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Simulation of sample paths for Gauss–Markov processes in the presence of a reflecting boundary

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Abstract: Algorithms for the simulation of sample paths of Gauss–Markov processes, restricted from below by particular time-dependent reflecting boundaries, are proposed. These algorithms are used to build the histograms of first passage time density through specified boundaries and for the estimation of related moments. Particular attention is dedicated to restricted Wiener and Ornstein–Uhlenbeck processes due to their central role in the class of Gauss–Markov processes.

1. Introduction

Diffusion and Gauss–Markov processes in the presence of a reflecting boundary are widely used for modeling continuous time phenomena in many scientific fields, such as neurosciences, mathematical biology, finance, and queueing systems. In some instances, these processes are derived as approximations of discrete-state Markovian models which, although more appropriate in describing the behavior of the real system, are often hard to study from both analytical and computational points of view (cf., for instance, Chen & Whitt, 1993; Harrison, 1985; Kushner, 2001).

Typical situations arise in population dynamics with immigration in which the total number of individuals is bound to take non-negative values, so it is necessary to impose a reflection condition at zero state (cf. Renshaw, 2011; Ricciardi, 1986 and references therein).

ABOUT THE AUTHORS

The research interest of the authors includes theory and simulation of stochastic processes with applications to biomathematical modeling and queueing systems. Specifically, they are involved in the development of analytical and computational methods for the solution of several theoretical and algorithmic problems suggested, or imposed, by the considered models. Their research activity has been particularly intense within the following themes: (i) the study of the first-passage time densities for diffusion processes and Gauss-Markov processes and their asymptotic behavior; (ii) the development of mathematical methods and tools mainly of a probabilistic and computational nature to analyze biological dynamics, among them the input-output behavior of neurons subject to random perturbations.

PUBLIC INTEREST STATEMENT

In a wide variety of different domains of applications, including finance, mathematical biology, physics, neurobiology, queueing systems and engineering, particular attention is dedicated to Gauss–Markov processes restricted from below by particular time-dependent reflecting boundaries. We formulate algorithms to simulate the sample paths of such stochastic processes. Furthermore, we construct histograms of first passage times for the restricted process. Particular attention has been dedicated to restricted Wiener and Ornstein–Uhlenbeck processes for their central role in the class of Gauss–Markov processes.

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In heavy traffic conditions, the dynamic of a queue can be approximated by a diffusion process with a zero reflecting boundary (cf. Di Crescenzo, Giorno, Nobile, & Ricciardi, 2003; Di Crescenzo & Nobile, 1995; Iglehart & Whitt, 1970; Reiman, 1984; Ward & Glynn, 2005). In the financial applications, diffusion models have been used to capture the stochastic movement of the short-term interest rate in the market, for which negative values are not allowed (cf. Cox, Ingersoll, & Ross, 1985; Goldstein & Keirstead, 1997; Han, Hu, & Lee, 2016; Linetsky, 2005).

In the neuronal context, the membrane potential can be modeled by diffusion or Gauss–Markov processes restricted by a lower reflecting boundary that can be interpreted as a neuronal reversal hyperpolarization potential (cf., for instance, Buonocore, Caputo, Nobile, & Pirozzi, 2014,2015; Ditlevsen & Lánský, 2006; Gerstner & Kistler, 2002; Giorno, Nobile, Pirozzi, & Ricciardi, 2003; Inoue & Doi, 2007; Lánský & Ditlevsen, 2008).

In the previous types of instances, the distribution of first passage times (FPT) is required to describe events such as the emptying of a queue, the breakdown of a device, the option pricing, and the firings of a neuron (see, Abrahams 1986; Blake & Lindsey, 1973; Buonocore, Caputo, Pirozzi, & Ricciardi, 2011; Darling & Siegert, 1953; Molini, Talkner, Katul, & Porporato, 2011). Despite the vastness and the relevance of the applications of the FPT, explicit analytic solutions for the FPT densities are not known except for very few cases (see, Abundo, 2014; Borodin & Salminen, 2002; Buonocore et al., 2014; Cox & Miller, 1970; Di Crescenzo, Giorno, & Nobile, 2016; Giorno, Nobile, & Ricciardi, 2011).

In diffusion and Gauss–Markov processes, restricted or not, the simulation plays an important role. Indeed, it allows to understand the basic properties of the models, to investigate the behavior of models under specific hypotheses, and to generate a random sample of data for parameter estimations (cf. Asmussen, Glynn, & Pitman, 1995; Devroye, 2010; D'Onofrio & Pirozzi, 2016; Glasserman, 2004; Higham, 2001; Iacus, 2008; Kloeden & Platen, 1999; Kroese, Taimre, & Botev, 2011; Lejay & Pichot, 2012; Tuerlinckx, Maris, Ratcliff, & De Boeck, 2001). A simulation approach is also required to construct the histogram of FPT pdf and estimates of the related moments (Giraudo, Sacerdote, & Zucca, 2001; Herrmann & Tanré, 2016; Taillefumier & Magnasco, 2010; Mousavi & Glynn, 2013).

In Buonocore et al. (2014), we have considered Gauss–Markov processes, restricted from below by particular time-dependent reflecting boundaries, and we have explicitly determined the transition probability density functions (pdf's), the conditional mean, and the second-order conditional moment. Furthermore, in some special cases, the FPT density through a time-dependent threshold has been computed using both asymptotic methods either numerical or simulation techniques. Instead, in Buonocore et al. (2015), the restricted Gauss–Markov processes have been used to construct inhomogeneous leaky integrate-and-fire stochastic models for single neurons activity in the presence of a lower reflecting boundary and periodic input signal.

The present paper deals with various aspects of the simulation of the restricted Gauss–Markov processes considered in Buonocore et al. (2014). The paper is organized as follows. In Section 2, we start considering a Gauss–Markov process $\{Y(t), t \in T\}$ conditioned on the initial state y at time τ and we give an exact method for its simulation at desired time instants. In Section 3, we take into consideration the restricted stochastic process $X(t)$, obtained considering $Y(t)$ in the presence of a special time-dependent lower reflecting boundary. The reflecting boundary takes the form $\nu(t) = m(t) + A h_1(t) + B h_2(t)$, where the real constants A and B are chosen in order to ensure that $y \geq \nu(\tau)$. Particular attention is dedicated to restricted Wiener and Ornstein–Uhlenbeck processes, in view of their potential interest in a wide variety of different applications domains (Di Crescenzo & Giorno, 2012; Giorno, Nobile, & di Cesare, 2012; Linetsky, 2005; Ricciardi & Sacerdote, 1987; Wonho, 2009). In Section 4, we formulate two algorithms to simulate the sample paths of the process $X(t)$, distinguishing the case in which $A = 0$ from the case $A \neq 0$. When $A = 0$, it is easy to realize the sample paths of $X(t)$, using a principle similar to that of reflection of Brownian motion. Instead, when $A \neq 0$, a different algorithm has been implemented that constitutes an extension to the

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Gauss–Markov processes of the one given in Asmussen et al. (1995) and Kroese et al. (2011) to simulate the sample paths of a reflected Wiener process with negative drift. Moreover, in Section 5, we use the algorithms proposed in Section 4 to construct the histogram of FPT density for the process $X(t)$. Finally, the R codes to generate sample paths and to simulate the first passage times for Wiener and Ornstein–Uhlenbeck processes are listed in Appendix 1.

Throughout this paper, the symbol $\overset{d}{=}$ denotes equality in distribution.

2. Gauss–Markov processes conditioned on the initial state

Let $m(t)$, $h_1(t)$, $h_2(t)$ be $C^1(T)$-class functions, where $C^1(T)$ denotes the set of continuously differentiable functions on T, with T continuous parameter set, such that $h_2(t) \neq 0$ and $r(t) = h_1(t)/h_2(t)$ is a non-negative and monotonically increasing function. Denoting by $\{W(t),\, t \geq 0\}$ a standard Wiener process, then [see, Abrahams and Thomas (1981), Di Nardo, Nobile, Pirozzi, and Ricciardi (2001), Mehr and McFadden (1965)]

$$Y(t) = m(t) + [y - m(\tau)]\frac{h_2(t)}{h_2(\tau)} + h_2(t)\, W[r(t) - r(\tau)] \qquad (t \geq \tau), \tag{1}$$

is a Gauss–Markov process conditioned to start from y at time τ. The pdf $f_Y(x, t)$ of $Y(t)$ is a normal density with mean and variance

$$E[Y(t)] \equiv M(t|y,\, \tau) = m(t) + \frac{h_2(t)}{h_2(\tau)}\big[y - m(\tau)\big],$$

$$\text{Var}[Y(t)] \equiv V(t|\tau) = h_2(t)\left[h_1(t) - \frac{h_2(t)}{h_2(\tau)}h_1(\tau)\right], \tag{2}$$

for $t,\, \tau \in T$ and $\tau < t$, respectively. In the sequel, we denote by

$$F_Y(x, t) = P\{Y(t) < x\} = \frac{1}{2}\left\{1 + \text{Erf}\left[\frac{x - M(t|y,\, \tau)}{\sqrt{2\, V(t|\tau)}}\right]\right\} \tag{3}$$

the transition distribution function of $Y(t)$, with $\text{Erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-z^2}\, dz$.

