Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 564363, 10 pages
doi:10.1155/2010/564363

Research Article

Multiplicity Results for a Perturbed Elliptic Neumann Problem

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Received 21 April 2010; Accepted 9 July 2010

Academic Editor: Pavel Drábek

The existence of three solutions for elliptic Neumann problems with a perturbed nonlinear term depending on two real parameters is investigated. Our approach is based on variational methods.

1. Introduction

Here and in the sequel, $\Omega \subset \mathbb{R}^n$ is a bounded open set, with a boundary of class C^1, $q \in L^\infty(\Omega)$ with $ess\inf_\Omega q > 0$, $p > n$; $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions.

The aim of this paper is to study the following perturbed boundary value problem with Neumann conditions:

$$\begin{gathered} -\Delta_p u + q(x)|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u), \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{$P_{\lambda,\mu}$}$$

where $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, ν is the outer unit normal to $\partial\Omega$, λ and μ are positive real parameters.

Nonlinear boundary value problems involving the p-Laplacian operator Δ_p (with $p \neq 2$) arise from a variety of physical problems. They are used in non-Newtonian fluids, reaction-diffusion problems, flow through porous media, and petroleum extraction (see, e.g., [1, 2]).

In the last years, several researchers have studied nonlinear problems of this type through different approaches. In [1], the authors have obtained results on the existence of a solution for the problem

$$\begin{gathered} -\Delta_p u + g(x,u) = f(u), \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

by using the perturbation result on sums of ranges of nonlinear accretive operators. Subsequently, Wei and Agarwal, in [2], have studied the same problem by developing some new techniques in the wake of [1]. Problem $(P_{\lambda,\mu})$, when $q = 0$, $\lambda = 1$ and g does not depend on u, has been studied in [3]. In this paper, the authors have obtained the existence of at least three solutions for small μ, by using Implicit Function Theorem and Morse Theory. By using variational methods and in particular critical point results given by Ricceri in [4], Faraci, in her nice paper [5], has dealt with a Neumann Problem involving the p-Laplacian (for any p) of type

$$\begin{gathered} -\Delta_p u + q(x)|u|^{p-2}u = f(x,u) + \mu g(x,u), \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

In particular, [5, Theorems 8, 9] assure the existence of three solutions for the problem given above.

In the present paper, we establish some results (Theorems 3.1, 3.2), which assure the existence of at least three weak solutions for the problem $(P_{\lambda,\mu})$. In particular the following result is a consequence of Theorem 3.2.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that*

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 0, \qquad \int_0^6 f(t)dt < 6\int_0^1 f(t)dt. \tag{1.3}$$

Then, for every $\lambda \in]3/4\int_0^1 f(t)dt, 9/2\int_0^6 f(t)dt[$ and for every positive continuous function $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ there exists $\delta^\star > 0$ such that, for each $\mu \in]0,\delta^*[$, the problem

$$\begin{gathered} -u'' + u = \lambda f(u) + \mu g(x,u) \quad \text{in }]0,1[, \\ u'(0) = u'(1) \end{gathered} \tag{1.4}$$

has at least three nonzero classical solutions.

With respect to [3, 5], we stress that our results hold under different assumptions (see Remarks 3.4 and 3.5). In particular, in Theorem 1.1, no asymptotic condition at infinity is required on the nonlinear term. We also point out that in Theorems 3.1 and 3.2, precise estimates of parameters λ and μ are given.

2. Preliminaries and Basic Notations

Our main tools are three critical point theorems that we recall here in a convenient form. The first has been obtained in [6], and it is a more precise version of Theorem 3.2 of [7]. The second has been established in [7].

Theorem 2.1 (see [6, Theorem 2.6]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0. \tag{2.1}$$

Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that:

(a_1) $\sup_{\Phi(x)\le r}\Psi(x)/r < \Psi(\overline{x})/\Phi(\overline{x})$;

(a_2) *for each $\lambda \in \Lambda_r :=]\Phi(\overline{x})/\Psi(\overline{x}),\ r/\sup_{\Phi(x)\le r}\Psi(x)[$ the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.2 (see [7, Corollary 3.1]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0. \tag{2.2}$$

Assume that there exist two positive constants r_1, r_2, and $\overline{x} \in X$, with $2r_1 < \Phi(\overline{x}) < r_2/2$, such that

