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Research Article

On the Increase in Signal Depth due to High-Order Effects in Micro- and Nanosized Deformable Conductors

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With regard to the transmission of a thermomechanical signal on extremely short temporal and spatial scales, which represents an issue particularly important dealing with micro- and nanosized electromechanical systems, it is well known that the so-called non-Fourier effects become not negligible. In addition, it has to be considered that the interaction among multiple energy carriers has, as a direct consequence, the involvement of high-order terms in the time-differential formulation of the dual-phase lag heat conduction constitutive equation linking the heat flux vector with the temperature variation gradient. Accepting that the deformations caused by the temperature variations are small enough to be modeled under the assumptions typical of the linear thermoelasticity, in the present article we take into account the highest Taylor expansion orders able to guarantee (under appropriate assumptions) stability conditions, thermodynamic consistency, and at the same time the existence of an influence domain of the external data linked to the energy transmission as thermal waves. To this aim, a cylindrical domain filled by an anisotropic and inhomogeneous thermoelastic material is investigated, although the results obtained will be independent from the considered geometry: for such a reason, we will be able to consider as illustrative examples some simulations referred to single-layer graphene and to show how the expansion orders selected strongly influence the domain of influence depth.

1. Introduction

It is well known that the Fourier heat conduction equation leads to a diffusive model which predicts that a thermal signal can propagate infinitely fast through the medium under investigation. Likewise, it is now clear that it turns out to be not applicable to ultrafast and ultrasmall contexts: dealing, for example, with micro- and nanosized electromechanical systems the so-called non-Fourier effects involving the coupled diffusion and the wavelike heat propagation become not negligible [1]. In this connection the phonon-electron coupling factor and, by extension, the related phase lags can represent thermophysical properties able to take into account also the microstructural interaction effects (see [2], Chapters 1, 5). Therefore, a deep understanding of the heat exchange mechanisms at micro-/nanoscales is today more than ever a topic of great technological interest: it is sufficient to think about its impact on the correct design of MEMS (microelectromechanical systems) and NEMS (nanoelectromechanical systems) or even, in general, to the continuous need for sensors and actuators miniaturization and to the growing interest in controlling the related heat transfer processes.

In this regard, we are convinced that the dual-phase lag (DPL) model, together with its time-differential formulation, turned out to be particularly suited to respond to this kind of needs, having its features in fact already deepened in a very wide number of works, of which [3–14] and the references therein just represent a selection also accounting for high expansion orders in thermoelasticity. The DPL constitutive equation puts in relation the temperature variation gradient $T_{,i}$ at a certain time $t + \tau_T$ with the heat flux vector q_i at a different time $t + \tau_q$. To the delay/relaxation times or phase lags involved τ_q (heat flux vector delay time) and τ_T (temperature variation gradient delay time), respectively, it is possible to give the following physical interpretations: τ_q is connected to the fast-transient effects of thermal inertia, while τ_T is able to take account of microstructural interactions such as phonon scattering or phonon-electron interactions. Specifically within the context of the time-differential formulation for the DPL constitutive law, it is then possible to consider high-order effects by stopping at appropriate Taylor series expansion orders n and m the following relation between the heat flux vector and the temperature variation gradient (the reader can refer, for instance, to [2], eq. (2.115) page 103, or even to [12], page 930):

$$q_i(\mathbf{x},t)+\frac{\tau_q}{1!}\frac{\partial q_i}{\partial t}(\mathbf{x},t)+\cdots+\frac{\tau_q^n}{n!}\frac{\partial^n q_i}{\partial t^n}(\mathbf{x},t)=-k_{ij}(\mathbf{x})\cdot\left[T_{,j}(\mathbf{x},t)+\frac{\tau_T}{1!}\frac{\partial T_{,j}}{\partial t}(\mathbf{x},t)+\cdots+\frac{\tau_T^m}{m!}\frac{\partial^m T_{,j}}{\partial t^m}(\mathbf{x},t)\right], \tag{1}$$

k_{ij} being the conductivity tensor. It is also worth emphasizing that, dealing with micro-/nanosystems, it seems more than reasonable to assume that the deformations caused by the temperature variations are small enough to be modeled under the assumptions typical of the linear thermoelasticity: such an observation makes therefore particularly fitting to the framework above described the studies reported in [10, 12–14] about the well-posedness question as well as the spatial behavior of the solutions for different linear thermoelastic models founded on constitutive equation (1).

Considering in more detail the possible choices of the Taylor series expansion orders n and m, we recall that Chiriţă et al. [11] showed very interesting results about some limitations in the application of constitutive equation (1), having proved in fact that for $n \geq 5$ or $m \geq 5$ the corresponding models unavoidably lead to instable mechanical systems; conversely, when the expansion orders are lower than or equal to four, the related models can be thermodynamically compatible, provided that suitable assumptions are made upon the delay times. This is the reason why in [12–14] the selection of the expansion orders has concerned the following choices: (a) $n=4$, $m=3$, (b) $n=3$, $m=2$, and (c) $n=m=3$. Moreover, in order to capture the second sound effect for the heat propagation at the micro-/nanoscale, and with reference to the selections just cited a, b, and c, we notice that only the choices a and b lead to models that can express a wavelike behavior, whereas c depicts a diffusive behavior. However, all of these expansion orders are able to account for the high-order effects in τ_q and τ_T linked to the thermal lagging phenomena and are closely related to the number of heat carriers involved (see again [2], p. 442).

Through this contribution we want to move forward and complete the study on the penetration depth of a thermomechanical signal in a micro-/nanosized deformable thermal conductor, considering the relationship between heat flux vector q_i and temperature variation gradient $T_{,i}$ given by (1) considered under the assumption a. This actually represents an advancement, since the corresponding study related to the comparison between the orders selection b and its counterpart $n=2$, $m=1$ has already been investigated in [14], showing unequivocally the depth increase of the thermoelastic signal with the rise of the expansion orders (provided that the condition $n=m+1$, able to guarantee the identification of an influence domain, is fulfilled). We take into account a cylindrical domain filled by an anisotropic and inhomogeneous thermoelastic material, although we will prove that the results obtained are valid regardless of the geometry considered: this feature will allow us to apply our analytical results to some simulations referred to single-layer graphene, showing that also in this case the influence of the expansion orders selected on the signal's depth is evident. An external disturbance is simulated through the application on the lower base of the cylinder of specific thermal and mechanical actions defined from the outside, and a suitable initial-boundary value problem is implemented and investigated in order to show the spatial behavior of the solution in terms of existence and extension of an influence domain of the assigned data.

2. A Suitably Defined Initial-Boundary Value Problem $\widehat{\mathscr{P}}$

Emphasizing once again that our results will be shown to be independent of the shape of the region considered, we take into consideration a right cylinder B of base $D(0)$ and height L and choose the referring axes system $Ox_1x_2x_3$ in such a way that $x_3 \in (0,L)$ and that the base $D(0)$ lies on the coordinate plane Ox_1x_2. We denote by $D(x_3)$ the plane cross section at distance x_3 from the base of the cylinder, assuming it smooth enough in order to allow the application of the divergence theorem; moreover, $B(x_3)$ denotes the portion of B comprised between the parallel cross sections $D(x_3)$ and $D(L)$. The cylinder is intended to be subjected to null initial conditions and null body force and heat supply, and also the boundary data are assumed null except for some prescribed nonzero actions insisting from the outside on the base $D(0)$: specifically, they are able to simulate the external thermomechanical perturbation of which the propagation depth will be estimated. Starting from an appropriately defined initial-boundary value problem, for which uniqueness and continuous dependence results from the given data have already been proved in [12, 13], we are able to show the spatial behavior of the solution with respect to the distance x_3 from the perturbed base $D(0)$.