In particular, hereafter, we consider the following Gauss–Markov process $Y(t)$ with continuous parameter set $T = (0, +\infty)$:

• **Wiener process:** Setting

$$m(t) = \mu\, t, \quad h_1(t) = \sigma^2\, t, \quad h_2(t) = 1 \quad (\mu \in \mathbb{R},\, \sigma \in \mathbb{R}^+), \tag{4}$$

from (1), one has:

$$Y(t) = y + \mu(t - \tau) + \sigma\, W(t - \tau) \qquad (t \geq \tau). \tag{5}$$

• **Ornstein-Uhlenbeck process:** Setting

$$m(t) = \frac{\beta}{\alpha}\big(e^{\alpha t} - 1\big), \quad h_1(t) = \frac{\sigma^2}{2\alpha}\big(e^{\alpha t} - e^{-\alpha t}\big), \quad h_2(t) = e^{\alpha t} \quad (\alpha,\, \beta \in \mathbb{R},\, \sigma \in \mathbb{R}^+), \tag{6}$$

from (1), it results:

$$Y(t) = y\, e^{\alpha(t-\tau)} + \frac{\beta}{\alpha}\big(e^{\alpha(t-\tau)} - 1\big) + \sigma\, W\left[\frac{e^{2\alpha(t-\tau)} - 1}{2\alpha}\right] \qquad (t \geq \tau). \tag{7}$$

2.1. Simulation of Gauss–Markov processes conditioned on the initial state

In this section, we first generate the sample paths of the conditioned Gauss–Markov process $Y(t)$, according to the stochastic equation (1), using an exact simulation method (cf., for instance, Kroese et al., 2011). Let s and t be two time instants, such that $t_0 < s < t$; by virtue of (1), we obtain the stochastic equation

$$Y(t) - Y(s)\frac{h_2(t)}{h_2(s)} = m(t) - m(s)\frac{h_2(t)}{h_2(s)} + h_2(t)\left\{W[r(t) - r(t_0)] - W[r(s) - r(t_0)]\right\}. \tag{8}$$

We now note that $W[r(t) - r(t_0)] - W[r(s) - r(t_0)]$ is characterized by a normal distribution with zero mean and variance $r(t) - r(s)$, i.e. $W[r(t) - r(t_0)] - W[r(s) - r(t_0)] \sim \mathcal{N}(0, r(t) - r(s))$. Hence, we can write

$$W[r(t) - r(t_0)] - W[r(s) - r(t_0)] = \sqrt{r(t) - r(s)}\,\xi_{s,t}, \tag{9}$$

with $\xi_{s,t} \sim \mathcal{N}(0, 1)$. Substituting (9) in (8), for $t_0 < s < t$, one has:

$$Y(t) = m(t) + \frac{h_2(t)}{h_2(s)}\left[Y(s) - m(s)\right] + h_2(t)\,\sqrt{r(t) - r(s)}\,\xi_{s,t}. \tag{10}$$

By applying (10) at times $s = t_{k-1}$ and $t = t_k$ for $k = 1, 2, \ldots, n$, we get

$$Y(t_k) = m(t_k) + \frac{h_2(t_k)}{h_2(t_{k-1})}\left[Y(t_{k-1}) - m(t_{k-1})\right] + h_2(t_k)\,\sqrt{r(t_k) - r(t_{k-1})}\,\xi_k, \tag{11}$$

where $\xi_1, \xi_2, \ldots, \xi_n$ is a sequence of independent and identically distributed (*iid*) standard normal random variables due to the independence of the increments of a Brownian motion for non-overlapping intervals.

In particular, by setting $t_k = \tau + kh$, where h is the time discretization step, for the Wiener process, Equation (11) becomes

$$Y(t_k) = Y(t_{k-1}) + \mu h + \sigma \sqrt{h}\,\xi_k \quad (k = 1, 2, \ldots, n), \tag{12}$$

whereas for the Ornstein–Uhlenbeck process Equation (11) is:

$$Y(t_k) = e^{\alpha h}Y(t_{k-1}) + \beta\frac{e^{\alpha h} - 1}{\alpha} + \sigma\sqrt{\frac{e^{2\alpha h} - 1}{2\alpha}}\,\xi_k \quad (k = 1, 2, \ldots, n). \tag{13}$$

The above exact method for simulate $Y(t)$ at the desired times $t_1, t_2, \ldots, t_n$ can be formulated as follows.

Algorithm 2.1. Let $t_0 < t_1 < \ldots < t_n$ be the set of distinct time instants for which the simulation of the conditioned Gauss-Markov process $Y(t)$ is desired.
STEP 1: Set $t_0 = \tau$ and $Y(t_0) = y$;
STEP 2: For $k = 1, 2, \ldots, n$ compute $Y(t_k)$ via (11), with $\xi_1, \xi_2, \ldots, \xi_n \overset{iid}{=} \mathcal{N}(0, 1)$.

In Buonocore et al. (2014), we have used the Algorithm 2.1 to generate sample paths of the membrane potential in a neuronal model based on a time inhomogeneous Ornstein–Uhlenbeck process.

3. Restricted Gauss–Markov processes conditioned on the initial state

In this section, we recall some results obtained in Buonocore et al. (2014); they are the starting point for the development of the simulation techniques considered in the next sections.

We denote by $\{X(t), t \in T\}$ the stochastic process with state space $[v(t), +\infty)$, obtained by considering $Y(t)$ in presence of a reflecting boundary

$$v(t) = m(t) + A\,h_1(t) + B\,h_2(t) \quad (A, B \in \mathbb{R}, \, t \in T). \tag{14}$$

The choice of the real constants A and B depends on the functions $m(t), h_1(t), h_2(t)$ and on the initial state y of the process $Y(t)$ at time τ; since $v(t)$ is a lower reflecting boundary, the real constants A and B are chosen in order to ensure that the starting point y of the process $X(t)$ is greater or equal to the reflecting boundary at the initial time τ, i.e. $y \geq v(\tau)$. As proved in Buonocore et al. (2014), the pdf of the conditioned process $\{X(t), t \in T\}$ is:

$$f_X(x, t) = f_Y(x, t) - \frac{\partial}{\partial x}\left[\exp\left\{-\frac{2A}{h_2(t)}\,[x - v(t)]\right\}P\{Y(t) \leq 2\,v(t) - x\}\right], \quad [x \geq v(t)]. \tag{15}$$

We recall that the normal pdf $f_Y(x, t)$, with mean and variance (2), satisfies the Fokker–Planck equation, with drift and infinitesimal variance

$$A_1(x, t) = m'(t) + [x - m(t)]\,\frac{h_2'(t)}{h_2(t)}, \quad A_2(t) = h_2^2(t)\,r'(t), \tag{16}$$

respectively, and the delta initial condition (cf. Di Nardo et al., 2001). Hence, for $x > v(t)$ and $y > v(\tau)$, the pdf $f_X(x, t)$, given in (15), satisfies the Fokker–Planck equation and the associated initial condition (see, Buonocore et al., 2014)

$$\frac{\partial f_X(x, t)}{\partial t} = -\frac{\partial}{\partial x}[A_1(x, t)f_X(x, t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[A_2(t)f_X(x,t)], \lim_{t \downarrow \tau} f_X(x, t) = \delta(x - y),$$

with the additional requirement that a reflecting condition is imposed on the boundary $v(t)$, i.e.

$$\lim_{x \downarrow v(t)}\left\{A_1(x, t)f_X(x, t) - \frac{A_2(t)}{2}\frac{\partial f_X(x, t)}{\partial x}\right\} - v'(t)f_X[v(t), t] = 0,$$

where the drift $A_1(x, t)$ and the infinitesimal variance $A_2(t)$ are given in (16).

Since $v(t)$ is a reflecting boundary, the total probability mass of $X(t)$ is preserved in $[v(t), +\infty)$, i.e.

$$\int\limits_{v(t)}^{+\infty} f_X(x, t)\, dx = 1 \quad [y \geq v(\tau)].$$

We remark that the boundary (14) includes several types of reflecting boundaries. In particular, for the Wiener process, from (4) and (14) one has

$$v(t) = (\mu + A\,\sigma^2)\,t + B \quad (t \geq \tau) \tag{17}$$

i.e. the reflecting boundary is linear. When $A = 0$, the angular coefficient of the reflection line coincides with the drift μ of the Wiener process. Instead, for the Ornstein–Uhlenbeck process, from (6) and (14) one obtains

$$v(t) = -\frac{\beta}{\alpha} + \left(\frac{\beta}{\alpha} + \frac{A\,\sigma^2}{2\,\alpha} + B\right)e^{\alpha t} - \frac{A\,\sigma^2}{2\,\alpha}e^{-\alpha t} \quad (t \geq \tau), \tag{18}$$

i.e. the reflecting boundary is an hyperbolic function.

4. Simulation of the restricted Gauss–Markov process

In this section, we formulate two different algorithms to simulate the sample paths of the process $X(t)$ originated from y at time τ, in the presence of the reflecting boundary $\nu(t)$, given in (14), with $y \geq \nu(\tau)$.

If $A = 0$, the problem is simple because the principle of reflection can be used (see Borodin & Salminen, 2002 for Brownian motion).

Proposition 4.1 Let

$$\nu(t) = m(t) + B\, h_2(t) \quad (B \in \mathbb{R},\ t \in T). \tag{19}$$

For $t \geq \tau$ and $y \geq \nu(\tau)$, the stochastic process $\{X_s(t),\ t \in T\}$, with

$$X_s(t) = \begin{cases} 2\nu(t) - Y(t), & Y(t) < \nu(t) \\ Y(t), & Y(t) \geq \nu(t) \end{cases} \tag{20}$$

is characterized by pdf:

$$f_{X_s}(x, t) = f_Y(x, t) + f_Y[2\nu(t) - x, t] \quad [x \geq \nu(t)]. \tag{21}$$

Hence, $X_s(t) \overset{d}{=} X(t)$ for all $t \in T$, where $X(t)$ is the Gauss–Markov process conditioned to start to y at time τ and restricted to $[\nu(t), +\infty)$, with the reflecting boundary $\nu(t)$ as in (19).

