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ROBOT'S FINGER AND EXPANSIONS IN NON-INTEGER BASES

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Abstract. We study a robot finger model in the framework of the theory of expansions in non-integer bases. We investigate the reachable set and its closure. A control policy to get approximate reachability is also proposed.

1. Introduction.

Aim of this paper is to give a model of a robot's finger in the framework of the theory of expansions in non-integer bases and to use the methods of this theory to study the reachability set and its closure. A discrete dynamical system models the position of the extremal junction of the finger, namely the reached point, starting with an initial point and a default angle (initial configuration). A *configuration* is a sequence of states of the system corresponding to a particular choice for the controls, while the union of all the possible states of the system is named *reachable set*. The closure of the reachable set is named *approximate reachable set*. The finger is composed by phalanxes, whose configurations can be studied by means of combinatorial analysis. The theory of expansions in non-integer bases reflects on the model through the arbitrariness of the number of phalanxes and it provides methods for the study of the asymptotic case, namely the limit case of infinite phalanxes. This approach allows us to construct an explicit binary control leading a phalanx to get close to any point in the approximate reachable set with a priori fixed precision. Our model includes two binary control parameters on every phalanx of the robot finger. The first control parameter rules the length of the phalanx, that can be either 0 or a fixed value, while the other control rules the angle between the current phalanx and the previous one. Such an angle can be either π, namely the phalanx is consecutive to the previous, or a fixed angle $\omega \in (0, \pi)$.

The ratio between any phalanx and the preceding is a constant $\rho < 1$. This assumption ensures the boundedness of the reachability set and the set of possible configurations to be self-similar. In particular the sub-configurations can be looked at as scaled miniatures with constant ratio ρ, also named *scaling factor*, of the whole structure. An important parameter of self-similar dynamics is the *branching factor*, i.e., the number of possible branches of a configuration. In our case the

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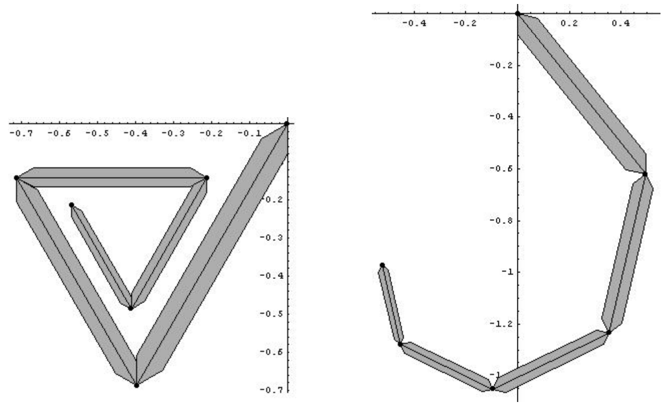


(A) $w = \pi/3$ and the ratio between phalanxes is $2^{1/3}$

(B) $w = \pi 6/7$ and the ratio between phalanxes is $2^{1/7}$

FIGURE 1. Two configurations with 5 phalanxes, fixed ω and fixed ratio between phalanxes ρ.

branching factor is 4, namely the cardinality of the possible couples of controls. In general, if for every given step the states do not overlap, the dimension of the approximate reachable set is given by the relation $D = \frac{B}{S}$ where B is the branching factor and S is the scaling factor; we refer to [9] for a discussion in the case of a robot-hand model whose approach is based on self-similar structures. For an overview of the general subject we refer to [16] and to [1] and to the references therein contained. The approximate reachable set in our model is intrinsically overlapping, hence in the analysis we adopt a different approach. By geometrical and combinatorial arguments, we show a condition ensuring the approximate reachable set to have dimension 2 when the rotation angle is $\omega = \pi/3$, namely to fill a region of the plane. This region is determined by a particular class of configurations, the *full-rotation configurations*.

We describe the obtained results and the organization of the paper. In Section 2 we introduce the model and we remark its relation with the theory of non-integer number systems. In Section 3 we focus on two particular configurations: the full-rotation and the full-extension configuration. We give a geometrical description of the convex hull of the reachable points corresponding to full-rotation configurations and we show a recursive relation describing the set of points that can be reached with full-extension configurations. In Section 4 we investigate the structure of the approximate reachable set by proving that it is a self-similar set and that it is the (unique) fixed point of a particular iterated function system, say $\mathcal{F}_{\rho,\omega}$. In general, neither the reachable set nor the approximate reachable set are explicitly known, nevertheless we show that the iteration $\mathcal{F}_{\rho,\omega}$ on the convex hull of the approximate reachable set yields approximations of the approximate reachable set whose accuracy is monotonically increasing with the number of iterations. Finally, for every point x belonging to the approximate reachable set, we define the *expansion* of x, namely a particular couple of control sequences ensuring the finger to reach an arbitrarily small neighborhood of x. Expansions are characterized and we introduce an algorithm generating them. The last section of this paper, Section 5, is devoted

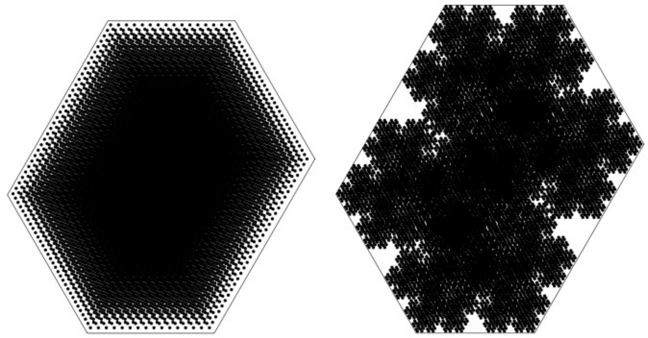


FIGURE 2. Full-rotation configurations.

to the particular case of $\omega = \pi/3$. We give an explicit description of the convex hull of the reachable points in the full-rotation, full-extension and general cases. We then study some convexity issues and we get the following results: the approximate reachable set is not a convex set, but if we restrict ourselves to the closure of the full-rotation configurations, a simple condition characterizes its convexity.

For an overview on the theoretical aspects of expansions in non-integer bases we refer to [15], [13] and [3]. In particular, expansions in non- integer bases were introduced in [15]. For the geometrical aspects of the expansions in complex base, namely the arguments that are more related to our problem, we refer to [5], [6],[7] and to [11]. We also mention the paper [2] where connections between control theory and expansions in non-integer bases are established.

2. **The model.** In this section we model a robot finger in the framework of non-integer bases theory. In our model the robot finger is composed by $n+1$ junctions and n phalanxes. We assume junctions and phalanxes to be thin, so to be respectively approximate with their middle axes and barycentres.

We also assume axes and barycentres to be coplanar and, by employing the isometry between $\mathbb{R}^2$, we use the symbols $x_0, \ldots, x_n \in \mathbb{C}$ to denote the position of the barycentres of the junctions, therefore the length l_k of the k-th phalanx is

$$l_k = |x_k - x_{k-1}|$$

and the configuration of robot finger is described by the system

$$\begin{cases} x_k = x_{k-1} + l_k e^{i\omega_k} \\ x_0 = l_0 e^{i\omega_0} \end{cases} \tag{1}$$

with $\omega_0, \ldots, \omega_k \in [0, 2\pi]$. We now introduce two binary control sequences $(u_k)_{k=1}^n$ and $(v_k)_{k=1}^n$, respectively ruling the length of each phalanx and the angle between two consecutive phalanxes. In particular, to represent phalanxes with a different length at each junction we assume

$$l_k = \frac{u_k}{\rho^k}, \tag{2}$$

with

$$u_k = \begin{cases} 1 & \text{extension} \\ 0 & \text{rotation or no motion} \end{cases} \tag{3}$$

We assume that the length of the phalanxes is decreasing, hence $\rho > 1$.

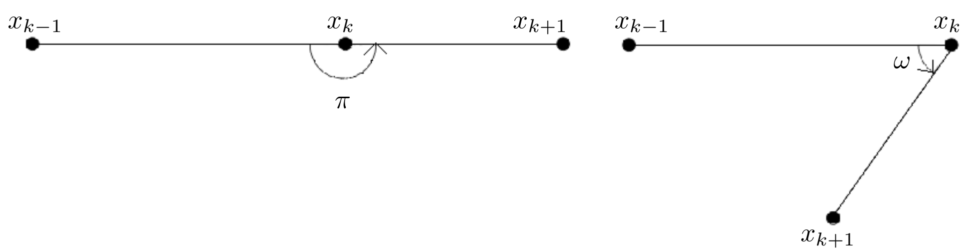
Remark 1. The length of the finger is finite for all n. Indeed
$$\sum_{k=1}^{n} l_k = \sum_{k=1}^{n} \frac{u_k}{\rho^k} < \frac{1}{\rho - 1}.$$

Example 1. If $\omega_k = 0$ for all k, then $x_k = x_0 + \sum_{j=0}^{k-1} l_j$.

Let us now focus on (v_k). In our model, the angle between two consecutive phalanxes is either π or a fixed $\omega \in (0, \pi)$. If $v_k = 0$ then the angle between the $k-1$-th phalanx and the k-th phalanx is π, while if $v_k = 1$ then the angle between the $k-1$-th phalanx and the k-phalanx is ω so that

$$v_k = \begin{cases} 1 & \text{rotation of the angle } \omega \\ 0 & \text{no rotation} \end{cases} \tag{4}$$

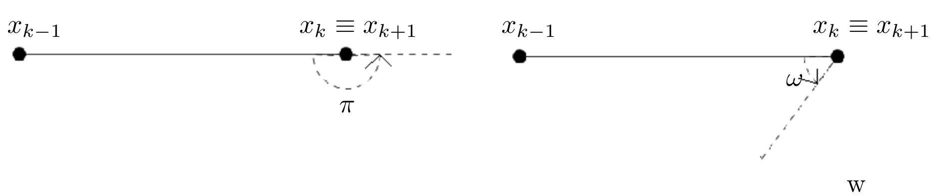
(see Figure 3).



(A) $v_{k+1} = 0$. (B) $v_{k+1} = 1$.

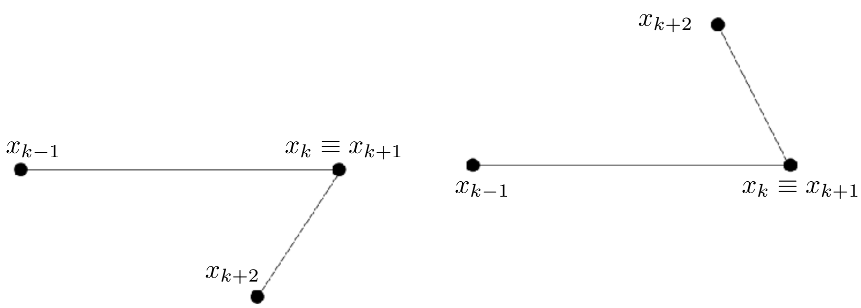
FIGURE 3. In both cases $u_{k+1} = 1$.

We also remark that in the case $u_{k+1} = 0$, namely when x_k and x_{k+1} coincide, the model keeps memory of the choice of v_{k+1}, affecting the successive rotations, see for instance Figures 4 and 5.



(A) $v_{k+1} = 0$; (B) $v_{k+1} = 1$.

FIGURE 4. In both cases $u_{k+1} = 0$.



(A) $v_{k+1} = 0$; (B) $v_{k+1} = 1$.

FIGURE 5. In both cases $u_{k+1} = 0$, $u_{k+2} = 1$ and $v_{k+2} = 1$.

As x_k is the vertex of the angle between the k-th phalanx and the $k+1$-th phalanx, we have the relations

$$\arg(x_{k+1} - x_k) - \arg(x_{k-1} - x_k) = v_{k+1}\omega + (1 - v_{k+1})\pi. \tag{5}$$

Remark 2. In the case $k = 0$, (5) holds by introducing $x_{-1} := 0$. Remark that this notation is consistent with (1), indeed $x_0 = x_{-1} + l_0 e^{i\omega_0}$.

We have the following relation between ω_k and the the first k control digits $(v_j)_{j=1}^k$.

Proposition 1. *Let $k \geq 0$ and $v_j \in \{0, 1\}$ for $j = 1, \ldots, k$. Then*

$$\omega_k = -\sum_{j=1}^{k} v_j(\pi - \omega) + \omega_0 + 2k\pi \tag{6}$$

Proof. By induction on k. If $k = 0$ then (6) is immediate. Assume now (6) as inductive hypothesis and consider $k \geq 1$. As (1) implies

$$x_{k+1} - x_k = l_{k+1} e^{i\omega_{k+1}};$$

we may deduce by (5) and (6)

$$\begin{aligned}
\omega_{k+1} &= \arg(x_{k+1} - x_k) \\
&= \arg(x_{k-1} - x_k) + v_{k+1}\omega + (1 - v_{k+1})\pi \\
&= \pi + \arg(x_k - x_{k-1}) + v_k\omega + (1 - v_k)\pi \\
&= \omega_k - v_{k+1}(\pi - \omega) + 2\pi \\
&= -\sum_{j=1}^{k+1} v_j(\pi - \omega) + \omega_0 + 2(k+1)\pi
\end{aligned}$$

hence the thesis. □

The following Corollary gives the case of *full rotation* or *full extensions* in a fixed direction ω_0.

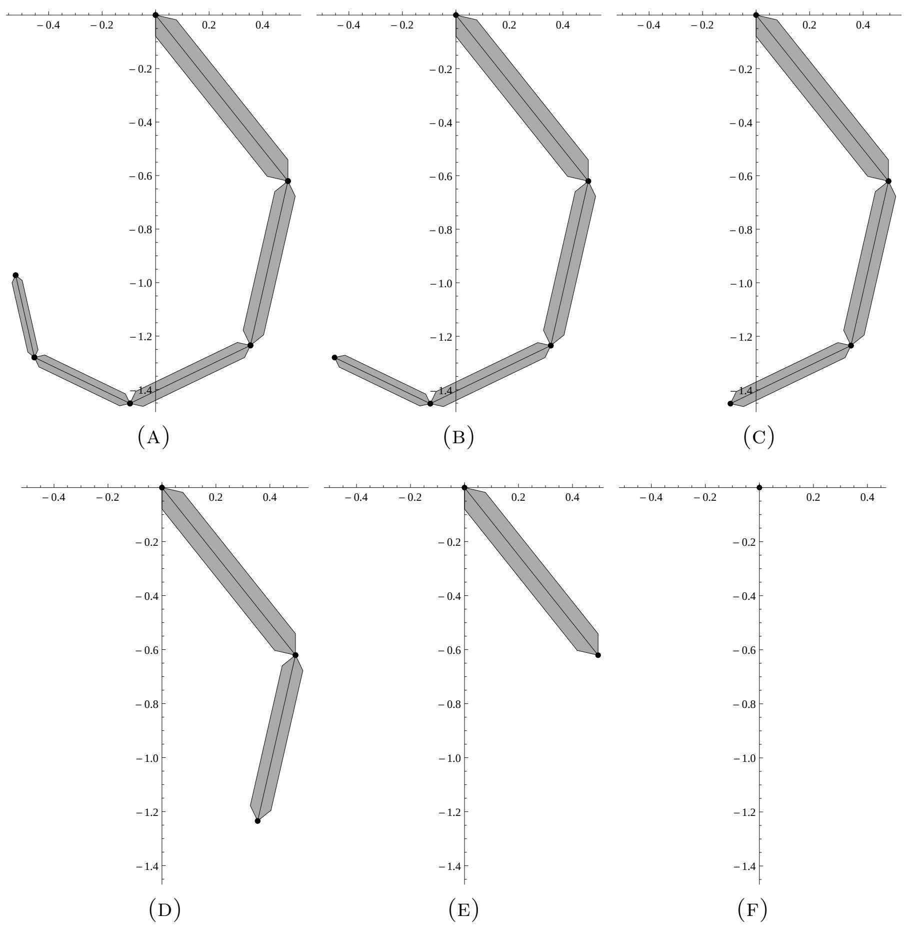


FIGURE 6. Various steps towards the initial condition $x_0 = 0$ of a robot finger with $\omega = 6\pi/7$ and $\rho = 2^{1/3}$.

Corollary 1. *Let $k \geq 1$. If $v_j = 1$ for every $j = 1, \ldots, k$, then*

$$\omega_k = -k(\pi - \omega) + \omega_0 + 2k\pi.$$

If $v_j = 0$ for every $j = 1, \ldots, k$, then $\omega_k = \omega_0$.

Generally

$$x_k = x_0 + \sum_{j=1}^{k} l_j e^{i\omega_j}$$

Using (2) and Proposition 1 we obtain the system

$$\begin{cases} x_k = x_0 + e^{i\omega_0} \displaystyle\sum_{j=1}^{k} \frac{u_j}{\rho^i} e^{-i \sum_{n=1}^{j} (\pi - \omega) v_n} \\ x_0 = l_0 e^{i\omega_0} \end{cases} \tag{7}$$

Remark 3. Due its linearity, the system is easily invertible (see Figure 6) so to get a dynamics driving the finger to retract its phalanxes:

$$\begin{cases} \overline{x}_j = \overline{x}_{j-1} - \dfrac{u_{k-j+1}}{\rho^{k-j+1}} e^{-i\sum_{n=1}^{k-j+1} v_n(\pi-\omega)}; \\ \overline{x}_0 = x_k. \end{cases} \tag{8}$$

By rewriting (7) in explicit form we have

$$x_k = x_0 + e^{i\omega_0} \sum_{j=1}^{k} \frac{u_j}{\rho^j} e^{i\sum_{n=1}^{j} i(1-v_n)(\pi-\omega)} e^{-i\ j(\pi-\omega)} \tag{9}$$

Then setting

$$\lambda := \rho e^{i(\pi-\omega)} \tag{10}$$

and

$$c_j = u_j e^{i\sum_{n=1}^{j}(1-v_n)(\pi-\omega)}, \tag{11}$$

we have

$$x_k = x_0 + e^{i\omega_0} \sum_{j=1}^{k} \frac{c_j}{\lambda^j} \tag{12}$$

In the analysis, we assume without loss of generality $x_0 = 0$ and $\omega_0 = 0$, then in what follows all reachability properties and notions of reachable sets will always be referred to the origin,

To make the finger able to reach with high precision a point in the reachable set, giving an appropriate sequence of 0 and 1, we need to analyze the behavior of x_k hence the sequence in non integer bases

$$\sum_{j=1}^{k} \frac{c_j}{\lambda^j}$$

for possible large number k.

