d+r}q^{r}\right)^{m-h}\frac{\mathscr{S}_h}{q^{2nlh}W_{2lh}^{(1)}}, \tag{54}$$

$$\mathscr{S}_h = \mathscr{S}_h\left(W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)}, q^{2nl}W_{2(l+r)}^{(2)}\right).$$

Moreover, if $\delta = 1$ and n is even, one gets

$$\sum_{k=1}^{n}(-1)^{k}\left(\frac{W_{2lk+r}^{(d)}}{q^{kl}}\right)^{2m} = \left(W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)}\right)\Sigma_{3,2m} \tag{55}$$

with

$$\begin{aligned}\Sigma_{3,2m} &= \Delta^{2m(d-2)}\sum_{h=1}^{m}\binom{2m}{m-h}\left((-1)^{d+r}q^{r}\right)^{m-h} \\ &\quad \cdot \frac{\mathscr{T}_h}{q^{2nlh}W_{2lh}^{(2)}}, \\ \mathscr{T}_h &= \mathscr{T}_h\left(W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)}, q^{2nl}W_{2(l+r)}^{(2)}\right), \\ \Delta^{2} &\mid W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)}.\end{aligned} \tag{56}$$

Finally, if $\delta = 1$ and n is odd, one obtains

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k}\left(\frac{W_{2lk+r}^{(d)}}{q^{kl}}\right)^{2m} &= -\Sigma_{4,2m} - \left(W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)}\right)\Sigma_{3,2m} \\ &\quad - \binom{2m}{m}\left((-1)^{d+r}q^{r}\Delta^{2(d-2)}\right)^{m}\end{aligned} \tag{57}$$

with

$$\Sigma_{4,2m} = 2\Delta^{2m(d-2)}\sum_{h=1}^{m}\binom{2m}{m-h}\left((-1)^{d+r}q^{r}\right)^{m-h}\frac{W^{(2)}_{2(l+r)h}}{W^{(2)}_{2lh}}. \tag{58}$$

Proof. According to the parity of $\delta+1$, the numerator $\mathscr{W}_{2h}$ of (22) in the identity (21) of Theorem 5 corresponds to (37) or (38) of Proposition 8 with $D=\delta+1$, $A=nl$, $B=l+r$, and $\gamma=n\delta+1$. In particular, taking into account the values of $\mathscr{U}$ defined in (23), we easily find identity (52) and the remaining identities (55) and (57), by means of equalities (37) and (38), respectively. Finally, from rule (6), we obviously have $\Delta^2 W^{(1)}_{2l(n+1)+2r}W^{(1)}_{2ln} = W^{(2)}_{2(l(2n+1)+r)} - q^{2nl}W^{(2)}_{2(l+r)}$. □

3. Applications and Divisibility Properties

From now on we suppose that $d\ge 1$ and $l>0$, $r\ge -2l$, in order to ensure that the functions $W^{(d)}_{2kl+r}$ are polynomials for all positive integers k. Clearly assigning different suitable values to the parameters δ, d, l, r in our formulas, we can easily obtain many interesting relations for polynomial sequences satisfying (1). In particular, for well-known sequences as Fibonacci, Lucas, Pell, and Chebyshev polynomials, having $q(x)=1$ or $q(x)=-1$, our general formulas (19) and (21) resume many identities related to their power sums, also with alternating sign. Obviously, these identities enclose also the similar identities on Fibonacci, Lucas, and Pell numbers when we evaluate the corresponding polynomial sequences in $x=1$. For instance, we easily retrieve the results presented in [4, 13] or the ones in [5], respectively, related to the sums of alternating sign powers of Fibonacci and Lucas numbers, to odd powers of Chebyshev polynomials, and to sum of powers of Fibonacci and Lucas polynomials. On the other hand, for polynomials with $q(x)\ne\pm1$, such as Jacobsthal $J_n(x)$ and Jacobsthal–Lucas polynomials $j_n(x)$, which have $p(x)=1$, $q(x)=2x$ and $d=1$ or $d=2$, we obtain identities concerning the sums of powers of the rational functions $J_{2kl+r}(x)/(2x)^{lk}$ and $j_{2kl+r}(x)/(2x)^{lk}$. We leave to the reader the exploration of the numerous variants arising from general formulas (19) and (21) by conveniently changing the involved parameters.