Proof From (1) and (20) we first note that

$$X_s(\tau) = \begin{cases} 2\,\nu(\tau) - y, & y < \nu(\tau) \\ \\ y, & y \geq \nu(\tau), \end{cases}$$

so that, recalling that $y \geq \nu(\tau)$, it follows $X_s(\tau) = y$. For $x \geq \nu(t)$ and $y \geq \nu(\tau)$, one has

$$\begin{aligned} P\{X_s(t) < x\} &= P\{X_s(t) < x,\ Y(t) < \nu(t)\} + P\{X_s(t) < x, Y(t) \geq \nu(t)\} \\ &= P\{2\nu(t) - x < Y(t) < \nu(t)\} + P\{\nu(t) \leq Y(t) < x\}. \end{aligned} \tag{22}$$

We now assume that $h_2(t) > 0$, so that recalling (1) and (19), from (22) one has:

$$\begin{aligned} P\{X_s(t) < x\} &= P\left\{ B - \frac{y - m(\tau)}{h_2(\tau)} \leq W[r(t) - r(\tau)] < \frac{x - m(t)}{h_2(t)} - \frac{y - m(\tau)}{h_2(\tau)} \right\} \\ &\quad + P\left\{ 2B - \frac{y - m(\tau)}{h_2(\tau)} - \frac{x - m(t)}{h_2(t)} < W[r(t) - r(\tau)] < B - \frac{y - m(\tau)}{h_2(\tau)} \right\}. \end{aligned} \tag{23}$$

We note that $\lim_{x \to +\infty} P\{X_s(t) < x\} = 1$. Furthermore, recalling that

$$P\{a < W[r(t) - r(\tau)] < b\} = \int_a^b \frac{1}{\sqrt{2\pi[r(t) - r(\tau)]}} \exp\left\{ -\frac{z^2}{2[r(t) - r(\tau)]} \right\} dz \tag{24}$$

for $a < b$, from (23) and (24), one has:

$$\begin{aligned} f_{X_s}(x, t) = \frac{dP\{X_s(t) < x\}}{dx} &= \frac{1}{\sqrt{2\pi V(t|\tau)}} \exp\left\{ -\frac{[x - M(t|y, \tau)]^2}{2V(t|\tau)} \right\} \\ &\quad + \frac{1}{\sqrt{2\pi V(t|\tau)}} \exp\left\{ -\frac{[2\nu(t) - x - M(t|y, \tau)]^2}{2V(t|\tau)} \right\} \quad (t \geq \tau), \end{aligned} \tag{25}$$

where $M(t|y, \tau)$ and $V(t|\tau)$ are given in (2). Hence, (21) follows immediately from (25). Proceeding in a similar way for $h_2(t) < 0$, relation (25) still holds. Since the right-hand side of (21) is equal to that of (15) with $A = 0$, then $X_s(t) \overset{d}{=} X(t)$ for all $t \in T$. □

Proposition 4.1 shows that, for all $t \geq \tau$, $X_s(t)$ is identically distributed as $X(t)$ originated at $X(\tau) = y \geq v(\tau)$ in the presence of the reflecting boundary (19). Hence, under the assumptions of Proposition 4.1, a method to simulate $X(t)$ at the desired times $t_1, t_2, \ldots, t_n$ can be expressed as follows.

Algorithm 4.1. Let $t_0 < t_1 < \ldots < t_n$ be the set of distinct time instants for which the simulation of the process $X(t)$ is required, with $\nu(t)$ given in (19).
STEP 1: Set $t_0 = \tau$ and $Y(t_0) = y$;
STEP 2: For $k = 1, 2, \ldots, n$ compute $Y(t_k)$ by means of Algorithm 2.1;
STEP 3: For $k = 1, 2, \ldots, n$ set

$$X_s(t_k) = \begin{cases} 2v(t_k) - Y(t_k), & Y(t_k) < v(t_k) \\ Y(t_k), & Y(t_k) \geq v(t_k). \end{cases} \tag{26}$$

In Buonocore et al. (2014), by means of the Algorithm 4.1, we have generated sample paths of the membrane potential in a neuronal model based on a special time inhomogeneous Ornstein–Uhlenbeck process restricted by a lower reflecting boundary.

In Figure 1, we plot a simulated sample path (green) of the Wiener process $Y(t)$ obtained with the Algorithm 2.1, by choosing $\mu = -0.1$ on the left ($\mu = 0.1$ on the right) and $\sigma^2 = 1$, $y = 3$, $\tau = 0$ in (5). Furthermore, making use of the Algorithm 4.1, we also include in the Figure 1 a simulated sample path (red) of the related restricted process $X(t)$ in $[v(t), +\infty)$, with $v(t) = \mu t + 1.0$ reflecting boundary. In this case, in (17), we set $A = 0$ and $B = 1.0$. The time discretization step is $h = 10^{-3}$. The R code of Figure 1(a) is given in Appendix 1.

Furthermore, in Figure 2, we plot a simulated sample path (green) of the Ornstein–Uhlenbeck process $Y(t)$ obtained with the Algorithm 2.1, by choosing $\alpha = -0.5$ on the left ($\alpha = 0.015$ on the right), $\beta = 1$ and $\sigma^2 = 1$, $y = 3$, $\tau = 0$ in (7). Moreover, by virtue of the Algorithm 4.1, we also include in Figure 2 a simulated sample path (red) of the related restricted process $X(t)$ in $[v(t), +\infty)$, with $v(t) = \beta(e^{\alpha t} - 1)/\alpha$ reflecting boundary, obtained by setting $A = B = 0$ in (18). The time discretization step is $h = 10^{-3}$. The R code of Figure 2(a) is given in Appendix 1.

The Wiener process $Y(t)$ and the related restricted process $X(t)$ do not admit a steady-state density, as highlighted by the behavior of the trajectory in Figure 1. Instead, as it is also evident in Figure 2(a), when $\alpha < 0$, the Ornstein–Uhlenbeck process $Y(t)$ and the related restricted process $X(t)$ admit the following steady-state densities:

Figure 1. Simulated sample paths of the Wiener process $Y(t)$ (green) with $m(t) = \mu t, h_1(t) = \sigma^2 t, h_2(t) = 1, y = 3, \tau = 0$, and of the restricted process $X(t)$ (red) in $[v(t), +\infty)$, with $v(t) = \mu t + 1.0$ reflecting boundary.
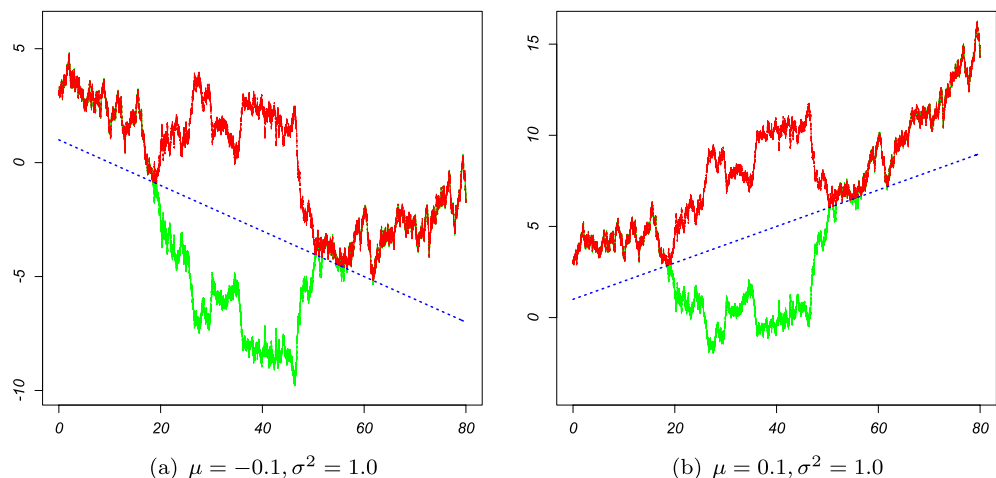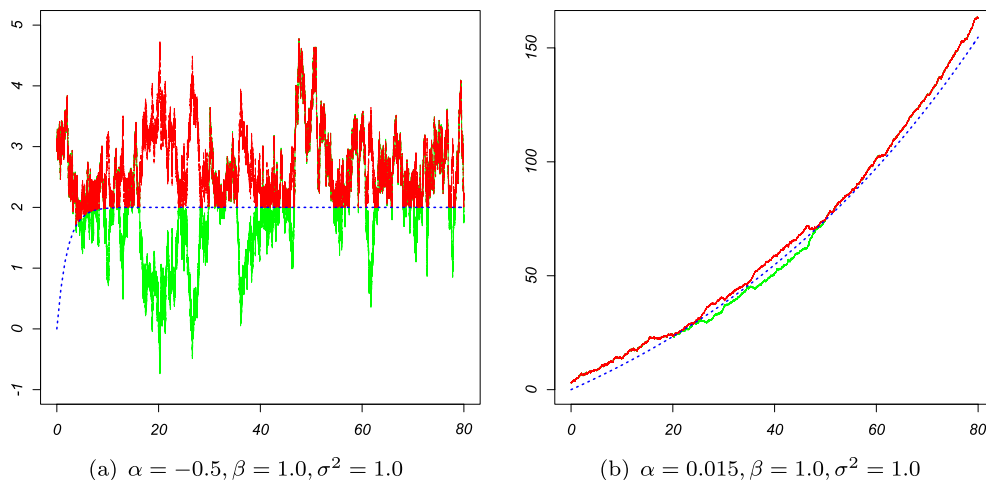

(a) $\mu = -0.1, \sigma^2 = 1.0$

(b) $\mu = 0.1, \sigma^2 = 1.0$

Figure 2. Simulated sample paths of the Ornstein–Uhlenbeck process Y(t) (green) with $m(t) = \beta(e^{\alpha t} - 1)/\alpha$, $h_1(t) = \sigma^2(e^{\alpha t} - e^{-\alpha t})/(2\alpha)$, $h_2(t) = e^{\alpha t}$, $y = 3$, $\tau = 0$, and of the restricted process X(t) (red) in $[v(t), +\infty)$, with $v(t) = \beta(e^{\alpha t} - 1)/\alpha$ reflecting boundary.

(a) $\alpha = -0.5, \beta = 1.0, \sigma^2 = 1.0$ (b) $\alpha = 0.015, \beta = 1.0, \sigma^2 = 1.0$

$$W_Y(x) = \lim_{t \to +\infty} f_Y(x, t) = \sqrt{\frac{|\alpha|}{\sigma^2 \pi}} \exp\left\{ -\frac{|\alpha|}{\sigma^2}\left(x - \frac{\beta}{|\alpha|}\right)^2 \right\} \quad (x \in \mathbb{R}),$$

$$W_X(x) = \lim_{t \to +\infty} f_X(x, t) = 2\,W_Y(x) \quad (x \geq \beta/|\alpha|).$$

Indeed, when $\alpha < 0$, by virtue of (6), from (2) one has $\lim_{t \to +\infty} E[Y(t)] = \beta/|\alpha|$ and $\lim_{t \to +\infty} \text{Var}[Y(t)] = \sigma^2/(2|\alpha|)$, so that $W_Y(x) = \lim_{t \to +\infty} f_Y(x, t)$ is a normal density with mean $\beta/|\alpha|$ and variance $\sigma^2/(2|\alpha|)$. Furthermore, recalling that $A = 0$ and $\alpha < 0$, from (18) one has $\lim_{t \to +\infty} v(t) = \beta/|\alpha|$, so that from (15) it follows $W_X(x) = \lim_{t \to +\infty} f_X(x, t) = \lim_{t \to +\infty}[f_Y(x, t) + f_Y(2v(t) - x, t)] = W_Y(x) + W_Y(2\beta/|\alpha| - x) = 2W_Y(x)$ for $x \geq \beta/|\alpha|$. Note that for the Ornstein–Uhlenbeck process, restricted or not, with $\alpha > 0$, no steady-state densities exist (see Figure 2(b)).