3. Remarkable configurations.

3.1. Full-rotation configurations.

In the *full rotation* configurations, the rotation controls v_k are constantly equal to 1. In view of the assumption

$$\omega_0 = x_0 = 0,$$

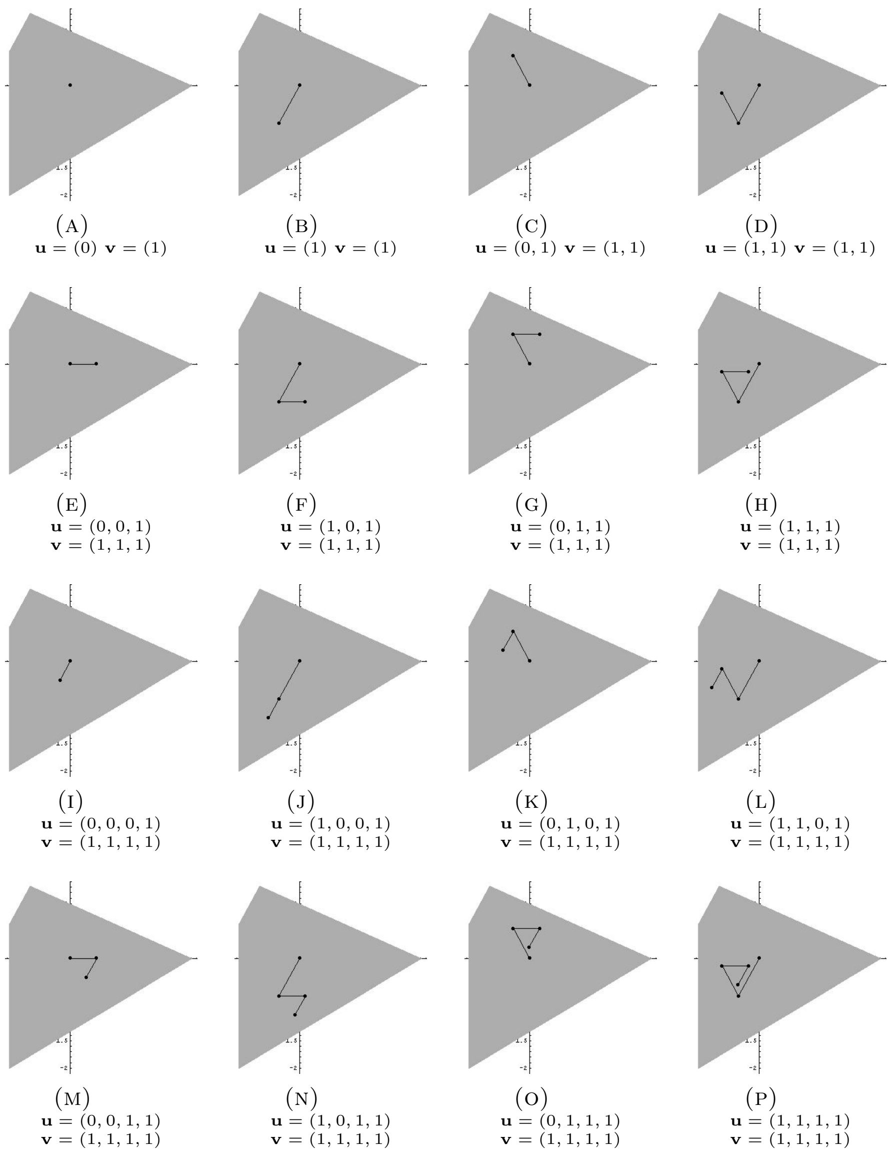
of Corollary 1 and of (12), the system (1) takes the form

$$x_k = \sum_{j=1}^{k} \frac{u_j}{\rho^j} e^{-i\ j(\pi-\omega)}$$

with $u_j \in \{0,1\}$. By setting $\lambda := \rho e^{i(\pi-\omega)}$, we get the more compact equation

$$x_k = \sum_{j=1}^{k} \frac{u_j}{\lambda^j}. \tag{13}$$

In other words the full rotation configuration are finite expansions in base λ and with alphabet $\{0,1\}$. In Figure 7 we show all the full rotation configurations of length 4 with $\rho = 2^{1/3}$ and $\omega = \pi/3$.



(A) $\mathbf{u} = (0)$ $\mathbf{v} = (1)$
(B) $\mathbf{u} = (1)$ $\mathbf{v} = (1)$
(C) $\mathbf{u} = (0, 1)$ $\mathbf{v} = (1, 1)$
(D) $\mathbf{u} = (1, 1)$ $\mathbf{v} = (1, 1)$

(E) $\mathbf{u} = (0, 0, 1)$ $\mathbf{v} = (1, 1, 1)$
(F) $\mathbf{u} = (1, 0, 1)$ $\mathbf{v} = (1, 1, 1)$
(G) $\mathbf{u} = (0, 1, 1)$ $\mathbf{v} = (1, 1, 1)$
(H) $\mathbf{u} = (1, 1, 1)$ $\mathbf{v} = (1, 1, 1)$

(I) $\mathbf{u} = (0, 0, 0, 1)$ $\mathbf{v} = (1, 1, 1, 1)$
(J) $\mathbf{u} = (1, 0, 0, 1)$ $\mathbf{v} = (1, 1, 1, 1)$
(K) $\mathbf{u} = (0, 1, 0, 1)$ $\mathbf{v} = (1, 1, 1, 1)$
(L) $\mathbf{u} = (1, 1, 0, 1)$ $\mathbf{v} = (1, 1, 1, 1)$

(M) $\mathbf{u} = (0, 0, 1, 1)$ $\mathbf{v} = (1, 1, 1, 1)$
(N) $\mathbf{u} = (1, 0, 1, 1)$ $\mathbf{v} = (1, 1, 1, 1)$
(O) $\mathbf{u} = (0, 1, 1, 1)$ $\mathbf{v} = (1, 1, 1, 1)$
(P) $\mathbf{u} = (1, 1, 1, 1)$ $\mathbf{v} = (1, 1, 1, 1)$

FIGURE 7. Full rotation configurations with $\rho = 2^{1/3}$ and $\omega = \pi/3$. The gray area represents the convex hull of the reachable points in the general case.

We denote $R_k^{(fr)}(\lambda)$ the reachable set in time k restricted to the full-rotation configurations, namely:

$$R_k^{(fr)}(\lambda) := \left\{ \sum_{j=1}^{k} \frac{u_j}{\lambda^j} \mid u_j \in \{0, 1\} \right\} \tag{14}$$

and we remark that the reachable sets are related by the recursive formula

$$\begin{cases} R^{(fr)}_{k+1}(\lambda) = R^{(fr)}_k(\lambda) \cup \left(R^{(fr)}_k(\lambda) + \frac{1}{\lambda^{k+1}} \right) \\ R^{(fr)}_0(\lambda) = 0. \end{cases} \tag{15}$$

We are interested in a qualitative study of $\operatorname{conv}(R^{(fr)}_k(\lambda))$, the convex hull of $R^{(fr)}_k(\lambda)$. To this end we note that $R^{(fr)}_k(\lambda)$ is a finite set (with at most 2^k elements), hence its convex hull is a convex polygon that we call $P_k(\lambda)$.

In view of (15) we have

$$\begin{cases} P_{k+1} = \operatorname{conv}\left(P_k \cup (P_k + \frac{1}{\lambda^{k+1}}) \right) \\ P_0 = 0. \end{cases} \tag{16}$$

namely for every $k \in \mathbb{N}$, P_{k+1} is the convex hull of the union of P_k with its translation $P_k + \frac{1}{\lambda^{k+1}}$. Hence it is useful for our purposes to set for every $\mathbf{u}, \mathbf{v} \in \mathbb{C}$

$$\mathbf{u} \cdot \mathbf{v} := |\mathbf{u}||\mathbf{v}| \cos(\arg \mathbf{u} - \arg \mathbf{v}) = \Re(\mathbf{u})\Re(\mathbf{v}) + \Im(\mathbf{u})\Im(\mathbf{v})$$

and the following result, whose proof can be found in [12].

Lemma 3.1. *Let P be a polygon with clock-wise ordered vertices $\mathbf{v}^1, \dots, \mathbf{v}^l$ and let*

$$\mathbf{n}_j := (\mathbf{v}^j - \mathbf{v}^{j-1})^\perp = e^{i\frac{\pi}{2}}(\mathbf{v}^j - \mathbf{v}^{j-1})$$

for $j = 1, \dots, l$. Assume the index operations to be performed modulus l so that $\mathbf{n}_1 = e^{i\frac{\pi}{2}}(\mathbf{v}^1 - \mathbf{v}^l)$. Then for every $\mathbf{t} \in \mathbb{C}$, $\mathbf{t} \neq 0$, there exists two indices $j_1, j_2 \in \{1, \dots, l\}$ such that

$$\mathbf{n}_{j_1-1} \cdot \mathbf{t} < 0 \quad \textit{and} \quad \mathbf{n}_{j_1} \cdot \mathbf{t} \geq 0 \tag{17}$$

$$\mathbf{n}_{j_2-1} \cdot \mathbf{t} > 0 \quad \textit{and} \quad \mathbf{n}_{j_2} \cdot \mathbf{t} \leq 0. \tag{18}$$

Moreover the convex hull of $P \cup (P + t)$ is a polygon whose vertices are:

$$\mathbf{v}^{j_1}, \dots, \mathbf{v}^{j_2}, \mathbf{v}^{j_2} + \mathbf{t}, \dots, \mathbf{v}^{j_1-1} + \mathbf{t}, \mathbf{v}^{j_1} + \mathbf{t}. \tag{19}$$

In particular l edges of $\operatorname{conv}(P \cup P + t)$ are parallel to the edges of P and 2 edges are parallel to $\mathbf{t}$.

Remark 4. As the index operations on the vertices are considered modulus l, if $j_1 > j_2$ the expression in (19) means

$$\mathbf{v}^{j_1}, \dots, \mathbf{v}^l, \mathbf{v}^1, \dots, \mathbf{v}^{j_2}, \mathbf{v}^{j_2} + \mathbf{t}, \dots, \mathbf{v}^{j_1-1} + \mathbf{t}, \mathbf{v}^{j_1} + \mathbf{t}. \tag{20}$$

conversely if $j_1 < j_2$ the extended version of (19) is

$$\mathbf{v}^{j_1}, \dots, \mathbf{v}^{j_2}, \mathbf{v}^{j_2} + \mathbf{t}, \dots, \mathbf{v}^l + \mathbf{t}, \mathbf{v}^1 + \mathbf{t}, \dots, \mathbf{v}^{j_1-1} + \mathbf{t}, \mathbf{v}^{j_1} + \mathbf{t}. \tag{21}$$

In this Section we use only the last statement of Lemma 3.1, namely the qualitative part of the result. The explicit characterization of the vertices of $\operatorname{conv}(P \cup (P + t))$ is used in Section 5, where the convex hull of the reachable full-rotation configuration is studied the particular case $\omega = \pi/3$.

We are now in position to prove

Theorem 3.2. *For every $k \in \mathbb{N}$ P_k is a polygon with $2k$ (possibly consecutive) pairwise parallel edges. Each couple of parallel edges is parallel to λ^{-j}, with $j = 1, \dots, k$.*

Proof. We remark that $P_1(\lambda) = \{0, \frac{1}{\lambda}\}$ can be looked at as a polygon with two vertices and with two overlapped and parallel to λ^{-1} edges. As

$$P_{j+1} = \text{conv}(P_j \cup (P_j + \frac{1}{\lambda^{j+1}}))$$

for every $j = 1, \ldots, k-1$, by iteratively applying Lemma 3.1 we get that P_k has pairwise parallel edges and every couple of edges is parallel either to λ^{-1} or to any of the successive translations, i.e., $\lambda^{-2}, \ldots, \lambda^{-k}$. □

3.2. **Full-extension configurations.** Full extensions configurations are characterized by the fact that the extension controls u_k are constantly equal to 1. In this case the set of reachable points is

$$R_k^{(fe)}(\rho,\omega) := \left\{ \sum_{j=1}^{k} \frac{1}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)} \mid v_n \in \{0,1\} \right\} \tag{22}$$

Proposition 2. *For every $\rho > 1$ and $\omega \in (0,\pi)$*

$$R_1^{(fe)} = \left\{ \frac{1}{\rho}, \frac{e^{-i(\pi-\omega)}}{\rho} \right\} \tag{23}$$

and for $k \geq 1$

$$R_{k+1}^{(fe)}(\rho,\omega) = \frac{1}{\rho}\left(1 + R_k^{(fe)}(\rho,\omega)\right) \bigcup \frac{e^{-i(\pi-\omega)}}{\rho}\left(1 + R_k^{(fe)}(\rho,\omega)\right). \tag{24}$$

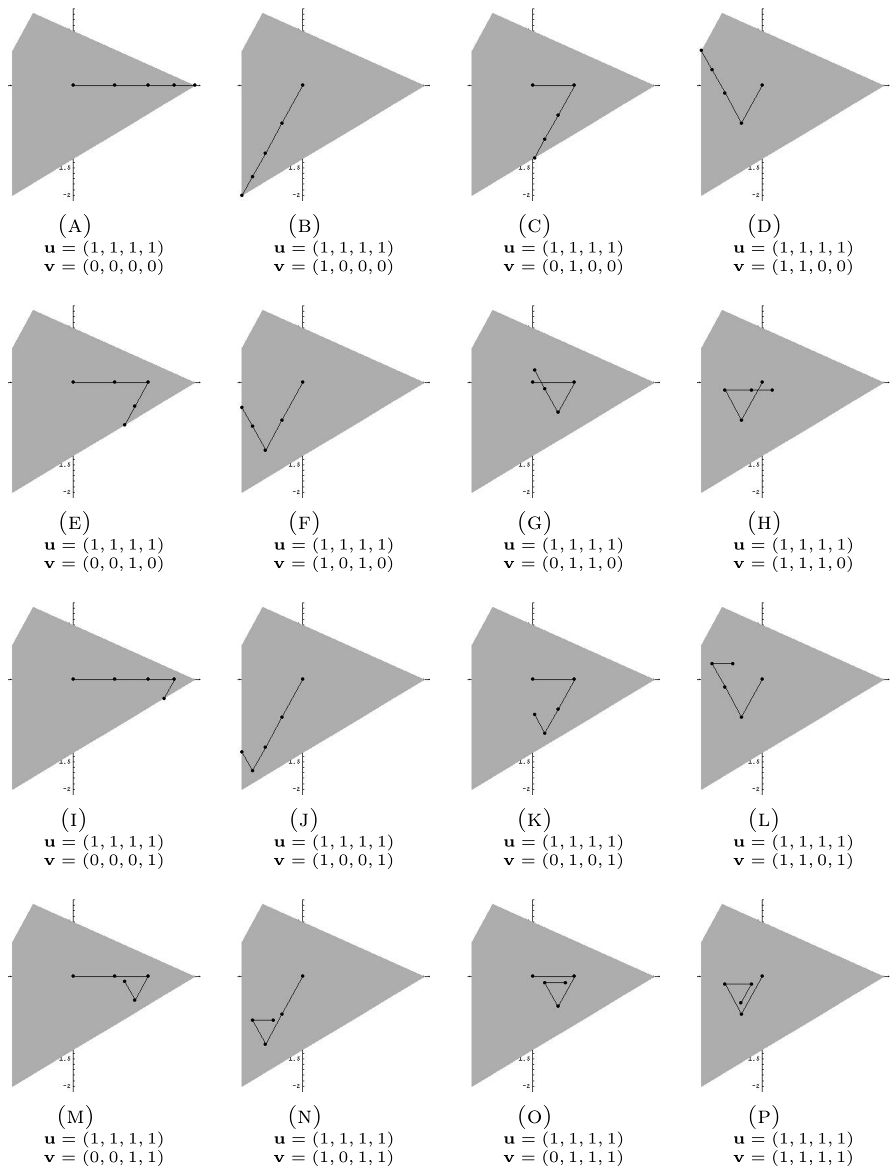
Proof. Equality (23) immediately follows by (22). Moreover any reachable point $x_{k+1} \in R_{k+1}^{(fe)}(\rho,\omega)$ satisfies

$$\begin{aligned}
x_{k+1} &= \sum_{j=1}^{k+1} \frac{1}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)} \\
&= \frac{e^{-i\ v_1(\pi-\omega)}}{\rho} + \sum_{j=2}^{k+1} \frac{1}{\rho^j} e^{-i\ v_1(\pi-\omega) - i\sum_{n=2}^{j} v_n(\pi-\omega)} \\
&= \frac{e^{-i\ v_1(\pi-\omega)}}{\rho} \left(1 + \sum_{j=2}^{k+1} \frac{1}{\rho^{j-1}} e^{-i\sum_{n=1}^{j-1} v_{n+1}(\pi-\omega)} \right) \\
&= \frac{e^{-i\ v_1(\pi-\omega)}}{\rho} \left(1 + \sum_{j=1}^{k} \frac{1}{\rho^{j}} e^{-i\sum_{n=1}^{j} v_{n+1}(\pi-\omega)} \right) \\
&= \frac{e^{-i\ v_1(\pi-\omega)}}{\rho} (1 + \tilde{x}_k)
\end{aligned}$$

where

$$\tilde{x}_k = \sum_{j=1}^{k+1} \frac{1}{\rho^j} e^{-i\sum_{n=1}^{j} \tilde{v}_n(\pi-\omega)} \in R_k^{(fe)}(\rho,\omega)$$

(in particular we set $\tilde{v}_n = v_{n+1}$). Thesis follows by taking into account both cases $v_1 = 0$ and $v_1 = 1$. □



(A) $\mathbf{u} = (1, 1, 1, 1)$ $\mathbf{v} = (0, 0, 0, 0)$

(B) $\mathbf{u} = (1, 1, 1, 1)$ $\mathbf{v} = (1, 0, 0, 0)$

(C) $\mathbf{u} = (1, 1, 1, 1)$ $\mathbf{v} = (0, 1, 0, 0)$

(D) $\mathbf{u} = (1, 1, 1, 1)$ $\mathbf{v} = (1, 1, 0, 0)$

(E) $\mathbf{u} = (1, 1, 1, 1)$ $\mathbf{v} = (0, 0, 1, 0)$

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FIGURE 8. Full extension configurations with $\rho = 2^{1/3}$ and $\omega = \pi/3$: the control digits u_k are constantly equal to one. We remark the self-intersecting configurations in (G) and (H).

