

Research Article

Common Fixed Points in a Partially Ordered Partial Metric Space

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In the first part of this paper, we prove some generalized versions of the result of Matthews in (Matthews, 1994) using different types of conditions in partially ordered partial metric spaces for dominated self-mappings or in partial metric spaces for self-mappings. In the second part, using our results, we deduce a characterization of partial metric 0-completeness in terms of fixed point theory. This result extends the Subrahmanyam characterization of metric completeness.

1. Introduction

In the mathematical field of domain theory, attempts were made in order to equip semantics domain with a notion of distance. In particular, Matthews [1] introduced the notion of a partial metric space as a part of the study of denotational semantics of data for networks, showing that the contraction mapping principle can be generalized to the partial metric context for applications in program verification. Moreover, the existence of several connections between partial metrics and topological aspects of domain theory has been lately pointed by other authors as O'Neill [2], Bukatin and Scott [3], Bukatin and Shorina [4], Romaguera and Schellekens [5], and others (see also [6–14] and the references therein).

After the result of Matthews [1], the interest for fixed point theory developments in partial metric spaces has been constantly growing, and many authors presented significant contributions in the directions of establishing partial metric versions of well-known fixed point theorems for the existence of fixed points, common fixed points, and coupled fixed points in classical metric spaces (see e.g., [15, 16]). Obviously, we cannot cite all these papers but we give only a partial list [17–49].

Recently, Romaguera [50] proved that a partial metric space (X, d) is 0-complete if and only if every p^s -Caristi mapping on X has a fixed point. In particular, the result of Romaguera extended Kirk's [51] characterization of metric completeness to a kind of complete partial metric spaces.

Successively, Karapinar in [36] extended the result of Caristi and Kirk [52] to partial metric spaces.

In the first part of this paper, following this research direction, we prove some generalized versions of the result of Matthews by using different types of conditions in ordered partial metric spaces for dominated self-mappings or in partial metric spaces for self-mappings. The notion of dominated mapping of economics, finance, trade, and industry is also applied to approximate the unique solution of nonlinear functional equations. In the second part, using the results obtained in the first part, we deduce a characterization of partial metric 0-completeness in terms of fixed point theory. This result extends the Subrahmanyam [53] characterization of metric completeness. For other characterizations of metric completeness in terms of fixed point theory, the reader can see, for example, [54, 55] and for partial metric completeness, [41].

2. Preliminaries

First, we recall some definitions and some properties of partial metric spaces that can be found in [1, 2, 40, 48, 50]. A partial metric on a nonempty set X is a function $p : X \times X \rightarrow [0, +\infty)$ such that for all $x, y, z \in X$

$$(p_1) \quad x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y),$$

$$(p_2) \quad p(x, x) \leq p(x, y),$$

$$(p_3) p(x, y) = p(y, x),$$

$$(p_4) p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X . It is clear that if $p(x, y) = 0$, then from (p_1) and (p_2) , it follows that $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair $([0, +\infty), p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, +\infty)$. Other examples of partial metric spaces which are interesting from a computational point of view can be found in [1].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p -balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\} \quad (1)$$

for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X , then the function $p^s : X \times X \rightarrow [0, +\infty)$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \quad (2)$$

is a metric on X .

Let (X, p) be a partial metric space. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow +\infty} p(x, x_n)$.

A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n, m \rightarrow +\infty} p(x_n, x_m)$.

A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p , to a point $x \in X$ such that $p(x, x) = \lim_{n, m \rightarrow +\infty} p(x_n, x_m)$.

A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n, m \rightarrow +\infty} p(x_n, x_m) = 0$. We say that (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p , to a point $x \in X$ such that $p(x, x) = 0$.

On the other hand, the partial metric space $(\mathbb{Q} \cap [0, +\infty), p)$, where \mathbb{Q} denotes the set of rational numbers and the partial metric p is given by $p(x, y) = \max\{x, y\}$, provides an example of a 0-complete partial metric space which is not complete.

It is easy to see that every closed subset of a complete partial metric space is complete.