For the investigation at issue, we consider it unnecessary to propose here a series of preliminary considerations already explained in [12, 13], but rather we start directly from a suitably transformed initial-boundary value problem $\widehat{\mathscr{P}}$ (inferable indifferently from the aforementioned articles) assuming for it null initial data, null body force and heat supply, and null boundary data, except for the abovementioned nonzero external actions insisting on the base $D(0)$ of the cylinder. The definitive formulation of our initial-boundary value problem is obtained through the *hat* integral operator defined, for any continuous function $g(t)$, as follows:

$$\begin{aligned}\widehat{g}(t)={}&\int_0^t\int_0^s\int_0^z\int_0^r g(\xi)\,d\xi\,dr\,dz\,ds\\&+\tau_q\int_0^t\int_0^s\int_0^z g(r)\,dr\,dz\,ds\\&+\frac{\tau_q^2}{2}\int_0^t\int_0^s g(z)\,dz\,ds+\frac{\tau_q^3}{6}\int_0^t g(s)\,ds\\&+\frac{\tau_q^4}{24}g(t),\end{aligned} \tag{2}$$

and that for the sake of brevity and with an obvious meaning of the Roman superscripts involved is shortened as

$$\widehat{g}(t) = g^{IV}(t) + \tau_q g^{III}(t) + \frac{\tau_q^2}{2} g^{II}(t) + \frac{\tau_q^3}{6} g^{I}(t) + \frac{\tau_q^4}{24} g(t). \tag{3}$$

In order to clarify the origin and significance of (3) and of its previous extended form (2), we believe it is appropriate to direct the reader towards [12], Section 3 *Description of the investigation strategy*. We recall that this kind of integral operator has been employed to obtain a number of results in terms of well-posedness and spatial behavior of the solutions for models based on the dual-phase lag as well as on the three-phase lag constitutive law (see at this regard, in addition to the articles already cited, also [15–17]).

By resorting to the linear thermoelastic theory under inhomogeneous and anisotropic assumptions, we recall that the convention for which a comma stands for partial differentiation with respect to the corresponding Cartesian coordinate will be employed, together with the following notations and properties: t_{ji} are the components of the stress tensor, ρ is the mass density, u_i are the components of the displacement vector, T is the temperature variation from the constant ambient temperature T_0 (with T_0 strictly positive), η is the entropy per unit mass, q_i are the components of the heat flux vector, ε_{ij} are the components of the strain tensor, and C_{ijkl}, β_{ij}, k_{ij} (k_{ij}, as already highlighted, represents the conductivity tensor), and a are constitutive tensors depending only on the spatial variable $\mathbf{x}$ and are characterized by the following symmetries:

$$\begin{aligned} C_{ijkl} &= C_{jikl} = C_{klij}, \\ \beta_{ij} &= \beta_{ji}, \\ k_{ij} &= k_{ji}. \end{aligned} \tag{4}$$

As anticipated, null initial conditions are assumed while, with regard to the boundary conditions, we set for any $t \geq 0$ the following:

(i) The lateral boundary conditions, i.e., when $(x_1, x_2) \in \partial D(x_3)$ with $x_3 \in (0, L)$:

$$\begin{aligned} t_{\alpha i}(x_1, x_2, x_3, t) n_\alpha &= 0 \\ \text{or } u_i(x_1, x_2, x_3, t) &= 0, \\ q_\alpha(x_1, x_2, x_3, t) n_\alpha &= 0 \\ \text{or } T(x_1, x_2, x_3, t) &= 0 \end{aligned} \tag{5}$$

(ii) The upper end boundary conditions, i.e., when $(x_1, x_2) \in D(L)$:

$$\begin{aligned} t_{3i}(x_1, x_2, L, t) &= 0 \\ \text{or } u_i(x_1, x_2, L, t) &= 0, \\ q_3(x_1, x_2, L, t) &= 0 \\ \text{or } T(x_1, x_2, L, t) &= 0 \end{aligned} \tag{6}$$

(iii) The lower base boundary conditions assimilable to appropriate actions insisting on $D(0)$, i.e., when $(x_1, x_2) \in D(0)$:

$$\begin{aligned} t_{3i}(x_1, x_2, 0, t) &= t_i(x_1, x_2, t) \\ \text{or } u_i(x_1, x_2, 0, t) &= v_i(x_1, x_2, t), \end{aligned} \tag{7}$$

$$\begin{aligned} q_3(x_1, x_2, 0, t) &= Q_3(x_1, x_2, t) \\ \text{or } T(x_1, x_2, 0, t) &= \Theta(x_1, x_2, t) \end{aligned} \tag{8}$$

where the functions $t_i(x_1, x_2, t)$, $v_i(x_1, x_2, t)$, $Q_3(x_1, x_2, t)$, and $\Theta(x_1, x_2, t)$ are assumed to be prescribed and sufficiently smooth on $D(0)$.

The initial-boundary value problem $\widehat{\mathscr{P}}$ (we recall, deduced indifferently from [12] or [13]) object of investigation is structured as follows:

The transformed equations of motion

$$\widehat{t}_{ji,j}(t) = \rho \frac{\partial^2 \widehat{u}_i}{\partial t^2}(t) \quad \text{in } B \times (0, +\infty) \tag{9}$$

The transformed equation of energy

$$\rho T_0 \frac{\partial \widehat{\eta}}{\partial t}(t) = -\widehat{q}_{i,i}(t) \quad \text{in } B \times (0, +\infty) \tag{10}$$

The transformed constitutive equations

$$\widehat{t}_{ij}(t) = C_{ijkl}\widehat{\varepsilon}_{kl}(t) - \beta_{ij}\widehat{T}(t) \quad \text{in } \overline{B} \times [0, +\infty), \tag{11}$$

$$\rho\widehat{\eta}(t) = \beta_{ij}\widehat{\varepsilon}_{ij}(t) + a\widehat{T}(t) \quad \text{in } \overline{B} \times [0, +\infty), \tag{12}$$

$$\widehat{q}_i(t) = -k_{ij}\left(T_{,j}^{IV}(t) + \tau_T T_{,j}^{III}(t) + \frac{\tau_T^2}{2} T_{,j}^{II}(t) + \frac{\tau_T^3}{6} T_{,j}^{I}(t)\right) \quad \text{in } \overline{B} \times [0, +\infty) \tag{13}$$

The transformed geometrical equations

$$\widehat{\varepsilon}_{ij}(t) = \frac{1}{2}\left(\widehat{u}_{i,j}(t) + \widehat{u}_{j,i}(t)\right) \quad \text{in } \overline{B} \times [0, +\infty) \tag{14}$$

The ordered array $\widehat{\mathscr{S}}(\mathbf{x}, t) = \{\widehat{u}_i, \widehat{T}, \widehat{\varepsilon}_{ij}, \widehat{t}_{ij}, \widehat{\eta}, \widehat{q}_i\}(\mathbf{x}, t)$ defined in $\overline{B} \times [0, +\infty)$ and associated with the set of transformed and nontrivial given data $\widehat{\mathscr{D}} = \{\widehat{t}_i, \widehat{v}_i, \widehat{Q}_3, \widehat{\Theta}\}$ is identified as the (unique (see [12])) solution of the initial-boundary value problem $\widehat{\mathscr{P}}$.