4. **Approximate reachability and control sequences.** We define approximate reachability set the following

$$R_\infty(\rho, \omega) := \overline{\left(\bigcup_{k=1}^{\infty} R_k(\rho, \omega) \right)} = \left\{ \sum_{j=1}^{\infty} \frac{u_j}{\rho^j} e^{-i \sum_{n=1}^{j} v_n (\pi - \omega)} \mid u_j, v_n \in \{0, 1\} \right\}. \quad (25)$$

Remark 5. $R_\infty(\rho,\omega)$ is a bounded set. Indeed if $x \in R_\infty(\rho,\omega)$ then

$$|x| = \left|\sum_{j=1}^{\infty} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)}\right| \leq \sum_{j=1}^{\infty} \left|\frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)}\right| = \sum_{j=1}^{\infty} \frac{u_j}{\rho^j} = \frac{1}{\rho-1}.$$

By simple computations we have the following result, stating that every point in the approximate reachable set may be reached by the finger with arbitrary precision.

Proposition 3. *Let $\rho > 1$, $\omega \in (0,\pi)$ and $x \in R_\infty(\rho,\omega)$. For every $k \geq 1$ there exists $x_k \in R_k(\rho,\omega)$ such that*

$$|x - x_k| \leq \frac{1}{\rho^k(\rho-1)}. \tag{26}$$

4.1. **Approximate reachable set.** We approach the study $R_\infty(\rho,\omega)$ by means of the Iterated Function System (IFS) theory. To consider the set of representable numbers as the attractor of an iterated function system is a rather classical approach in the study of expansions in non-integer bases (see for instance [7]). Relations between representability in non-integer bases and discrete control systems where established by Y. Chitour and B. Piccoli in [2], where the reachability of linear discrete control systems is discussed by means of results coming from the theory of expansions in non-integer bases. In this section we extend the idea of Chitour and Piccoli in two different directions: we consider a different linear dynamics, indeed both rotation and translation are controlled, and we deep the relation between discrete control systems and expansions in non-integer bases by using fractal geometry arguments.

For a general introduction on fractal geometry and, in particular on fractals generated by iterated function systems, we refer to [4]. We recall that an IFS $\mathcal{F}$ is a finite set of contractive maps, namely

$$\mathcal{F} = \{f_h : \mathbb{C} \to \mathbb{C} | h = 1,\dots,H\}$$

and for every $x, y \in \mathbb{C}$ and $h = 1,\dots,H$

$$|f_h(x) - f_h(y)| \leq c_h|x-y|$$

for some $0 < c_h < 1$. The *Hutchinson operator* acts on the power set of $\mathbb{C}$ as follows

$$\mathcal{F}(X) := \bigcup_{h=1}^{H} f_h(X) = \bigcup_{h=1}^{H} \bigcup_{x\in X} f_h(x).$$

We now state an adapted version of Hutchinson's classical theorem, originally proved in [10].

Theorem 4.1. *Let $\mathcal{F}$ be an iterated function system. There exists a unique closed bounded set R, the* attractor*, such that $R = \mathcal{F}(R)$. For an arbitrary $X \subset \mathbb{C}$ let $\mathcal{F}^k(X) = \mathcal{F}(\mathcal{F}^{k-1}(X))$. Then for closed bounded X,*

$$R = \lim_{k\to\infty} \mathcal{F}^k(X).$$

in the Hausdorff metric.

Attractors can be constructed by remarking that if $X \subset \mathbb{C}$ satisfies

$$\mathcal{F}(X) \subset X \tag{27}$$

then

$$R = \bigcap_{k=1}^{\infty} \mathcal{F}^k(X),$$

namely the sequence $\mathcal{F}^k(X)$ provides increasingly good approximations of R [4].

Remark 6. For linear iterated function systems, $X = conv(R)$ satisfies (27). Indeed if $\tilde{x} \in \mathcal{F}(\text{conv}(R))$ then $\tilde{x} = f_h(x)$ for some $h = 1, \dots, H$ and some $x \in \text{conv}(R)$. In particular

$$x = f_h(\alpha x_1 + (1-\alpha)x_2)$$

for some $\alpha \in [0,1]$ and $x_1, x_2 \in R$ such that $x = \alpha x_1 + (1-\alpha)x_2$. As f_h is a linear map we also have

$$\tilde{x} = \alpha f_h(x_1) + (1-\alpha) f_h(x_2)$$

and setting $\tilde{x}_1 := f_h(x_1)$ and $\tilde{x}_2 := f_h(x_2)$ we may rewrite the above equality as follows

$$\tilde{x} = \alpha \tilde{x}_1 + (1-\alpha)\tilde{x}_2.$$

Since $x_1, x_2 \in R = \mathcal{F}(R)$ then $\tilde{x}_1, \tilde{x}_2 \in R$, and consequently $\tilde{x}$ is a convex combination of elements of R, namely $\tilde{x} \in \text{conv}(R)$.

Remark 6, together with the following Proposition, motivates our interest on the convex hull of the reachable set (see Figure 9).

Proposition 4. *For every $\rho > 1$ and $\omega \in (0, \pi)$, the approximate reachability set $R_\infty(\rho, \omega)$ is the (unique) fixed point of the IFS*

$$\mathcal{F}_{\rho,\omega} = \{f_h : \mathbb{C} \to \mathbb{C} \mid h = 1, \dots, 4\}$$

where

$$\begin{array}{ll} f_1 : x \mapsto \frac{1}{\rho}x & f_2 : x \mapsto \frac{e^{-i(\pi-\omega)}}{\rho}x \\ f_3 : x \mapsto \frac{1}{\rho}(x+1) & f_4 : x \mapsto \frac{e^{-i(\pi-\omega)}}{\rho}(x+1). \end{array} \tag{28}$$

Proof. $R_\infty(\rho, \omega)$ is the fixed point of $\mathcal{F}_{\rho,\omega}$ if and only if

$$R_\infty(\rho,\omega) = \bigcup_{h=1}^{4} f_h(R_\infty(\rho,\omega)).$$

We prove the above equality by double inclusion. First of all remark that for every $h = 1, \dots, 4$

$$f_h(x) = \frac{e^{-iv(\pi-\omega)}}{\rho}(u+x) \tag{29}$$

for some $u, v \in \{0, 1\}$. In particular

- $h = 1$ if and only if $u = v = 0$;
- $h = 2$ if and only if $u = 0$ and $v = 1$;
- $h = 3$ if and only if $u = 1$ and $v = 0$;
- $h = 4$ if and only if $u = v = 1$.

Now, let $x \in R_\infty(\rho,\omega)$. We have

$$\begin{aligned} x &= \sum_{j=1}^{\infty} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)} \\ &= \frac{e^{-iv_1(\pi-\omega)}}{\rho} \left(u_1 + \sum_{j=1}^{\infty} \frac{u_{j+1}}{\rho^j} e^{-i\sum_{n=1}^{j} v_{n+1}(\pi-\omega)} \right) \end{aligned}$$

Therefore, setting $u = u_1$ and $v = v_1$ in (29), we obtain

$$x = f_h \left(\sum_{j=1}^{\infty} \frac{u_{j+1}}{\rho^j} e^{-i\sum_{n=1}^{j} v_{n+1}(\pi-\omega)} \right)$$

for some $h = 1, \dots, 4$. As

$$\sum_{j=1}^{\infty} \frac{u_{j+1}}{\rho^j} e^{-i\sum_{n=1}^{j} v_{n+1}(\pi-\omega)} \in R_\infty(\rho,\omega).$$

we get $x \in \bigcup_{h=1}^4 f_h(R_\infty(\rho,\omega))$ and, by the arbitrariness of x,

$$R_\infty(\rho,\omega) \subseteq \bigcup_{h=1}^{4} f_h(R_\infty(\rho,\omega)).$$

Now consider

$$\tilde{x} \in \bigcup_{h=1}^{4} f_h(R_\infty(\rho,\omega))$$

so that

$$\tilde{x} = f_h(x)$$

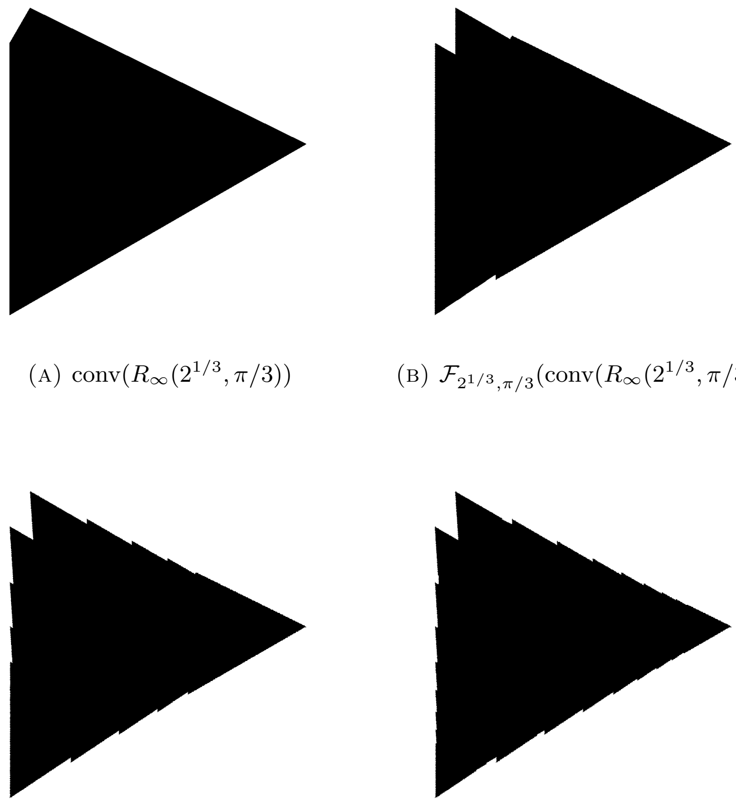
for some $h = 1, \dots, 4$ and some $x \in R_\infty(\rho,\omega)$. In view of (29) and of the definition of $R_\infty(\rho,\omega)$, for appropriate $u, v \in \{0,1\}$ the above equality is equivalent to

$$\begin{aligned} \tilde{x} &= \frac{e^{-iv(\pi-\omega)}}{\rho}(u + x) \\ &= \frac{e^{-iv(\pi-\omega)}}{\rho} \left(u + \sum_{j=1}^{\infty} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)} \right) \\ &= \sum_{j=1}^{\infty} \frac{\tilde{u}_j}{\rho^j} e^{-i\sum_{n=1}^{j} \tilde{v}_n(\pi-\omega)} \end{aligned}$$

where $\tilde{u}_1 = u$, $\tilde{v}_1 = v$ and, for every $j > 1$, $\tilde{u}_j = u_{j-1}$ and $\tilde{v}_j = v_{j-1}$. Hence $\tilde{x} \in R_\infty(\rho,\omega)$ and, by the arbitrariness of $\tilde{x}$, we get the inclusion

$$\bigcup_{h=1}^{4} f_h(R_\infty(\rho,\omega)) \subseteq R_\infty(\rho,\omega)$$

and hence the thesis. □



(A) $\mathrm{conv}(R_\infty(2^{1/3},\pi/3))$

(B) $\mathcal{F}_{2^{1/3},\pi/3}(\mathrm{conv}(R_\infty(2^{1/3},\pi/3)))$

(C) $\mathcal{F}^4_{2^{1/3},\pi/3}(\mathrm{conv}(R_\infty(2^{1/3},\pi/3)))$

(D) $\mathcal{F}^8_{2^{1/3},\pi/3}(\mathrm{conv}(R_\infty(2^{1/3},\pi/3)))$

FIGURE 9. Various approximations of $R_\infty(2^{1/3},\pi/3)$. The set $\mathrm{conv}(R_\infty(2^{1/3},\pi/3))$ is explicitly characterized in Section 5.

4.2. **Expansions of a reachable point.** In Proposition 3 we showed that for every $x \in R_\infty(\rho,\omega)$ there exists an arbitrary close reachable point. In particular for every given $\varepsilon > 0$, if $k \in \mathbb{N}$ satisfies

$$\frac{1}{\rho^k(\rho-1)} < \epsilon$$

then there exists $x_k \in R_k(\rho,\omega)$ satisfying

$$|x - x_k| < \varepsilon. \tag{30}$$

Our goal is to determine some control sequences $(u_j)_{j=1}^k$ and $(v_j)_{j=1}^k$ depending on x such that x_k in (30) satisfies

$$x_k = \sum_{j=1}^{k} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\rho)}.$$

To this end we focus on the particular class of *expansions*.

Definition 4.2. Let $x \in R_\infty(\rho,\omega)$. A couple of infinite binary control sequences $\big((u_j)_{j=1}^\infty,(v_j)_{j=1}^\infty\big)$ is an *expansion* of x if

$$x=\sum_{j=1}^{\infty}\frac{u_j}{\rho^j}e^{-i\sum_{n=1}^{j}v_n(\pi-\omega)}. \tag{31}$$

Remark 7. If $\big((u_j)_{j=1}^\infty,(v_j)_{j=1}^\infty\big)$ is an expansion of x then

$$\left|x-\sum_{j=1}^{\infty}\frac{u_j}{\rho^j}e^{-i\sum_{n=1}^{j}v_n(\pi-\omega)}\right|\leq\frac{1}{\rho^k(\rho-1)}<\varepsilon$$

for every $\varepsilon>0$ and any sufficiently large k.

We want to find sufficient and necessary conditions for a couple of control sequences to be an expansion of a given $x\in R_\infty(\rho,\omega)$. To this end, we recall the definitions

$$\begin{array}{ll} f_1: x\mapsto \frac{1}{\rho}x & f_2: x\mapsto \frac{e^{-(\pi-\omega)i}}{\rho}x \\ f_3: x\mapsto \frac{1}{\rho}(x+1) & f_4: x\mapsto \frac{e^{-(\pi-\omega)i}}{\rho}(x+1) \end{array} \tag{32}$$

and we introduce the *decision function*

$$d:\{1,2,3,4\}\to\{0,1\}\times\{0,1\}$$

such that

$$d(h)=(u,v)$$

if and only if

$$f_h(x)=\frac{e^{-iv\omega}}{\rho}(u+x).$$

Remark 8. We have

$$\begin{array}{ll} d(1)=(0,0); & d(2)=(1,0); \\ d(3)=(1,0); & d(4)=(1,1). \end{array}$$

Lemma 4.3. *Let $k\geq 1$, $h_1,\dots,h_k\in\{1,2,3,4\}$, $(u_j,v_j)=d(h_j)$ for $j=1,\dots,k$ and $r\in\mathbb{C}$. Then*

$$f_{h_1}\circ\cdots\circ f_{h_k}(r)=\sum_{j=1}^{k}\frac{u_j}{\rho^j}e^{-i\sum_{n=1}^{j}v_n(\pi-\omega)}+\frac{r}{\rho^k}e^{-i\sum_{n=1}^{k}v_n(\pi-\omega)} \tag{33}$$

Proof. By induction on k. If $k=1$ then it follows by the definition of decision function that for every $h_1=\{1,2,3,4\}$

$$f_{h_1}(r)=\frac{u_1}{\rho}e^{-iv_1(\pi-\omega)}+\frac{r}{\rho}e^{-iv_1(\pi-\omega)} \tag{34}$$

where $(u_1, v_1) = d(h_1)$. Fix now $k \geq 1$ and assume (33) as inductive hypothesis. For every $h_{k+1} \in \{1, 2, 3, 4\}$

$$\begin{aligned}
f_{h_1} \circ \cdots \circ f_{h_k} \circ f_{h_{k+1}}(r) &= f_{h_1} \circ \cdots \circ f_{h_k}(f_{h_{k+1}}(r)) \\
&= f_{h_1} \circ \cdots \circ f_{h_k}\left(\frac{u_{k+1}}{\rho} e^{-iv_{k+1}(\pi-\omega)} + \frac{r}{\rho} e^{-iv_{k+1}(\pi-\omega)}\right) \\
&= \sum_{j=1}^{k} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)} \\
&\quad + \frac{1}{\rho^k}\left(\frac{u_{k+1}}{\rho} e^{-iv_{k+1}(\pi-\omega)} + \frac{r}{\rho} e^{-iv_{k+1}(\pi-\omega)}\right) e^{-i\sum_{n=1}^{k} v_n(\pi-\omega)} \\
&= \sum_{j=1}^{k+1} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)} + \frac{r}{\rho^{k+1}} e^{-i\sum_{n=1}^{k+1} v_n(\pi-\omega)}.
\end{aligned}$$

This proves the inductive step and, hence, the thesis. □

Theorem 4.4. *With the same notations of (32) and of Remark 8 Let $\rho > 1$, $\omega \in (0, \pi)$, $x \in R_\infty(\rho, \omega)$ and $(u_j)_j^\infty$ and $(v_j)_j^\infty$ be two infinite binary sequences. Set for every $j \geq 1$*

$$h_j := d^{-1}(u_j, v_j),$$

$$r_0 := x$$

and

$$r_j := f_{h_j}^{-1}(r_{j-1})$$

Then $\left((u_j)_{j=1}^\infty, (v_j)_{j=1}^\infty\right)$ is an expansion of x, namely

$$x = \sum_{j=1}^{\infty} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\rho)}, \tag{35}$$

if and only if $(r_j)_{j=0}^\infty$ is bounded. In particular if (35) holds, then $r_j \in R_\infty(\rho, \omega)$ for every $j \geq 0$.