If $A \neq 0$, the problem is more complex since one cannot use, as in Proposition 4.1, the principle of reflection.

Proposition 4.2   We consider the stochastic process $\{X_s(t), t \in T\}$, such that

$$X_s(t) = m(t) + Ah_1(t) + \delta h_2(t)\left[Z[r(t)] + \max\left\{ 0, \widetilde{B} + \sup_{r(\tau) \leq s \leq r(t)} [-Z(s)] \right\} \right] \tag{27}$$

where

$$Z[r(t)] = \widetilde{A}[r(t) - r(\tau)] + \delta\frac{y - m(\tau) - Ah_1(\tau)}{h_2(\tau)} + W[r(t) - r(\tau)]. \tag{28}$$

In (27) and (28), we assume that $\delta = 1$, $\widetilde{A} = -A$, $\widetilde{B} = B$ when $h_2(t) > 0$ for all $t \in T$, whereas $\delta = -1$, $\widetilde{A} = A$, $\widetilde{B} = -B$ when $h_2(t) < 0$ for all $t \in T$.

Hence, for all $t \in T$

$$X_s(t) \stackrel{d}{=} X(t).$$

Proof   We assume that $h_2(t) > 0$ for all $t \in T$, so that $\delta = 1$, $\widetilde{A} = -A$, $\widetilde{B} = B$. Then, making use of (27) and (28), for $x \geq v(t)$ and $y \geq v(\tau)$ one has:

$$P\{X_s(t) < x\}$$

$$= P\left\{ Z[r(t)] + \max\left\{ 0, B + \sup_{r(\tau) \leq s \leq r(t)} [-Z(s)] \right\} < \frac{x - m(t) - Ah_1(t)}{h_2(t)} \right\}, \tag{29}$$

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where $Z(t)$ denotes a Wiener process with drift $-A$ and unitary infinitesimal variance, conditioned on $z_\tau = \{y - m[r^{-1}(\tau)] - Ah_1[r^{-1}(\tau)]\}/h_2[r^{-1}(\tau)]$ at time τ. We now consider the process

$$\hat{Z}(t) = Z(t) + \max\left\{0, B + \sup_{\tau \leq s \leq t}[-Z(s)]\right\} \quad (t \geq \tau), \tag{30}$$

conditioned on $\hat{Z}(\tau) = z_\tau \geq B$ (cf., for instance, Asmussen et al., 1995). For all $t \geq \tau$, the distribution function of $\hat{Z}(t)$ is:

$$P\left\{\hat{Z}(t) < z\right\} = P\left\{Z(t) < z, \sup_{\tau \leq s \leq t}[-Z(s)] \leq -B\right\}$$
$$+ P\left\{Z(t) + B + \sup_{\tau \leq s \leq t}[-Z(s)] < z, \sup_{\tau \leq s \leq t}[-Z(s)] > -B\right\} \quad (z \geq B). \tag{31}$$

We note that

$$P\left\{Z(t) < z, \sup_{\tau \leq s \leq t}[-Z(s)] \leq -B\right\} = P\{Z(t) < z\} - P\left\{Z(t) < z, \sup_{\tau \leq s \leq t}[-Z(s)] > -B\right\}$$
$$= P\{Z(t) < z\} - P\left\{Z(t) < z, \inf_{\tau \leq s \leq t}[Z(s)] < B\right\}$$
$$= P\{Z(t) < z\} - \int_{-\infty}^{B} da \int_{-\infty}^{a} \varphi(a, b, t)\, db - \int_{B}^{z} da \int_{-\infty}^{B} \varphi(a, b, t)\, db,$$

$$P\left\{Z(t) + B + \sup_{\tau \leq s \leq t}[-Z(s)] < z, \sup_{\tau \leq s \leq t}[-Z(s)] > -B\right\}$$
$$= P\left\{Z(t) + B - \inf_{\tau \leq s \leq t}[Z(s)] < z, \inf_{\tau \leq s \leq t}[Z(s)] < B\right\}$$
$$= \int_{-\infty}^{z} da \int_{a-z+B}^{a} \varphi(a, b, t)\, db - \int_{B}^{z} da \int_{B}^{a} \varphi(a, b, t)\, db, \tag{32}$$

where $\varphi(a, b, t)$ is the joint density of $Z(t)$ and $\inf_{\tau \leq s \leq t}[Z(s)]$ (cf., for instance, Borodin & Salminen, 2002):

$$\varphi(a, b, t) = \frac{2(a + z_\tau - 2b)e^{-2A(b-z_\tau)}}{(t-\tau)\sqrt{2\pi(t-\tau)}} \exp\left\{-\frac{[a + z_\tau - 2b + A(t-\tau)]^2}{2(t-\tau)}\right\} \quad (b \leq z_\tau, a \geq b).$$

From (31) and (32), one can derive the pdf of $\hat{Z}(t)$:

$$f_{\hat{Z}}(z, t) = \frac{\partial P\left\{\hat{Z}(t) < z\right\}}{\partial z} = f_Z(z, t) - \int_{-\infty}^{B} \varphi(z, b, t)\, db + \int_{-\infty}^{z} \varphi(a, a - z + B, t)\, da$$

$$= \frac{1}{\sqrt{2\pi(t-\tau)}} \exp\left\{-\frac{[z - z_\tau + A(t-\tau)]^2}{2(t-\tau)}\right\}$$
$$+ \frac{e^{2A(z_\tau - B)}}{\sqrt{2\pi(t-\tau)}} \exp\left\{-\frac{[z + z_\tau - 2B + A(t-\tau)]^2}{2(t-\tau)}\right\}$$
$$+ Ae^{-2A(z-B)}\left[1 - \mathrm{Erf}\left(\frac{z + z_\tau - 2B - A(t-\tau)}{\sqrt{2(t-\tau)}}\right)\right] \quad (z \geq B, z_\tau \geq B). \tag{33}$$

The right-hand side of (33) identifies the transition density of a Wiener process $\widehat{W}(t)$ with drift $-A$, unitary infinitesimal variance, restricted to the interval $[B, +\infty)$ with a reflecting boundary in B, conditioned on $z_\tau \geq B$ at time τ (cf., for instance, Cox & Miller, 1970). The distribution function of $\widehat{W}(t)$ is

$$
\begin{aligned}
P\{\widehat{W}(t) < z\} = \frac{1}{2} &\left[1 + \mathrm{Erf}\left(\frac{z - z_\tau + A(t - \tau)}{\sqrt{2(t - \tau)}} \right) \right] \\
&- \frac{1}{2} e^{-2A(z-B)} \left[1 + \mathrm{Erf}\left(\frac{2B - z + A(t - \tau) - z_\tau}{\sqrt{2(t - \tau)}} \right) \right] \quad (z \geq B).
\end{aligned}
\tag{34}
$$

Now, recalling (34), Equation (29) becomes:

$$
\begin{aligned}
P\{X_s(t) < x\} &= P\left\{ \widehat{Z}[r(t)] < \frac{x - m(t) - Ah_1(t)}{h_2(t)} \right\} \\
&= \frac{1}{2} \left[1 + \mathrm{Erf}\left(\frac{x - M(t|y, \tau)}{\sqrt{2V(t|\tau)}} \right) \right] - \frac{1}{2} \exp\left\{ -\frac{2A[x - v(t)]}{h_2(t)} \right\} \\
&\quad \times \left[1 + \mathrm{Erf}\left(\frac{2v(t) - x - M(t|y, \tau)}{\sqrt{2V(t|\tau)}} \right) \right],
\end{aligned}
\tag{35}
$$

for $x \geq v(t)$ and $y \geq v(\tau)$, with $M(t|y, \tau)$ and $V(t|\tau)$ given in (2). The right-hand side of (35) identifies with the transition distribution function of the conditioned Gauss–Markov process $X(t)$, restricted to $[v(t), +\infty)$ in the presence of the reflecting boundary $v(t)$ as in (14). Hence, for $h_2(t) > 0$, $X_s(t) \overset{d}{=} X(t)$ for all $t \in T$. This completes the proof in the case $h_2(t) > 0$. The proof when $h_2(t) < 0$ for all $t \in T$ follows in a similar way. □

Proposition 4.2 allows to construct an exact algorithm for generating $X_s(t)$ at the specified time instants. To this end, we assume that $h_2(t) > 0$ for all $t \in T$. We set $t = t_k = \tau + kh$ ($k = 0, 1, \ldots, n$; $h > 0$), where h denotes the time discretization step, and let

$$
M_k := \max\left\{ 0, B + \sup_{r(\tau) \leq s \leq r(t_k)} [-Z(s)] \right\} \quad (k = 0, 1, 2, \ldots, n).
\tag{36}
$$

Since $r(t)$ is a non-negative and monotonically increasing function, we note that the following relation holds:

$$
M_k = \max\left\{ M_{k-1}, B + \sup_{r(t_{k-1}) \leq s \leq r(t_k)} [-Z(s)] \right\} \quad (k = 1, 2, \ldots, n).
\tag{37}
$$

By setting $t = t_k = \tau + kh$ in (27), one has:

$$
X_s(t_k) = m(t_k) + Ah_1(t_k) + h_2(t_k) Z[r(t_k)] + h_2(t_k) M_k,
\tag{38}
$$

so that, making use of (37) in (38), one obtains:

$$
X_s(t_k) = \max\left\{ m(t_k) + Ah_1(t_k) + h_2(t_k) Z[r(t_k)] + h_2(t_k) M_{k-1}, S_k \right\},
\tag{39}
$$

where for $k = 1, 2, \ldots, n$ it results:

$$
S_k = m(t_k) + Ah_1(t_k) + h_2(t_k) Z[r(t_k)] + h_2(t_k) \left\{ B + \sup_{r(t_{k-1}) \leq s \leq r(t_k)} [-Z(s)] \right\}.
\tag{40}
$$