Now, we point out some divisibility properties for polynomials obtained by multiplying our sum of powers with other suitable polynomials, generalizing the results on the so-called Melham sums, or Melham convolutions, introduced in [3] and studied in [5–9]. At the same time, we will show that the divisibility properties conjectured in [3] and proved in [5, 9] are only special cases of analogous properties for polynomials defined by recurrence (1) and simple consequences of Corollary 9.

3.1. Odd Powers. From (19) and (48), we recognize that the polynomial

$$q^{nl(2m+1)}\prod_{j=0}^{m}W^{(\beta)}_{l(2j+1)}\sum_{k=1}^{n}(-1)^{\delta k}\left(\frac{W^{(d)}_{2lk+r}}{q^{lk}}\right)^{2m+1} \tag{59}$$

is equal to

$$q^{nl(2m+1)}\prod_{j=0}^{m}W^{(\beta)}_{l(2j+1)}\Bigg((-1)^{\delta n}\cdot\Delta^{(2m+1)(d-2)-\alpha}\sum_{h=0}^{m}\binom{2m+1}{m-h}\left((-1)^{d+r}q^{r}\right)^{m-h}\cdot\mathscr{W}_{2h+1}\Bigg), \tag{60}$$

or equivalently to the product

$$q^{nl(2m+1)}\prod_{j=0}^{m}W^{(\beta)}_{l(2j+1)}\left(W^{(\alpha+\beta)}_{l(2n+1)+r}+(-1)^{n\delta+1}q^{nl}W^{(\alpha+\beta)}_{(l+r)}\right)\cdot F, \tag{61}$$

where

$$F = \frac{\Sigma_{1,2m+1}}{q^{nl}W^{(\alpha+\beta+1)}_{l+r}} + \left(W^{(\alpha+\beta)}_{l(2n+1)+r}+(-1)^{n\delta+1}q^{nl}W^{(\alpha+\beta)}_{l+r}\right)\Sigma_{2,2m+1}. \tag{62}$$

We need to pay attention in order to state the correct divisibility property involving the polynomial $W^{(\alpha+\beta)}_{l(2n+1)+r}+(-1)^{n\delta+1}q^{nl}W^{(\alpha+\beta)}_{(l+r)}$. Indeed, if $d\ge3$, the factor $\Delta^{(2m+1)(d-2)-\alpha}$ in (60) is a polynomial because $(2m+1)(d-2)-\alpha$ is a positive even integer (we recall that, by definition, $\alpha\in\{1,2\}$ and $\alpha\equiv d \bmod 2$), so we may state that

$$\left(W^{(\alpha+\beta)}_{l(2n+1)+r}+(-1)^{n\delta+1}q^{nl}W^{(\alpha+\beta)}_{(l+r)}\right)\mid q^{nl(2m+1)}\prod_{j=0}^{m}W^{(\beta)}_{l(2j+1)}\cdot\sum_{k=1}^{n}(-1)^{\delta k}\left(\frac{W^{(d)}_{2lk+r}}{q^{lk}}\right)^{2m+1}, \tag{63}$$

but when $d=2$ or $d=1$, the term $\Delta^{(2m+1)(d-2)-\alpha}$ is not a polynomial since it has a negative even exponent. Thus, in these two cases, we have to take into account the fact that the presence of this term is balanced by the even powers of Δ nested in the sum on the right member of (19). In particular, from Corollary 9, we know

$$\Delta^2\mid\left(W^{(\alpha+\beta)}_{l(2n+1)+r}+(-1)^{n\delta+1}q^{nl}W^{(\alpha+\beta)}_{(l+r)}\right) \tag{64}$$

and it is easy to verify that Δ^2 is the greatest positive even power of Δ which is a polynomial factor of $W^{(\alpha+\beta)}_{l(2n+1)+r}+(-1)^{n\delta+1}q^{nl}W^{(\alpha+\beta)}_{(l+r)}$. Therefore, for $d\in\{1,2\}$, the correct statement is