3. The Domain of Influence Theorem

Referring to the above defined three-dimensional region $B(x_3)$ and using (9)-(12) and (14) together with the divergence theorem and the null upper end and lateral boundary conditions, we get

$$\frac{d}{dt}\int_{B(x_3)}\frac{1}{2}\left(\rho\frac{\partial\hat{u}_i}{\partial t}\frac{\partial\hat{u}_i}{\partial t}+C_{ijkl}\hat{\varepsilon}_{ij}\hat{\varepsilon}_{kl}+a\hat{T}^2\right)dv$$
$$=\int_{D(x_3)}\left(\frac{1}{T_0}\hat{q}_3\hat{T}-\hat{t}_{3i}\frac{\partial\hat{u}_i}{\partial t}\right)da+\frac{1}{T_0}\int_{B(x_3)}\hat{q}_i\hat{T}_{,i}dv, \tag{15}$$

and multiplying (15) by $e^{-\delta t}$ we receive

$$\frac{d}{dt}\int_{B(x_3)}\frac{e^{-\delta t}}{2}\left(\rho\frac{\partial\hat{u}_i}{\partial t}\frac{\partial\hat{u}_i}{\partial t}+C_{ijkl}\hat{\varepsilon}_{ij}\hat{\varepsilon}_{kl}+a\hat{T}^2\right)dv$$
$$+\delta\int_{B(x_3)}\frac{e^{-\delta t}}{2}\left(\rho\frac{\partial\hat{u}_i}{\partial t}\frac{\partial\hat{u}_i}{\partial t}+C_{ijkl}\hat{\varepsilon}_{ij}\hat{\varepsilon}_{kl}+a\hat{T}^2\right)dv$$
$$=\int_{D(x_3)}e^{-\delta t}\left(\frac{1}{T_0}\hat{q}_3\hat{T}-\hat{t}_{3i}\frac{\partial\hat{u}_i}{\partial t}\right)da$$
$$+\frac{1}{T_0}\int_{B(x_3)}e^{-\delta t}\hat{q}_i\hat{T}_{,i}dv. \tag{16}$$

The next step is to make explicit, with the aid of (3) and (13), the last integral term of (16) through suitable handling of the partial differentiations with respect to the time variable and an appropriate manipulation of the coefficients. The aim is to obtain an expression in which only products between temperature variation gradients characterized by common superscripts are present and, as highlighted in [13], this will result in the emergence of time derivatives up to the order four:

$$\begin{aligned}
\frac{1}{T_0}\int_{B(x_3)}e^{-\delta t}\hat{q}_i\hat{T}_{,i}dv &= -\frac{\tau_q^4}{48T_0}\frac{d^4}{dt^4}\int_{B(x_3)}e^{-\delta t}k_{ij}T^{IV}_{,j}T^{IV}_{,i}dv\\
&-\left(\frac{\tau_q^4}{12T_0}\delta+\frac{\tau_T^3+\tau_q^3}{12T_0}\right)\frac{d^3}{dt^3}\int_{B(x_3)}e^{-\delta t}k_{ij}T^{IV}_{,j}T^{IV}_{,i}dv-\frac{\tau_T\tau_q^4}{48T_0}\frac{d^3}{dt^3}\int_{B(x_3)}e^{-\delta t}k_{ij}T^{III}_{,j}T^{III}_{,i}dv\\
&-\left(\frac{\tau_q^4}{8T_0}\delta^2+\frac{\tau_T^3+\tau_q^3}{4T_0}\delta+\frac{\tau_T^2+\tau_q^2}{4T_0}\right)\frac{d^2}{dt^2}\int_{B(x_3)}e^{-\delta t}k_{ij}T^{IV}_{,j}T^{IV}_{,i}dv\\
&-\left(\frac{\tau_T\tau_q^4}{16T_0}\delta+\frac{\tau_T\tau_q^3+\tau_T^3\tau_q}{12T_0}-\frac{\tau_q^4}{12T_0}\right)\frac{d^2}{dt^2}\int_{B(x_3)}e^{-\delta t}k_{ij}T^{III}_{,j}T^{III}_{,i}dv-\frac{\tau_T^2\tau_q^4}{96T_0}\frac{d^2}{dt^2}\int_{B(x_3)}e^{-\delta t}k_{ij}T^{II}_{,j}T^{II}_{,i}dv\\
&-\left(\frac{\tau_q^4}{12T_0}\delta^3+\frac{\tau_T^3+\tau_q^3}{4T_0}\delta^2+\frac{\tau_T^2+\tau_q^2}{2T_0}\delta+\frac{\tau_T+\tau_q}{2T_0}\right)\frac{d}{dt}\int_{B(x_3)}e^{-\delta t}k_{ij}T^{IV}_{,j}T^{IV}_{,i}dv\\
&-\left(\frac{\tau_T\tau_q^4}{16T_0}\delta^2+\frac{\tau_T\tau_q^3+\tau_T^3\tau_q-\tau_q^4}{6T_0}\delta+\frac{\tau_T\tau_q^2+\tau_T^2\tau_q}{4T_0}-\frac{\tau_T^3+\tau_q^3}{4T_0}\right)\frac{d}{dt}\int_{B(x_3)}e^{-\delta t}k_{ij}T^{III}_{,j}T^{III}_{,i}dv\\
&-\left(\frac{\tau_T^2\tau_q^4}{48T_0}\delta+\frac{\tau_T^2\tau_q^3+\tau_T^3\tau_q^2}{24T_0}-\frac{\tau_T\tau_q^4}{16T_0}\right)\frac{d}{dt}\int_{B(x_3)}e^{-\delta t}k_{ij}T^{II}_{,j}T^{II}_{,i}dv-\frac{\tau_T^3\tau_q^4}{288T_0}\frac{d}{dt}\int_{B(x_3)}e^{-\delta t}k_{ij}T^{I}_{,j}T^{I}_{,i}dv\\
&-\left(\frac{\tau_q^4}{48T_0}\delta^4+\frac{\tau_T^3+\tau_q^3}{12T_0}\delta^3+\frac{\tau_T^2+\tau_q^2}{4T_0}\delta^2+\frac{\tau_T+\tau_q}{2T_0}\delta+\frac{1}{T_0}\right)\int_{B(x_3)}e^{-\delta t}k_{ij}T^{IV}_{,j}T^{IV}_{,i}dv\\
&-\left(\frac{\tau_T\tau_q^4}{48T_0}\delta^3+\frac{\tau_T\tau_q^3+\tau_T^3\tau_q-\tau_q^4}{12T_0}\delta^2+\frac{\tau_T\tau_q^2+\tau_T^2\tau_q-\tau_T^3-\tau_q^3}{4T_0}\delta+\frac{\tau_T\tau_q}{T_0}-\frac{\tau_T^2+\tau_q^2}{2T_0}\right)\int_{B(x_3)}e^{-\delta t}k_{ij}T^{III}_{,j}T^{III}_{,i}dv\\
&-\left(\frac{\tau_T^2\tau_q^4}{96T_0}\delta^2+\frac{2\tau_T^2\tau_q^3+2\tau_T^3\tau_q^2-3\tau_T\tau_q^4}{48T_0}\delta+\frac{\tau_T^2\tau_q^2}{4T_0}+\frac{\tau_q^4}{24T_0}-\frac{\tau_T\tau_q^3+\tau_T^3\tau_q}{6T_0}\right)\int_{B(x_3)}e^{-\delta t}k_{ij}T^{II}_{,j}T^{II}_{,i}dv\\
&-\left(\frac{\tau_T^3\tau_q^4}{288T_0}\delta+\frac{\tau_T^3\tau_q^3}{36T_0}-\frac{\tau_T^2\tau_q^4}{48T_0}\right)\int_{B(x_3)}e^{-\delta t}k_{ij}T^{I}_{,j}T^{I}_{,i}dv.
\end{aligned} \tag{17}$$

At this point, in order to proceed, we need to set a threshold value for the parameter δ, which we select as follows:

$$\delta=2M=4\max\left\{\frac{\tau_T+\tau_q}{\tau_T\tau_q}-\frac{2}{\tau_T+\tau_q},\frac{3}{\tau_T}\right\}. \tag{18}$$

It is fundamental to underline that δ, chosen in this way, not only results to be strictly positive but also makes the right-hand side of (17) nonpositive. Then (as already mentioned in [14] for the corresponding case relative to the orders selection $n = 3$, $m = 2$, and confirming once again the validity of the model) we underline that the above threshold value (18) able to guarantee the nonpositivity of the RHS of (17) coincides with what was found for the present model $n = 4$, $m = 3$ in [13], p. 837, investigating the corresponding continuous dependence issue.