By virtue of (38), M_{k-1} can be expressed in terms of $X_s(t_{k-1})$ as:

$$
M_{k-1} = \frac{X_s(t_{k-1}) - m(t_{k-1}) - Ah_1(t_{k-1}) - h_2(t_{k-1}) Z[r(t_{k-1})]}{h_2(t_{k-1})}
\tag{41}
$$

for $k = 1, 2, \ldots, n$. Substituting (28) with $t = t_{k-1}$, $\delta = 1$ and $\widetilde{A} = -A$ in (41), the first term of the maximum in (39) becomes:

$$
\begin{aligned}
m(t_k) &+ A h_1(t_k) + h_2(t_k) Z[r(t_k)] + h_2(t_k) M_{k-1} \\
&= \frac{h_2(t_k)}{h_2(t_{k-1})} X_s(t_{k-1}) + m(t_k) - \frac{h_2(t_k)}{h_2(t_{k-1})} m(t_{k-1}) \\
&\quad + h_2(t_k)\{W[r(t_k) - r(t_0)] - W[r(t_{k-1}) - r(t_0)]\}.
\end{aligned}
\tag{42}
$$

Now we want to simulate the random variable S_k that appears in (40). Let $Z_1(t) = -Z(t)$ for all $t \in T$, with $Z(t)$ given in (28). Hence,

$$
\begin{aligned}
P\{S_k &\leq x | Z_1[r(t_k)] = z\} \\
&= P\left\{ \sup_{r(t_{k-1}) \leq s \leq r(t_k)} [Z_1(s)] \leq \frac{x + h_2(t_k) z - v(t_k)}{h_2(t_k)} \Big| Z_1[r(t_k)] = z \right\}.
\end{aligned}
\tag{43}
$$

We now use the well-known result for the supremum of the Wiener process (cf. Borodin & Salminen, 2002) with drift μ, infinitesimal variance σ^2, starting from z_0 at time ϑ_0:

$$
P\left\{ \sup_{\vartheta_0 \leq s \leq \vartheta} Z(s) \leq b | Z(\vartheta) = a \right\} = 1 - \exp\left\{ -\frac{2(b-a)(b-z_0)}{\sigma^2(\vartheta - \vartheta_0)} \right\} \quad (a \leq b,\ b \geq z_0),
\tag{44}
$$

i.e. an exponential distribution that does not depend on μ. Making use of (44) with $\mu = A$, $\sigma^2 = 1$, $a = z$, $b = [x + h_2(t_k) z - v(t_k)]/h_2(t_k)$, $z_0 = Z_1[r(t_{k-1})]$, $\vartheta = r(t_k)$ and $\vartheta_0 = r(t_{k-1})$, from (43) one obtains:

$$
P\{S_k \leq x | Z_1[r(t_k)] = z\} = 1 - \exp\left\{ -2 \frac{\frac{x - v(t_k)}{h_2(t_k)} \left(\frac{x + h_2(t_k) z - v(t_k)}{h_2(t_k)} - z_0 \right)}{r(t_k) - r(t_{k-1})} \right\}
\tag{45}
$$

for $x \geq v(t_k)$ and $b \geq z_0$. Let $\widetilde{S}_k$ be the random variable obtained by conditioning S_k with the requirements that $Z_1[r(t_{k-1})] = z_0$ and $Z_1[r(t_k)] = z$. Hence, applying the inverse transform algorithm, we set

$$
1 - \exp\left\{ -\frac{2[\widetilde{S}_k - v(t_k)][\widetilde{S}_k + h_2(t_k) z - v(t_k) - z_0 h_2(t_k)]}{h_2^2(t_k)[r(t_k) - r(t_{k-1})]} \right\} = U_k,
$$

with $U_k \sim \mathcal{U}(0, 1)$, and solve for $\widetilde{S}_k$ in terms of random variable U_k. Hence,

$$
\widetilde{S}_k = v(t_k) + h_2(t_k) \frac{-(z - z_0) \pm \sqrt{(z - z_0)^2 - 2[r(t_k) - r(t_{k-1})] \ln(1 - U_k)}}{2}.
\tag{46}
$$

Since we have imposed that $Z_1[r(t_{k-1})] = z_0$ and $Z_1[r(t_k)] = z$, by virtue of (28), the value $-(z - z_0)$ in (46) can be obtained by simulation of the following random variable

$$
\begin{aligned}
Z_1[r(t_{k-1})] - Z_1[r(t_k)] &= Z[r(t_k)] - Z[r(t_{k-1})] \\
&= -A[r(t_k) - r(t_{k-1})] + W[r(t_k) - r(\tau)] - W[r(t_{k-1}) - r(\tau)] \\
&= \sqrt{r(t_k) - r(t_{k-1})} \left[-A\sqrt{r(t_k) - r(t_{k-1})} + \xi_k \right],
\end{aligned}
\tag{47}
$$

where the last identity follows from (9), with $\xi_k \sim \mathcal{N}(0, 1)$. Hence, making use of (46) and (47), the random variable S_k in (40) can be thus simulated as

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$$S_k = v(t_k) + h_2(t_k)\sqrt{r(t_k) - r(t_{k-1})}$$

$$\times \frac{-A\sqrt{r(t_k) - r(t_{k-1})} + \xi_k \pm \sqrt{\left[-A\sqrt{r(t_k) - r(t_{k-1})} + \xi_k\right]^2 - 2\ln(1 - U_k)}}{2}. \tag{48}$$

Since $h_2(t) > 0$ for all $t \in T$, it ensures that $S_k \geq v(t_k)$ is necessary to take the positive square root in (48). The case $h_2(t) < 0$ for all $t \in T$ follows in a similar way. Then, the following algorithm can be implemented:

Algorithm 4.2. Let $t_0 < t_1 < \ldots < t_n$, with $t_k = t_0 + k\,h$ $(h > 0, k = 1, 2, \ldots, n)$, be the set of equidistant time instants for which the simulation of $X(t)$ is required, with $\nu(t)$ given in (14).
STEP 1: Set $t_0 = \tau$ and $X_s(t_0) = y$;
STEP 2: For $k = 1, 2, \ldots, n$ set

$$X_s(t_k) = \max\left\{ \frac{h_2(t_k)}{h_2(t_{k-1})}X_s(t_{k-1}) + m(t_k) - \frac{h_2(t_k)}{h_2(t_{k-1})}m(t_{k-1}) \right.$$
$$\left. + \delta\, h_2(t_k)\sqrt{r(t_k) - r(t_{k-1})}\,\xi_k,\ v(t_k) + \delta\, h_2(t_k)\sqrt{r(t_k) - r(t_{k-1})}H_k \right\}, \tag{49}$$

where

$$H_k = \frac{\widetilde{A}\sqrt{r(t_k) - r(t_{k-1})} + \xi_k + \sqrt{\left[\widetilde{A}\sqrt{r(t_k) - r(t_{k-1})} + \xi_k\right]^2 - 2\ln(1 - U_k)}}{2}, \tag{50}$$

with $U_1, U_2, \ldots, U_n \overset{iid}{=} \mathcal{U}(0, 1)$ and $\xi_1, \xi_2, \ldots, \xi_n \overset{iid}{=} \mathcal{N}(0, 1)$. In (49) and (50), we assume that $\delta = 1, \widetilde{A} = -A$ when $h_2(t) > 0$ for all $t \in T$, whereas $\delta = -1, \widetilde{A} = A$ when $h_2(t) < 0$ for all $t \in T$.

Note that, by virtue of (9), the first term of the maximum in (49) follows from (42). Furthermore, the second term of the maximum in (49) follows from (48).

We remark that the Algorithm 4.2 is a generalization to restricted Gauss–Markov processes of the simulation algorithm given in Asmussen et al. (1995) and Kroese et al. (2011) for a Brownian motion with a negative drift and unit infinitesimal variance in the presence of a reflecting boundary in zero state.

In Figure 3, we plot a simulated sample path of the Wiener process $Y(t)$ obtained with Algorithm 2.1, by choosing $\mu = -1.0, \sigma^2 = 1.0, y = 3, \tau = 0$ in (5), and a simulated sample path of the related restricted process $X(t)$ in $[v(t), +\infty)$, with $v(t)$ given in in (17). In Figure 3(a), we set $A = 0.5$ and $B = 1.0$, so that from (17) it follows that $v(t) = -0.5\,t + 1.0$, whereas in Figure 3(b) we set $A = 1.5$ and $B = 1.0$, so that from (17) one has $v(t) = 0.5\,t + 1.0$. The sample path of $X(t)$ is obtained with the Algorithm 4.2. The time discretization step is $h = 10^{-3}$. The R code of Figure 3(a) is given in Appendix 1.

Furthermore, in Figure 4, we plot a simulated sample path of the Ornstein–Uhlenbeck process $Y(t)$ achieved with Algorithm by choosing $\alpha = -0.5, \beta = 1.0, \sigma^2 = 1.0, y = 3, \tau = 0$ in (7), and a simulated sample path of the related restricted process $X(t)$ in $[v(t), +\infty)$, with the reflecting boundary $v(t)$ given in (18). In Figure 4(a), we set $A = 0.35$ and $B = 1.35$, so that from (18) it follows that $v(t) = 2 - e^{-t/2} + 0.35\,e^{t/2}$; instead, in Figure 4(b) we set $A = -0.01$ and $B = 2.19$, so that from (18) one has $v(t) = 2 + 0.2\,e^{-t/2} - 0.01\,e^{t/2}$. The sample path of $X(t)$ is obtained via the Algorithm 4.2. The time discretization step is $h = 10^{-3}$. The R code of Figure 4(a) is given in Appendix 1.