$$\frac{W^{(\alpha+\beta)}_{l(2n+1)+r}+(-1)^{n\delta+1}q^{nl}W^{(\alpha+\beta)}_{(l+r)}}{\Delta^2}\mid q^{nl(2m+1)}\prod_{j=0}^{m}W^{(\beta)}_{l(2j+1)}\cdot\sum_{k=1}^{n}(-1)^{\delta k}\left(\frac{W^{(d)}_{2lk+r}}{q^{lk}}\right)^{2m+1} \tag{65}$$

in order to preserve in the factor F the correct amount of positive even powers of Δ balancing the term $\Delta^{(2m+1)(d-2)-\alpha}$.

Moreover we observe that when $r = 0$ and d odd, that is, $\alpha = 1$, the following binomial identity

$$\sum_{h=0}^{m} \binom{2m+1}{m-h} (-1)^{d(m-h)} (2h+1) = \frac{(-1)^{md}}{2} \left(1 + (-1)^{d}\right) (2m+1) \binom{2m}{m} \tag{66}$$

ensures that $\Sigma_{1,2m+1} = 0$ since $W^{(\alpha+\beta+1)}_{(l+r)(2h+1)} / W^{(\beta)}_{l(2h+1)} = W^{(\beta+2)}_{l(2h+1)} / W^{(\beta)}_{l(2h+1)} = \Delta^2$. Thus, with a similar reasoning as before we have for ≥ 3, d odd,

$$\begin{aligned} &\left(W^{(\beta+1)}_{l(2n+1)} + (-1)^{n\delta+1} q^{nl} W^{(\beta+1)}_{l}\right)^2 \mid q^{nl(2m+1)} \prod_{j=1}^{m} W^{(\beta)}_{l(2j+1)} \\ &\cdot \sum_{k=1}^{n} (-1)^{\delta k} \left(\frac{W^{(d)}_{2lk}}{q^{lk}}\right)^{2m+1} \end{aligned} \tag{67}$$

and for $d = 1$

$$\begin{aligned} &\frac{\left(W^{(\beta+1)}_{l(2n+1)} + (-1)^{n\delta+1} q^{nl} W^{(\beta+1)}_{l}\right)^2}{\Delta^4} \mid q^{nl(2m+1)} \prod_{j=1}^{m} W^{(\beta)}_{l(2j+1)} \\ &\cdot \sum_{k=1}^{n} (-1)^{\delta k} \left(\frac{W^{(1)}_{2lk}}{q^{lk}}\right)^{2m+1}. \end{aligned} \tag{68}$$

We resume all these results in the following theorem.

Theorem 11. *For integers d, l, r, m, t, n, δ, with $d \geq 1$, $l > 0$, $r \geq -2l$, $m \geq 0$, $t, n \geq 1$, and $\delta \in \{0,1\}$, considering a polynomial sequence $(W^{(d)}_i(x))_{i=0}^{+\infty}$ satisfying recurrence (1), we have the following divisibility relations:*

$$\begin{aligned} &\left(W^{(\alpha+\beta)}_{l(2n+1)+r} + (-1)^{n\delta+1} q^{nl} W^{(\alpha+\beta)}_{(l+r)}\right) \mid q^{nl(2m+1)} \prod_{j=0}^{m} W^{(\beta)}_{l(2j+1)} \\ &\cdot \sum_{k=1}^{n} (-1)^{\delta k} \left(\frac{W^{(d)}_{2lk+r}}{q^{lk}}\right)^{2m+1}, \quad \forall d \geq 3 \end{aligned} \tag{69}$$

$$\begin{aligned} &\frac{W^{(\alpha+\beta)}_{l(2n+1)+r} + (-1)^{n\delta+1} q^{nl} W^{(\alpha+\beta)}_{(l+r)}}{\Delta^2} \mid q^{nl(2m+1)} \prod_{j=0}^{m} W^{(\beta)}_{l(2j+1)} \\ &\cdot \sum_{k=1}^{n} (-1)^{\delta k} \left(\frac{W^{(d)}_{2lk+r}}{q^{lk}}\right)^{2m+1}, \quad \textit{if } d \in \{1,2\} \end{aligned} \tag{70}$$

where $\alpha, \beta \in \{1,2\}$ and $\alpha \equiv d \bmod 2$, $\beta \equiv \delta + l + 1 \bmod 2$.

