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Procedia Structural Integrity 2 (2016) 2873–2880

Structural Integrity Procedia

www.elsevier.com/locate/procedia

21st European Conference on Fracture, ECF21, 20-24 June 2016, Catania, Italy

A nonlinear procedure for the analysis of RC beams

Patrizia Bernardi, Roberto Cerioni, Elena Michelini*

DICATeA, University of Parma, Parco Area delle Scienze 181/A, 43124 Parma, Italy

Abstract

This work deals with the development of a computational method for the nonlinear analysis of reinforced concrete beams subjected to general loading and constraint conditions, able to catch crack formation and propagation. To this aim, a layered beam finite element is developed. The displacement field along beam axis and height is modelled through polynomial functions, whose number of terms is varied based on the complexity of the considered problem. The mechanical nonlinearity of the material is taken into account by implementing a smeared constitutive model for cracked reinforced concrete elements. The effectiveness of the proposed procedure, which can be applied to the analysis of both new and existing buildings, is proved through comparison with significant experimental data from technical literature, relative to both statically determinate and indeterminate beams.

Peer-review under responsibility of the Scientific Committee of ECF21.

Keywords: Reinforced concrete; layered beam finite element; nonlinear behavior; smeared model; crack width

1. Introduction

The prediction of the behavior of reinforced concrete (RC) structures is a quite complex task, since mechanical nonlinearity should be taken into account even under low loads, due to crack formation. At ultimate limit state, this aspect, combined to the interaction between materials, largely influences the structural global behavior as well as failure conditions, especially in terms of ductility.

In case of RC beams subjected to general loading and constraint conditions, the development of efficient numerical methods, based on a simplification of the actual displacement field and able of correctly describing the mechanical behavior of the material, is even less straightforward. Further modeling difficulties arise indeed from the need of

* Corresponding author. Tel.: +39-0521-905709; fax: +39-0521-905924.
E-mail address: elena.michelini@unipr.it

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Peer review under responsibility of the Scientific Committee of ECF21.
10.1016/j.prostr.2016.06.359

correctly representing the influence exerted by the interaction between axial - shear forces (N-V) and/or between bending moment - shear force (M-V) on the structural response, as well as by the presence of "discontinuity" regions near constraints or concentrated loads, where plane section hypothesis is no longer valid. Moreover, the possible development of diagonal shear cracks determines a variation of the strain field along the cross-section height, which can be hardly caught by traditional computational methods, often requiring more sophisticated approaches such as bi-dimensional analyses, especially in case of elements characterized by a low slenderness ratio.

To solve this problem, several approaches have been proposed in the past, characterized by different levels of detailing and computational efficiency (among others, e.g. Haddadin et al. (1971), Vecchio and Collins (1988), Izzuddin et al. (1994), Di Prisco and Gambarova (1995), Manfredi and Pecce (1998), Sanches Jr. and Venturini (2007), Oliveira et al. (2008), Contraffatto et al. (2010)). Within this context, Belletti et al. (2002) developed a sectional analysis method for the modeling of the progressive behavior up to failure of statically determinate RC beams subjected to bending, shear and axial force. This approach was based on an extension of the kinematic hypotheses of classic beam theory, by considering the normal strain along beam height direction in addition to global deformations (axial strain, curvature and shear strain), so to realistically simulate the element crack pattern, as well as the strain field in the stirrups and in the concrete struts of the web. Mechanical nonlinearity was taken into account by implementing a smeared crack constitutive model for RC elements, named PARC (Belletti et al. (2001)). Starting from this approach, a computational method for the analysis of RC beams subjected to general loading and constraint conditions is developed herein, based on the introduction of a layered beam finite element (FE), whose displacement field along both beam axis and height is modelled by means of polynomial functions. Also in this case, the local behavior of reinforced concrete in each layer is described through PARC model. The solution of the nonlinear problem is carried out numerically through an incremental-iterative procedure based on a discretization of the analyzed beam in layered finite elements. The reliability and the potentiality of the proposed computational method is proved by comparisons with classic and well-documented experimental tests reported in the literature (Leonhardt and Walther (1962) and Leonhardt et al. (1964), for statically determinate and indeterminate beams, respectively).