(b_1) $\sup_{x\in\Phi^{-1}(]-\infty,r_1[)}\Psi(x)/r_1 < (2/3)(\Psi(\overline{x})/\Phi(\overline{x}))$;

(b_2) $\sup_{x\in\Phi^{-1}(]-\infty,r_2[)}\Psi(x)/r_2 < (1/3)(\Psi(\overline{x})/\Phi(\overline{x}))$;

(b_3) *for each $\lambda \in \Lambda'_{r_1,r_2} :=](3/2)(\Phi(\overline{x})/\Psi(\overline{x})),\ \min\{r_1/\sup_{x\in\Phi^{-1}(]-\infty,r_1[)}\Psi(x), r_2/2\sup_{x\in\Phi^{-1}(]-\infty,r_2[)}\Psi(x)\}[$ and for every $x_1, x_2 \in X$, which are local minima for the functional $\Phi - \lambda\Psi$, and such that $\Psi(x_1) \ge 0$ and $\Psi(x_2) \ge 0$ one has $\inf_{t\in[0,1]}\Psi(tx_1 + (1-t)x_2) \ge 0$.*

Then, for each $\lambda \in \Lambda'_{r_1,r_2}$ the functional $\Phi - \lambda\Psi$ admits three critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

Now we recall some basic definitions and notations.

A function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is called an L^1-Carathéodory function if $x \to h(x,t)$ is measurable for all $t \in \mathbb{R}$, $t \to h(x,t)$ is continuous for almost every $x \in \Omega$, for all $M > 0$ one has $\sup_{|t|\le M}|h(x,t)| \in L^1(\Omega)$. Clearly, if h is continuous in $\overline{\Omega} \times \mathbb{R}$, then it is L^1-Carathéodory.

We also recall that a weak solution of the problem $(P_{\lambda,\mu})$ is any $u \in W^{1,p}(\Omega)$, such that

$$\begin{aligned}&\int_\Omega \Big(|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\Big)dx + \int_\Omega \Big(q(x)|u(x)|^{p-2}u(x)v(x)\Big)dx\\&\quad - \int_\Omega \big[\lambda f(x,u(x)) + \mu g(x,u(x))\big]v(x)dx = 0, \quad \forall v \in W^{1,p}(\Omega).\end{aligned} \tag{2.3}$$

Put

$$k = \sup_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega}|u(x)|}{\left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega q(x)|u(x)|^p dx\right)^{1/p}}. \tag{2.4}$$

If Ω is convex, an explicit upper bound for the constant k is

$$k \le 2^{(p-1)/p} \max\left\{ \left(\frac{1}{\int_\Omega q(x)dx}\right)^{1/p}, \frac{\operatorname{diam}(\Omega)}{n^{1/p}} \left(\frac{p-1}{p-n} m(\Omega)\right)^{(p-1)/p} \frac{\|q\|_\infty}{\int_\Omega q(x)dx} \right\} \tag{2.5}$$

(see, e.g., [8, Remark 1]).

Put $F(x,\xi) = \int_0^\xi f(x,t)dt$ for all $(x,\xi) \in \Omega \times \mathbb{R}$ and $G(x,\xi) = \int_0^\xi g(x,t)dt$ for all $(x,\xi) \in \Omega \times \mathbb{R}$.

Moreover, set $G^c := \int_\Omega \max_{|\xi|\le c} G(x,\xi)dx$ for all $c > 0$ and $G_d := \inf_{\Omega\times[0,d]} G$ for all $d > 0$. Clearly, $G^c \ge 0$ and $G_d \le 0$.

3. Main Results

In this section, we present our main results on the existence of at least three weak solutions for the problem $(P_{\lambda,\mu})$.

In order to introduce our first result, fixing $c, d > 0$ such that $\|q\|_1 d^p / \int_\Omega F(x,d)dx < c^p/k^p \int_\Omega \max_{|\xi|\le c} F(x,\xi)dx$ and picking $\lambda \in \Lambda :=]\|q\|_1 d^p / p\int_\Omega F(x,d)dx, c^p/pk^p \int_\Omega \max_{|\xi|\le c} F(x,\xi)dx[$, put

$$\delta := \min\left\{ \frac{c^p - \lambda p k^p \int_\Omega \max_{|\xi|\le c} F(x,\xi)dx}{pk^p G^c}, \frac{d^p\|q\|_1 - p\lambda \int_\Omega F(x,d)dx}{|\Omega| p G_d} \right\}, \tag{3.1}$$