Furthermore, when $r = 0$

$$\begin{aligned} &\left(W^{(\beta+1)}_{l(2n+1)} + (-1)^{n\delta+1} q^{nl} W^{(\beta+1)}_{l}\right)^2 \mid q^{nl(2m+1)} \prod_{j=1}^{m} W^{(\beta)}_{l(2j+1)} \\ &\cdot \sum_{k=1}^{n} (-1)^{\delta k} \left(\frac{W^{(2t+1)}_{2lk}}{q^{lk}}\right)^{2m+1}, \\ &\frac{\left(W^{(\beta+1)}_{l(2n+1)} + (-1)^{n\delta+1} q^{nl} W^{(\beta+1)}_{l}\right)^2}{\Delta^4} \mid q^{nl(2m+1)} \prod_{j=1}^{m} W^{(\beta)}_{l(2j+1)} \\ &\cdot \sum_{k=1}^{n} (-1)^{\delta k} \left(\frac{W^{(1)}_{2lk}}{q^{lk}}\right)^{2m+1}. \end{aligned} \tag{71}$$

We observe that the previous theorem allows us to give a refinement for Corollary 7 in [13]. Indeed, if we consider Chebyshev polynomials, that is, $p(x) = 2x$, $q(x) = -1$, $\Delta^2 = 4(x^2 - 1)$, and $(W^{(1)}_i(x))_{i=0}^{+\infty} = (U_{i-1}(x))_{i=0}^{+\infty}$, $(W^{(2)}_i(x))_{i=0}^{+\infty} = (2T_i(x))_{i=0}^{+\infty}$, with the choices $\delta = 1$, $l = 1$, we have $\beta = 1$ and, thanks to rule (6),

$$\begin{aligned} \frac{\left(2T_{2n+1}(x) - 2T_1(x)\right)^2}{\Delta^4} &= \frac{\left(\Delta^2 U_n(x) U_{n-1}(x)\right)^2}{\Delta^4} \\ &= U_n^2(x) U_{n-1}^2(x). \end{aligned} \tag{72}$$

So, omitting the trivial factor $(-1)^n = q^{n(2m+1)}$, relation (71) becomes

$$U_n^2(x) U_{n-1}^2(x) \mid \prod_{j=1}^{m} U_{2j}(x) \sum_{k=1}^{n} \left(U_{2k-1}(x)\right)^{2m+1} \tag{73}$$

or equivalently (since $2 \mid U_{2k-1}(x)$ for all $k \geq 1$ and $T_1(x) = x$)

$$\left(T_{2n+1}(x) - x\right)^2 \mid \Delta^4 \prod_{j=1}^{m} U_{2j}(x) \sum_{k=1}^{n} \left(U_{2k-1}(x)\right)^{2m+1}. \tag{74}$$

Finally we point out that, unfortunately, the statement of Corollary 7 in [13] is wrong. It asserts (in our notations) that for integers $m \geq 0$, $n \geq 1$

$$\begin{aligned} &\prod_{j=0}^{m} U_{2j}(x) \sum_{k=1}^{n} \left(U_{2k-1}(x)\right)^{2m+1} \\ &= \left(T_{2n+1}(x) - x\right) Q_{2m}\left(x, T_{2n+1}\right), \end{aligned} \tag{75}$$

where $Q_{2m}(x, y)$ is an integer coefficients polynomial of two variables with degree $2m$ in y. But a simple calculation when $m = 0$, $n = 1$ shows that the left member of (75) is $U_0(x) U_1(x) = 2x$, which obviously can not be divided by the factor $T_3(x) - x = 4x^3 - 4x$ in the right member. Equality (75) also fails in the case ≥ 1. For instance, when $m = 1$ and $n = 1$, we get