2. Theoretical formulation of the layered beam finite element

According to the proposed computational method, RC beams are discretized through a mesh of layered beam finite elements (Fig. 1), whose displacement field is expressed according to the following general relations:

$$
\begin{aligned}
u(x,y) &= u_0(x) + U(x,y) \\
v(x,y) &= v_0(x) + V(x,y)
\end{aligned}
\tag{1}
$$

where u and v represent the displacements of an arbitrary point of the beam, having coordinates (x, y), while u_0 and v_0 are the corresponding displacements at beam centreline, which is assumed coincident with the x-axis, as depicted in Figure 1. Generally speaking, $U(x,y)$ and $V(x,y)$ are generic functions describing displacement variation along beam height and satisfying the condition $U(x,0) = V(x,0) = 0$; in this work, a polynomial expression is chosen, whose number of terms is made varying on the basis of the complexity of the considered problem, so having:

$$
U(x,y) = \sum_{i=1}^{N} u_i(x) \cdot y^i; \quad V(x,y) = \sum_{j=1}^{M} v_j(x) \cdot y^j \ . \tag{2}
$$

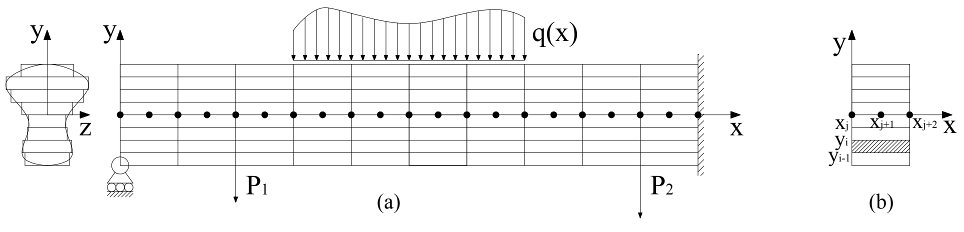

Fig. 1. (a) Finite element discretization of a RC beam subjected to general load and constraint conditions; (b) layered beam finite element.

In this way, the only unknowns of the problem are the generalized displacements u_N and v_M, which represent the coefficients of the adopted series expansion. By posing:

$$\mathbf{S}(x,y)=\{u(x,y),v(x,y)\}^T , \tag{3}$$

Equations (1) can be rewritten in matrix form as:

$$\mathbf{S}(x,y)=\mathbf{Y}(y)\,\mathbf{s}(x), \tag{4}$$

where $\mathbf{Y}(y)$ is a 2x(N+M+2) matrix, having the following structure:

$$\mathbf{Y}(y)=\begin{bmatrix} 1 & y & y^2 & \dots & y^N & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & y & y^2 & \dots & y^M \end{bmatrix}, \tag{5}$$

while $\mathbf{s}(x) = \{u_0(x),\dots, u_N(x), v_0(x),\dots, v_M(x)\}^T$ is the vector containing the generalized displacements. This latter can be in turn expressed as a function of the nodal unknowns – grouped in vector $\mathbf{S_e}$ – by using proper shape functions, which depend on the specific type of finite element adopted in the mesh:

$$\mathbf{s}(x)=\mathbf{N}(x)\mathbf{S_e}. \tag{6}$$

The displacement field in matrix form, to be used in the solution of the finite element problem, is then obtained by simply substituting Equation (6) into Equation (4):

$$\mathbf{S}(x,y)=\mathbf{Y}(y)\mathbf{N}(x)\mathbf{S_e}. \tag{7}$$

Under the hypothesis of small displacements, the strain-displacement relation can be expressed in the form:

$$\boldsymbol{\varepsilon}=[\partial]\mathbf{S}(x,y)=[\partial]\mathbf{Y}(y)\mathbf{N}(x)\mathbf{S}_e=\mathbf{B}(x,y)\mathbf{S_e}, \tag{8}$$