Proof. Assume (35). By definition of r_k and by Lemma 4.3 for every $k \geq 1$

$$\begin{aligned}
r_k &= f_{h_k}^{-1} \circ \cdots f_{h_1}^{-1}(x) \\
&= (f_{h_1} \circ \cdots \circ f_{h_k})^{-1}(x) \\
&= \left(x - \sum_{j=1}^{k} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)}\right) \rho^k e^{-i\sum_{n=0}^{k} v_n(\pi-\omega)} \\
&= \sum_{j=1}^{\infty} \frac{u_{j+k}}{\rho^j} e^{-i\sum_{n=1}^{j} v_{n+k}(\pi-\omega)}
\end{aligned}$$

hence for every $k \geq 1$,

$$r_k \in R_\infty(\rho, \omega)$$

and, also in view of Remark 5, we get that $(r_k)_{j=0}^\infty$ is bounded.

Suppose now that $(r_k)_{k=0}^\infty$ is bounded. By the relation

$$r_k = f_{h_k}^{-1} \circ \cdots f_{h_1}^{-1}(x)$$

and by Lemma 4.3

$$x = f_{h_1} \circ \cdots f_{h_k}(r_k) = \sum_{j=1}^{k} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)} + \frac{r_k}{\rho^k} e^{-i\sum_{n=0}^{k} v_n(\pi-\omega)}.$$

Therefore the boundedness of $(r_k)_{k=0}^{\infty}$ implies

$$\lim_{k\to\infty} \sum_{j=1}^{k} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)} + \frac{r_k}{\rho^k} e^{-i\sum_{n=0}^{k} v_n(\pi-\omega)} = \sum_{j=1}^{\infty} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)}.$$

and, consequently, (35). □

4.3. **Control algorithms.** In view of Theorem 4.4, to construct a couple of control sequences approximating $x \in R_\infty(\rho,\omega)$ we may proceed as follows. We set $r_0 = x$ and we remark that $R_\infty(\rho,\omega) = \mathcal{F}_{\rho,\omega}(R_\infty(\rho,\omega))$ implies the existence of $h_1 \in \{1,2,3,4\}$ such that

$$r_0 \in f_{h_1}(R_\infty(\rho,\omega))$$

and we define

$$r_1 := f_{h_1}^{-1}(r_0).$$

By construction, $r_1 \in R(\rho,\infty)$ therefore there exists h_2 such that $r_1 \in f_{h_2}(R_\infty(\rho,\omega))$ and we may define

$$r_2 := f_{h_2}^{-1}(r_0).$$

By iterating this argument we get the sequences $(r_j)_{j=1}^{\infty}$ and $(h_j)_{j=1}^{\infty}$ with the following properties:

$$r_j = f_{h_j}^{-1}(r_{j-1})$$

and $r_{j-1} \in R_\infty(\rho,\omega)$ for every $j \geq 0$. By Theorem 4.4, if $(u_j, v_j) = d(h_j)$ for every $j \geq 1$ then

$$x = \sum_{j=1}^{\infty} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\rho)}.$$

Remark 9. If $h', h'' \in \{1,\dots,4\}$ the intersection

$$f_{h'}(R_\infty(\rho,\omega)) \cap f_{h''}(R_\infty(\rho,\omega))$$

could be not empty. In particular, if

$$r_{j-1} \in f_{h'}(R_\infty(\rho,\omega)) \cap f_{h''}(R_\infty(\rho,\omega)),$$

then h_j is arbitrarily chosen in $\{h', h''\}$. This ambiguity possibly gives rise to different expansions and to infinite different algorithms to generate expansions. The described method is a generalization of a classical techniques for expansions in non-integer bases in the unidimensional case.

We now construct an expansion where, at every step, the choice to not rotate and to not extend the phalanxes is privileged. We set

$$r_0 = x; \tag{36}$$

$$h_j = \begin{cases} 1 \text{ if } r_{j-1} \in f_1(R_\infty(\rho,\omega)); \\ 2 \text{ if } r_{j-1} \in f_2(R_\infty(\rho,\omega)) \setminus f_1^{-1}(R_\infty(\rho,\omega)); \\ 3 \text{ if } r_{j-1} \in f_3(R_\infty(\rho,\omega)) \setminus \bigcup_{h=1}^{2} f_h^{-1}(R_\infty(\rho,\omega)); \\ 4 \text{ otherwise} \end{cases} \tag{37}$$

$$r_j = f_{h_j}^{-1}(r_{j-1}); \tag{38}$$

$$(u_j, v_j) = d(h_j) \tag{39}$$

By construction $r_j \in R_\infty(\rho,\omega)$ hence, by Theorem 4.4,

$$x = \sum_{j=1}^{\infty} \frac{u_j}{\rho^j} e^{\sum_{n=1}^{j} v_n(\pi-\rho)i}. \tag{40}$$

Moreover (37) implies that, when it is possible, null controls are chosen.

5. **Case $\omega = \pi/3$.** Before we discuss this model case we wish to point out that the results obtained here can be generalized to the case $\omega = q\pi$ with fixed $q \in \mathbb{Q}$, nevertheless this would involve more sophisticated combinatorial arguments. The aim of this paper is mainly to show the connections between the model and the theory of expansions in non-integer bases and we privileged a model case in order to make the computations more handly. The case $\omega = q\pi$ with fixed $q \in \mathbb{Q}$ will be object of a further study.

In general the configurations that can be obtained in our model are combinations of rotations of the angle $\pi - \omega$ and translations, in particular every extended phalanx is parallel to an integer power of $e^{i(\pi-\omega)}$ on the complex plane (see (1) and Proposition 1 and recall the assumption $\omega_0 = 0$). By choosing $\omega = q\pi$ with $q \in \mathbb{Q}$, the number of possible directions for phalanx is finite and, in particular, if $\omega = \pi/3$ the possible directions are $e^{i2\pi/3}$, $e^{i4\pi/3}$ and $e^{i2\pi}$. Since we assumed $\omega \in (0,\pi)$, this case realizes the minimal number of possible different directions and numerical simulations suggest that it provides the minimal number of extremal points for the convex hull of the reachable set.

Here we obtain an explicit description of the extremal points of the convex hull of the reachable set and a convexity condition for the set of points that can be reached by full-rotation configurations.

5.1. **The full rotation case with $\omega = \pi/3$.** We now discuss the particular case of $\omega = \pi/3$, so that fixing $\rho > 1$

$$\lambda = \rho e^{i\frac{2\pi}{3}}$$

and, in particular

$$\lambda^3 = \rho^3.$$

We recall the notations

$$R_k^{(fr)}(\lambda) := \left\{ \sum_{j=1}^{k} \frac{u_j}{\lambda^j} \mid u_j \in \{0,1\} \right\}$$

and

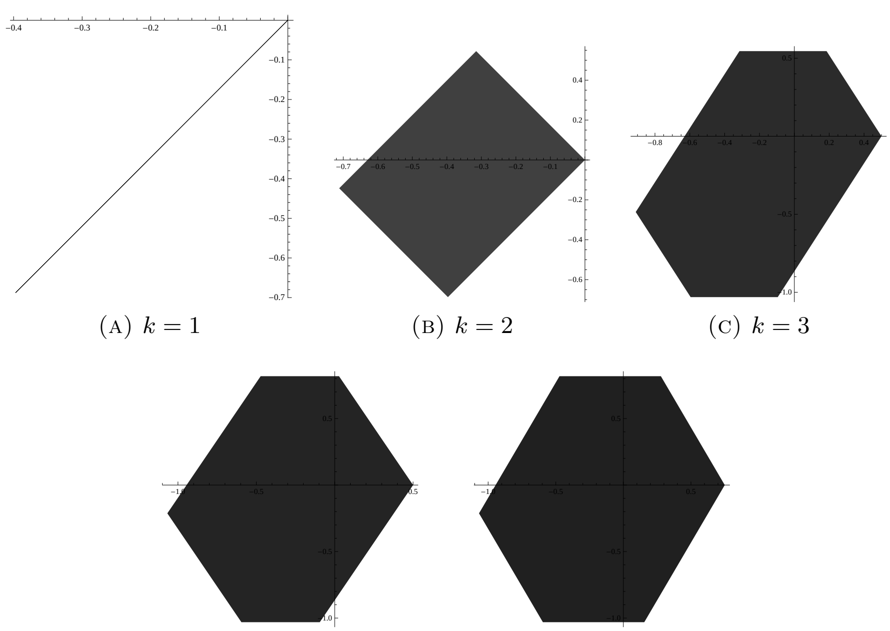
$$P_k(\lambda) := \operatorname{conv}(R_k^{(fr)}(\lambda))$$

In general a polygon on the complex plane is a simple finite chain of edges whose endpoints are the vertices. An *extremal vertex* of a polygon is a vertex which is the common point of two edges not lying on the same line. Clearly a polygon is characterized by the ordered list of its extremal vertices, hence we introduce the notation

$$V_k(\lambda) := \{\mathbf{u}^j \mid \mathbf{u}^j \text{ is an extremal vertex of } P_k(\lambda)\}.$$

For $k = 0, \ldots, 5$ we computed $V_k(\lambda)$ by hand:

$$\begin{aligned}
V_0(\lambda) &= \{0\} \\
V_1(\lambda) &= \left\{0, \frac{1}{\lambda}\right\} \\
V_2(\lambda) &= \left\{0, \frac{1}{\lambda}, \frac{1}{\lambda} + \frac{1}{\lambda^2}, \frac{1}{\lambda^2}\right\} \\
V_3(\lambda) &= \left\{\frac{1}{\rho^3}, \frac{1}{\rho^3} + \frac{1}{\lambda}, \frac{1}{\lambda}, \frac{1}{\lambda} + \frac{1}{\lambda^2}, \frac{1}{\lambda^2}, \frac{1}{\lambda^2} + \frac{1}{\rho^3}\right\} && (41) \\
V_4(\lambda) &= \left\{\frac{1}{\rho^3}, \frac{1}{\rho^3} + \frac{1}{\rho^3\lambda}, \frac{1}{\rho^3\lambda}, \frac{1}{\rho^3\lambda} + \frac{1}{\lambda^2}, \frac{1}{\lambda^2}, \frac{1}{\lambda^2} + \frac{1}{\rho^3}\right\} && (42) \\
V_5(\lambda) &= \left\{\frac{1}{\rho^3}, \frac{1}{\rho^3} + \frac{1}{\rho^3\lambda}, \frac{1}{\rho^3\lambda}, \frac{1}{\rho^3\lambda} + \frac{1}{\rho^3\lambda^2}, \frac{1}{\rho^3\lambda^2}, \frac{1}{\rho^3\lambda^2} + \frac{1}{\rho^3}\right\} && (43) \\
&= \frac{1}{\rho^3}\left\{1, 1 + \frac{1}{\lambda}, \frac{1}{\lambda}, \frac{1}{\lambda} + \frac{1}{\lambda^2}, \frac{1}{\lambda^2}, \frac{1}{\lambda^2} + 1\right\},
\end{aligned}$$



(A) $k = 1$ (B) $k = 2$ (C) $k = 3$

(D) $k = 4$ (E) $k = 5$

FIGURE 10. $P_k(\lambda) = \mathrm{conv}(V_k)$ with $k = 1, \ldots, 5$ and $\lambda = 2^{1/3}e^{i2\pi/3}$.

while for larger k's we have

Theorem 5.1. *Let $k \geq 1$,*

$$\mathsf{S}_{k_0,k,\rho^3} := \sum_{j=k_0}^{k} \frac{1}{\rho^{3j}}$$

and

$$a_k := \mathsf{S}_{1,k,\rho^3}, \quad b_k := \frac{1}{\lambda}\mathsf{S}_{1,k-1,\rho^3}, \quad c_k := \frac{1}{\lambda^2}\mathsf{S}_{1,k-1,\rho^3}$$

Then

$$V_{3k}(\lambda) = \{a_k,\ a_k + b_k,\ b_k,\ b_k + c_k,\ c_k,\ c_k + a_k\} \tag{44}$$

$$V_{3k+1}(\lambda) = \{a_k,\ a_k + b_{k+1},\ b_{k+1},\ b_{k+1} + c_k,\ c_k,\ c_k + a_k\} \tag{45}$$

$$V_{3k+2}(\lambda) = \{a_k,\ a_k + b_{k+1},\ b_{k+1},\ b_{k+1} + c_{k+1},\ c_{k+1},\ c_{k+1} + a_k\} \tag{46}$$

Proof. By induction on k. The inductive base $k = 1$ is given by (41),(42) and (43). Assume now (44),(45) and (46) as inductive hypothesis. To prove the inductive step we need to show

$$V_{3(k+1)}(\lambda) = \{a_{k+1},\ a_{k+1} + b_{k+1},\ b_{k+1},\ b_{k+1} + c_{k+1},\ c_{k+1},\ c_{k+1} + a_{k+1}\} \tag{47}$$

$$V_{3(k+1)+1}(\lambda) = \{a_{k+1},\ a_{k+1} + b_{k+2},\ b_{k+2},\ b_{k+2} + c_{k+1},\ c_{k+1},\ c_{k+1} + a_{k+1}\} \tag{48}$$

$$V_{3(k+1)+2}(\lambda) = \{a_{k+1},\ a_{k+1} + b_{k+2},\ b_{k+2},\ b_{k+2} + c_{k+2},\ c_{k+2},\ c_{k+2} + a_{k+1}\} \tag{49}$$

Since the proves of (47), (48) and (49) are similar, we focus only on (47). Call $\mathbf{u}^1_{3k+2}, \dots, \mathbf{u}^6_{3k+2}$ the extremal vertices of $P_{3k+2}(\lambda)$, so that by inductive hypothesis

$$\mathbf{u}^1_{3k+2} = a_k$$
$$\mathbf{u}^2_{3k+2} = a_k + b_{k+1}$$
$$\dots$$
$$\mathbf{u}^6_{3k+2} = c_{k+1} + a_k,$$

and consider the normal vectors

$$\mathbf{n}^1_{3k+2} := e^{-i\frac{\pi}{2}}(\mathbf{u}^1_{3k+2} - \mathbf{u}^6_{3k+2})$$

and for $j > 1$

$$\mathbf{n}^j_{3k+2} := e^{-i\frac{\pi}{2}}(\mathbf{u}^j_{3k+2} - \mathbf{u}^{j-1}_{3k+2}).$$

In view of the recursive relation

$$P_{3(k+1)}(\lambda) = \operatorname{conv}\left(P_{3k+2} \cup \left(P_{3k+2}(\lambda) + \frac{1}{\lambda^{3(k+1)}}\right)\right)$$

we want to apply Lemma 3.1 to have a list of vertices of $P_{3(k+1)}(\lambda)$. Then we select from such a list the extremal vertices, so to get $V_{3(k+1)}(\lambda)$. In view of Lemma 3.1, we need to find a couple of indices j_1 and j_2 such that, setting $\mathbf{t} := \frac{1}{\lambda^{3(k+1)}} = \frac{1}{\rho^{3(k+1)}}$,

$$\mathbf{n}^{j_1-1}_{3k+2} \cdot \mathbf{t} < 0 \quad \text{and} \quad \mathbf{n}^{j_1}_{3k+2} \cdot \mathbf{t} \geq 0 \tag{50}$$

$$\mathbf{n}^{j_2-1}_{3k+2} \cdot \mathbf{t} > 0 \quad \text{and} \quad \mathbf{n}^{j_2}_{3k+2} \cdot \mathbf{t} \leq 0. \tag{51}$$

We have that $j_1 = 3$ and $j_2 = 6$. Indeed

$$\mathbf{n}_{3k+2}^2 \cdot \mathbf{t} = \mathsf{S}_{1,k,\rho^3} \left(\frac{-\sqrt{3}}{2} - \frac{1}{2} i \right) \cdot \frac{1}{\rho^{3(k+1)}} > 0; \tag{52}$$

$$\mathbf{n}_{3k+2}^3 \cdot \mathbf{t} = -i \ \mathsf{S}_{i,k,\rho^3} \cdot \frac{1}{\rho^{3(k+1)}} = 0; \tag{53}$$

and

$$\mathbf{n}_{3k+2}^5 \cdot \mathbf{t} = \mathsf{S}_{1,k,\rho^3} \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right) \cdot \frac{1}{\rho^{3(k+1)}} < 0; \tag{54}$$

$$\mathbf{n}_{3k+2}^6 \cdot \mathbf{t} = i \ \mathsf{S}_{i,k,\rho^3} \cdot \frac{1}{\rho^{3(k+1)}} = 0. \tag{55}$$