$$\begin{aligned} 8x^3\left(4x^2 - 1\right) &= U_0(x) U_2(x) U_1^3(x) \\ &= \left(T_3(x) - x\right) Q_2\left(x, T_3(x)\right) \\ &= 4x\left(x^2 - 1\right) Q_2\left(x, 4x^3 - x\right) \end{aligned} \tag{76}$$

and clearly $4x(x^2-1) \nmid 8x^3(4x^2-1)$. The correct statement is the one given by (70), which reveals that when m, n are positive integers, we need Δ^2 as extra factor in the left member of (75) in order to obtain a real equality.

3.2. Even Powers. When we deal with sums involving even powers, considering identity (21) and the results of Corollary 10, from (52), we easily find that for the difference

$$\begin{aligned}\mathscr{D}_d &= \sum_{k=1}^{n}\left(\frac{W_{2lk+r}^{(d)}}{q^{kl}}\right)^{2m} \\ &\quad - \left(\frac{W_{2(l(2n+1)+r)}^{(1)} - q^{2nl}W_{2(l+r)}^{(1)}}{q^{2nl}W_{2(l+r)}^{(1)}}\Sigma_{1,2m}\right. \\ &\quad \left. + n\binom{2m}{m}\left((-1)^{d+r}q^{r}\Delta^{2(d-2)}\right)^{m}\right)\end{aligned} \quad (77)$$

the following divisibility relation holds for all $d \geq 2$:

$$\begin{aligned}&W_{2(l(2n+1)+r)}^{(1)}\left(W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)}\right) \mid \\ &q^{2mnl}\prod_{j=1}^{m}W_{2lj}^{(1)}\mathscr{D}_d,\end{aligned} \quad (78)$$

and if $d = 1$

$$\begin{aligned}&W_{2(l(2n+1)+r)}^{(1)}\left(W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)}\right) \mid \\ &\Delta^{2m}q^{2mnl}\prod_{j=1}^{m}W_{2lj}^{(1)}\mathscr{D}_1,\end{aligned} \quad (79)$$

since in this case we need to balance the factor Δ^{-2m} in $\mathscr{D}_1$.

When $\delta = 1$ and n is even, from (55) with $d \geq 2$, we have

$$\begin{aligned}&\left(W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)}\right) \mid q^{2mnl}\prod_{j=1}^{m}W_{2lj}^{(2)} \\ &\cdot \sum_{k=1}^{n}(-1)^k\left(\frac{W_{2lk+r}^{(d)}}{q^{kl}}\right)^{2m}.\end{aligned} \quad (80)$$

In the case $d = 1$, we have to consider the presence of the factor Δ^{-2m} in the term $\Sigma_{3,2m}$, so we apply the same reasoning we have done in the previous subsection, in order to preserve the positive even powers of Δ nested in $(W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)})\Sigma_{3,2m}$. Thus, since from Corollary 10 we know that $\Delta^2 \mid W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)}$, we find the divisibility relation for n even

$$\begin{aligned}&\frac{W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)}}{\Delta^2} \\ &\quad = W_{2(l(n+1)+r)}^{(1)}W_{2nl}^{(1)} \mid q^{2mnl}\prod_{j=1}^{m}W_{2lj}^{(2)} \\ &\quad \cdot \sum_{k=1}^{n}(-1)^k\left(\frac{W_{2lk+r}^{(1)}}{q^{kl}}\right)^{2m}.\end{aligned} \quad (81)$$

In the last case $\delta = 1$, n odd, corresponding to equality (57), if we consider the sum

$$\begin{aligned}\mathscr{S}_d &= \sum_{k=1}^{n}(-1)^k\left(\frac{W_{2lk+r}^{(d)}}{q^{kl}}\right)^{2m} + \Sigma_{4,2m} \\ &\quad + \binom{2m}{m}\left((-1)^{d+r}q^{r}\Delta^{2(d-2)}\right)^{m},\end{aligned} \quad (82)$$

we have

$$\begin{aligned}&W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)} \mid q^{2mnl}\prod_{j=1}^{m}W_{2lj}^{(2)}\mathscr{S}_d, \\ &W_{2(l(2n+1)+r)}^{(2)} - q^{2nl}W_{2(l+r)}^{(2)} \mid \Delta^{2m}q^{2mnl}\prod_{j=1}^{m}W_{2lj}^{(2)}\mathscr{S}_1,\end{aligned} \quad (83)$$

since we need to balance the factor Δ^{-2m} in $\mathscr{S}_1$.