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Figure 3. Simulated sample paths of the Wiener process *Y(t)* (green) given in (5), with $\mu = -1.0, \sigma^2 = 1.0, y = 3$, $\tau = 0$, and of the restricted process *X(t)* (red) in $[\nu(t), +\infty)$, with $\nu(t) = a\,t + 1.0$ reflecting boundary.
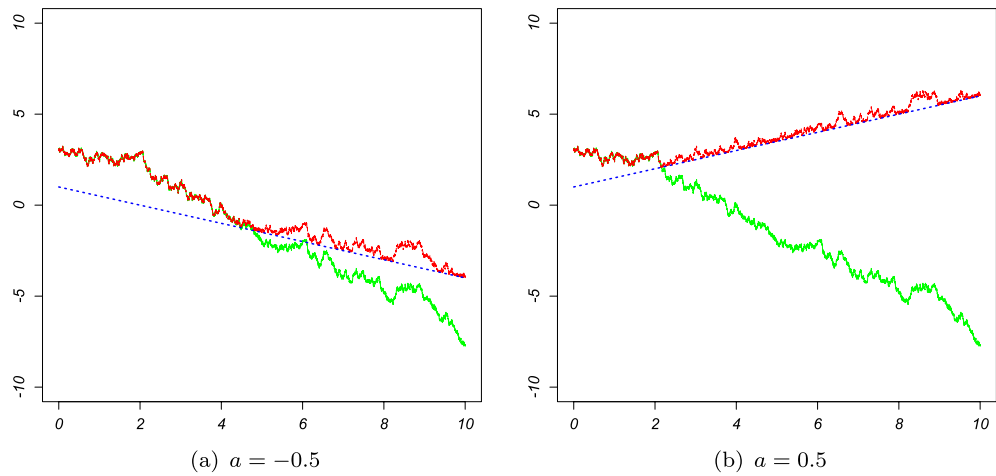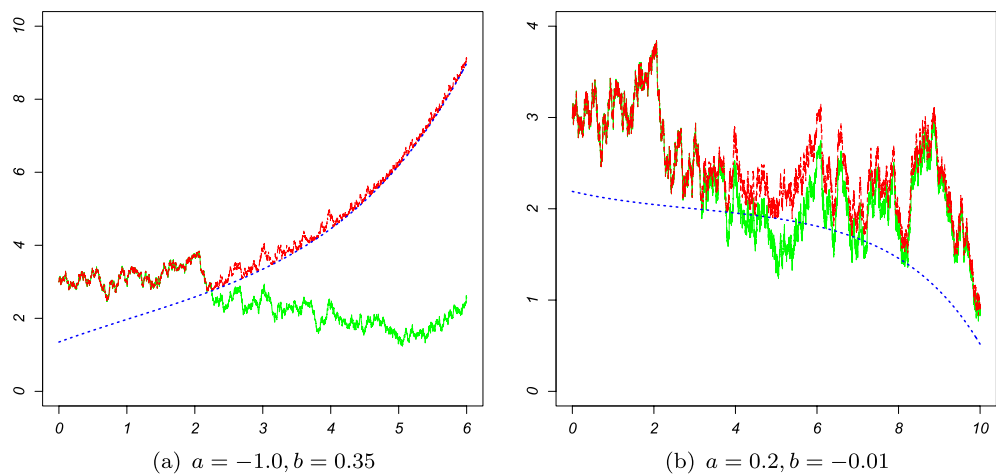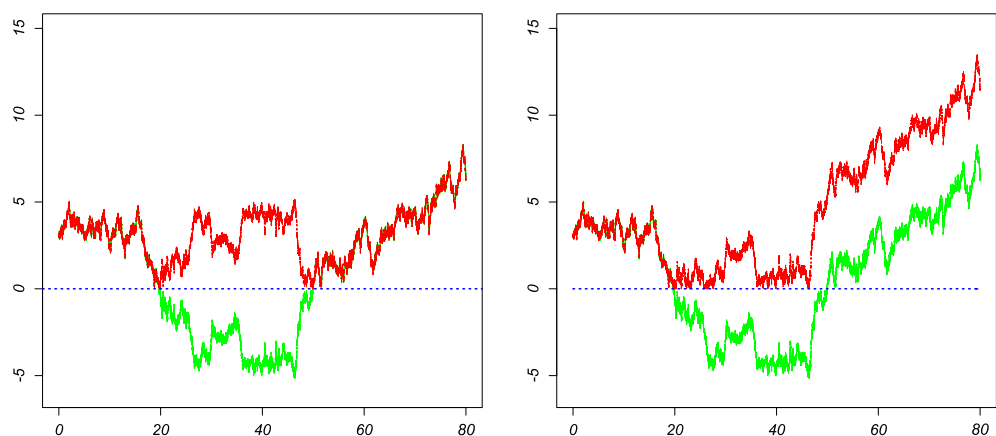

(a) $a = -0.5$

(b) $a = 0.5$

Figure 4. Simulated sample paths of the Ornstein–Uhlenbeck process *Y(t)* (green) given in (7) with $\alpha = -0.5, \beta = 1.0, \sigma^2 = 1.0, y = 3$, $\tau = 0$, and of the restricted process *X(t)* (red) in $[\nu(t), +\infty)$, with the reflecting boundary $\nu(t) = a\,e^{-t/2} + b\,e^{t/2} + 2$.



(a) $a = -1.0, b = 0.35$

(b) $a = 0.2, b = -0.01$

Figure 5. Simulated sample paths of the Wiener process *Y(t)* (green) given in (5) with $\mu = 0, \sigma^2 = 1.0, y = 3, \tau = 0$, and of the restricted process *X(t)* (red) in $[0, +\infty)$, with the reflecting boundary $\nu(t) = 0$, using Algorithm 4.1 on the left and Algorithm 4.2 on the right.

In Figures 5 and 6, we show that, starting from the same realization of the Wiener and Ornstein–Uhlenbeck processes *Y(t)* (green), respectively, Algorithms 4.1 and 4.2 provide for $A = 0$ and $B = 0$ different trajectories for the processes *X(t)* (red). However, as proved in Propositions 4.1 and 4.2, the simulated restricted processes are characterized by the same pdf. We finally note that for $A = 0$, the Algorithm 4.2 has an higher computational cost than that of the Algorithm 4.1. Specifically, for the comparison of different numerical or simulation methods, the analysis of their accuracy as function

Figure 6. Simulated sample paths of the Ornstein–Uhlenbeck process Y(t) (green) given in (7) with $\alpha = -0.1$, $\beta = 0.0$, $\sigma^2 = 1.0$, $y = 3$, $\tau = 0$, and of the restricted process X(t) (red) in $[0, +\infty)$, with the reflecting boundary $v(t) = 0$, using Algorithm 4.1 on the left and Algorithm 4.2 on the right.
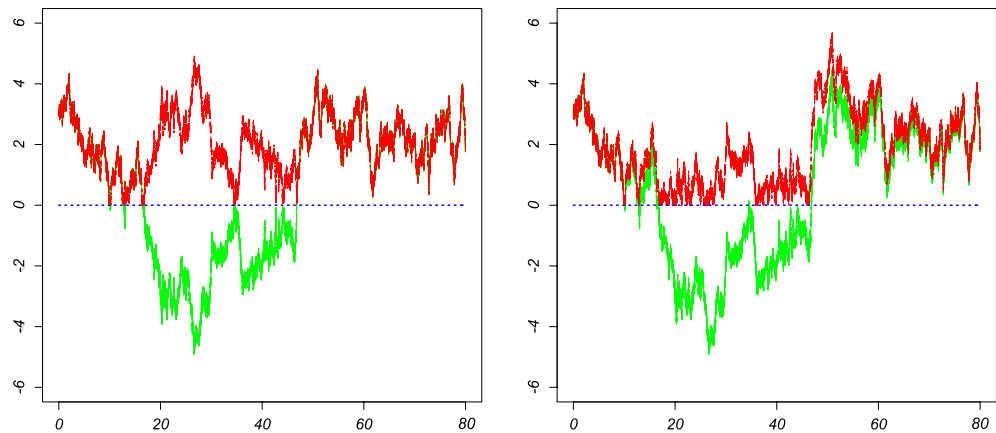
of the discretization step, as well as the study of their numerical complexity, we refer to the books (Asmussen & Glynn, 2007; Kloeden & Platen, 1999; Kroese et al., 2011).

5. First passage time simulation of restricted Gauss–Markov process

In this section, we consider the first passage time problem for the Gauss–Markov process and for the related restricted process.

5.1. FPT simulation of the conditioned Gauss–Markov processes

For the process Y(t), the FPT problem through a boundary $S(t) \in C^1(T)$ can be considered. Let

$$\mathcal{T}_Y = \inf_{t \geq \tau} \{t : Y(t) > S(t)\}, \quad Y(\tau) = y < S(\tau), \quad \tau, t \in T \tag{51}$$

be the random variable that denotes the FPT of the Gauss–Markov process Y(t) from $Y(\tau) = y$ to the continuous boundary S(t). Let

$$g_Y[S(t), t] = \frac{\partial}{\partial t} P(\mathcal{T}_Y < t) \tag{52}$$

denotes the FPT pdf. In order to simulate $g_Y[S(t), t]$, we generate sample paths of the Gauss–Markov process Y(t) using the Algorithm 2.1. Then, the method for the simulation of FPT for the process Y(t) through S(t) can be made as follows:

Algorithm 5.1. Let $t_0 < t_1 < \ldots < t_n$ be the set of distinct time instants for which the simulation of the conditioned Gauss-Markov process $Y(t)$ is desired.
STEP 1: Set $t_0 = \tau$, $Y(t_0) = y$, $k = 1$;
STEP 2: Generate $\xi_k \sim \mathcal{N}(0, 1)$ and compute $Y(t_k)$ via Algorithm 2.1;
STEP 3: If $Y(t_k) \geq S(t_k)$ collect t_k as a first passage time and stop, else $k \leftarrow k + 1$ and go to Step 2.

By implementing the previous procedure for *N* times, one obtains a collection of *N* simulated first passage times of Y(t) through S(t). Hence, an estimation of the FPT pdf can be achieved by the histogram of such first passage times.