By Lemma 3.1 the (possibly not extremal) vertices of $P_{3(k+1)}$ are

$$\mathbf{u}_{3k+2}^3, \ \mathbf{u}_{3k+2}^4, \ \mathbf{u}_{3k+2}^5, \ \mathbf{u}_{3k+2}^6,$$

$$\mathbf{u}_{3k+2}^6 + \mathbf{t}, \ \mathbf{u}_{3k+2}^1 + \mathbf{t}, \ \mathbf{u}_{3k+2}^2 + \mathbf{t}, \ \mathbf{u}_{3k+2}^3 + \mathbf{t}.$$

Since the order on the vertices of a polygon is invariant for circular shift, we may rearrange the above vertices in the following manner

$$\mathbf{u}_{3k+2}^1 + \mathbf{t}, \ \mathbf{u}_{3k+2}^2 + \mathbf{t}, \ \mathbf{u}_{3k+2}^3 + \mathbf{t}, \mathbf{u}_{3k+2}^3,$$

$$\mathbf{u}_{3k+2}^4, \ \mathbf{u}_{3k+2}^5, \ \mathbf{u}_{3k+2}^6, \mathbf{u}_{3k+2}^6 + \mathbf{t}.$$

By a direct computation we have that the only not-extremal vertices are $\mathbf{u}_{3k+2}^3 + \mathbf{t}$ and $\mathbf{u}_{3k+2}^6$ because they are convex combinations of other vertices. Hence,

$$\begin{aligned} V_{3(k+1)}(\lambda) &= \{ \mathbf{u}_{3k+2}^1 + \mathbf{t}, \mathbf{u}_{3k+2}^2 + \mathbf{t}, \mathbf{u}_{3k+2}^3, \mathbf{u}_{3k+2}^4, \ \mathbf{u}_{3k+2}^5, \mathbf{u}_{3k+2}^6 + \mathbf{t} \} \\ &= \left\{ a_k + \frac{1}{\rho^{3(k+1)}}, \ a_k + b_{k+1} + \frac{1}{\rho^{3(k+1)}}, \ b_{k+1}, \ b_{k+1} + c_{k+1}, \right. \\ &\qquad \left. c_{k+1}, \ c_{k+1} + a_k + \frac{1}{\rho^{3(k+1)}} \right\} \\ &= \{ a_{k+1}, \ a_{k+1} + b_{k+1}, \ b_{k+1}, \ b_{k+1} + c_{k+1}, \ c_{k+1}, \ c_{k+1} + a_{k+1} \}. \end{aligned}$$

□

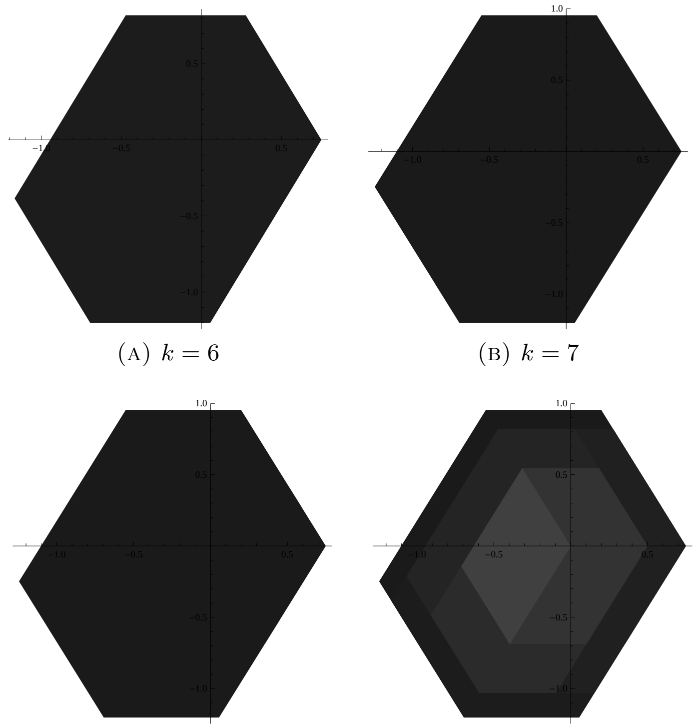
5.2. **A full extension case with $\omega = \pi/3$.** We assume the full extension condition $u_k = 1$ for every $k \in \mathbb{N}$ and, as in Section 5.1, that $w = \pi/3$. We provide an explicit characterization of the convex hull of $R_k^{(fe)}(\rho, \omega)$.

Lemma 5.2. *Let*

$$\operatorname{conv}(R_k^{(fe)}(\rho, \pi/3)) = \operatorname{conv}(\{ \mathbf{v}_k^j \mid j = 1, \dots, J \}) \tag{56}$$

then

$$\operatorname{conv}(R_{k+1}^{(fe)}(\rho, \pi/3)) = \operatorname{conv} \left(\left\{ \frac{1}{\rho}(1 + \mathbf{v}_k^j) \right\} \bigcup \left\{ \frac{e^{-i\frac{2\pi}{3}}}{\rho}(1 + \mathbf{v}_k^j) \right\} \right) \tag{57}$$



(A) $k = 6$ (B) $k = 7$

(C) $k = 8$ (D) $k = 2, \dots, 6$

FIGURE 11. (A), (B) and (C) respectively represent $P_k(2^{1/3}e^{i2\pi/3}) = \mathrm{conv}(V_k)$ with $k = 6, 7, 8$. In (D) $P_k(2^{1/3}e^{i2\pi/3})$ with $k = 2, \dots, 6$ are overlapped, so that the clearest area is $P_2(2^{1/3}e^{i2\pi/3})$ and the underlying darker areas range between $k = 3$ and $k = 6$.

Proof. By Proposition 2

$$R^{(fe)}_{k+1}(\rho, \pi/3) = \frac{1}{\rho}\left(1 + R^{(fe)}_k(\rho, \pi/3)\right) \bigcup \frac{e^{-i\frac{2\pi}{3}}}{\rho}\left(1 + R^{(fe)}_k(\rho, \pi/3))\right) \tag{58}$$

hence the inclusion

$$\mathrm{conv}(R^{(fe)}_{k+1}(\rho, \pi/3)) \supseteq \mathrm{conv}\left(\left\{\frac{1}{\rho}(1 + \mathbf{v}^j_k)\right\} \bigcup \left\{\frac{e^{-i\frac{2\pi}{3}}}{\rho}(1 + \mathbf{v}^j_k)\right\}\right)$$

is immediate, because since for every $j = 1, \dots, J$

$$\mathbf{v}^k_j \in R^{(fe)}_k(\rho, \pi/3)$$

then

$$\frac{1}{\rho}(1 + \mathbf{v}^j_k),\ \frac{e^{-i\frac{2\pi}{3}}}{\rho}(1 + \mathbf{v}^j_k) \in R^{(fe)}_{k+1}(\rho, \pi/3)$$

for every $j = 1, \dots, J$. Define now

$$\{\mathbf{z}^j \mid j = 1, \dots, 2J\} := \left\{\frac{1}{\rho}(1 + \mathbf{v}^j_k)\right\} \bigcup \left\{\frac{e^{-i\frac{2\pi}{3}}}{\rho}(1 + \mathbf{v}^j_k)\right\}.$$

and fix $x \in \mathrm{conv}(R^{(fe)}_{k+1}(\rho, \pi/3))$, so that

$$x = \alpha x_1 + (1-\alpha)x_2 \tag{59}$$

for some $\alpha \in [0,1]$ and some $x_1, x_2 \in R^{(fe)}_{k+1}$. Now, (56) implies that

$$x_1 = \frac{e^{-iv_1 \frac{2\pi}{3}}}{\rho}(1+\tilde{x}_1)$$

for an appropriate $v_1 \in \{0,1\}$. As

$$\tilde{x}_1 = \sum_{j=1}^{J} \mu^1_j \mathbf{v}^j$$

for some $\mu^1_1, \dots, \mu^1_J \geq 0$ with $\sum_{j=1}^J \mu^1_j = 1$, then for an appropriate $J(x_1) \subset \{1, \dots, 2J\}$

$$x_1 = \sum_{j \in J(x_1)} \mu^1_j \mathbf{z}^j$$

Similarly there exist $\mu^2_1, \dots, \mu^2_J \geq 0$ with $\sum_{j=1}^J \mu^2_j = 1$ such that

$$x_2 = \sum_{j \in J(x_2)} \mu^2_j \mathbf{z}^j$$

for some $J(x_2) \subset \{1, \dots, J\}$. Hence setting for every $j = 1, \dots, 2J$

$$\mu_j := \begin{cases} \alpha\mu^1_j + (1-\alpha)\mu^2_j & \text{if } j \in J(x_1) \cap J(x_2) \\ \alpha\mu^1_j & \text{if } j \in J(x_1) \setminus J(x_2) \\ (1-\alpha)\mu^2_j & \text{if } j \in J(x_2) \setminus J(x_1) \\ 0 & \text{otherwise} \end{cases}$$

we have $\mu_1, \dots, \mu_{2J} \geq 0$, $\sum_{j=1}^{2J} \mu_j = 1$ and, in view of (59),

$$x = \sum_{j=1}^{2J} \mu_j \mathbf{z}^j.$$

Therefore $x \in \mathrm{conv}(\{\mathbf{z}^j \mid j = 1, \dots, 2J\})$ and thesis follows by the arbitrariness of x. □

Theorem 5.3. *For every $\rho > 1$ and for every $k \geq 1$ then the convex hull of $R^{(fe)}_k(\rho, \pi/3)$ is a (possibly degenerate) triangle, whose vertices are*

$$\mathbf{v}^1_k := \sum_{j=1}^{k} \frac{1}{\rho^j}, \quad \mathbf{v}^2_k := \sum_{j=1}^{k} \frac{1}{\rho^j} e^{-2\pi i/3} \quad \textit{and} \quad \mathbf{v}^3_k := \frac{1}{\rho} e^{-2\pi i/3} + \sum_{j=2}^{k} \frac{1}{\rho^j} e^{-4\pi i/3}.$$

Proof. By induction on k. If $k = 1$ then

$$R^{(fe)}_1(\rho, \pi/3) = \left\{ \frac{1}{\rho}, \frac{1}{\rho} e^{-\frac{2\pi}{3}} \right\} = \{\mathbf{v}^1_1, \mathbf{v}^2_1 = \mathbf{v}^3_1\}$$

hence the base of the induction is immediate. Consider now $k > 2$. By inductive hypothesis

$$\mathrm{conv}(R^{(fe)}_k(\rho, \pi/3)) = \mathrm{conv}(\{\mathbf{v}^1_k, \mathbf{v}^2_k, \mathbf{v}^3_k\}) \tag{60}$$

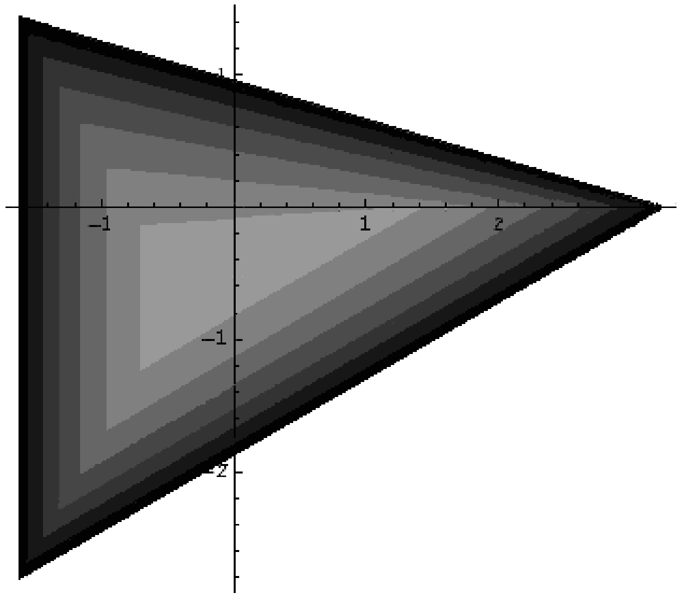


FIGURE 12. Convex hull of $R_k^{(fe)}(2^{1/3}, \pi/3)$ with $k = 2, \ldots, 8$.

hence by Lemma 5.2

$$\operatorname{conv}(R_{k+1}^{(fe)}(\rho, \pi/3)) = \operatorname{conv}\left(\frac{1}{\rho}(1 + \{\mathbf{v}_k^1, \mathbf{v}_k^2, \mathbf{v}_k^3\}) \right. \\ \left. \bigcup \frac{1}{\rho} e^{-2\pi i/3}(1 + \{\mathbf{v}_k^1, \mathbf{v}_k^2, \mathbf{v}_k^3\})\right). \quad (61)$$

By the definition of $\mathbf{v}_k^1, \mathbf{v}_k^2$ and of $\mathbf{v}_k^3$ we may rewrite the above equality

$$\operatorname{conv}(R_{k+1}^{(fe)}(\rho, \pi/3)) = \operatorname{conv}\left(\{\mathbf{v}_{k+1}^1,\ \mathbf{v}_{k+1}^2,\ \mathbf{v}_{k+1}^3, \right. \\ \left. \frac{1}{\rho}(\mathbf{v}_k^2 + 1),\ \frac{1}{\rho}(\mathbf{v}_k^3 + 1),\ \frac{e^{-\frac{2\pi i}{3}}}{\rho}(\mathbf{v}_k^3 + 1)\}\right) \quad (62)$$

therefore to complete the proof of the inductive step, we need to prove that the reachable points

$$\frac{1}{\rho}(\mathbf{v}_k^2 + 1),\ \frac{1}{\rho}(\mathbf{v}_k^3 + 1) \text{ and } \frac{e^{-\frac{2\pi i}{3}}}{\rho}(\mathbf{v}_k^3 + 1)$$

can be written as a convex combination of $\mathbf{v}_{k+1}^1, \mathbf{v}_{k+1}^2$ and $\mathbf{v}_{k+1}^3$. Thesis hence follows by following relations

- if $\alpha = \frac{1}{\rho}\left(\sum_{j=1}^{k+1} \frac{1}{\rho^j}\right)^{-1}$ then

$$\frac{1}{\rho}(\mathbf{v}_k^2 + 1) = \alpha \mathbf{v}_{k+1}^1 + (1 - \alpha)\mathbf{v}_{k+1}^2; \quad (63)$$

- if $\alpha = \sum_{j=3}^{k+1} \frac{1}{\rho^j} \left(\sum_{j=1}^{k+1} \frac{1}{\rho^j} \right)^{-1}$ and $\beta = \sum_{j=3}^{k+1} \frac{1}{\rho^j} \left(\sum_{j=2}^{k+1} \frac{1}{\rho^j} \right)^{-1}$ then

$$\frac{1}{\rho}(\mathbf{v}_k^3 + 1) = \alpha \mathbf{v}_{k+1}^1 + (1-\alpha-\beta)\mathbf{v}_{k+1}^2 + \beta \mathbf{v}_{k+1}^3; \tag{64}$$

- if $\alpha = \sum_{j=3}^{k+1} \frac{1}{\rho^j} \left(\sum_{j=1}^{k+1} \frac{1}{\rho^j} \right)^{-1}$ and $\beta = \frac{1}{\rho} \left(\sum_{j=1}^{k} \frac{1}{\rho^j} \right)^{-1}$ then

$$\frac{e^{-\frac{2\pi i}{3}}}{\rho}(\mathbf{v}_k^3 + 1) = \alpha \mathbf{v}_{k+1}^1 + (1-\alpha-\beta)\mathbf{v}_{k+1}^2 + \beta \mathbf{v}_{k+1}^3. \tag{65}$$

□

5.3. **General configurations with $\omega = \pi/3$.** In this section we investigate the convex hull of the reachable configurations in the general case by assuming $\omega = \frac{\pi}{3}$. We recall that any reachable point of the system (1) is of the form

$$x_k = \sum_{j=1}^{k} \frac{u_j}{\rho^j} e^{-i(\pi-\omega)\sum_{n=1}^{j} v_n}$$

with $u_j, v_j \in \{0,1\}$. We now show that a reachable point can be expressed by a recursive formula.

Proposition 5. *For every $\rho > 1$, $\omega \in [0,\pi)$ and $k \geq 0$, if $x_{k+1} \in R_{k+1}(\rho,\omega)$ then*

$$x_{k+1} = \frac{e^{-iv_1(\pi-\omega)}}{\rho}(u_1 + \tilde{x}_k) \tag{66}$$

for some $\tilde{x}_k \in R_k(\rho,\omega)$ and $u_1, v_1 \in \{0,1\}$.