When $r = 0$, taking into account the following binomial identity,

$$\begin{aligned}&\sum_{h=1}^{m}\binom{2m}{m-h}(-1)^{d(m-h)} \\ &\quad = \frac{1}{2}\left(\left(1+(-1)^d\right)^{2m} - (-1)^{md}\binom{2m}{m}\right),\end{aligned} \quad (84)$$

we also obtain further simplifications of the previous relations.

Indeed, when d is odd, from (84), (53), and (58), we get

$$\begin{aligned}\Sigma_{1,2m} &= \frac{(-1)^{m+1}\Delta^{2m(d-2)}}{2}\binom{2m}{m}, \\ \Sigma_{4,2m} &= (-1)^{m+1}\Delta^{2m(d-2)}\binom{2m}{m}.\end{aligned} \quad (85)$$

In particular, the right members of equalities (55) and (57) differ only by a minus sign; thus we have for all positive integers n and t

$$\begin{aligned}&\left(W_{2l(2n+1)}^{(2)} - q^{2nl}W_{2l}^{(2)}\right) \mid q^{2mnl}\prod_{j=1}^{m}W_{2lj}^{(2)} \\ &\cdot \sum_{k=1}^{n}(-1)^k\left(\frac{W_{2lk}^{(2t+1)}}{q^{kl}}\right)^{2m}, \\ &\frac{W_{2l(2n+1)}^{(2)} - q^{2nl}W_{2l}^{(2)}}{\Delta^2} = W_{2l(n+1)}^{(1)}W_{2nl}^{(1)} \mid q^{2mnl}\prod_{j=1}^{m}W_{2lj}^{(2)} \\ &\cdot \sum_{k=1}^{n}(-1)^k\left(\frac{W_{2lk}^{(1)}}{q^{kl}}\right)^{2m}.\end{aligned} \quad (86)$$

We resume all these results in the following theorem.

Theorem 12. *For integers d, l, r, m, t, n, δ, with $d \geq 1$, $l > 0$, $r \geq -2l$, $m \geq 0$, $t, n \geq 1$, and $\delta \in \{0,1\}$, considering a polynomial sequence $(W_i^{(d)}(x))_{i=0}^{+\infty}$ satisfying recurrence (1), one has the following divisibility relations:*

$$\begin{aligned}
&W^{(1)}_{2(l(2n+1)+r)}\left(W^{(2)}_{2(l(2n+1)+r)} - q^{2nl}W^{(2)}_{2(l+r)}\right) \mid \\
&\quad q^{2mnl}\prod_{j=1}^{m} W^{(1)}_{2lj}\mathcal{D}_d, \\
&\left(W^{(2)}_{2(l(2n+1)+r)} - q^{2nl}W^{(2)}_{2(l+r)}\right) \mid q^{2mnl}\prod_{j=1}^{m} W^{(2)}_{2lj} \\
&\quad \cdot \sum_{k=1}^{n}(-1)^k\left(\frac{W^{(d)}_{2lk+r}}{q^{kl}}\right)^{2m}, \quad n \text{ odd}, \\
&W^{(2)}_{2(l(2n+1)+r)} - q^{2nl}W^{(2)}_{2(l+r)} \mid q^{2mnl}\prod_{j=1}^{m} W^{(2)}_{2lj}\mathcal{S}_d, \quad n \text{ even}, \\
&W^{(1)}_{2(l(2n+1)+r)}\left(W^{(2)}_{2(l(2n+1)+r)} - q^{2nl}W^{(2)}_{2(l+r)}\right) \mid \\
&\quad \Delta^{2m}q^{2mnl}\prod_{j=1}^{m} W^{(1)}_{2lj}\mathcal{D}_1, \\
&\frac{W^{(2)}_{2(l(2n+1)+r)} - q^{2nl}W^{(2)}_{2(l+r)}}{\Delta^2} = W^{(1)}_{2(l(n+1)+r)}W^{(1)}_{2nl} \mid \\
&\quad q^{2mnl}\prod_{j=1}^{m} W^{(2)}_{2lj}\sum_{k=1}^{n}(-1)^k\left(\frac{W^{(1)}_{2lk+r}}{q^{kl}}\right)^{2m}, \quad n \text{ odd} \\
&W^{(2)}_{2(l(2n+1)+r)} - q^{2nl}W^{(2)}_{2(l+r)} \mid \Delta^{2m}q^{2mnl}\prod_{j=1}^{m} W^{(2)}_{2lj}\mathcal{S}_1, \\
&\qquad n \text{ even}, \\
&\qquad \forall d \geq 2,
\end{aligned} \tag{87}$$