5.2. FPT simulation of the restricted Gauss–Markov process

For the process X(t), restricted to $[v(t), +\infty)$ with the reflecting boundary $v(t)$ given in (14), the FPT problem through a boundary $S(t) > v(t)$ can be considered. To this purpose, let S(t) be a $C^1(T)$-class function, such that $S(t) > v(t)$ for all $t \in T$. For $y \geq v(\tau)$, we denote by

$$\mathcal{T}_X = \inf_{t \geq \tau} \{t : X(t) > S(t)\}, \quad v(\tau) \leq X(\tau) = y < S(\tau), \quad \tau, t \in T \tag{53}$$

the random variable FPT of $X(t)$ from $X(\tau) = y$ to the boundary $S(t)$. Further, let

$$g_X[S(t), t] = \frac{\partial}{\partial t} P(\mathcal{T}_X \le t) \quad [v(\tau) \le y < S(\tau)] \tag{54}$$

be the FPT pdf.

As proved in Buonocore et al. (2014), if $v(t) = m(t) + B\,h_2(t)$ and $S(t) = m(t) + D\,h_2(t)$, with $B, D \in \mathbb{R}$ and $(D - B)\,h_2(\tau) > 0$, one has

$$
\begin{aligned}
g_X[S(t), t] = {} & \frac{S(\tau) - y}{r(t) - r(\tau)} \frac{h_2(t)}{h_2(\tau)} \frac{dr(t)}{dt} f_Y[S(t), t] \\
& \times \left\{ 1 + 2 \sum_{n=1}^{+\infty} (-1)^n \exp\left\{ -\frac{2(D - B)^2 n^2}{r(t) - r(\tau)} \right\} \left[\cosh\left(\frac{2\,n(D - B)}{r(t) - r(\tau)} \frac{S(\tau) - y}{h_2(\tau)} \right) \right. \right. \\
& \left. \left. - \frac{2\,n\,(D - B)h_2(\tau)}{S(\tau) - y} \sinh\left(\frac{2\,n(D - B)}{r(t) - r(\tau)} \frac{S(\tau) - y}{h_2(\tau)} \right) \right] \right\} \quad [v(\tau) \le y < S(\tau)],
\end{aligned}
\tag{55}
$$

where $f_Y(x, t)$ is the pdf of $Y(t)$. Furthermore, if $A_1(x, t)$, $A_2(t)$, $v(t)$, and $S(t)$ are not depend on time t, then the FPT moments through a constant boundary for the process $X(t)$ can be obtained making use of the Siegert formula (cf. Ricciardi, Di Crescenzo, Giorno, & Nobile, 1999; Siegert, 1951). Indeed, setting $A_1(x, t) \equiv A_1(x)$, $A_2(t) \equiv \sigma^2$, $v(t) = v$, and $S(t) = S(v \le y < S)$, one has:

$$t_n(S|y) := \int_0^{+\infty} t^n\, g_X(S, t|y, 0)\, dt = n \int_y^S dz\, h(z) \int_v^z s(u)\, t_{n-1}(S|u)\, du \tag{56}$$

$$(n = 1, 2, \ldots)$$

with $t_0(S|y) = 1$, and where

$$h(x) = \exp\left\{ -\frac{2}{\sigma^2} \int^x A_1(z)\, dz \right\}, \qquad s(x) = \frac{2\,h^{-1}(x)}{\sigma^2}.$$

Explicit expressions of moments for the Ornstein–Uhlenbeck process with a constant reflecting boundary are given in Inoue, Sato, and Ricciardi (1997).

The sample paths of the restricted process $X(t)$ are constructed taking into account the presence of the reflecting boundary $v(t)$ given in (14), using one of the algorithms described in Section 4. Then, the method for the simulation of FPT for the restricted process $X(t)$ through $S(t)$ can be carried out as follows:

Algorithm 5.2. Let $t_0 < t_1 < \ldots < t_n$ be the set of distinct time instants for which the simulation of the conditioned Gauss-Markov process $Y(t)$ is desired.
STEP 1: Set $t_0 = \tau$, $Y(t_0) = X(t_0) = y$, $k = 1$;
STEP 2: Generate $\xi_k \sim \mathcal{N}(0, 1)$ and

- compute $Y(t_k)$ by means of Algorithm 2.1;
- compute $X(t_k)$ by means of the Algorithm 4.1 when $A = 0$ and of the Algorithm 4.2 when $A \ne 0$.

STEP 3: If $X(t_k) \ge S(t_k)$ collect t_k as a first passage time and stop, else $k \leftarrow k + 1$ and go to Step 2.

By implementing the previous procedure for *N* times, one obtains a collection of *N* simulated first passage times $X(t)$ through $S(t)$. Hence, an estimation of the FPT pdf can be achieved by the histogram of such first passage times.

Figure 7. Histograms of FPT pdf through the boundary $S = 4.5$ **for the Wiener process** $X(t)$ **with** $\mu = 0, \sigma^2 = 1.0, y = 3.0, \tau = 0$ **restricted to** $[0, +\infty)$**, with the reflecting boundary in zero, simulated using Algorithm 4.1 on the left and Algorithm 4.2 on the right. The exact FPT density (red curve), obtained via (55), is superimposed on the histograms.**
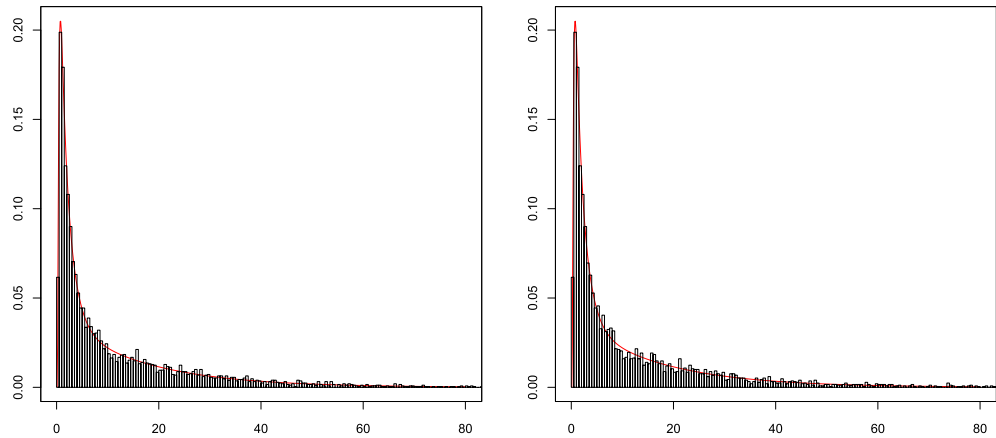

Table 1. For the restricted Wiener process $X(t)$ of Figure 7, the values of $t_1(S|y)$, $V(S|y)$, and $C(S|y)$ are compared with the related simulated estimated values

|	$t_1(S|y)$	$V(S|y)$	$C(S|y)$
Exact values	11.25	219.375	1.31656
Algorithm 4.1	11.34716	214.6468	1.291145
Algorithm 4.2	11.54748	230.397	1.314471

For the restricted Wiener process $X(t)$, with $\mu = 0, \sigma^2 = 1.0, y = 3.0, \tau = 0$, in Figure 7 we plot the histograms of a collection of $N = 5000$ simulated FPT through the boundary $S = 4.5$, obtained making use of Algorithm 4.1 on the left and of Algorithm 4.2 on the right; in this case, we set $A = 0$ and $B = 0$ in (17). In Figure 7, the histograms of the FPT densities are compared with the exact FPT density (red curve), obtained via (55). It appears from Figure 7 that the FPT histograms fit the exact FPT density reasonably well. Furthermore, in Table 1, we compare the FPT mean $t_1(S|y)$, the FPT variance $V(S|y) = t_2(S|y) - [t_1(S|y)]^2$, and the FPT coefficient of variation $C(S|y) = \sqrt{V(S|y)}/t_1(S|y)$, derived from (56), with the sample mean, variance, and coefficient of variation obtained from the collection of $N = 5000$ simulated FPT of Figure 7. The R codes of Figure 7(a) and (b) are given in Appendix 1.

For the restricted Ornstein–Uhlenbeck process $X(t)$, with $\alpha = -0.1, \beta = 0.0, \sigma^2 = 1.0, y = 3.0, \tau = 0$, in Figure 8 we plot the histograms of the FPT pdf through the boundary $S = 4.5$ obtained making use of Algorithm 4.1 on the left and of Algorithm 4.2 on the right; furthermore, note that $A = B = 0$ in (18). We have considered a collection of $N = 5000$ simulated FPT. In this case, there is no closed-form expression for the FPT density and the simulation tool is necessary. Furthermore, the FPT mean $t_1(S|y)$, the FPT variance $V(S|y)$, and the FPT coefficient of variation $C(S|y)$ can be obtained from (56). Hence, in Table 2 are reported $t_1(S|y)$, $V(S|y)$, and $C(S|y)$ and the related sample values, obtained from the collection of $N = 5000$ simulated FPT of Figure 8.

Table 2. For the restricted Ornstein–Uhlenbeck process $X(t)$ of Figure 8, the exact values of $t_1(S|y)$, $V(S|y)$, and $C(S|y)$ are compared with the related simulated estimated values

	$t_1(S|y)$	$V(S|y)$	$C(S|y)$
Exact values	33.6361	1670.25	1.21502
Algorithm 4.1	34.68059	1785.15	1.21829
Algorithm 4.2	33.47736	1577.127	1.186266

Figure 8. Histograms of FPT pdf through the boundary $S = 4.5$ **for the Ornstein–Uhlenbeck process** $X(t)$ **with** $\alpha = -0.1$, $\beta = 0.0$, $\sigma^2 = 1.0$, $y = 3.0$, $\tau = 0$ **restricted to** $[0, +\infty)$, **with the reflecting boundary in zero, simulated using Algorithm 4.1 on the left and Algorithm 4.2 on the right.**
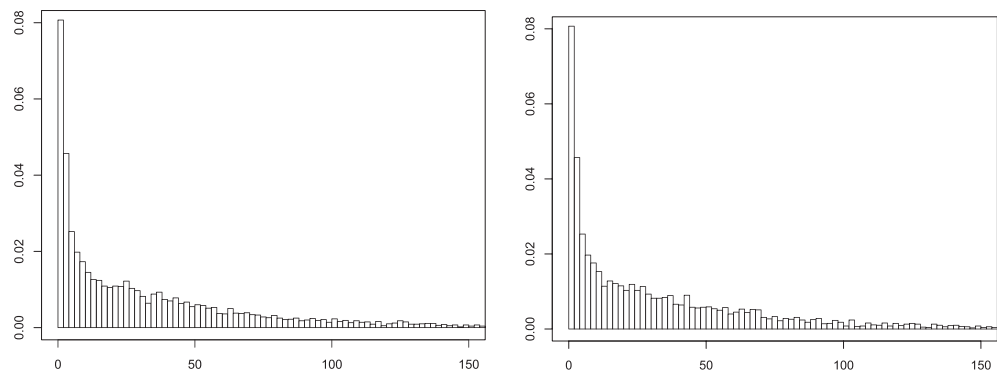
6. Concluding remarks