Lemma 1 (see [1, 40]). *Let (X, p) be a partial metric space. Then*

- (a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s) .
- (b) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim_{n \rightarrow +\infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n \rightarrow +\infty} p(x_n, x) = \lim_{n, m \rightarrow +\infty} p(x_n, x_m). \quad (3)$$

The following lemma is obvious.

Lemma 2. *Let (X, p) be a partial metric space and $\{x_n\} \subset X$. If $x_n \rightarrow x \in X$ and $p(x, x) = 0$, then $\lim_{n \rightarrow +\infty} p(x_n, z) = p(x, z)$ for all $z \in X$.*

Define $p(x, A) = \inf\{p(x, a) : a \in A\}$. Then $a \in \overline{A} \Leftrightarrow p(a, A) = p(a, a)$, where \overline{A} denotes the closure of A (for details see [22, Lemma 1]). From

$$p^s(x, a) = 2p(x, a) - p(x, x) - p(a, a) \leq 2p(x, a) \quad (4)$$

for every $a \in A$, we deduce that $p^s(x, A) \leq 2p(x, A)$.

Let X be a nonempty set and $T, f : X \rightarrow X$. The mappings T, f are said to be weakly compatible if they commute at their coincidence points (i.e., $Tfx = fTx$ whenever $Tx = fx$). A point $y \in X$ is called a point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$.

Lemma 3 (see [56]). *Let X be a nonempty set and the mappings $T, f : X \rightarrow X$ have a unique point of coincidence v in X . If T and f are weakly compatible, then T and f have a unique common fixed point.*

Let X be a nonempty set. If (X, p) is a partial metric space and (X, \preceq) is a partially ordered set, then (X, p, \preceq) is called a partially ordered partial metric space. $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. Let (X, \preceq) be a partially ordered set and $T, f : X \rightarrow X$ two mappings. T is called an f -dominated mapping if $Tx \preceq fx$ for every $x \in X$.

3. Main Results

Let (X, p) be a partial metric space and $T, f : X \rightarrow X$ be such that $TX \subset fX$. For every $x_0 \in X$ we consider the sequence $\{x_n\} \subset X$ defined by $fx_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ and we say that $\{Tx_n\}$ is a T - f -sequence of the initial point x_0 (see [57]).

Denote with Ψ the family of non-decreasing functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$ such that $\psi(t) > 0$ and $\lim_{n \rightarrow +\infty} \psi^n(t) = 0$ for each $t > 0$, where ψ^n is the n th iterate of ψ .

Lemma 4. *For every function $\psi \in \Psi$, the following holds, if for each $t > 0$, $\lim_{n \rightarrow +\infty} \psi^n(t) = 0$ then $\psi(t) < t$.*

The following theorem is one of our main results, and it ensures the existence of a common fixed point for two self-mappings in the setting of partially ordered partial metric spaces.

Theorem 5. *Let (X, p, \preceq) be a partially ordered partial metric space and $T, f : X \rightarrow X$ two mappings such that $TX \subset fX$. Assume that there exists $\psi \in \Psi$ such that*

$$p(Tx, Ty) \leq \max\{\psi(p(fx, fy)), \psi(p(fx, Tx)), \psi(p(fy, Ty))\} \quad (5)$$

for all $x, y \in X$ with fx and fy comparable. If the following conditions hold:

- (i) T is a f -dominated mapping,
- (ii) either TX or fX is a 0-complete subspace of X ,

(iii) for a non-increasing sequence $\{fx_n\} \subset X$ converging to $fu \in X$, we have $fu \leq fx_n$ for all $n \in \mathbb{N}$ and $ffu \leq fu$,

then T and f have a point of coincidence. Moreover, if T and f are weakly compatible, then T and f have a common fixed point.

Proof. Let $x_0 \in X$ be fixed and $\{Tx_n\}$ be a T - f -sequence of the initial point x_0 . As $fx_{n+1} = Tx_n \leq fx_n$ for all $n \in \mathbb{N}$, then the sequence $\{Tx_n\}$ is non-increasing.