$$\overline{\delta} := \min\left\{ \delta, \frac{1}{\max\left\{0, p|\Omega|k^p \limsup_{|\xi|\to+\infty}\left[\left(\sup_{x\in\Omega} G(x,\xi)\right)/\xi^p\right]\right\}} \right\}, \tag{3.2}$$

where we read $r/0 = +\infty$ so that, for instance, $\overline{\delta} = +\infty$ when $\limsup_{|\xi|\to+\infty} (\sup_{x\in\Omega} G(x,\xi)/\xi^p) \le 0$ and $G_d = G^c = 0$.

Theorem 3.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function, and put $F(x,\xi) = \int_0^\xi f(x,t)dt$ for all $(x,\xi) \in \Omega \times \mathbb{R}$. Assume that there exist two positive constants c, d, with $c < (k\|q\|_1^{1/p})d$, such that*

(i) $\int_\Omega \max_{|\xi|\le c} F(x,\xi)dx / c^p < \dfrac{1}{k^p\|q\|_1} \dfrac{\int_\Omega F(x,d)dx}{d^p}$, *where k is given by* (2.4)*;*

(ii) $\limsup_{|\xi|\to+\infty} [(\sup_{x\in\Omega} F(x,\xi))/\xi^p] \le 0$.

Then, for every $\lambda \in \Lambda :=]\|q\|_1 d^p/p\int_\Omega F(x,d)dx, c^p/pk^p\int_\Omega \max_{|\xi|<c}F(x,\xi)dx[$ *and for every* $L^1(\Omega)$*-Carathédory function* $g:\Omega\times\mathbb{R}\to\mathbb{R}$ *such that*

(iii) $\limsup_{|\xi|\to+\infty}[(\sup_{x\in\Omega}G(x,\xi))/\xi^p] < +\infty$, *where* $G(x,\xi) = \int_0^\xi g(x,t)dt$ *for all* $(x,\xi)\in\Omega\times\mathbb{R}$,

there exists $\overline{\delta}>0$ *given by* (3.2) *such that, for each* $\mu\in[0,\overline{\delta}[$, *Problem* $(P_{\lambda,\mu})$ *has at least three weak solutions.*

Proof. Fix λ, g, and μ as in the conclusion. Take $X = W^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega q(x)|u(x)|^p dx\right)^{1/p}. \tag{3.3}$$

On the space $C^0(\overline{\Omega})$, we consider the norm $\|u\|_\infty := \sup_{x\in\overline{\Omega}}|u(x)|$. Since $p>n$, X is compactly embedded in $C^0(\overline{\Omega})$, we have

$$\|u\|_\infty \le k\|u\|. \tag{3.4}$$

Put, for each $u\in X$,

$$\begin{gathered}\Phi(u) = \frac{1}{p}\|u\|^p,\\ \Psi(u) = \int_\Omega\left[F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))\right]dx.\end{gathered}\tag{3.5}$$

Since the critical points of the functional $\Phi-\lambda\Psi$ on X are weak solutions of problem $(P_{\lambda,\mu})$, our aim is to apply Theorem 2.1 to Φ and Ψ. To this end, taking into account that the regularity assumptions of Theorem 2.1 on Φ and Ψ are satisfied, we will verify (a_1) and (a_2).

Put $r = 1/p(c/k)^p$ taking into account (3.4), one has

$$\sup_{\Phi(u)\le r}\Psi(u) = \sup_{\Phi(u)\le r}\int_\Omega\left[F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))\right]dx \le \int_\Omega \max_{|\xi|\le c}F(x,\xi)dx + \frac{\mu}{\lambda}G^c. \tag{3.6}$$

Now, fix $\overline{u}=d$. Clearly $\overline{u}\in X$ and moreover $\Phi(\overline{u})>r$. One has

$$\begin{gathered}\Psi(\overline{u}) = \int_\Omega F(x,d)dx + \frac{\mu}{\lambda}\int_\Omega G(x,d)dx,\\ \Phi(\overline{u}) = \frac{1}{p}\|\overline{u}\|^p = \frac{1}{p}d^p\|q\|_1.\end{gathered}\tag{3.7}$$