We insert now relation (17) into (16) and then, in view of the null initial conditions, it is sufficient to integrate the expression obtained four times with respect to the time variable to get

$$\begin{aligned}
&\int_0^t\int_0^s\int_0^z\int_{B(x_3)}\frac{e^{-\delta r}}{2}\left(\rho\frac{\partial\hat{u}_i}{\partial r}\frac{\partial\hat{u}_i}{\partial r}+C_{ijkl}\hat{\varepsilon}_{ij}\hat{\varepsilon}_{kl}+a\hat{T}^2\right)dv\,dr\,dz\,ds\\
&+\int_0^t\int_0^s\int_0^z\int_0^r\int_{B(x_3)}\delta\frac{e^{-\delta\xi}}{2}\left(\rho\frac{\partial\hat{u}_i}{\partial\xi}\frac{\partial\hat{u}_i}{\partial\xi}+C_{ijkl}\hat{\varepsilon}_{ij}\hat{\varepsilon}_{kl}+a\hat{T}^2\right)dv\,d\xi\,dr\,dz\,ds+\frac{\tau_q^4}{48T_0}\int_{B(x_3)}e^{-\delta t}k_{ij}T^{IV}_{,j}T^{IV}_{,i}dv\\
&+\left(\frac{\tau_q^4}{12T_0}\delta+\frac{\tau_T^3+\tau_q^3}{12T_0}\right)\int_0^t\int_{B(x_3)}e^{-\delta s}k_{ij}T^{IV}_{,j}T^{IV}_{,i}dv\,ds+\frac{\tau_T\tau_q^4}{48T_0}\int_0^t\int_{B(x_3)}e^{-\delta s}k_{ij}T^{III}_{,j}T^{III}_{,i}dv\,ds\\
&+\left(\frac{\tau_q^4}{8T_0}\delta^2+\frac{\tau_T^3+\tau_q^3}{4T_0}\delta+\frac{\tau_T^2+\tau_q^2}{4T_0}\right)\int_0^t\int_0^s\int_{B(x_3)}e^{-\delta z}k_{ij}T^{IV}_{,j}T^{IV}_{,i}dv\,dz\,ds\\
&+\left(\frac{\tau_T\tau_q^4}{16T_0}\delta+\frac{\tau_T\tau_q^3+\tau_T^3\tau_q}{12T_0}-\frac{\tau_q^4}{12T_0}\right)\int_0^t\int_0^s\int_{B(x_3)}e^{-\delta z}k_{ij}T^{III}_{,j}T^{III}_{,i}dv\,dz\,ds+\frac{\tau_T^2\tau_q^4}{96T_0}\int_0^t\int_0^s\int_{B(x_3)}e^{-\delta z}k_{ij}T^{II}_{,j}T^{II}_{,i}dv\,dz\,ds\\
&+\left(\frac{\tau_q^4}{12T_0}\delta^3+\frac{\tau_T^3+\tau_q^3}{4T_0}\delta^2+\frac{\tau_T^2+\tau_q^2}{2T_0}\delta+\frac{\tau_T+\tau_q}{2T_0}\right)\int_0^t\int_0^s\int_0^z\int_{B(x_3)}e^{-\delta r}k_{ij}T^{IV}_{,j}T^{IV}_{,i}dv\,dr\,dz\,ds\\
&+\left(\frac{\tau_T\tau_q^4}{16T_0}\delta^2+\frac{\tau_T\tau_q^3+\tau_T^3\tau_q-\tau_q^4}{6T_0}\delta+\frac{\tau_T\tau_q^2+\tau_T^2\tau_q}{4T_0}-\frac{\tau_T^3+\tau_q^3}{4T_0}\right)\int_0^t\int_0^s\int_0^z\int_{B(x_3)}e^{-\delta r}k_{ij}T^{III}_{,j}T^{III}_{,i}dv\,dr\,dz\,ds\\
&+\left(\frac{\tau_T^2\tau_q^4}{48T_0}\delta+\frac{\tau_T^2\tau_q^3+\tau_T^3\tau_q^2}{24T_0}-\frac{\tau_T\tau_q^4}{16T_0}\right)\int_0^t\int_0^s\int_0^z\int_{B(x_3)}e^{-\delta r}k_{ij}T^{II}_{,j}T^{II}_{,i}dv\,dr\,dz\,ds\\
&+\frac{\tau_T^3\tau_q^4}{288T_0}\int_0^t\int_0^s\int_0^z\int_{B(x_3)}e^{-\delta r}k_{ij}T^{I}_{,j}T^{I}_{,i}dv\,dr\,dz\,ds\\
&+\left(\frac{\tau_q^4}{48T_0}\delta^4+\frac{\tau_T^3+\tau_q^3}{12T_0}\delta^3+\frac{\tau_T^2+\tau_q^2}{4T_0}\delta^2+\frac{\tau_T+\tau_q}{2T_0}\delta+\frac{1}{T_0}\right)\int_0^t\int_0^s\int_0^z\int_0^r\int_{B(x_3)}e^{-\delta\xi}k_{ij}T^{IV}_{,j}T^{IV}_{,i}dv\,d\xi\,dr\,dz\,ds\\
&+\left(\frac{\tau_T\tau_q^4}{48T_0}\delta^3+\frac{\tau_T\tau_q^3+\tau_T^3\tau_q-\tau_q^4}{12T_0}\delta^2+\frac{\tau_T\tau_q^2+\tau_T^2\tau_q-\tau_T^3-\tau_q^3}{4T_0}\delta+\frac{\tau_T\tau_q}{T_0}-\frac{\tau_T^2+\tau_q^2}{2T_0}\right)\\
&\times\int_0^t\int_0^s\int_0^z\int_0^r\int_{B(x_3)}e^{-\delta\xi}k_{ij}T^{III}_{,j}T^{III}_{,i}dv\,d\xi\,dr\,dz\,ds\\
&+\left(\frac{\tau_T^2\tau_q^4}{96T_0}\delta^2+\frac{2\tau_T^2\tau_q^3+2\tau_T^3\tau_q^2-3\tau_T\tau_q^4}{48T_0}\delta+\frac{\tau_q^4}{24T_0}+\frac{\tau_T^2\tau_q^2}{4T_0}-\frac{\tau_T\tau_q^3+\tau_T^3\tau_q}{6T_0}\right)\\
&\times\int_0^t\int_0^s\int_0^z\int_0^r\int_{B(x_3)}e^{-\delta\xi}k_{ij}T^{II}_{,j}T^{II}_{,i}dv\,d\xi\,dr\,dz\,ds\\
&+\left(\frac{\tau_T^3\tau_q^4}{288T_0}\delta+\frac{\tau_T^3\tau_q^3}{36T_0}-\frac{\tau_T^2\tau_q^4}{48T_0}\right)\int_0^t\int_0^s\int_0^z\int_0^r\int_{B(x_3)}e^{-\delta\xi}k_{ij}T^{I}_{,j}T^{I}_{,i}dv\,d\xi\,dr\,dz\,ds\\
&=\int_0^t\int_0^s\int_0^z\int_0^r\int_{D(x_3)}e^{-\delta\xi}\left(\frac{1}{T_0}\hat{q}_3\hat{T}-\hat{t}_{3i}\frac{\partial\hat{u}_i}{\partial\xi}\right)da\,d\xi\,dr\,dz\,ds.
\end{aligned}\tag{19}$$

Subsequently, to the unique solution $\widehat{\mathcal{S}}(\mathbf{x},t)$ of the initial-boundary value problem $\widehat{\mathcal{P}}$ we associate, for every possible value of the variable x_3, the following functional $\mathcal{J}_\delta(x_3,t)$:

$$\mathcal{J}_\delta(x_3,t) = -\int_0^t\int_0^s\int_0^z\int_0^r\int_{D(x_3)} e^{-\delta\xi}\left(\widehat{t}_{3i}\frac{\partial\widehat{u}_i}{\partial\xi} - \frac{1}{T_0}\widehat{q}_3\widehat{T}\right)da\,d\xi\,dr\,dz\,ds \quad \forall t\geq 0. \tag{20}$$

Some important features of the above functional $\mathcal{J}_\delta(x_3,t)$ can then be easily shown.