If $Tx_n = Tx_{n-1} = fx_n$ for some $n \in \mathbb{N}$, then $y = Tx_n = fx_n$ is a point of coincidence of T and f . Suppose that $Tx_n \neq Tx_{n-1}$ for all $n \in \mathbb{N}$. Since fx_n and fx_{n+1} are comparable for all $n \in \mathbb{N}$, we have

$$\begin{aligned} & p(Tx_{n+1}, Tx_n) \\ & \leq \max\{\psi(p(fx_{n+1}, fx_n)), \psi(p(fx_{n+1}, Tx_{n+1})), \\ & \quad \psi(p(fx_n, Tx_n))\} \\ & = \max\{\psi(p(Tx_n, Tx_{n-1})), \psi(p(Tx_{n+1}, Tx_n))\}. \end{aligned} \tag{6}$$

If $\max\{\psi(p(Tx_n, Tx_{n-1})), \psi(p(Tx_{n+1}, Tx_n))\} = \psi(p(Tx_{n+1}, Tx_n))$ from

$$\begin{aligned} p(Tx_{n+1}, Tx_n) & \leq \psi(p(Tx_{n+1}, Tx_n)) \\ & < p(Tx_{n+1}, Tx_n), \end{aligned} \tag{7}$$

we obtain a contradiction and so

$$\begin{aligned} & \max\{\psi(p(Tx_n, Tx_{n-1})), \psi(p(Tx_{n+1}, Tx_n))\} \\ & = \psi(p(Tx_n, Tx_{n-1})). \end{aligned} \tag{8}$$

Then, we have

$$p(Tx_{n+1}, Tx_n) \leq \psi^n(p(Tx_1, Tx_0)), \quad \forall n \in \mathbb{N} \tag{9}$$

and hence

$$\lim_{n \rightarrow +\infty} p(Tx_{n+1}, Tx_n) = 0. \tag{10}$$

Fix $\varepsilon > 0$ and we choose $n(\varepsilon) \in \mathbb{N}$ such that

$$p(Tx_m, Tx_{m+1}) < \varepsilon - \psi(\varepsilon) \tag{11}$$

for all $m \geq n(\varepsilon)$. Let $m \geq n(\varepsilon)$ and we show that

$$p(Tx_m, Tx_{n+1}) < \varepsilon, \quad \forall n \geq m. \tag{12}$$

Clearly, (12) is true for $n = m$. Suppose that (12) holds for some $n \geq m$, as fx_i and fx_j are comparable for all $i, j \in \mathbb{N}$, then

$$\begin{aligned} & p(Tx_m, Tx_{n+2}) \\ & \leq p(Tx_m, Tx_{m+1}) + p(Tx_{m+1}, Tx_{n+2}) \\ & \leq p(Tx_m, Tx_{m+1}) \\ & \quad + \max\{\psi(p(Tx_m, Tx_{n+1})), \psi(p(Tx_m, Tx_{m+1})), \\ & \quad \psi(p(Tx_{n+1}, Tx_{n+2}))\} \\ & < \varepsilon - \psi(\varepsilon) + \max\{\psi(\varepsilon), \psi(\varepsilon - \psi(\varepsilon))\} = \varepsilon. \end{aligned} \tag{13}$$

This implies that (12) holds for $n + 1$ and by induction, it holds for all $n \geq m$. From (12), we deduce that there exists

$$\lim_{n, m \rightarrow +\infty} p(Tx_n, Tx_m) = 0 \tag{14}$$

and hence $\{Tx_n\}$ is a 0-Cauchy sequence.

Suppose that TX is a 0-complete subspace of (X, p) , then there exists $y \in TX \subset fX$ such that

$$p(y, y) = \lim_{n \rightarrow +\infty} p(Tx_n, y) = \lim_{n, m \rightarrow +\infty} p(Tx_n, Tx_m) = 0. \tag{15}$$

This holds also if fX is 0-complete with $y \in fX$.