Hence,

$$\begin{gathered}\frac{\Psi(\overline{u})}{\Phi(\overline{u})} \geq \frac{p}{\|q\|_1}\frac{\int_\Omega F(x,d)dx}{d^p} + \frac{p|\Omega|}{\|q\|_1}\frac{\mu}{\lambda}\frac{G_d}{d^p},\\ \frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} \leq pk^p\frac{\int_\Omega \max_{|\xi|\leq c}F(x,\xi)dx}{c^p} + pk^p\frac{\mu}{\lambda}\frac{G^c}{c^p}.\end{gathered} \tag{3.8}$$

Since $\mu < \delta$, one has

$$\mu < \frac{c^p - \lambda pk^p\int_\Omega \max_{|\xi|<c}F(x,\xi)dx}{pk^pG^c}, \text{ this means } pk^p\frac{\int_\Omega \max_{|\xi|\leq c}F(x,\xi)dx}{c^p} + pk^p\frac{\mu}{\lambda}\frac{G^c}{c^p} < \frac{1}{\lambda}. \tag{3.9}$$

Furthermore, $\mu < (d^p\|q\|_1 - p\lambda\int_\Omega F(x,d)dx)/p|\Omega|G_d$; this means that $(p/\|q\|_1)(\int_\Omega F(x,d)dx/d^p) + (p|\Omega|/\|q\|_1)(\mu/\lambda)(G_d/d^p) > 1/\lambda$.

Then,

$$pk^p\frac{\int_\Omega \max_{|\xi|\leq c}F(x,\xi)dx}{c^p} + pk^p\frac{\mu}{\lambda}\frac{G^c}{c^p} < \frac{1}{\lambda} < \frac{p}{\|q\|_1}\frac{\int_\Omega F(x,d)dx}{d^p} + \frac{|\Omega|p}{\|q\|_1}\frac{\mu}{\lambda}\frac{G_d}{d^p}. \tag{3.10}$$

Hence, from (3.8) and (3.10), condition (a_1) of Theorem 2.1 is verified.

Finally, since $\mu < \overline{\delta}$, we can fix $l > 0$ such that $\limsup_{|\xi|\to+\infty}[(\sup_{x\in\Omega}G(x,\xi))/\xi^p] < l$ and $\mu l < 1/|\Omega|pk^p$. Therefore, there exists a function $h \in L^1(\Omega)$ such that

$$G(x,\xi) \leq l\xi^p + h(x) \tag{3.11}$$

for each $(x,\xi) \in \Omega\times\mathbb{R}$.

Now, fix $0 < \varepsilon < (1/p|\Omega|k^p\lambda) - (\mu l/\lambda)$; from (ii) there is a function $h_\varepsilon \in L^1(\Omega)$ such that

$$F(x,\xi) \leq \varepsilon\xi^p + h_\varepsilon(x) \tag{3.12}$$

for each $(x,\xi) \in \Omega\times\mathbb{R}$. It follows that, for each $u \in X$,

$$\Phi(u) - \lambda\Psi(u) \geq \left(\frac{1}{p} - \lambda|\Omega|k^p\varepsilon - \mu|\Omega|k^p l\right)\|u\|^p - \lambda\|h_\varepsilon\|_1 - \mu\|h\|_1. \tag{3.13}$$

This leads to the coercivity of $\Phi - \lambda\Psi$, and condition (a_2) of Theorem 2.1 is verified. Since, from (3.8) and (3.10),

$$\lambda \in \left]\frac{\Phi(\overline{u})}{\Psi(\overline{u})}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\right[, \tag{3.14}$$

Theorem 2.1 assures the existence of three critical points for the functional $\Phi - \lambda\Psi$, and the proof is complete. □

Now, we state a variant of Theorem 3.1. Here no asymptotic condition on g is requested; on the other hand, the functions f, g are supposed to be nonnegative.