Lemma 1. *Under the hypotheses $\rho, a > 0$, C_{ijkl}, k_{ij} positive definite tensors and selecting for δ the threshold value defined in (18), $\mathcal{J}_\delta(x_3,t)$ is not increasing with respect to the x_3 variable, or in other words $\partial\mathcal{J}_\delta/\partial x_3 \leq 0$. Moreover, $\mathcal{J}_\delta(x_3,t)$ can be identified as a measure for the (unique) solution $\widehat{\mathcal{S}}(\mathbf{x},t)$ of the initial-boundary value problem $\widehat{\mathcal{P}}$, in the sense that it turns out to be not negative for every possible value of the variables x_3 and t.*

Proof. In order to prove that $\partial\mathcal{J}_\delta/\partial x_3 \leq 0$, it is sufficient to remember how the volume $B(x_3)$ is structured and evaluate, starting from (19), the partial derivative of $\mathcal{J}_\delta(x_3,t)$ with respect to the x_3 variable: it is therefore evident that, under the hypotheses at issue, $\partial\mathcal{J}_\delta/\partial x_3$ is not positive. Consequently, in view of the null upper end boundary conditions and of definition (20) of $\mathcal{J}_\delta(x_3,t)$ we have

$$\mathcal{J}_\delta(x_3,t) \geq \mathcal{J}_\delta(L,t) = 0, \tag{21}$$

for every possible value of x_3 and t. Specifically, it is easy to prove that the condition $\mathcal{J}_\delta(x_3,t) = 0$ necessarily correlates to a null solution for the considered initial-boundary value problem. □

The following theorem can then be proved, concerning the solution $\mathcal{S}(\mathbf{x},t)$ of the initial-boundary value problem $\mathcal{P}$ (no *hat*) from which the problem $\widehat{\mathcal{P}}$ derives, in view of the considered hypothesis of linearity (see again [12] or [13] for details).

Theorem 2 (existence of a domain of influence). *Let $\mathcal{S}(\mathbf{x},t) = \{u_i, T, \varepsilon_{ij}, t_{ij}, \eta, q_i\}(\mathbf{x},t)$ be the solution of the initial-boundary value problem $\mathcal{P}$. Let ρ and a be greater than zero, let C_{ijkl} and k_{ij} be positive definite tensors, and let δ be identified with the threshold value (18). Moreover, suppose that the cylinder B is subjected on its lower base $D(0)$ to the external actions described through (7) and (8). Then there exists a positive constant σ, dimensionally expressed by a speed, such that*

$$\mathcal{S}(\mathbf{x},t) = \{u_i, T, \varepsilon_{ij}, t_{ij}, \eta, q_i\}(\mathbf{x},t) = 0 \quad \text{for every } \mathbf{x}\in B \text{ such that } x_3 \geq \sigma t; \tag{22}$$

i.e., the whole activity in the cylinder B due to the presence of the assigned data acting on its base $D(0)$ vanishes at distances from $D(0)$ greater than or equal to σt.

Proof. Avoiding, just for reasons of synthesis, reporting the explicit expression of $\partial\mathcal{J}_\delta/\partial x_3$, anyway easily obtainable from (19), let us estimate instead the partial derivative of $\mathcal{J}_\delta(x_3,t)$ with respect to the time variable t. To do this, we recall that

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2 \quad \forall a,b\in\mathbb{R},\ \forall\varepsilon>0, \tag{23}$$

and, thus, from the definition (20) we get

$$\left|\frac{\partial}{\partial t}\mathcal{J}_\delta(x_3,t)\right| \leq \frac{1}{2}\int_0^t\int_0^s\int_0^z\int_{D(x_3)} e^{-\delta r}\left[\frac{\varepsilon_1}{\rho}\widehat{t}_{3i}\widehat{t}_{3i} + \frac{\rho}{\varepsilon_1}\frac{\partial\widehat{u}_i}{\partial r}\frac{\partial\widehat{u}_i}{\partial r} + \frac{\varepsilon_2}{aT_0^2}\widehat{q}_3^2 + \frac{a}{\varepsilon_2}\widehat{T}^2\right]da\,dr\,dz\,ds \quad \forall\varepsilon_1,\varepsilon_2>0. \tag{24}$$

In order to estimate the right-hand side of (24) (in particular, the terms containing $\widehat{t}_{3i}\widehat{t}_{3i}$ and $\widehat{q}_3^2$) we first take into account constitutive equation (11) and denote by μ_M, positive and constant, the greatest elastic modulus of the constitutive tensor C_{ijkl} such that for every ξ_{ij} it is $C_{ijkl}\xi_{ij}\xi_{kl} \leq \mu_M\xi_{ij}\xi_{ij}$. In view of the arithmetic-geometric mean inequality, the following relation is valid:

$$\widehat{t}_{3i}\widehat{t}_{3i} \leq \widehat{t}_{ij}\widehat{t}_{ij} \leq (1+\varepsilon)\mu_M C_{ijkl}\widehat{\varepsilon}_{ij}\widehat{\varepsilon}_{kl} + \left(1+\frac{1}{\varepsilon}\right)B^2\widehat{T}^2, \quad \forall\varepsilon>0 \text{ and where } B^2 = \beta_{ij}\beta_{ij} > 0. \tag{25}$$

On the other side, recalling constitutive equation (13), distributing $(-k_{ij}\widehat{q}_i)$ with respect to the terms of the sum, and then applying the Cauchy-Schwarz inequality we can write

$$\begin{aligned}\widehat{q}_i\widehat{q}_i &= -k_{ij}\widehat{q}_i\left(T^{IV}_{,j}(t) + \tau_T T^{III}_{,j}(t) + \frac{\tau_T^2}{2}T^{II}_{,j}(t) + \frac{\tau_T^3}{6}T^{I}_{,j}(t)\right) \leq (k_{rs}k_{rs})^{1/2}\left(T^{IV}_{,j}\widehat{q}_i T^{IV}_{,j}\widehat{q}_i\right)^{1/2} \\ &+ \left(\tau_T^2 k_{rs}k_{rs}\right)^{1/2}\left(T^{III}_{,j}\widehat{q}_i T^{III}_{,j}\widehat{q}_i\right)^{1/2} + \left(\frac{\tau_T^4}{4}k_{rs}k_{rs}\right)^{1/2} \cdot\left(T^{II}_{,j}\widehat{q}_i T^{II}_{,j}\widehat{q}_i\right)^{1/2} + \left(\frac{\tau_T^6}{36}k_{rs}k_{rs}\right)^{1/2}\left(T^{I}_{,j}\widehat{q}_i T^{I}_{,j}\widehat{q}_i\right)^{1/2} \\ &= (\widehat{q}_i\widehat{q}_i)^{1/2}\left[(k_{rs}k_{rs})^{1/2}\left(T^{IV}_{,j}T^{IV}_{,j}\right)^{1/2} + \tau_T(k_{rs}k_{rs})^{1/2}\left(T^{III}_{,j}T^{III}_{,j}\right)^{1/2}\right.\end{aligned}$$

$$+\frac{\tau_T^2}{2}\left(k_{rs}k_{rs}\right)^{1/2}\left(T_{,j}^{II}T_{,j}^{II}\right)^{1/2} + \frac{\tau_T^3}{6}\left(k_{rs}k_{rs}\right)^{1/2}\left(T_{,j}^{I}T_{,j}^{I}\right)^{1/2}\Bigg], \tag{26}$$

and then through squaring we get

$$\widehat{q}_3^2 \le \widehat{q}_i\widehat{q}_i \le 4\left[k_{rs}k_{rs}\left(T_{,j}^{IV}T_{,j}^{IV}\right) + \tau_T^2 k_{rs}k_{rs}\left(T_{,j}^{III}T_{,j}^{III}\right) + \frac{\tau_T^4}{4}k_{rs}k_{rs}\left(T_{,j}^{II}T_{,j}^{II}\right) + \frac{\tau_T^6}{36}k_{rs}k_{rs}\left(T_{,j}^{I}T_{,j}^{I}\right)\right]. \tag{27}$$