Let $u \in X$ be such that $y = fu$. We show that y is a point of coincidence of T and f . If not, we have $p(fu, Tu) > 0$. This implies that there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} & \max\{\psi(p(Tx_{n-1}, fu)), \psi(p(Tx_{n-1}, Tx_n)), \psi(p(fu, Tu))\} \\ & = \psi(p(fu, Tu)), \end{aligned} \tag{16}$$

for every $n \geq n_0$. By condition (iii), fx_n and fu are comparable for every $n \in \mathbb{N}$ and hence, by condition (5) with $x = x_n$ and $y = u$, we deduce that

$$\begin{aligned} & p(Tx_n, Tu) \\ & \leq \max\{\psi(p(fx_n, fu)), \psi(p(fx_n, Tx_n)), \\ & \quad \psi(p(fu, Tu))\} \\ & = \max\{\psi(p(Tx_{n-1}, fu)), \psi(p(Tx_{n-1}, Tx_n)), \\ & \quad \psi(p(fu, Tu))\} \\ & = \psi(p(fu, Tu)), \end{aligned} \tag{17}$$

for every $n \geq n_0$. Letting $n \rightarrow +\infty$ in the previous inequality and using Lemma 2, we obtain

$$p(fu, Tu) \leq \psi(p(fu, Tu)) < p(fu, Tu), \tag{18}$$

which implies $p(fu, Tu) = 0$, that is, $Tu = fu$. Thus, we have shown that $y = fu = Tu$ is a point of coincidence of T and f . If T and f are weakly compatible, then $Ty = Tfu = fTu = fy$. By condition (iii), $fy = ffu \leq fu$, that is, fy and fu are comparable. Using the contractive condition (5), we get

$$\begin{aligned} & p(Tu, Ty) \\ & \leq \max\{\psi(p(fu, fy)), \psi(p(fu, Tu)), \psi(p(fy, Ty))\} \\ & = \psi(p(fu, fy)) < p(fu, fy), \end{aligned} \tag{19}$$

which implies $Ty = Tu = y$ and hence y is a common fixed point of T and f . \square

We shall give a sufficient condition for the uniqueness of the common fixed point in Theorem 5.

Theorem 6. *Let all the conditions of Theorem 5 be satisfied. If the following condition holds:*

(iv) for all $x, y \in fX$ there exists $v_0 \in X$ such that $fv_0 \leq x, fv_0 \leq y$ and $\lim_{n \rightarrow +\infty} p(Tv_{n-1}, Tv_n) = 0$, where $\{Tv_n\}$ is the T - f -sequence of the initial point v_0 ,

then T and f have a unique common fixed point.

Proof. Let z, w be two common fixed points of T and f with $z \neq w$. If z and w are comparable, then using the contractive condition (5), we deduce that $z = w$. If z and w are not comparable, then there exists $v_0 \in X$ such that $fv_0 \leq z = fz, fv_0 \leq w = fw$. As T is a f -dominated mapping, we get that

$$fv_1 = Tv_0 \leq fv_0 \leq z = fz. \tag{20}$$

To continue, we obtain

$$fv_{n+1} = Tv_n \leq fv_n \leq z = fz \quad \forall n \in \mathbb{N} \tag{21}$$

and hence fv_n and fz are comparable.

Using the contractive condition (5) with $x = v_n$ and $y = z$, we get

$$\begin{aligned} p(Tv_n, Tz) &\leq \max \{ \psi(p(fv_n, fz)), \psi(p(fv_n, Tv_n)), \\ &\quad \psi(p(fz, Tz)) \} \\ &= \max \{ \psi(p(Tv_{n-1}, Tz)), \psi(p(Tv_{n-1}, Tv_n)), \\ &\quad \psi(p(Tz, Tz)) \} \end{aligned} \tag{22}$$

for all $n \in \mathbb{N}$. Since, the contractive condition (5) ensures that $p(Tz, Tz) = 0$, we have

$$p(Tv_n, Tz) \leq \max \{ \psi(p(Tv_{n-1}, Tz)), \psi(p(Tv_{n-1}, Tv_n)) \}. \tag{23}$$