where

$$\begin{aligned}
\mathcal{D}_d &= \sum_{k=1}^{n}\left(\frac{W^{(d)}_{2lk+r}}{q^{kl}}\right)^{2m} \\
&\quad - \left(\frac{W^{(1)}_{2(l(2n+1)+r)} - q^{2nl}W^{(1)}_{2(l+r)}}{q^{2nl}W^{(1)}_{2(l+r)}}\Sigma_{1,2m}\right. \\
&\quad \left. + n\binom{2m}{m}\left((-1)^{d+r}q^r\Delta^{2(d-2)}\right)^m\right),
\end{aligned} \tag{88}$$

$$\begin{aligned}
\mathcal{S}_d &= \sum_{k=1}^{n}(-1)^k\left(\frac{W^{(d)}_{2lk+r}}{q^{kl}}\right)^{2m} + \Sigma_{4,2m} \\
&\quad + \binom{2m}{m}\left((-1)^{d+r}q^r\Delta^{2(d-2)}\right)^m.
\end{aligned}$$

Furthermore, if $r = 0$

$$\begin{aligned}
&\left(W^{(2)}_{2l(2n+1)} - q^{2nl}W^{(2)}_{2l}\right) \mid q^{2mnl}\prod_{j=1}^{m} W^{(2)}_{2lj} \\
&\quad \cdot \sum_{k=1}^{n}(-1)^k\left(\frac{W^{(2t+1)}_{2lk}}{q^{kl}}\right)^{2m}, \\
&\frac{W^{(2)}_{2l(2n+1)} - q^{2nl}W^{(2)}_{2l}}{\Delta^2} = W^{(1)}_{2l(n+1)}W^{(1)}_{2nl} \mid q^{2mnl}\prod_{j=1}^{m} W^{(2)}_{2lj} \\
&\quad \cdot \sum_{k=1}^{n}(-1)^k\left(\frac{W^{(1)}_{2lk}}{q^{kl}}\right)^{2m}.
\end{aligned} \tag{89}$$

4. Conclusions

We have found formulas which give a clear, interesting, and comprehensive synthesis of many identities related to power sums of well-known families of polynomials and, in general, for a wide class of functions satisfying recurrence (1). We think that these formulas, beyond their intrinsic beauty, could be useful in applications. Indeed many of the polynomials satisfying recurrence (1), such as Chebyshev and Jacobsthal–Lucas polynomials, have several applications in physics and engineering, for example, concerning solutions of partial and ordinary differential equations or good approximations of functions using polynomials. In some calculations power sums may arise and our formulas give a simpler form to these sums (also with alternating sign), providing relatively easy summations avoiding powers of polynomials. Moreover divisibility relations between polynomials, which also generalize the ones considered by Melham for Fibonacci and Lucas numbers in [3], could give other additional information related to the knowledge of the factorization of the power sums and convolutions involved. Finally, thanks to Girard–Waring formulas, we have also proved the equivalents of the two conjectures presented in [3] for all the polynomial sequences defined by (1).

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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