In addition, identifying with k_m the smallest (and strictly positive) eigenvalue of the conductivity tensor k_{ij}, the following relations are valid:

$$\begin{aligned}
T_{,j}^{IV}T_{,j}^{IV} &\le \frac{1}{k_m}k_{ij}T_{,j}^{IV}T_{,i}^{IV},\\
T_{,j}^{III}T_{,j}^{III} &\le \frac{1}{k_m}k_{ij}T_{,j}^{III}T_{,i}^{III},\\
T_{,j}^{II}T_{,j}^{II} &\le \frac{1}{k_m}k_{ij}T_{,j}^{II}T_{,i}^{II},\\
T_{,j}^{I}T_{,j}^{I} &\le \frac{1}{k_m}k_{ij}T_{,j}^{I}T_{,i}^{I},
\end{aligned} \tag{28}$$

and so from (24) it follows that

$$\begin{aligned}
\left|\frac{\partial}{\partial t}\mathcal{J}_\delta\left(x_3,t\right)\right| \le{}& \frac{1}{2}\int_0^t\int_0^s\int_0^z\int_{D(x_3)} e^{-\delta r}\left\{\frac{\rho}{\varepsilon_1}\frac{\partial\widehat{u}_i}{\partial r}\frac{\partial\widehat{u}_i}{\partial r} + \frac{\varepsilon_1\left(1+\varepsilon\right)\mu_M}{\rho}C_{ijkl}\widehat{\varepsilon}_{ij}\widehat{\varepsilon}_{kl} + \left[\frac{1}{\varepsilon_2} + \frac{B^2\varepsilon_1}{a\rho}\left(1+\frac{1}{\varepsilon}\right)\right]\right.\\
&\cdot a\widehat{T}^2 + \left(\frac{4\varepsilon_2}{T_0}\frac{k_{rs}k_{rs}}{ak_m}\right)\left(\frac{1}{T_0}k_{ij}T_{,j}^{IV}T_{,i}^{IV} + \frac{\tau_T^2}{T_0}k_{ij}T_{,j}^{III}T_{,i}^{III} + \frac{\tau_T^4}{4T_0}k_{ij}T_{,j}^{II}T_{,i}^{II}\right.\\
&\left.\left. + \frac{\tau_T^6}{36T_0}k_{ij}T_{,j}^{I}T_{,i}^{I}\right)\right\} da\,dr\,dz\,ds \qquad \forall\varepsilon,\varepsilon_1,\varepsilon_2 > 0.
\end{aligned} \tag{29}$$

Assuming again for δ the threshold value (18) and reconsidering the partial derivative $\partial\mathcal{J}_\delta/\partial x_3$, but retaining this time only the terms with a number of integrals equal to that characterizing (29), we reach the following estimate:

$$\begin{aligned}
-\frac{\partial}{\partial x_3}\mathcal{J}_\delta\left(x_3,t\right) \ge{}& \frac{1}{2}\int_0^t\int_0^s\int_0^z\int_{D(x_3)} e^{-\delta r}\left\{\rho\frac{\partial\widehat{u}_i}{\partial r}\frac{\partial\widehat{u}_i}{\partial r} + C_{ijkl}\widehat{\varepsilon}_{ij}\widehat{\varepsilon}_{kl} + a\widehat{T}^2 + \left[\frac{\tau_q^4}{6}\delta^3 + \frac{\tau_T^3+\tau_q^3}{2}\delta^2\right.\right.\\
&\left. + \left(\tau_T^2+\tau_q^2\right)\delta + \tau_T + \tau_q\right]\frac{1}{T_0}k_{ij}T_{,j}^{IV}T_{,i}^{IV} + \left(\frac{\tau_T\tau_q^4}{8}\delta^2 + \frac{\tau_T\tau_q^3+\tau_T^3\tau_q-\tau_q^4}{3}\delta + \frac{\tau_T\tau_q^2+\tau_T^2\tau_q}{2} - \frac{\tau_T^3+\tau_q^3}{2}\right)\\
&\cdot\frac{1}{T_0}k_{ij}T_{,j}^{III}T_{,i}^{III} + \left(\frac{\tau_T^2\tau_q^4}{24}\delta + \frac{\tau_T^2\tau_q^3+\tau_T^3\tau_q^2}{12} - \frac{\tau_T\tau_q^4}{8}\right)\frac{1}{T_0}k_{ij}T_{,j}^{II}T_{,i}^{II} + \frac{\tau_T^3\tau_q^4}{144}\frac{1}{T_0}\\
&\left.\cdot k_{ij}T_{,j}^{I}T_{,i}^{I}\right\} da\,dr\,dz\,ds \ge 0.
\end{aligned} \tag{30}$$

At this point it is sufficient to define

$$\begin{aligned}
\Psi = \max\Bigg\{&\frac{12}{2\tau_q^4\delta^3 + 6\left(\tau_T^3+\tau_q^3\right)\delta^2 + 12\left(\tau_T^2+\tau_q^2\right)\delta + 12\left(\tau_T+\tau_q\right)},\\
&\frac{24\tau_T^2}{3\tau_T\tau_q^4\delta^2 + 8\left(\tau_T\tau_q^3+\tau_T^3\tau_q-\tau_q^4\right)\delta + 12\left(\tau_T\tau_q^2+\tau_T^2\tau_q-\tau_T^3-\tau_q^3\right)}, \frac{6\tau_T^4}{\tau_T^2\tau_q^4\delta + 2\left(\tau_T^2\tau_q^3+\tau_T^3\tau_q^2\right) - 3\tau_T\tau_q^4}, \frac{4\tau_T^3}{\tau_q^4}\Bigg\}\sup_{D(x_3)}\frac{k_{rs}k_{rs}}{ak_m}
\end{aligned} \tag{31}$$

and to impose the equality between the following coefficients, which is a consequence of the comparison between the estimates involving the partial derivatives of $\mathcal{J}_\delta(x_3,t)$ with respect to x_3 and t. We denote them by σ (strictly positive) and set

$$\frac{1}{\varepsilon_1} = \frac{\varepsilon_1\left(1+\varepsilon\right)\mu_M}{\rho} = \frac{1}{\varepsilon_2} + \frac{B^2\varepsilon_1}{a\rho}\left(1+\frac{1}{\varepsilon}\right) = \frac{4\varepsilon_2}{T_0}\Psi = \sigma. \tag{32}$$

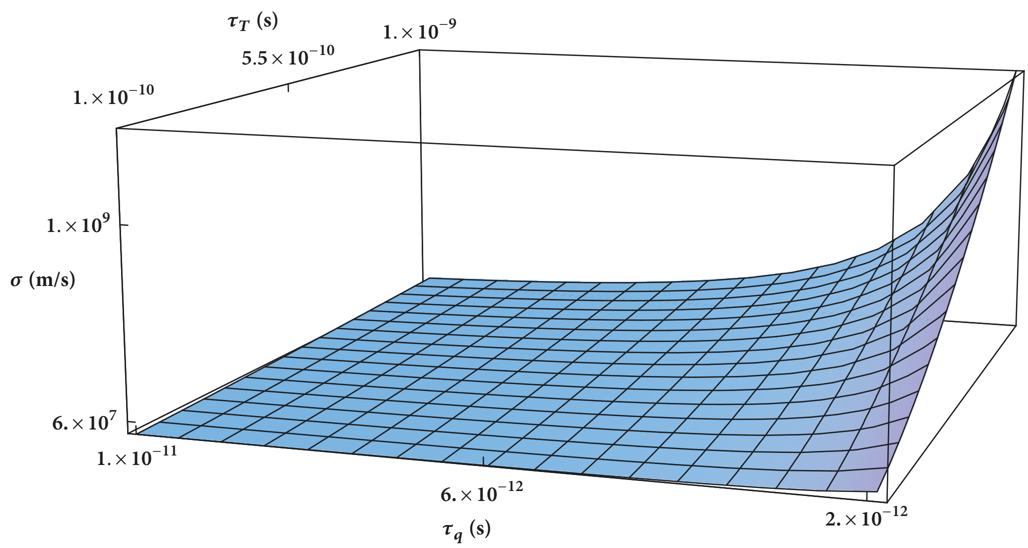

Figure 1: The behavior of the parameter σ with respect to both the relaxation times τ_T and τ_q.