Now, by condition (iv), $\lim_{n \rightarrow +\infty} p(Tv_{n-1}, Tv_n) = 0$ and hence for n sufficiently large, we have

$$p(Tv_n, Tz) \leq \psi(p(Tv_{n-1}, Tz)). \tag{24}$$

Without loss of generality, assuming that (24) holds for all $n \in \mathbb{N}$, it follows that

$$p(Tv_n, Tz) \leq \psi^n(p(Tv_0, Tz)). \tag{25}$$

Now, letting $n \rightarrow +\infty$ in (25), we obtain

$$\lim_{n \rightarrow +\infty} p(Tv_n, Tz) = 0. \tag{26}$$

With similar arguments, we deduce that $\lim_{n \rightarrow +\infty} p(Tw, Tv_n) = 0$. Hence

$$\begin{aligned} 0 < p(w, z) = p(Tw, Tz) \\ &\leq p(Tw, Tv_n) + p(Tv_n, Tz) \longrightarrow 0 \end{aligned} \tag{27}$$

as $n \rightarrow +\infty$, which is a contradiction. Thus T and f have a unique common fixed point. \square

As a consequence of Theorem 5, we state the following result.

Theorem 7. Let (X, p) be a partial metric space and $T, f : X \rightarrow X$ two mappings such that $TX \subset fX$. Assume that there exists $\psi \in \Psi$ such that

$$\begin{aligned} p(Tx, Ty) \\ \leq \max \{ \psi(p(fx, fy)), \psi(p(fx, Tx)), \psi(p(fy, Ty)) \} \end{aligned} \tag{28}$$

for all $x, y \in X$. If TX or fX is a 0-complete subspace of X , then T and f have a unique point of coincidence. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Proof. Proceeding as in the proof of Theorem 5, we get that T and f have a unique point of coincidence and, by Lemma 3, T and f have a unique common fixed point. \square

From Theorem 7, we can deduce the following corollaries.

Corollary 8 (Banach type). Let (X, p) be a partial metric space and $T, f : X \rightarrow X$ two mappings such that $TX \subset fX$. Assume that

$$p(Tx, Ty) \leq kp(fx, fy) \tag{29}$$

for all $x, y \in X$, where $0 \leq k < 1$. If TX or fX is a 0-complete subspace of X , then T and f have a unique point of coincidence. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Corollary 9 (Bianchini type). Let (X, p) be a partial metric space and let $T, f : X \rightarrow X$ be two mappings such that $TX \subset fX$. Assume that

$$p(Tx, Ty) \leq k \max \{ p(fx, Tx), p(fy, Ty) \} \tag{30}$$

for all $x, y \in X$, where $0 \leq k < 1$. If TX or fX is a 0-complete subspace of X , then T and f have a unique point of coincidence. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Corollary 10 (Reich type [58]). Let (X, p) be a partial metric space and let $T, f : X \rightarrow X$ be two mappings such that $TX \subset fX$. Assume that

$$p(Tx, Ty) \leq ap(fx, fy) + bp(fx, Tx) + cp(fy, Ty) \tag{31}$$

for all $x, y \in X$, where $a, b, c \geq 0$ and $a + b + c < 1$. If TX or fX is a 0-complete subspace of X , then T and f have a unique point of coincidence. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

The following example shows that there exist mappings that satisfy the contractive condition (28), but are not quasi-contractions [59].

Example 11. Consider the set $X = \{1, 2, 3\}$ and the function $p : X \times X \rightarrow [0, +\infty)$ given by $p(1, 2) = p(2, 3) = 1, p(1, 3) = 3/2, p(1, 1) = p(3, 3) = 1/2, p(2, 2) = 0$,

and $p(x, y) = p(y, x)$. Obviously, p is a partial metric on X , but it is not a metric (since $p(x, x) \neq 0$ for $x = 1$ and $x = 3$). Clearly, (X, p) is a 0-complete partial metric space. Let $T, f : X \rightarrow X$ be defined by $T1 = 2, T2 = 2, T3 = 1$ and $fx = x$ for every $x \in X$. Take $\psi(t) = (2/3)t$ for every $t \geq 0$.