Moreover if $\omega = \frac{\pi}{3}$

$$\Re(x_{k+1}) = \frac{1}{\rho}\left(1 - \frac{3}{2}v_1\right)(u_1 + \Re(\tilde{x}_k)) + \frac{\sqrt{3}}{2\rho} v_1 \Im(\tilde{x}_k). \tag{67}$$

and

$$\Im(x_{k+1}) = \frac{1}{\rho}\left(1 - \frac{3}{2}v_1\right)\Im(\tilde{x}_k) - \frac{\sqrt{3}}{2\rho} v_1 \left(u_1 + \Re(\tilde{x}_k)\right). \tag{68}$$

Proof. If $x_{k+1} \in R_{k+1}(\rho,\omega)$ then

$$\begin{aligned} x_{k+1} &= \sum_{j=1}^{k+1} \frac{u_j}{\rho^j} e^{-i\sum_{n=1}^{j} v_n(\pi-\omega)} \\ &= \frac{u_1}{\rho} e^{-iv_1(\pi-\omega)} + \sum_{j=2}^{k+1} \frac{u_j}{\rho^j} e^{-i\sum_{n=2}^{j} v_n(\pi-\omega)} e^{-iv_1(\pi-\omega)} \\ &= \frac{e^{-iv_1(\pi-\omega)}}{\rho}(u_1 + \sum_{j=1}^{k} \frac{\tilde{u}_j}{\rho^j} e^{-i\sum_{n=1}^{j} \tilde{v}_n(\pi-\omega)}) \\ &= \frac{e^{-iv_1(\pi-\omega)}}{\rho}(u_1 + \tilde{x}_k) \end{aligned}$$

with $\tilde{x}_k \in R_k(\rho,\omega)$ and $u_1, v_1 \in \{0,1\}$.

Equalities (67) and (68) follow by the relations

$$\Re(xy) = \Re(x)\Re(y) - \Im(x)\Im(y)$$

and

$$\Im(xy) = \Re(x)\Im(y) + \Im(x)\Re(y);$$

indeed

$$\begin{aligned}\Re(x_{k+1}) &= \Re\left(\frac{e^{-\frac{2\pi}{3}iv_1}}{\rho}\right)\Re\left(u_1+\tilde{x}_k\right) - \Im\left(\frac{e^{-\frac{2\pi}{3}iv_1}}{\rho}\right)\Im\left(u_1+\tilde{x}_k\right)\\ &= \frac{1}{\rho}\left(1-\frac{3}{2}v_1\right)\left(u_1+\Re(\tilde{x}_k)\right)+\frac{\sqrt{3}}{2\rho}v_1\Im(\tilde{x}_k).\end{aligned}$$

and

$$\begin{aligned}\Im(x_{k+1}) &= \Re\left(\frac{e^{-\frac{2\pi}{3}iv_1}}{\rho}\right)\Im(u_1+\tilde{x}_k) + \Im\left(\frac{e^{-\frac{2\pi}{3}iv_1}}{\rho}\right)\Re(u_1+\tilde{x}_k)\\ &= \frac{1}{\rho}\left(1-\frac{3}{2}v_1\right)\Im(\tilde{x}_k)-\frac{\sqrt{3}}{2\rho}v_1(u_1+\Re(\tilde{x}_k)).\end{aligned}$$

□

Next result is a preliminary approximation of $\mathrm{conv}(R_k(\rho,\omega))$ and before stating it we introduce the following notation

$$\mathsf{S}_{k_0,k_1} := \sum_{j=k_0}^{k_1}\frac{1}{\rho^j}. \tag{69}$$

Remark 10. With notation given in (69), by Theorem 5.3 the vertices $\mathbf{v}_k^1, \mathbf{v}_k^2$ and $\mathbf{v}_k^3$ of the convex hull of $R_k^{(fe)}(\rho,\pi/3)$ satisfy

$$\mathbf{v}_k^1 = \mathsf{S}_{1,k};\quad \mathbf{v}_k^2 = \mathsf{S}_{1,k}e^{-i\frac{2\pi}{3}};\quad \mathbf{v}_k^3 = \frac{e^{-i\frac{2\pi}{3}}}{\rho}+\mathsf{S}_{2,k}e^{-i\frac{4\pi}{3}}.$$

Proposition 6. *Let $\rho>1$, and for every $k\geq 1$ define*

$$\mathbf{w}_k := \mathsf{S}_{1,k}e^{-i\frac{4\pi}{3}}.$$

Then

$$R_k(\rho,\omega)\subset T_k(\rho,\omega) := \mathrm{conv}(\{\mathbf{v}_k^1,\mathbf{v}_k^2,\mathbf{w}_k\}). \tag{70}$$

Proof. By induction on k. If $k=1$ then

$$R_1(\rho,\omega) = \left\{0,\frac{1}{\rho},\frac{e^{-i\frac{2\pi}{3}}}{\rho}\right\} = \left\{0,\mathbf{v}_1^1,\mathbf{v}_1^2\right\}$$

hence we just need to check that $0\in T_1(\rho,\omega)$. If $\alpha=\beta=\frac{1}{3}$ then

$$\alpha\mathbf{v}_1^1+\beta\mathbf{v}_1^2+(1-\alpha-\beta)\mathbf{w}_1 = 0;$$

hence 0 is a convex combination of the vertices of $T^k(\rho,\omega)$ and

$$R_1(\rho,\omega)\subset T_1(\rho,\omega).$$

Now assume (70) as inductive hypothesis and consider $x_{k+1} \in R_{k+1}(\rho,\omega)$. By Proposition 5 and by (70) we have that for some $\tilde{x}_k \in R_k(\rho,\omega)$, some $u_1, v_1 \in \{0,1\}$ and some $\alpha_k, \beta_k \in [0,1]$ with $\alpha_k + \beta_k \in [0,1]$:

$$\begin{aligned} x_{k+1} &= \frac{e^{-iv_1\frac{2\pi}{3}}}{\rho}(u_1 + \tilde{x}_k) \\ &= \frac{e^{-iv_1\frac{2\pi}{3}}}{\rho}\left(u_1 + \alpha_k \mathbf{v}_k^1 + \beta_k \mathbf{v}_k^2 + (1-\alpha_k-\beta_k)\mathbf{w}_k\right). \end{aligned}$$

To complete the proof we show that x_{k+1} can be written as a convex combination of $\mathbf{v}_{k+1}^1, \mathbf{v}_{k+1}^2$ and $\mathbf{w}_{k+1}$, namely we need to show the real solutions α and β of the equation

$$\begin{aligned} &\alpha \mathbf{v}_{k+1}^1 + \beta \mathbf{v}_{k+1}^2 + (1-\alpha-\beta)\mathbf{w}_{k+1} = \\ &\qquad = \frac{e^{-iv_1\frac{2\pi}{3}}}{\rho}\left(u_1 + \alpha_k \mathbf{v}_k^1 + \beta_k \mathbf{v}_k^2 + (1-\alpha_k-\beta_k)\mathbf{w}_k\right) \end{aligned}$$

to satisfy

$$\alpha, \beta, \alpha+\beta \in [0,1]. \tag{71}$$

By a direct computation we have

- if $v_1 = 0$ then

$$\alpha = \frac{1}{3}(2u_1+1)\frac{\rho^k(\rho-1)}{\rho^{k+1}(\rho-1)} + \frac{\rho^k-1}{\rho^{k+1}-1}\alpha_k$$

$$\beta = \frac{1}{3}(1-u_1)\frac{\rho^k(\rho-1)}{\rho^{k+1}(\rho-1)} + \frac{\rho^k-1}{\rho^{k+1}-1}\beta_k$$

  and $\alpha, \beta \in [0,1]$ because $u_1 \in \{0,1\}$, $\alpha_k, \beta_k \in [0,1]$ and $\rho > 1$. Moreover

$$1-\alpha-\beta = \frac{1}{3}(1-u_1)\frac{\rho^k(\rho-1)}{\rho^{k+1}(\rho-1)} + \frac{\rho^k-1}{\rho^{k+1}-1}(1-\alpha_k-\beta_k) \in [0,1]. \tag{72}$$

- if $v_1 = 1$ then, also remarking

$$e^{-i\frac{2\pi}{3}}\mathbf{v}_k^1 = \mathbf{v}_k^2,$$

$$e^{-i\frac{2\pi}{3}}\mathbf{v}_k^2 = \mathbf{v}_k^3$$

  and

$$e^{-i\frac{2\pi}{3}}\mathbf{v}_k^3 = \mathbf{v}_k^1,$$

  we get

$$\alpha = \frac{1}{3}(1-u_1)\frac{\rho^k(\rho-1)}{\rho^{k+1}(\rho-1)} + \frac{\rho^k-1}{\rho^{k+1}-1}(1-\alpha_k-\beta_k)$$

$$\beta = \frac{1}{3}(2u_1+1)\frac{\rho^k(\rho-1)}{\rho^{k+1}(\rho-1)} + \frac{\rho^k-1}{\rho^{k+1}-1}\alpha_k$$

  and, consequently $\alpha, \beta, 1-\alpha-\beta \in [0,1]$.

□

Corollary 2. *For every* $x_k \in R(\rho, \pi/3)$*:*

$$-\frac{1}{2}\mathsf{S}_{1,k} \leq \Re(x_k) \leq \mathsf{S}_{1,k}; \tag{73}$$

$$-\frac{\sqrt{3}}{2}\mathsf{S}_{1,k} \leq \Im(x_k) \leq \frac{\sqrt{3}}{2}\mathsf{S}_{1,k} \tag{74}$$

and

$$\frac{\sqrt{3}}{2}\Re(x_k) + \frac{1}{2}\Im(x_k) \leq 0. \tag{75}$$

Proof. By Proposition 6,

$$x_k \in T^k(\rho, \pi/3)$$

and, in particular,

$$x_k = \alpha\mathsf{S}_{1,k} + \beta\mathsf{S}_{1,k}e^{-i\frac{2\pi}{3}} + (1-\alpha-\beta)\mathsf{S}_{1,k}e^{-i\frac{4\pi}{3}}$$

for some $\alpha, \beta, \alpha+\beta \in [0,1]$. Therefore

$$\Re(x_k) = -\frac{1}{2}\mathsf{S}_{1,k} + \frac{3}{2}\alpha\mathsf{S}_{1,k}$$

and, by choosing respectively $\alpha = 0$ and $\alpha = 1$ we obtain the inequalities in (73). Moreover

$$\Im(x_k) = \frac{\sqrt{3}}{2}\mathsf{S}_{1,k} - \sqrt{3}\left(\beta + \frac{\alpha}{2}\right)\mathsf{S}_{1,k}$$

and the relations $\alpha \geq 0$ and $0 \leq \alpha + \beta \leq 1$ imply (74). Finally $\beta \geq 0$ implies

$$\begin{aligned}\frac{\sqrt{3}}{2}\Re(x_k) + \frac{1}{2}\Im(x_k) &= \frac{\sqrt{3}}{2}\left(-\frac{1}{2}\mathsf{S}_{1,k} + \frac{3}{2}\alpha\mathsf{S}_{1,k}\right) \\ &\quad + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\mathsf{S}_{1,k} - \sqrt{3}\left(\beta + \frac{\alpha}{2}\right)\mathsf{S}_{1,k}\right) \\ &= -\beta\frac{\sqrt{3}}{2}\mathsf{S}_{1,k} \\ &\leq 0.\end{aligned}$$

□

Next result refines the upper bound of $\Im(x_k)$

Corollary 3. *For every* $k \geq 2$ *and* $x_k \in R_k(\rho, \pi/3)$

$$\Im(x_k) \leq \frac{\sqrt{3}}{2}\mathsf{S}_{2,k}. \tag{76}$$

Proof. By Proposition 5, for every $k \geq 1$

$$x_{k+1} = \frac{e^{-iv_1\frac{2\pi}{3}}}{\rho}(u_1 + \tilde{x}_k)$$

for some $\tilde{x}_k \in R_k(\rho, \pi/3)$ and some $u_1, v_1 \in \{0,1\}$ and, in particular,

$$\Im(x_{k+1}) = \frac{1}{\rho}(1 - \frac{3}{2}v_1)\Im(\tilde{x}_k) - \frac{\sqrt{3}}{2\rho}v_1(u_1 + \Re(\tilde{x}_k)).$$

Then Corollary 2 implies that if $v_1 = 0$ then

$$\Im(x_{k+1}) = \frac{1}{\rho}\Im(\tilde{x}_k) \leq \frac{\sqrt{3}}{2\rho}\mathsf{S}_{1,k} = \frac{\sqrt{3}}{2}\mathsf{S}_{2,k+1}.$$

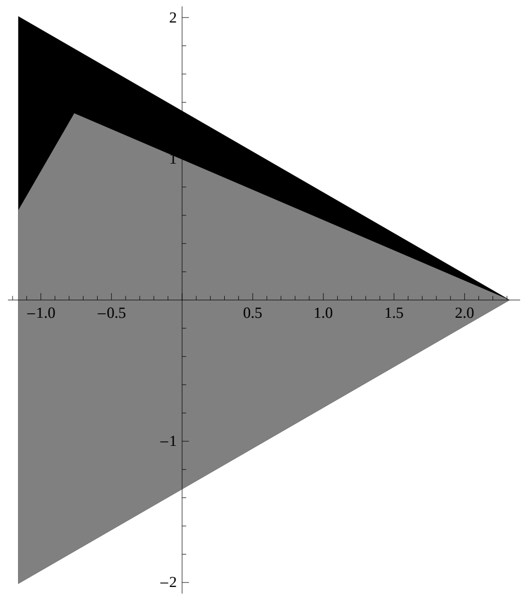


FIGURE 13. The gray area is the convex hull of $R_4(2^{1/3}, \pi/3)$, the underlying black area is the triangle $T_4(2^{1/3}, \pi/3)$.

and if $v_1 = 1$ then

$$\begin{aligned}\Im(x_{k+1}) &= -\frac{1}{2}\Im(\tilde{x}_k) - \frac{\sqrt{3}}{2}(u_1 + \Re(\tilde{x}_k)) \\ &\le \frac{\sqrt{3}}{4\rho}\mathsf{S}_{1,k} + \frac{\sqrt{3}}{4\rho}\mathsf{S}_{1,k} \\ &= \frac{\sqrt{3}}{2}\mathsf{S}_{2,k+1}.\end{aligned}$$

□

Remark 11. The estimation

$$\Im(x_k) \le \frac{\sqrt{3}}{2}\mathsf{S}_{2,k}.$$

is sharp, indeed

$$\mathsf{S}_{2,k}e^{-\frac{4\pi i}{3}} \in R_k(\rho, \pi/3)$$

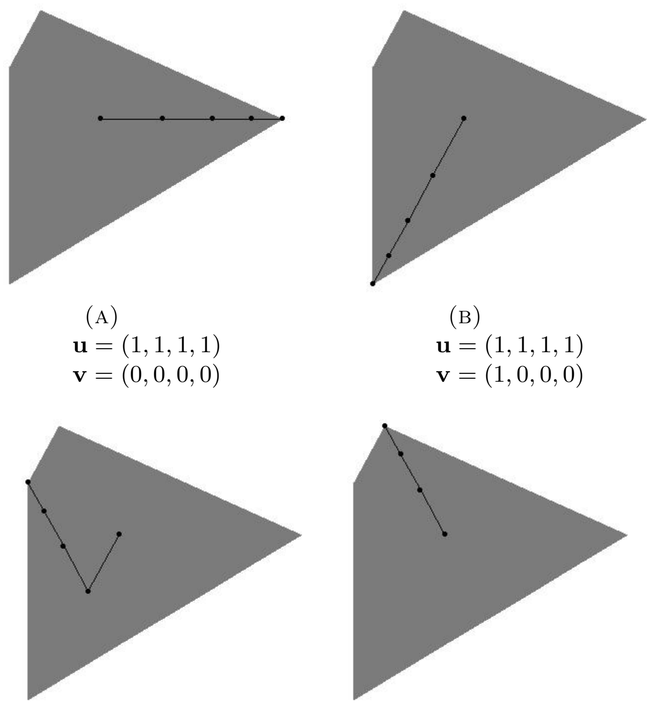
and

$$\Im(\mathsf{S}_{2,k}e^{-\frac{4\pi i}{3}}) = \frac{\sqrt{3}}{2}\mathsf{S}_{2,k}.$$

Experimental data suggest the vertices of the convex hull of $R_k(\rho, \pi/3)$ to be $\mathbf{v}_k^1$, $\mathbf{v}_k^2$, $\mathbf{v}_k^3$ and the following

$$\mathbf{v}_k^4 := \mathsf{S}_{2,k}e^{-\frac{4\pi}{3}i}; \tag{77}$$

Remark 12. The vertex $\mathbf{v}_k^1$ is reached by the all-zero rotation control sequence $v_j = 0$, $j = 1, \dots, k$ and the all-one extension control sequence $u_j = 1$, with $j = 1, \dots, k$. The vertex $\mathbf{v}_k^2$ corresponds to the rotation control sequence $v_1 = 1$



(A) $\mathbf{u} = (1,1,1,1)$ $\mathbf{v} = (0,0,0,0)$

(B) $\mathbf{u} = (1,1,1,1)$ $\mathbf{v} = (1,0,0,0)$

(C) $\mathbf{u} = (1,1,1,1)$ $\mathbf{v} = (1,1,0,0)$

(D) $\mathbf{u} = (0,1,1,1)$ $\mathbf{v} = (1,1,0,0)$

FIGURE 14. Extremal configurations.

and $v_j = 0$ for $j = 2, \dots, k$ and to the all-one extension control sequence. The vertex $\mathbf{v}_k^3$ is reached if $v_1 = v_2 = 1$, $v_j = 0$ for $j = 3, \dots, k$ and to the all-one extension control sequence. Finally we get the vertex $\mathbf{v}_k^4$ if $v_1 = v_2 = 1$, $v_j = 0$ for $j = 3, \dots, k$ and $u_1 = 0$ and $u_j = 1$ for $j = 2, \dots, k$.

Moreover the vertices $\mathbf{v}_k^1, \dots, \mathbf{v}_k^4$ are clock-wise ordered on the complex plane.