It follows that

$$\varepsilon_1 = \sqrt{\frac{\rho}{(1+\varepsilon)\,\mu_M}},$$
$$\varepsilon_2 = \frac{T_0}{4\Psi}\sqrt{\frac{(1+\varepsilon)\,\mu_M}{\rho}}, \tag{33}$$

ε being the positive root of the algebraic equation

$$\varepsilon^2 + \left(1 - \frac{T_0 B^2 + 4\rho a \Psi}{a T_0 \mu_M}\right)\varepsilon - \frac{B^2}{a\mu_M} = 0. \tag{34}$$

With these choices, comparing relations (29) and (30) we reach the following differential inequality, valid for all $(x_3, t) \in (0, L) \times (0, \infty)$:

$$\left|\frac{\partial}{\partial t}\mathcal{I}_\delta(x_3, t)\right| + \sigma \frac{\partial}{\partial x_3}\mathcal{I}_\delta(x_3, t) \le 0. \tag{35}$$

Such a differential inequality can be integrated, for example, as in [17] in order to easily deduce that

$$\mathcal{I}_\delta(x_3, t) = 0 \quad \textit{for every} \quad (x_3, t) \in (\sigma t, L) \times (0, \infty), \tag{36}$$

a condition that evidently completes the proof of the theorem. The existence of an influence domain of the external data with respect to the x_3 coordinate remains therefore proved. □

4. Final Remarks and Illustrative Simulations

It is worth noting that the investigation techniques used in this article, although deriving from the classical linear thermoelasticity, turn out to be particularly suitable for the handling of innovative thermomechanical models such as the one here treated: in this regard it is not trivial to underline that, again in the linear thermoelastic field, there are numerous examples in the literature concerning the study of mixed initial-boundary value problems also involving very peculiar material structures (see, for instance, [18, 19]). We also report that, with reference to the investigation of the spatial behavior of the solutions for time-differential DPL deformable thermal conductors, the analysis proposed in this article has been further deepened in [20], currently in press.

In order to provide useful elements for a complete understanding of the heat exchange mechanisms able to take into account also elastic deformations on the micro-/nanoscale, the influence of the DPL high-order effects up to $n = 4$, $m = 3$ (directly related to the number of heat carriers) on the depth of the influence domain has been taken into account with the aid of a properly formulated initial-boundary value problem. To this aim a cylindrical domain B has been considered, although the resulting parameter σ (to be understood as the signal transmission speed) able to completely describe the abovementioned domain depth results indeed to be

(i) independent of the shape of the region B

(ii) depending only on the features of the selected material, among which there are the constitutive coefficients and tensors and the relaxation times

Imagining the lower base of the deformable, anisotropic, and inhomogeneous cylinder B excited by an appropriate thermomechanical perturbation, the spatial behavior of the solution in terms of existence and extension of an influence domain has been investigated.

In the attempt to apply the result obtained to a concrete estimate, and in continuity with what was done in [14], a homogeneous and isotropic single-layer graphene sheet at the temperature of 300 K has been considered, showing a first graphic simulation in which the signal transmission speed σ is given with respect to both the relaxation times τ_q and τ_T (Figure 1). The material features of the considered single-layer

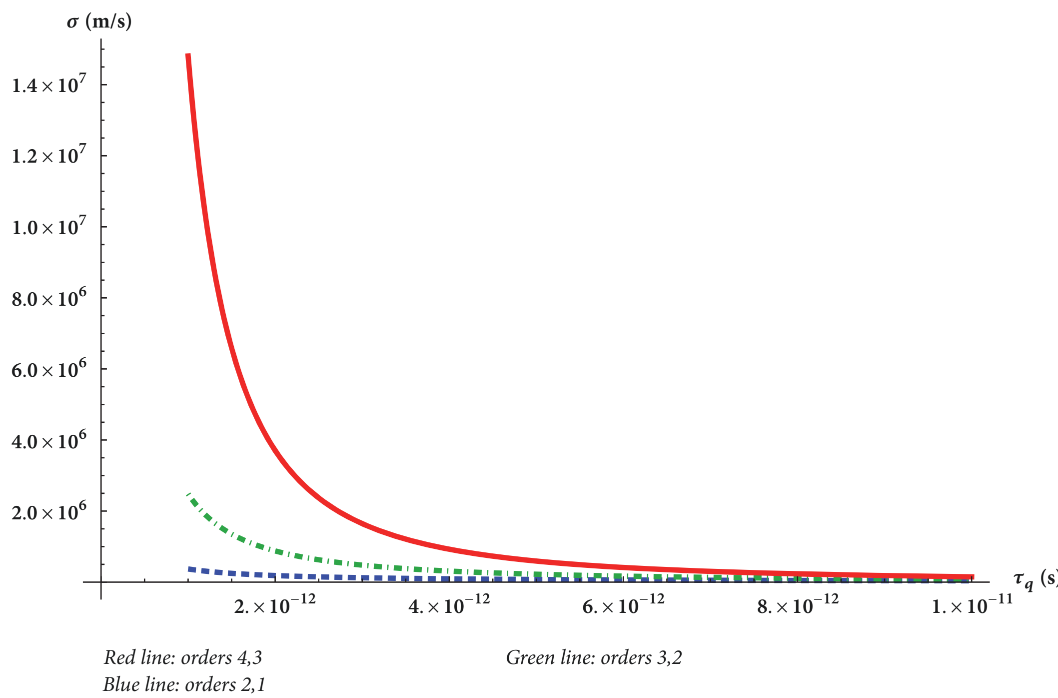

FIGURE 2: The behavior of the parameter σ for the expansion orders 4, 3 (continuous red line) with respect to the relaxation time τ_q, having selected $\tau_T = 2 \times 10^{-11} s$, in comparison with the expansion orders 3, 2 (dash-dot green line) and 2, 1 (dashed blue line).

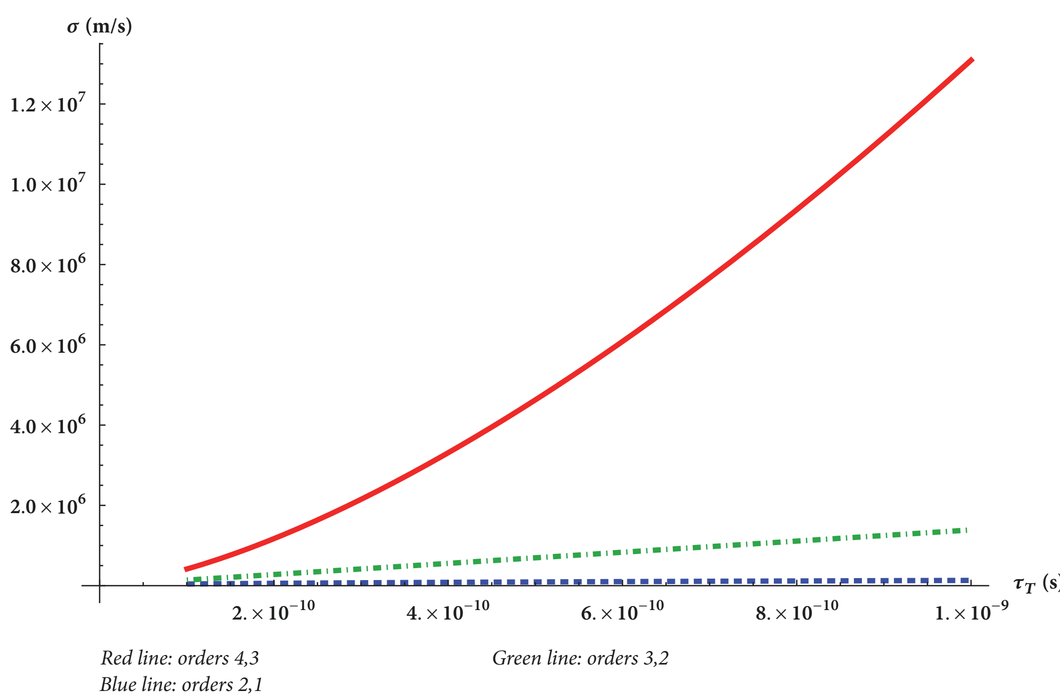

FIGURE 3: The behavior of the parameter σ for the expansion orders 4, 3 (continuous red line) with respect to the relaxation time τ_T, having selected $\tau_q = 2 \times 10^{-11} s$, in comparison with the expansion orders 3, 2 (dash-dot green line) and 2, 1 (dashed blue line).

graphene are summarized in Table 1 (in this regard, the reader can refer to [21] and to the references therein).