Fix $c_1, d, c_2 > 0$ such that $3\|q\|_1 d^p/2\int_\Omega F(x,d)dx < 1/k^p \min\{c_1^p/\int_\Omega F(x,c_1)dx,\ c_2^p/2\int_\Omega F(x,c_2)dx\}$ and picking

$$\lambda \in \Lambda := \left]\frac{3\|q\|_1 d^p}{2p\int_\Omega F(x,d)dx}, \frac{1}{pk^p}\min\left\{\frac{c_1^p}{\int_\Omega F(x,c_1)dx}, \frac{c_2^p}{2\int_\Omega F(x,c_2)dx}\right\}\right[, \tag{3.15}$$

put

$$\delta^* := \min\left\{\frac{c_1^p - \lambda pk^p\int_\Omega F(x,c_1)dx}{pk^p G^{c_1}}, \frac{c_2^p - 2\lambda pk^p\int_\Omega F(x,c_2)dx}{2pk^p G^{c_2}}\right\}. \tag{3.16}$$

Theorem 3.2. *Assume that there exist three positive constants c_1, c_2, d, with $2^{1/p}c_1 < k\|q\|_1^{1/p}d < 2^{-1/p}c_2$, such that*

(j) $f(x,\xi) \geq 0$ *for each* $(x,\xi) \in \Omega \times [0,c_2]$;

(jj) $\int_\Omega F(x,c_1)dx/c_1^p < (2/3k^p\|q\|_1)(\int_\Omega F(x,d)dx/d^p)$;

(jjj) $\int_\Omega F(x,c_2)dx/c_2^p < (1/3k^p\|q\|_1)(\int_\Omega F(x,d)dx/d^p)$.

Then, for every

$$\lambda \in \Lambda := \left]\frac{3\|q\|_1 d^p}{2p\int_\Omega F(x,d)dx}, \frac{1}{pk^p}\min\left\{\frac{c_1^p}{\int_\Omega F(x,c_1)dx}, \frac{c_2^p}{2\int_\Omega F(x,c_2)dx}\right\}\right[\tag{3.17}$$

and for every nonnegative L^1-Carathédory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\delta^* > 0$ given by (3.16) *such that, for each $\mu \in]0, \delta^*[$, the problem $(P_{\lambda,\mu})$ has at least three weak solutions u_i, $i = 1,2,3$, such that*

$$0 \leq u_i(x) < c_2, \quad \forall x \in \Omega,\ i = 1,2,3. \tag{3.18}$$

Proof. Without loss of generality, we can assume $f(x,t) \geq 0$ for all $(x,t) \in \Omega \times \mathbb{R}$. Fix λ, g, and μ as in the conclusion and take X, Φ and Ψ as in the proof of Theorem 3.1.

We observe that the regularity assumptions of Theorem 2.2 on Φ and Ψ are satisfied.

Then, our aim is to verify (b_1) and (b_2). To this end, put $\overline{u}$ as in Theorem 3.1, $r_1 = (1/p)(c_1/k)^p$, and $r_2 = (1/p)(c_2/k)^p$. Therefore, one has $2r_1 < \Phi(\overline{u}) < r_2/2$ and, since $\mu < \delta^*$ and $G_d = 0$, one has

$$\begin{aligned}
\frac{\sup_{\Phi(u)<r_1}\Psi(u)}{r_1} &\leq pk^p \frac{\int_\Omega F(x,c_1)dx}{c_1^p} + pk^p \frac{\mu}{\lambda}\frac{G^{c_1}}{c_1^p} \\
&< \frac{1}{\lambda} < \frac{2p}{3\|q\|_1}\frac{\int_\Omega F(x,d)dx}{d^p} + \frac{\mu}{\lambda}\frac{2|\Omega|p}{3\|q\|_1}\frac{G_d}{d^p} \leq \frac{2}{3}\frac{\Psi(\overline{u})}{\Phi(\overline{u})}, \\
\frac{2\sup_{\Phi(u)<r_2}\Psi(u)}{r_2} &\leq 2pk^p \frac{\int_\Omega F(x,c_2)dx}{c_2^p} + 2pk^p \frac{\mu}{\lambda}\frac{G^{c_2}}{c_2^p} \\
&< \frac{1}{\lambda} < \frac{2p}{3\|q\|_1}\frac{\int_\Omega F(x,d)dx}{d^p} + \frac{\mu}{\lambda}\frac{2|\Omega|p}{3\|q\|_1}\frac{G_d}{d^p} \leq \frac{2}{3}\frac{\Psi(\overline{u})}{\Phi(\overline{u})}.
\end{aligned} \tag{3.19}$$

Therefore, (b_1) and (b_2) of Theorem 2.2 are verified.

Finally, we verify that $\Phi - \lambda\Psi$ satisfies assumption (b_3) of Theorem 2.2.