First, we will check that T and f satisfy the contractive condition (28). If $x, y \in \{1, 2\}$, then $p(Tx, Ty) = p(2, 2) = 0$ and (28) trivially holds. Let, for example, $y = 3$, then we have the following three cases:

$$\begin{aligned} p(T1, T3) &= p(2, 1) = 1 \leq \frac{2}{3} \cdot \frac{3}{2} \\ &= \max \{ \psi(p(1, 3)), \psi(p(1, 2)), \psi(p(3, 1)) \}; \\ p(T2, T3) &= p(2, 1) = 1 \leq \frac{2}{3} \cdot \frac{3}{2} \\ &= \max \{ \psi(p(2, 3)), \psi(p(2, 2)), \psi(p(3, 1)) \}; \\ p(T3, T3) &= p(1, 1) = \frac{1}{2} < \frac{2}{3} \cdot \frac{3}{2} \\ &= \max \{ \psi(p(3, 3)), \psi(p(3, 1)) \}. \end{aligned} \tag{32}$$

Thus, all the conditions of Theorem 7 are satisfied and the existence of a common fixed point of T and f (which is 2) follows. The same conclusion cannot be obtained by the main results from [59]. Indeed, using $p^s(a, b) = 2p(a, b) - p(a, a) - p(b, b)$, and then taking p^s instead of p , $x = 1, y = 3$ in (5), we obtain

$$\begin{aligned} L &= p^s(T1, T3) = p^s(2, 1) = \frac{3}{2}, \\ R &= \max \{ \psi(p^s(1, 3)), \psi(p^s(1, T1)), \psi(p^s(3, T3)), \\ &\quad \psi(p^s(1, T3)), \psi(p^s(3, T1)) \} \\ &= \frac{2}{3} \max \{ p^s(1, 3), p^s(1, 2), p^s(3, 1), p^s(1, 1), p^s(3, 2) \} \\ &= \frac{2}{3} \max \left\{ 2, \frac{3}{2}, 2, 0, \frac{3}{2} \right\} = \frac{4}{3}. \end{aligned} \tag{33}$$

Since $L > R$, the conclusion follows.

The following example shows that there exist mappings that satisfy the contractive condition (5), but do not satisfy the contractive condition (28).

Example 12. Let $X = [0, 2]$ be endowed with the partial metric

$$p(x, y) = \begin{cases} |x - y| & \text{if } x, y \in [0, 1], \\ \max \{x, y\} & \text{if } \{x, y\} \cap (1, 2] \neq \emptyset. \end{cases} \tag{34}$$

Clearly, (X, p) is a 0-complete partial metric space. Let $T, f : X \rightarrow X$ be defined by

$$Tx = \begin{cases} x & \text{if } x \in [0, 1], \\ \frac{1+x}{2} & \text{if } x \in (1, 2] \end{cases} \tag{35}$$

and $fx = x$ for each $x \in X$. As T and f have many common fixed points (each $x \in [0, 1]$ is a common fixed point), then it is immediate to show that T and f do not satisfy the contractive condition (28).

If (X, p) is ordered by

$$x \leq y \iff (x = y) \text{ or } (x, y \in (1, 2], x \leq y), \tag{36}$$

then T and f satisfy the contractive condition (5) where $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is defined by

$$\psi(t) = \begin{cases} \frac{t}{2} & \text{if } t \in [0, 1], \\ \frac{1+t}{2} & \text{if } t \in (1, 2]. \end{cases} \tag{37}$$

Using Theorem 5, we deduce that T and f have a common fixed point.

4. Completeness in Partial Metric Spaces and Fixed Points

In this section, we characterize those partial metric spaces for which every Bianchini mapping has a fixed point in the style of Subrahmanyam characterization of metric completeness. This will be done by means of the notion of 0-completeness which was introduced by Romaguera in [50].