Our goal is to prove $\mathbf{v}_k^1, \dots, \mathbf{v}_k^4$ to be the vertices of $\mathrm{conv}(R_k(\rho, \pi/3))$, namely

$$\mathrm{conv}(\{\mathbf{v}_k^1, \mathbf{v}_k^2, \mathbf{v}_k^3, \mathbf{v}_k^4\}) = \mathrm{conv}(R_k(\rho, \pi/3)) \tag{78}$$

So far, excepting Theorem 5.1, our argument for the study of convex hulls consisted in a trivial inductive application of the definition of convex set, namely we showed an explicit convex combination for every reachable point. In this case, this approach leads to involved formulas and consequent long computations, hence we prefer an indirect method. First remark that the vertices $\mathbf{v}_k^1, \dots, \mathbf{v}_k^4$ are reachable points, thus

$$\mathrm{conv}(\{\mathbf{v}_k^1, \mathbf{v}_k^2, \mathbf{v}_k^3, \mathbf{v}_k^4\}) \subseteq \mathrm{conv}(R_k(\rho, \pi/3)).$$

To prove the other inclusion

$$\mathrm{conv}(R_k(\rho, \pi/3)) \subseteq \mathrm{conv}(\{\mathbf{v}_k^1, \mathbf{v}_k^2, \mathbf{v}_k^3, \mathbf{v}_k^4\}) \tag{79}$$

we need to show that every reachable point belongs to $\mathrm{conv}(\{\mathbf{v}_k^1, \mathbf{v}_k^2, \mathbf{v}_k^3, \mathbf{v}_k^4\})$. The convex hull of a finite set of complex numbers is in general a polygon on the complex plane and we may employ the isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$ to have access to results related to the Point-In-Polygon Problem (see for instance [8] and [14]). A polygon can be looked at as the limited intersection of a finite number of half-planes, whose generating lines are given by the edges, namely the differences between two consecutive vertices. The exterior criterion states that a point belongs to a convex polygon if and only if it belongs to the "same side" of each half-plane. Moreover, if the vertices are clock-wise ordered, the point must belong to the right side of each half-plane. This can be formalized by introducing the so called edge-function: let be $\mathbf{v}^1$ and $\mathbf{v}^2$ a couple of clock-wise ordered consecutive vertices and $x \in \mathbb{R}^2$

$$H_{\mathbf{v}^1,\mathbf{v}^2}(x) := (x - \mathbf{v}^2) \cdot (\mathbf{v}^2 - \mathbf{v}^1)^{\perp}.$$

We have that

$$H_{\mathbf{v}^1,\mathbf{v}^2}(x) = \begin{cases} < 0 & \text{if } x \text{ belongs to the right side of the half-plane generated by } \mathbf{v}_1 \text{ and } \mathbf{v}_2 \\ = 0 & \text{if } x \text{ belongs to the line intersecting both } \mathbf{v}_1 \text{ and } \mathbf{v}_2 \\ > 0 & \text{if } x \text{ belongs to the left side of the half-plane generated by } \mathbf{v}_1 \text{ and } \mathbf{v}_2 \end{cases} \tag{80}$$

hence a point belongs to the polygon if $H_{\mathbf{v}^1,\mathbf{v}^2}(x) \geq 0$ for every couple of clock-wise, consecutive vertices $\mathbf{v}^1$ and $\mathbf{v}^2$.

In the complex plane geometry, the vector $\mathbf{n}$ is normal to $\mathbf{e}$ if the difference between their arguments is $\frac{\pi}{2}$, namely

$$\mathbf{n} = e^{\frac{\pi}{2} i}\mathbf{e}.$$

We also recall that the scalar product on the complex plane is given by

$$\mathbf{u} \cdot \mathbf{v} = \Re(\mathbf{u})\Re(\mathbf{v}) + \Im(\mathbf{u})\Im(\mathbf{v}).$$

We adapt the exterior criterion to our case as follows.

Proposition 7. *Define*

$$\begin{aligned} \mathbf{n}_k^1 &:= e^{\frac{\pi}{2} i}(\mathbf{v}_k^1 - \mathbf{v}_k^4); \quad & \mathbf{n}_k^2 &:= e^{\frac{\pi}{2} i}(\mathbf{v}_k^2 - \mathbf{v}_k^1); \\ \mathbf{n}_k^3 &:= e^{\frac{\pi}{2} i}(\mathbf{v}_k^3 - \mathbf{v}_k^2); \quad & \mathbf{n}_k^4 &:= e^{\frac{\pi}{2} i}(\mathbf{v}_k^4 - \mathbf{v}_k^3). \end{aligned} \tag{81}$$

We have

$$x \in \operatorname{conv}(\{\mathbf{v}_k^1, \mathbf{v}_k^2, \mathbf{v}_k^3, \mathbf{v}_k^4\})$$

if and only if for every $h = 1, \ldots, 4$

$$(x - \mathbf{v}_k^h) \cdot \mathbf{n}_k^h \leq 0.$$

Next results state that the reachable points x_k satisfy the conditions of Proposition 7.

Lemma 5.4. *For every* $\rho > 1$, $k \geq 1$ *and* $x_k \in \operatorname{conv}(R_k(\rho, \pi/3))$

$$(x_k - \mathbf{v}_k^1) \cdot \mathbf{n}_k^1 \leq 0.$$

Proof. For simplicity we consider x_{k+1} with $k \geq 0$ instead of the equivalent case of x_k with $k \geq 1$. First of all recall from Proposition 5

$$x_{k+1} = \frac{e^{-iv_1 \frac{2\pi}{3}}}{\rho}(u_1 + \tilde{x}_k)$$

for some $\tilde{x}_k \in R_k(\rho, \pi/3)$ and some $u_1, v_1 \in \{0,1\}$. By the definition of $\mathbf{v}_{k+1}^1$

$$\Re(\mathbf{v}_{k+1}^1) = \mathsf{S}_{1,k+1}; \tag{82}$$

$$\Im(\mathbf{v}_{k+1}^1) = 0 \tag{83}$$

and by the definition of $\mathbf{n}_{k+1}^1$

$$\mathbf{n}_{k+1}^1 = i\left(\mathsf{S}_{1,k} - \mathsf{S}_{2,k}e^{-i\frac{4\pi}{3}}\right)$$

and

$$\Re(\mathbf{n}_{k+1}^1) = \frac{\sqrt{3}}{2}S_{2,k+1}; \tag{84}$$

$$\Im(\mathbf{n}_{k+1}^1) = S_{1,k+1} + \frac{1}{2}S_{2,k+1}. \tag{85}$$

We now distinguish the cases $v_1 = 0$ and $v_1 = 1$. If $v_1 = 0$ then by Proposition 5

$$\Re(x_{k+1}) = \frac{1}{\rho}\left(u_1 + \Re(\tilde{x}_k)\right) \tag{86}$$

$$\Im(x_{k+1}) = \frac{1}{\rho}\Im(\tilde{x}_k). \tag{87}$$

This, together with $u_1 \in \{0,1\}$, (75) in Corollary 2 and Corollary 3, implies

$$\begin{aligned}(x_{k+1} - \mathbf{v}_{k+1}^1)\cdot\mathbf{n}_{k+1}^1 =& u_1\frac{\sqrt{3}}{2}\mathsf{S}_{2,k+1} + \frac{1}{\rho}\left(\frac{\sqrt{3}}{2}\Re(\tilde{x}_k) + \frac{1}{2}\Im(\tilde{x}_k)\right)\mathsf{S}_{2,k+1}\\ &+ \frac{1}{\rho}\Im(\tilde{x}_k)\mathsf{S}_{2,k+1} - \frac{\sqrt{3}}{2}\mathsf{S}_{1,k+1}\mathsf{S}_{2,k+1}\\ \le& \frac{\sqrt{3}}{2}\mathsf{S}_{2,k+1} + \frac{\sqrt{3}}{2}\mathsf{S}_{2,k} - \frac{\sqrt{3}}{2}\mathsf{S}_{1,k+1}\mathsf{S}_{2,k+1}\\ =& -\frac{\sqrt{3}}{2\rho^{k+2}}\mathsf{S}_{2,k+1}\\ <& 0.\end{aligned}$$

If $v_1 = 1$ then by Proposition 5

$$\Re(x_{k+1}) = -\frac{1}{2\rho}\left(u_1 + \Re(\tilde{x}_k)\right) + \frac{\sqrt{3}}{2}\Im(\tilde{x}_k) \tag{88}$$

$$\Im(x_{k+1}) = -\frac{1}{2\rho}\Im(\tilde{x}_k) - \frac{\sqrt{3}}{2}\left(u_1 + \Re(\tilde{x}_k)\right). \tag{89}$$

and by Corollary 2 and $u_1 \in \{0,1\}$

$$\begin{aligned}(x_{k+1} - \mathbf{v}_{k+1}^1)\cdot\mathbf{n}_{k+1}^1 =& -\frac{\sqrt{3}}{4\rho}(3\mathsf{S}_{1,k+1} + \mathsf{S}_{2,k+1})(u_1 + \Re(\tilde{x}_k))\\ &+ \frac{1}{4\rho^2}\Im(\tilde{x}_k) - \frac{\sqrt{3}}{2}\mathsf{S}_{1,k+1}\mathsf{S}_{2,k+1}\\ \le& \frac{\sqrt{3}}{8\rho}(3\mathsf{S}_{1,k+1} + \mathsf{S}_{2,k+1})\mathsf{S}_{1,k} + \frac{\sqrt{3}}{8\rho^2}\mathsf{S}_{1,k} - \frac{\sqrt{3}}{2}\mathsf{S}_{1,k+1}\mathsf{S}_{2,k+1}\\ =& 0.\end{aligned}$$

□

Lemma 5.5. *For every $\rho > 1$, $k \geq 1$ and $x_k \in \operatorname{conv}(R_k(\rho, \pi/3))$*

$$(x_k - \mathbf{v}_k^2) \cdot \mathbf{n}_k^2 \leq 0 \tag{90}$$

and

$$(x_k - \mathbf{v}_k^3) \cdot \mathbf{n}_k^3 \leq 0. \tag{91}$$

Proof. By Lemma 6, x_k belongs to $T_k(\rho, \pi/3)$, the triangle whose vertices are $\mathbf{w}_k^1 = \mathbf{v}_k^1$, $\mathbf{w}_k^2 = \mathbf{v}_k^2$ and

$$\mathbf{w}_k^3 = \mathsf{S}_{1,k} e^{-i\frac{4\pi}{3}} = \mathbf{v}_k^3 + \frac{\sqrt{3}}{\rho} i.$$

Therefore (90) is immediate. Now

$$\mathbf{n}_k^3 = e^{i\frac{\pi}{2}}(\mathbf{v}_k^3 - \mathbf{v}_k^2) = e^{i\frac{\pi}{2}}\left(\mathbf{w}_k^3 - \mathbf{w}_k^2 - \frac{\sqrt{3}}{\rho}\right) = e^{i\frac{\pi}{2}}\frac{\mathsf{S}_{2,k}}{\mathsf{S}_{1,k}}(\mathbf{w}_k^3 - \mathbf{w}_k^2) \tag{92}$$

therefore

$$(x - \mathbf{w}_k^3) \cdot \mathbf{n}_k^3 \leq 0;$$

moreover

$$\mathbf{n}_k^3 = e^{i\frac{\pi}{2}}\left(\frac{e^{-i\frac{2\pi}{3}}}{\rho} + \mathsf{S}_{2,k}e^{-i\frac{4\pi}{3}} - \mathsf{S}_{1,k}e^{-i\frac{2\pi}{3}}\right) = -\frac{\sqrt{3}}{\rho}\mathsf{S}_{2,k}$$

and, consequently,

$$(x_k - \mathbf{v}_k^3) \cdot \mathbf{n}_k^3 = (x_k - \mathbf{w}_k^3) \cdot \mathbf{n}_k^3 + i\frac{\sqrt{3}}{\rho} \cdot \mathbf{n}_k^3 \leq i\frac{\sqrt{3}}{\rho} \cdot \mathbf{n}_k^3 = 0.$$

□

Lemma 5.6. *For every $\rho > 1$, $k \geq 1$ and $x_k \in \operatorname{conv}(R_k(\rho, \pi/3))$*

$$(x_k - \mathbf{v}_k^4) \cdot \mathbf{n}_k^4 \leq 0.$$

Proof. The proof is very similar to the proof of Lemma 5.4. We consider x_{k+1} with $k \geq 0$ instead of the equivalent case of x_k with $k \geq 1$ and we recall from Proposition 5

$$x_{k+1} = \frac{e^{-iv_1\frac{2\pi}{3}}}{\rho}(u_1 + \tilde{x}_k)$$

for some $\tilde{x}_k \in R(\rho, \pi/3)$ and some $u_1, v_1 \in \{0, 1\}$. By the definition of $\mathbf{v}_{k+1}^4$

$$\Re(\mathbf{v}_{k+1}^1) = -\frac{1}{2}\mathsf{S}_{2,k+1}; \tag{93}$$

$$\Im(\mathbf{v}_{k+1}^1) = \frac{\sqrt{3}}{2}\mathsf{S}_{2,k+1}; \tag{94}$$

$$\tag{95}$$

and by the definition of $\mathbf{n}_{k+1}^4$

$$\mathbf{n}_{k+1}^4 = -\frac{\sqrt{3}}{2\rho} + i\frac{1}{2\rho}$$

We now distinguish the cases $v_1 = 0$ and $v_1 = 1$. If $v_1 = 0$ then by Proposition 5

$$\Re(x_{k+1}) = \frac{1}{\rho}\left(u_1 + \Re(\tilde{x}_k)\right) \tag{96}$$

$$\Im(x_{k+1}) = \frac{1}{\rho}\Im(\tilde{x}_k) \tag{97}$$

and this, together with Corollary 2, implies

$$\begin{aligned}(x_{k+1} - \mathbf{v}_{k+1}^4) \cdot \mathbf{n}_{k+1}^4 =& \frac{\sqrt{3}}{2\rho^2}\Re(\tilde{x}_k) + \frac{1}{2\rho^2}\Im(\tilde{x}_k) - \frac{\sqrt{3}}{2}\mathsf{S}_{2,k+1} \\ \leq& \frac{\sqrt{3}}{4\rho^2}\mathsf{S}_{1,k} + \frac{\sqrt{3}}{4\rho^2}\mathsf{S}_{1,k} - \frac{\sqrt{3}}{2\rho}\mathsf{S}_{2,k+1} \\ =& 0.\end{aligned}$$

If $v_1 = 1$ then, again by Proposition 5

$$\Re(x_{k+1}) = \frac{1}{\rho}\left(u_1 + \Re(\tilde{x}_k)\right) \tag{98}$$

$$\Im(x_{k+1}) = \frac{1}{\rho}\Im(\tilde{x}_k) \tag{99}$$

and, consequently,

$$\begin{aligned}(x_{k+1} - \mathbf{v}_{k+1}^4) \cdot \mathbf{n}_{k+1}^4 =& \frac{\sqrt{3}u_1}{2\rho^2} - \frac{1}{\rho^2}\Im(\tilde{x}_k) - \frac{\sqrt{3}}{2}\mathsf{S}_{2,k+1} \\ \leq& \frac{\sqrt{3}}{2\rho^2}\mathsf{S}_{1,k} - \frac{\sqrt{3}}{2\rho^2}\mathsf{S}_{1,k} \\ =& 0.\end{aligned}$$

□

In view of the above reasonings we may finally conclude

Theorem 5.7. *For every $\rho > 1$ and for every $k \geq 1$*

$$\operatorname{conv}(R_k(\rho, \pi/3)) = \operatorname{conv}(\{\mathbf{v}_k^1, \mathbf{v}_k^2, \mathbf{v}_k^3, \mathbf{v}_k^4\}). \tag{100}$$

5.4. **Approximate reachability for $\omega = \pi/3$.** The study of $R_\infty(\rho, \omega)$ is rather complicated even in the case $\omega = \pi/3$: we give a non-convexity result and an approximation from above of $R_\infty(\rho, \pi/3)$. For $h = 1, \dots, 4$ define

$$\mathbf{v}_\infty^h := \lim_{k\to\infty} \mathbf{v}_k^h$$

so that

$$\mathbf{v}_\infty^1 = \frac{1}{\rho - 1}, \qquad \mathbf{v}_\infty^2 = \frac{e^{-i\frac{2\pi}{3}}}{\rho - 1},$$
$$\mathbf{v}_\infty^3 = \frac{e^{-\frac{2\pi}{3}i}}{\rho} + \frac{e^{-i\frac{4\pi}{3}}}{\rho(\rho - 1)}, \qquad \mathbf{v}_\infty^4 = \frac{e^{-i\frac{4\pi}{3}}}{\rho(\rho - 1)}.$$

Letting $k \to \infty$ in Theorem 5.7, we get

$$\operatorname{conv}(R_\infty(\rho, \pi/3)) = \operatorname{conv}\left(\left\{\mathbf{v}_\infty^1,\ \mathbf{v}_\infty^2,\ \mathbf{v}_\infty^3,\ \mathbf{v}_\infty^4\right\}\right) \tag{101}$$

and, in particular

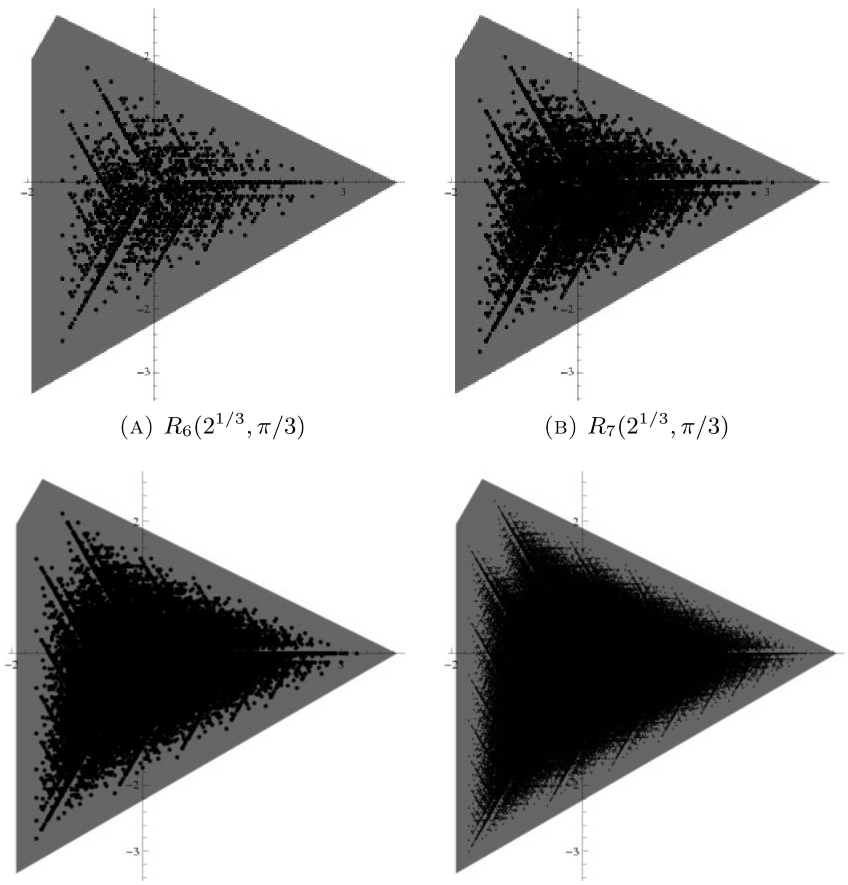
$$R_\infty(\rho, \pi/3) \subseteq \operatorname{conv}\left(\left\{\mathbf{v}_\infty^1,\ \mathbf{v}_\infty^2,\ \mathbf{v}_\infty^3,\ \mathbf{v}_\infty^4\right\}\right).$$

We now prove that the above inclusion is always strict.