Let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem $(P_{\lambda,\mu})$. For every positive parameter λ, μ, and for every $(x,t) \in \Omega \times [0,+\infty[$ one has $(\lambda f + \mu g)(x,t) \geq 0$, hence, owing to the Weak Maximum Principle (see for instance [9]) we obtain $u_1(x) \geq 0$, $u_2(x) \geq 0$, for all $x \in \Omega$. Then, it follows that $su_1 + (1-s)u_2 \geq 0$, for all $s \in [0,1]$, and that $(\lambda f + \mu g)(x, su_1 + (1-s)u_2) \geq 0$, and, hence, $\Psi(su_1 + (1-s)u_2) \geq 0$, for all $s \in [0,1]$.

From Theorem 2.2, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are weak solutions of $(P_{\lambda,\mu})$ and the conclusion is achieved. □

Example 3.3. Consider the following problem

$$\begin{aligned}
-\Delta_3 u + |u|u = 320|x|^2\frac{u^3}{1+u^8} + \mu\left(|x|^2+1\right)|u|^3, \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega,
\end{aligned} \tag{3.20}$$

where $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$. Then, owing to Theorem 3.2, for each $\mu \in]0, 10^{-3}[$, the problem given above has at least three weak solutions. It is enough to choose, for instance, $c_1 = 10^{-6}$, $c_2 = 10^2$ and $d = 1$.

Remark 3.4. We observe that [5, Theorem 8] cannot be applied to the problem of Example 3.3 since the assumption

(N_6') there exist $s > 1, \alpha > 0, \beta \in L^1(\Omega)$ with $s < p$ such that $|G(x,u)| \leq \alpha|u|^s + \beta(x)$ does not hold. Moreover, also [5, Theorem 9] cannot be applied since condition

(N_7'') $\limsup_{u\to 0^+}(\inf_{x\in\Omega}\int_0^u g(x,t)dt/|u|^p) = +\infty$ is not verified.

Finally, we observe that [5, Theorem 7] and [5, Theorem 9] ensure only two solutions and two nonzero solutions, respectively.

Furthermore, contrary to theorems in [5], owing to our results, we have precise values of μ for which the problem admits solutions.

On the other hand, we observe that in [5], the case $p \leq n$ is investigated too.

Remark 3.5. It is easy to verify that our results and theorems in [3] are mutually independent. In particular, we remark that none of theorems in [3] can be applied to the problem in Example 3.3. We also observe that in [3] no estimate of small μ is given.

Proof of Theorem 1.1. Our aim is to apply Theorem 3.2 by choosing $n = 1, p = 2,$ and $q = 1$. Put $d = 1$ and $c_2 = 6$. Therefore, taking into account that $k = \sqrt{2}$, one has

$$\frac{3\|q\|_1 d^p}{2pF(d)} = \frac{3}{4\int_0^1 f(t)dt}, \qquad \frac{1}{pk^p}\frac{c_2^p}{2F(c_2)} = \frac{9}{2\int_0^6 f(t)dt}. \tag{3.21}$$

Moreover, since $\lim_{t\to 0^+}(F(t)/t^2) = 0$, there is a positive constant $c_1 < 1$, such that $(F(c_1)/c_1^2) < (1/3)\int_0^1 f(t)dt$ and $c_1^2/F(c_1) > 18/\int_0^6 f(t)dt$. Hence, a simple computation shows that all assumptions of Theorem 3.2 are satisfied, and the conclusion follows. □

Remark 3.6. We explicitly observe that in Theorem 3.2, as well as in Theorem 1.1, no asymptotic condition at infinity on the nonlinear term is requested.

Example 3.7. The function $\overline{f} : \mathbb{R} \to \mathbb{R}$

$$\overline{f}(x) = \begin{cases} 0, & t \leq \dfrac{1}{2}, \\ 2t - 1, & \dfrac{1}{2} < t \leq 1, \\ 33t - 32t, & 1 < t \leq \dfrac{263}{256}, \\ \dfrac{1}{8}, & \dfrac{263}{256} < t \leq 6, \\ e^t - e^6 + \dfrac{1}{8}, & t > 6, \end{cases} \tag{3.22}$$

satisfies all assumptions of Theorem 1.1.

Remark 3.8. We explicitly observe that the very nice Theorem 1 of [10] cannot be applied to the function of Example 3.7 since $\lim_{t\to+\infty}(\overline{f}(t)/t) = +\infty$.

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