Let (X, p) be a partial metric space and $T : X \rightarrow X$ be a mapping. We recall that T is a Bianchini [60] mapping if

$$p(Tx, Ty) \leq k \max \{ p(x, Tx), p(y, Ty) \} \tag{38}$$

for all $x, y \in X$, where $0 \leq k < 1$.

Theorem 13. *Let (X, p) be a partial metric space. If every mapping $T : X \rightarrow X$ satisfying the following conditions:*

- (i) $p(Tx, Ty) \leq \lambda \max \{ p(x, Tx), p(y, Ty) \}$ for all $x, y \in X$, for a fixed $\lambda > 0$,
- (ii) TX is countable

has a fixed point, then (X, p) is 0-complete.

Proof. Suppose that there is a 0-Cauchy sequence $\{x_n\}$ of distinct points in (X, p) which is not convergent in (X, p^s) . We put $A = \{x_n\}$ and we note that for every $x \notin A$, we have $p^s(x, A) > 0$.

Now, $0 < p^s(x, A) \leq 2p(x, A)$ implies that $p(x, A) > 0$. Since $\{x_n\}$ is a 0-Cauchy sequence in (X, p) , there exists a least positive integer $N(x)$ such that

$$\begin{aligned} p(x_m, x_n) \\ < \lambda p(x, A) \leq \lambda p(x, x_l), \quad l \in \mathbb{N}, \forall m, n \geq N(x). \end{aligned} \tag{39}$$

In particular

$$\begin{aligned} p(x_m, x_{N(x)}) \\ < \lambda p(x, A \setminus \{x\}) \leq \lambda p(x, x_{N(x)}), \quad \forall m \geq N(x). \end{aligned} \tag{40}$$

For fixed $n \in \mathbb{N}$, since $p^s(x_n, A \setminus \{x_n\}) > 0$ and so $p(x_n, A \setminus \{x_n\}) > 0$, there is $n'(n) > n$ such that

$$p(x_m, x_{n'}) < \lambda p(x_n, A \setminus \{x_n\}) \leq \lambda p(x_n, x_{n'}), \quad \forall m \geq n'. \quad (41)$$

Now, let $T : X \rightarrow X$ be defined by

$$Tx = \begin{cases} x_{N(x)} & \text{if } x \notin A, \\ x_{n'} & \text{if } x (= x_n) \in A. \end{cases} \quad (42)$$

From the definition of T , we deduce that T satisfies the condition (ii). On the other hand, T satisfies also the condition (i). In fact, (i) is verified by assuming $Tx = x_n$ and $Ty = x_m$, and noting that

$$p(x_m, x_n) < \begin{cases} \lambda p(y, A \setminus \{y\}) \leq \lambda p(y, Ty) & \text{if } n \geq m, \\ \lambda p(x, A \setminus \{x\}) \leq \lambda p(x, Tx) & \text{if } n < m. \end{cases} \quad (43)$$

It is clear that T has not fixed points since $x_{n'} \neq x_n, n = 1, 2, \dots$. Thus, the assumption that there is a 0-Cauchy sequence $\{x_n\}$ which is not convergent in (X, p^s) leads to a contradiction to Theorem 13 and thereby establishes the same. \square

If in Theorem 13 we choose $\lambda \in [0, 1)$, by Corollary 9, we obtain the following characterization of 0-completeness for partial metric spaces.

Theorem 14. *A partial metric space (X, p) is 0-complete if and only if every mapping $T : X \rightarrow X$ satisfying the following conditions:*

- (i) $p(Tx, Ty) \leq \lambda \max\{p(x, Tx), p(y, Ty)\}$ for all $x, y \in X$, for a fixed $\lambda \in [0, 1)$,
- (ii) TX is countable

has a fixed point.

In Theorem 14, the class of mappings satisfying (i) and (ii) can be replaced by the class of mappings satisfying (ii) and the following condition:

- (i) $p(Tx, Ty) \leq \lambda[p(x, Tx) + p(y, Ty)]$ for all $x, y \in X$, for a fixed $\lambda \in [0, 1/2)$.

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