In order to integrate and facilitate a direct comparison with the results shown in [14], the same significant combination of intervals has been considered for τ_q and for τ_T: $1.0 \times 10^{-12}\, s \leq \tau_q \leq 1.0 \times 10^{-11}\, s$, $1.0 \times 10^{-10}\, s \leq \tau_T \leq 1.0 \times 10^{-9}\, s$. Subsequently, in Figures 2 and 3 the value of σ is given separately with respect to τ_q and τ_T, showing in parallel also the homologous behaviors obtained in [14] relative to the expansion orders $n = 3$, $m = 2$ and $n = 2$, $m = 1$. The value

Table 1

Mass density	$2.35 \times 10^3\ kg\ m^{-3}$
Thermal conductivity	$3.10 \times 10^3\ W\ m^{-1}K^{-1}$
Specific heat capacity	$7.70 \times 10^2\ J\ kg^{-1}K^{-1}$
Elastic modulus	$1.05 \times 10^{12}\ N\ m^{-2}$
Shear modulus	$4.45 \times 10^{11}\ N\ m^{-2}$
Thermal expansion coefficient	$-4.00 \times 10^{-6}\ K^{-1}$

$2 \times 10^{-11} s$ has been selected to fix, alternately, τ_T and τ_q in Figures 2 and 3, being possible to appreciate how the depth of the signal is further increased if $n = 4$, $m = 3$ are selected as done in the present article.

Again as highlighted in [14] for the model $n = 3$, $m = 2$, but in this case even more markedly, we can observe in Figure 1 that, for increasing values of τ_T, a significantly shorter relaxation time τ_q makes the penetration depth of the signal extremely wide: also in this case, in fact, in view of the expansion orders involved, a suitable combination of values for the phase lags makes the model tend to assume almost a diffusive behavior. For the fixed value of τ_T (Figure 2) a sudden enlargement of the amplitude of the influence domain for decreasing values of τ_q is detectable. We recall that this is due to the fact that the transmission speed becomes extremely high, bringing the behavior of the model very close to the classical thermoelasticity (let us recall the paradox of the infinite propagation speed). Also in Figure 3, after having selected $\tau_q = 2 \times 10^{-11}\ s$, the direct proportionality between σ and τ_T is preserved.

In conclusion, we have proved that the latter feature highlighted by Tzou in [2] (Chapter 12) up to the orders selection $n = 2$, $m = 1$ "*As a general trend, the high-order waves and diffusion (corresponding to higher-order effects in* τ_T *and* τ_q*) gradually increase the temperature level in the heat-affected zone. When thermal waves are activated* [as in the case here considered] … *the physical domain of the heat-affected zone gradually increases as well*" is preserved also in the context of the linear thermoelasticity if we cross the sequence of expansion orders $n = 2, m = 1; n = 3, m = 2; n = 4, m = 3$. We also specify that all the proposed simulations, performed through *Wolfram Mathematica 9*, were conducted under the assumption of flux-precedence.

Data Availability

The main results shown in the manuscript are analytical. With regard to the proposed illustrative simulations, the values of the material parameters considered can be easily found in the literature. For this reason, no information seems to be necessary regarding the availability of specific data to support the results.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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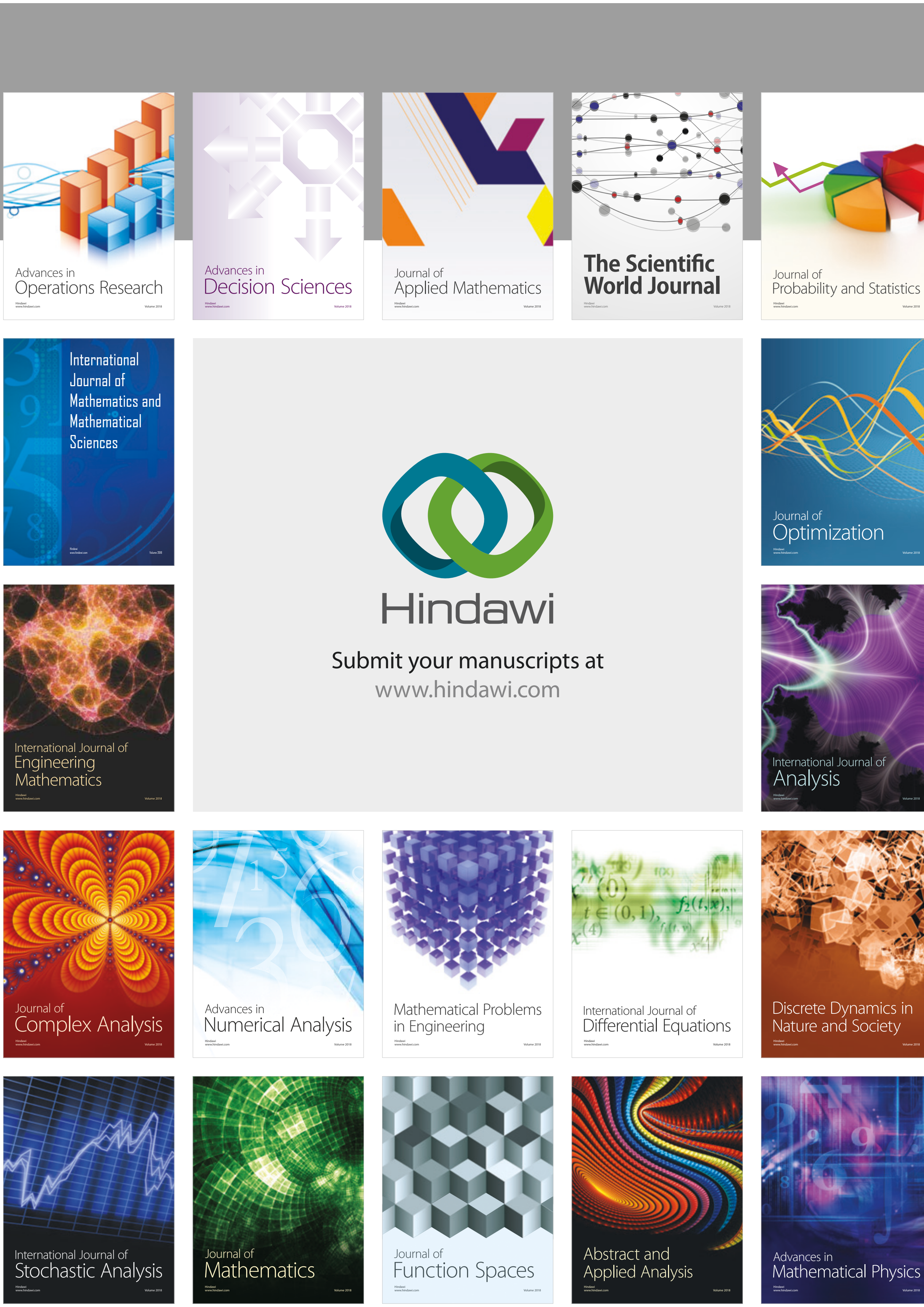
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