Proposition 8. *For every $\rho > 1$, $R_\infty(\rho, \pi/3)$ is not convex.*

Proof. Fix $\alpha \in (0, 1)$ and consider

$$x_\alpha := \alpha\mathbf{v}_\infty^3 + (1 - \alpha)\mathbf{v}_\infty^4 \in \operatorname{conv}(R_\infty(\rho, \pi/3)).$$



(A) $R_6(2^{1/3}, \pi/3)$ (B) $R_7(2^{1/3}, \pi/3)$

(C) $R_8(2^{1/3}, \pi/3)$ (D) $R_{10}(2^{1/3}, \pi/3)$

FIGURE 15. Gray areas represent $\mathrm{conv}(R_\infty(2^{1/3}, \pi/3))$.

We want to show that $x_\alpha \notin R_\infty(\rho, \pi/3)$. In order to get a contradiction, suppose on the contrary that $x_\alpha \in R_\infty(\rho, \pi/3)$. By Proposition 4

$$R_\infty(\rho, \pi/3) = \mathcal{F}_{\rho,\pi/3}(R_\infty(\rho, \pi/3)) = \bigcup_{h=1}^{4} f_h(R_\infty(\rho, \pi/3))$$

where for every $h = 1, \ldots, 4$

$$f_h(x) = \frac{e^{-iv\frac{2\pi}{3}}}{\rho}(u + x)$$

for appropriate $v, u \in \{0, 1\}$. Hence $x_\alpha \in R_\infty(\rho, \pi/3)$ implies the existence of $x \in R_\infty(\rho, \pi/3)$ such that

$$x_\alpha = \frac{e^{-iv\frac{2\pi}{3}}}{\rho}(u + x) \tag{102}$$

for some $u, v \in \{0,1\}$. By solving the above equation with respect to x and by recalling the definition of x_α, we obtain

$$x = \alpha e^{-i(1-v)\frac{2\pi}{3}} + \frac{1}{\rho - 1} e^{-i(2-v)\frac{2\pi}{3}} - u. \tag{103}$$

As we assumed $x \in R_\infty(\rho, \pi/3)$ then

$$x \in \mathrm{conv}(R_\infty(\rho, \pi/3)) = \mathrm{conv}(\{\mathbf{v}^1_\infty, \mathbf{v}^2_\infty, \mathbf{v}^3_\infty, \mathbf{v}^4_\infty\})$$

and in particular

$$\begin{aligned} (x - \mathbf{v}^1_\infty) \cdot \mathbf{n}^1_\infty &\le 0 \\ (x - \mathbf{v}^2_\infty) \cdot \mathbf{n}^2_\infty &\le 0 \\ (x - \mathbf{v}^3_\infty) \cdot \mathbf{n}^3_\infty &\le 0 \\ (x - \mathbf{v}^4_\infty) \cdot \mathbf{n}^4_\infty &\le 0. \end{aligned} \tag{104}$$

But

- if $v = 0$ in (103) then $(x - \mathbf{v}^4_\infty) \cdot \mathbf{n}^4_\infty > 0$;
- if $v = 1$ and $u = 0$ in (103) then $(x - \mathbf{v}^2_\infty) \cdot \mathbf{n}^2_\infty > 0$;
- if $v = 1$ and $u = 1$ in (103) then $(x - \mathbf{v}^3_\infty) \cdot \mathbf{n}^3_\infty > 0$

and we get the required contradiction. □

We want to show a (rough) approximation from below of $R_\infty(\rho, \pi/3)$. To this end, consider the full rotation case and recall the notation $\lambda = \rho e^{\frac{2\pi}{3} i}$. Letting $k \to \infty$ in (14) we have

$$R^{(fr)}_\infty = \left\{ \sum_{j=1}^{\infty} \frac{u_j}{\lambda^j} \mid u_j \in \{0,1\} \right\}. \tag{105}$$

Define now

$$\begin{aligned} \mathbf{u}^1_\infty &:= \frac{1}{\rho^3 - 1}; & \mathbf{u}^2_\infty &:= \frac{1}{\rho^3 - 1}(1 + \lambda); \\ \mathbf{u}^3_\infty &:= \frac{1}{\rho^3 - 1}\lambda = \lambda \mathbf{u}^1_\infty; & \mathbf{u}^4_\infty &:= \frac{1}{\rho^3 - 1}(\lambda + \lambda^2) = \lambda \mathbf{u}^2_\infty; \\ \mathbf{u}^5_\infty &:= \frac{1}{\rho^3 - 1}\lambda^2 = \lambda \mathbf{u}^3_\infty; & \mathbf{u}^6_\infty &:= \frac{1}{\rho^3 - 1}(\lambda^2 + 1); \end{aligned}$$

By Theorem 5.1 we have

$$\mathrm{conv}(R^{(fr)}_\infty) = \mathrm{conv}\left(\left\{\mathbf{u}^h_\infty \mid h = 1, \dots, 6\right\}\right) =: X^{(fr)}_\infty. \tag{106}$$

Remark 13. $X^{(fr)}_\infty$ is a hexagon on the complex plane whose edges are pairwise parallel.

Our goal is to show a necessary and sufficient condition on ρ to have $R^{(fr)}_\infty = X^{(fr)}_\infty$, so that

$$X^{(fr)}_\infty \subset R_\infty(\rho, \pi/3).$$

Our approach is based on the uniqueness of the fixed point of IFSs. First of all remark

$$R^{(fr)}_\infty = \frac{1}{\lambda} R^{(fr)}_\infty \cup \frac{1}{\lambda}(R^{(fr)}_\infty + 1).$$

In other words $R^{(fr)}_\infty$ is the (unique) fixed point of the IFS

$$\mathcal{F}^{(fr)}_{\rho,\pi/3} := \left\{ g_1 : x \mapsto \frac{1}{\lambda} x,\ g_2 : x \mapsto \frac{1}{\lambda}(1 + x) \right\}.$$

Remark 14. By the definition given in (28)-Proposition 4, $g_1 = f_2$ and $g_2 = f_4$.

We have

Lemma 5.8. *For every* $\rho > 1$

$$R_\infty^{(fr)} = X_\infty^{(fr)}$$

if and only if

$$\mathcal{F}_{\rho,\pi/3}^{(fr)}(X_\infty^{(fr)}) = X_\infty^{(fr)}. \qquad (107)$$

Proof. Thesis immediately follows by the fact that $R_\infty^{(fr)}$ is the unique fixed point of $\mathcal{F}_{\rho,\pi/3}^{(fr)}$. □

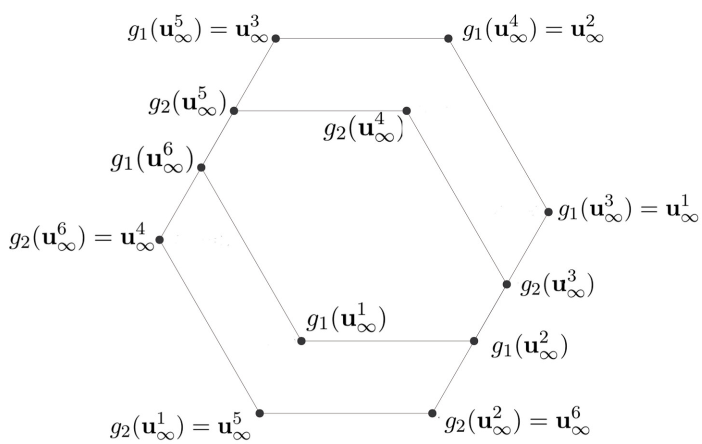


FIGURE 16. $g_1(X_\infty^{(fr)}) \cup g_2(X_\infty^{(fr)})$ with $\rho = 2^{1/3} - 0.1$

Theorem 5.9. $R_\infty^{(fr)} = X_\infty^{(fr)}$ *if and only if* $\rho \leq 2^{1/3}$.

Proof. By Lemma 5.8 and by the definition of $\mathcal{F}_{\rho,\pi/3}^{(fr)}$, it suffices to show that

$$g_1(X_\infty^{(fr)}) \cup g_2(X_\infty^{(fr)}) = X_\infty^{(fr)}. \qquad (108)$$

Remark that $g_1(X_\infty^{(fr)})$ and $g_2(X_\infty^{(fr)})$ are two hexagons on the complex plane, whose mutual positions are described in Figures 16, 17 and 18: we just need to show that the situation therein represented is general. By the definition of g_1 and g_2, we have that for every $\rho > 1$

- by a direct computation

$$\begin{aligned} g_1(\mathbf{u}_\infty^3) &= \mathbf{u}_\infty^1 \qquad & g_2(\mathbf{u}_\infty^1) &= \mathbf{u}_\infty^5 \\ g_1(\mathbf{u}_\infty^4) &= \mathbf{u}_\infty^2 \qquad & g_2(\mathbf{u}_\infty^2) &= \mathbf{u}_\infty^6 \\ g_1(\mathbf{u}_\infty^5) &= \mathbf{u}_\infty^3 \qquad & g_2(\mathbf{u}_\infty^6) &= \mathbf{u}_\infty^4 \end{aligned}$$

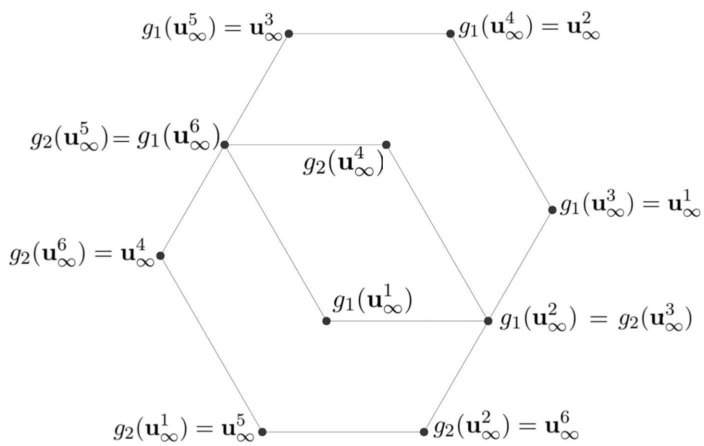


FIGURE 17. $g_1(X^{(fr)}_\infty) \cup g_2(X^{(fr)}_\infty)$ with $\rho = 2^{1/3}$

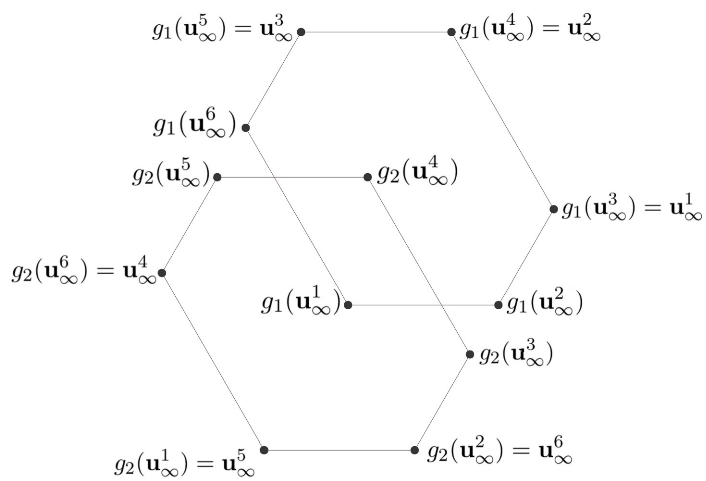


FIGURE 18. $g_1(X^{(fr)}_\infty) \cup g_2(X^{(fr)}_\infty)$ with $\rho = 2^{1/3} + 0.1$

- $g_1(\mathbf{u}^1_\infty)$ and $g_2(\mathbf{u}^4_\infty)$ are internal points of $X^{(fr)}_\infty$, indeed setting

$$\alpha := \frac{\rho - 1}{\rho} \qquad \beta := \frac{\rho - 1}{\rho^2}$$

we have

$$\alpha, \beta, 1 - \alpha - \beta \in (0, 1)$$

and

$$g_1(\mathbf{u}^1_\infty) = \frac{1}{\rho^3 - 1}\frac{1}{\lambda} = \alpha\mathbf{u}^1_\infty + \beta\mathbf{u}^3_\infty + (1 - \alpha - \beta)\mathbf{u}^5_\infty,$$

$$g_2(\mathbf{u}_\infty^4) = \frac{1}{\rho^3 - 1}(1+\lambda) + \frac{1}{\lambda} = (1-\alpha-\beta)\mathbf{u}_\infty^1 + \alpha\mathbf{u}_\infty^3 + \beta\mathbf{u}_\infty^5;$$

- the points $g_1(\mathbf{u}_\infty^3) = \mathbf{u}_\infty^1$, $g_1(\mathbf{u}_\infty^2)$, $g_2(\mathbf{u}_\infty^3)$ and $g_2(\mathbf{u}_\infty^2) = \mathbf{u}_\infty^6$ are aligned, indeed setting

$$\gamma := \frac{1}{\rho^3} \tag{109}$$

we have

$$\begin{aligned} g_1(\mathbf{u}_\infty^2) &= (1-\gamma)\mathbf{u}_\infty^1 + \gamma\mathbf{u}_\infty^6 \\ g_2(\mathbf{u}_\infty^3) &= \gamma\mathbf{u}_\infty^1 + (1-\gamma)\mathbf{u}_\infty^6 \end{aligned} \tag{110}$$

- the points $g_1(\mathbf{u}_\infty^5) = \mathbf{u}_\infty^3$, $g_1(\mathbf{u}_\infty^6)$, $g_2(\mathbf{u}_\infty^5)$ and $g_2(\mathbf{u}_\infty^6) = \mathbf{u}_\infty^4$ are aligned, indeed

$$\begin{aligned} g_1(\mathbf{u}_\infty^6) &= \gamma\mathbf{u}_\infty^3 + (1-\gamma)\mathbf{u}_\infty^4 \\ g_2(\mathbf{u}_\infty^5) &= (1-\gamma)\mathbf{u}_\infty^3 + \gamma\mathbf{u}_\infty^4. \end{aligned} \tag{111}$$

By the reasonings above (108) holds if and only if the following pairs of edges are overlapped:

- the edge with endpoints $\mathbf{u}_\infty^1$ and $g_1(\mathbf{u}_\infty^2)$ and the edge with endpoints $g_2(\mathbf{u}_\infty^3)$ and $\mathbf{u}_\infty^6$;
- the edge with endpoints $\mathbf{u}_\infty^3$, $g_1(\mathbf{u}_\infty^5)$ and the edge with endpoints $g_2(\mathbf{u}_\infty^6)$ and $\mathbf{u}_\infty^4$;

In view of (110) and (111) this is equivalent to have

$$\gamma = \frac{1}{\rho^3} \geq \frac{1}{2}$$

and, consequently, to

$$\rho \leq 2^{1/3}.$$

□

Corollary 4. *If $\rho \leq 2^{1/3}$ then for every $y \in X_\infty^{(fr)}$ and for every $\varepsilon > 0$ there exists $x_k \in R_k(\rho, \pi/3)$ such that*

$$|y - x| < \varepsilon.$$

Proof. Fix $\varepsilon > 0$ and consider $k \in \mathbb{N}$ such that

$$\frac{1}{\rho^k(\rho-1)} < \varepsilon.$$

By Theorem 5.9 we have

$$y \in X_\infty^{(fr)} = R_\infty^{(fr)}(\rho, \pi/3) \subset R_\infty(\rho, \pi/3).$$

Hence thesis follows by Proposition 3. □

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