In conclusion, we propose algorithms for the simulation of the Gauss–Markov processes conditioned to start to y at time τ and restricted to $[\nu(t), +\infty]$ with the time-dependent lower reflecting boundary $\nu(t) = m(t) + A\,h_1(t) + B\,h_2(t)$, where the real constants A and B are chosen in order to ensure that $y \geq \nu(\tau)$. Specifically, we formulate two algorithms to simulate the sample paths of the process $X(t)$ for both cases $A = 0$ and $A \neq 0$. When $A = 0$, the sample paths of the restricted Gauss–Markov processes are obtained using a reflection principle type (see Algorithm 4.1), whereas if $A \neq 0$, the sample paths are constructed via Algorithm 4.2. Finally, we use the algorithms proposed to construct the histogram of FPT density for the restricted process. Particular attention has been dedicated to restricted Wiener and Ornstein–Uhlenbeck processes for their central role in the class of Gauss–Markov processes.

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Appendix 1

We now list the R codes used in the previous sections to generate the sample paths and to simulate the first passage times for Wiener and Ornstein–Uhlenbeck processes.

```
> # R code of Figure 1(a)  - Algorithm 4.1 - Wiener process
> tau=0.0
> n<-8*10^4
> h<-10^{-3}
> t<-seq(tau,tau+n*h,h)
> mu<--0.1
> sigma<-sqrt(1.0)
> a<--0.1
> b<-1.0
> Y<-numeric(n+1)
> X<-numeric(n+1)
> set.seed(123)
> Z<-rnorm(n+1)
> Y[1]<-3.0
> X[1]<-3.0
> r<-n+1
> for(k in 2:r){
+ Y[k]<-Y[k-1]+mu*h+sigma*sqrt(h)*Z[k]
+ X[k]<-max(Y[k],2*(a*(tau+k*h)+b)-Y[k])
+ }
> matplot(t,data.frame(Y,X),type="l",xlim=c(0,n*h),ylim=c(-10,6),
+ xlab="␣",ylab="␣",col=c("green","red"),cex.axis=1.2,font.axis=3)
> curve(a*x+b,lty=3,col="blue",lwd=2,add=TRUE)
```

```
> # R code of Figure 2(a)  - Algorithm 4.1 - Ornstein-Uhlenbeck process
> tau=0.0
> n<-8*10^4
> h<-10^{-3}
> t<-seq(tau,tau+n*h,h)
> alpha<--0.5
> beta<-1.0
> sigma<-sqrt(1.0)
> Y<-numeric(n+1)
> X<-numeric(n+1)
> set.seed(123)
> Z<-rnorm(n+1)
> Y[1]<-3.0
> X[1]<-3.0
> r<-n+1
> for(k in 2:r){
+ Y[k]<-exp(alpha*h)*Y[k-1]+(beta/alpha)*(exp(alpha*h)-1)+
+ sigma*sqrt((exp(2*alpha*h)-1)/(2*alpha))*Z[k]
+ X[k]<-max(Y[k],2*(beta/alpha)*(exp(alpha*k*h)-1)-Y[k])
+ }
> matplot(t,data.frame(Y,X),type="l",xlim=c(0,n*h),ylim=c(-1,5),xlab="␣",
+ ylab="␣",col=c("green","red"),cex.axis=1.2,font.axis=3)
> curve((beta/alpha)*(exp(alpha*x)-1),lty=3,col="blue",lwd=2,add=TRUE)
```

```
> # R code of Figure 3(a)  - Algorithm 4.2 - Wiener process
> tau=0.0
> n<-10^4
> h<-10^{-3}
> t<-seq(tau,tau+n*h,h)
> mu<--1.0
> sigma<-sqrt(1.0)
> a<--0.5
> b<-1.0
> Ag<-(a-mu)/sigma^2
> Y<-numeric(n+1)
> X<-numeric(n+1)
> R<-numeric(n+1)
> H<-numeric(n+1)
> M<-numeric(n+1)
> set.seed(123)
> Z<-rnorm(n+1)
> set.seed(124)
> U<-runif(n+1)
> Y[1]<-3.0
> X[1]<-3.0
> r<-n+1
> for(k in 2:r){
+ Y[k]<-Y[k-1]+mu*h+sigma*sqrt(h)*Z[k]
+ R[k]<--Ag*sigma*sqrt(h)+Z[k]
+ H[k]<-(R[k]+sqrt((R[k])^2-2*log(1-U[k])))/2.0
+ X[k]<-max(X[k-1]+mu*h+sigma*sqrt(h)*Z[k],a*(tau+k*h)+b+sigma*sqrt(h)*H[k])
+ }
> matplot(t,data.frame(Y,X),type="l",xlim=c(0,n*h),ylim=c(-10,10),xlab="␣",
+ ylab="␣",col=c("green","red"),cex.axis=1.2,font.axis=3)
> curve(a*x+b,lty=3,col="blue",lwd=2,add=TRUE)
```

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```
> # R code of Figure 4(a)  - Algorithm 4.2 - Ornstein-Uhlenbeck process
> tau=0.0
> n<-6*10^3
> h<-10^{-3}
> t<-seq(tau,tau+n*h,h)
> alpha<--0.5
> beta<-1.0
> sigma<-sqrt(1.0)
> a<--1.0
> b<-0.35
> c<--beta/alpha
> Ag<--2.0*alpha*b/sigma^2
> Bg<-a+b+c
> Y<-numeric(n+1)
> X<-numeric(n+1)
> R<-numeric(n+1)
> H<-numeric(n+1)
> M<-numeric(n+1)
> set.seed(123)
> Z<-rnorm(n+1)
> set.seed(124)
> U<-runif(n+1)
> Y[1]<-3.0
> X[1]<-3.0
> r<-n+1
> for(k in 2:r){
+ Y[k]<-exp(alpha*h)*Y[k-1]+(beta/alpha)*(exp(alpha*h)-1)+
+ sigma*sqrt((exp(2*alpha*h)-1)/(2*alpha))*Z[k]
+ R[k]<--Ag*sigma*sqrt((exp(-2*alpha*(k-1)*h)-exp(-2*alpha*k*h))/(2*alpha))+
+ Z[k]
+ H[k]<-(R[k]+sqrt((R[k])^2-2*log(1-U[k])))/2.0
+ X[k]<-max(exp(alpha*h)*X[k-1]+beta*(exp(alpha*h)-1)/alpha+
+ sigma*sqrt((exp(2*alpha*h)-1)/(2*alpha))*
+ Z[k],a*exp(alpha*k*h)+b*exp(-alpha*k*h)+c+
+ sigma*sqrt((exp(2*alpha*h)-1)/(2*alpha))*H[k])
+ }
> matplot(t,data.frame(Y,X),type="l",xlim=c(0,n*h),ylim=c(0,10),xlab="␣",
+ ylab="␣",col=c("green","red"),cex.axis=1.2,font.axis=3)
> curve(a*exp(alpha*x)+b*exp(-alpha*x)+c,lty=3,col="blue",lwd=2,add=TRUE)
```

```
> # R code of Figure 7(a)  - Algorithm 5.1 - FPT for Wiener process
> S<- 4.5
> size<-5000
> tk<-0
> n<-10^6
> h<-10^{-3}
> t<-seq(0,n*h,h)
> mu<-0.
> sigma<-sqrt(1)
> nu<-0.
> Y<-numeric(n+1)
> X<-numeric(n+1)
> tpWPR<-numeric(size)
> Y[1]<-3.0
> X[1]<-3.0
> for(i in 1:size){
+ set.seed(i)
+ Z<-rnorm(n+1)
+ k<-1
+ while(X[k]<S){
+ k<-k+1
+ Y[k]<-Y[k-1]+sigma*sqrt(h)*Z[k]
+ X[k]<-max(Y[k],2*nu-Y[k])
+ }
+ tk<-tk+1
+ tpWPR[tk]<-k*h
+ }
> hist(tpWPR,freq=F,ylim=c(0,0.20), xlim=c(0,80), main="␣", breaks=200,
+ xlab="␣", ylab="␣")
> box()
```

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```
> # R code of Figure 7(b)  - Algorithm 5.2 - FPT for Wiener process
> S<- 4.5
> size<-5000
> tk<-0
> tpWNPR<-numeric(size)
> n<-10^6
> h<-10^{-3}
> t<-seq(0,n*h,h)
> mu<-0.
> sigma<-sqrt(1.0)
> a<-0.
> b<-0.
> Ag<-(a-mu)/sigma^2
> Y<-numeric(n+1)
> X<-numeric(n+1)
> R<-numeric(n+1)
> H<-numeric(n+1)
> M<-numeric(n+1)
> for(i in 1:size){
+ set.seed(i)
+ Z<-rnorm(n+1)
+ set.seed(i+1)
+ U<-runif(n+1)
+ Y[1]<-3.0
+ X[1]<-3.0
+ k<-1
+ while(X[k]<S){
+ k<-k+1
+ Y[k]<-Y[k-1]+mu*h+sigma*sqrt(h)*Z[k]
+ R[k]<--Ag*sigma*sqrt(h)+Z[k]
+ H[k]<-(R[k]+sqrt((R[k])^2-2*log(1-U[k])))/2.0
+ X[k]<-max(X[k-1]+mu*h+sigma*sqrt(h)*Z[k],a*k*h+b+sigma*sqrt(h)*H[k])
+ }
+ tk<-tk+1
+ tpWNPR[tk]<-k*h
+ }
> hist(tpWNPR,freq=F,ylim=c(0,0.20), xlim=c(0,80), main="␣", breaks=200,
+ xlab="␣", ylab="␣")
> box()
```

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