$\mathbf{B}(x,y)=[\partial]\ \mathbf{Y}(y)\ \mathbf{N}(x)$ being the strain-displacement matrix. The stress field in the i^{th} layer can be finally evaluated through the classic equation:

$$\boldsymbol{\sigma}_i=\mathbf{D}_i\ \boldsymbol{\varepsilon}_i \tag{9}$$

where $\mathbf{D}_i$ represents the stiffness matrix of the material forming the i^{th} layer, in the global coordinate system x-y. In order to account for RC mechanical nonlinearity, the latter is here evaluated by properly implementing PARC model. This model is based on a smeared-fixed crack approach and its theoretical basis, deduced for RC membrane elements subjected to general in-plane stresses, can be found in details in Belletti et al. (2001) and in Cerioni et al. (2008), Bernardi et al. (2016) in more recently improved formulations. According to this model, in the uncracked stage, concrete is treated as an elastic nonlinear orthotropic continuum, with orthotropic axes coincident with principal stress directions. When the maximum tensile stress at an integration point exceeds concrete tensile strength, a cracked stage is assumed. The strain components are then determined in the local crack coordinate system on the basis of the crack opening w, of the crack slip v and of the strain of the concrete strut between cracks ε_{c2}. The stiffness matrix at each integration point is then formulated by taking into account the contributions due to the main phenomena occurring after cracking, such as aggregate interlock, aggregate bridging, dowel action and tension stiffening between cracks. More details relative to the constitutive laws adopted for the modeling of each single contribution can be found in Belletti et al. (2001), to which reference is made.

3. Description of the adopted computational method

In order to apply the above described layered finite element to the analysis of RC beams subjected to general loading and constraint conditions, the computational method discussed in the following is implemented into a FORTRAN routine, working under loading control. The main input data requested by the program are relative to the desired general control parameters (polynomial function order, type and number of elements in the mesh, number of layers in each element, constraint type and position), to the geometric and mechanical characteristics of the examined beam, as well as to the applied external loads. In this work, an isoparametric layered beam element with three nodes is adopted, with the intermediate node placed in a central position (Fig. 1b). For each element of the FE mesh, the program evaluates the corresponding stiffness matrix $\mathbf{K}_e$ through the following general relation:

$$\mathbf{K}_e = \int_{V_e} \mathbf{B}(x,y)^T \mathbf{D}\, \mathbf{B}(x,y) dV = \int_{V_e} \left([\partial]\mathbf{Y}(y)\mathbf{N}(x)\right)^T \mathbf{D} \left([\partial]\mathbf{Y}(y)\mathbf{N}(x)\right)\, dV\,, \tag{10}$$

which can be rewritten, taking advantage of the subdivision into layers (Fig. 1b), in the form:

$$\begin{aligned}\mathbf{K}_e &= \sum_{i=1}^{n^\circ layers} \int_{V_{ei}} \left([\partial]\mathbf{Y}(y)\mathbf{N}(x)\right)^T \mathbf{D}_i \left([\partial]\mathbf{Y}(y)\mathbf{N}(x)\right)\, dV = \sum_{i=1}^{n^\circ layers} \int_{y_{i-1}}^{y_i} \left(\int_{S_{ei}} \mathbf{B}(x,y)^T \mathbf{D}_i\, \mathbf{B}(x,y)\, dS \right) dy \\ &= \sum_{i=1}^{n^\circ layers} b_i \int_{y_{i-1}}^{y_i} \left(\int_{x_j}^{x_{j+2}} \mathbf{B}(x,y)^T \mathbf{D}_i\, \mathbf{B}(x,y) dx \right) dy\end{aligned} \tag{11}$$

by first transforming the volume integral into a line integral on beam height and into a surface integral in the (x, y) plane and then further decomposing the surface integral into two line integrals along x and z directions. In Equation (11), x_j represents the coordinate of the first node of the considered element (Fig. 1b), while b_i is the beam cross-section width, which can be generally variable from one layer to the other. The model is indeed applicable to beams having a generic cross-section, by approximating its boundary through straight-line segments and by assigning to each layer an average width, as depicted in Figure 1a.

For computation ease, Equation (11) is first rewritten in the intrinsic coordinate system (ξ, η) of the element:

$$\mathbf{K}_e = \sum_{i=1}^{n^\circ layers} b_i \int_{-1}^{1} \left(\int_{-1}^{1} \mathbf{B}(\xi,\eta)^T \mathbf{D}_i\, \mathbf{B}(\xi,\eta) \det[J] \frac{y_i - y_{i-1}}{2} d\xi \right) d\eta \tag{12}$$

being $(y_i - y_{i-1})/2$ the derivative of the global coordinate y with respect to η and $\det[J]$ the Jacobian determinant; subsequently it is evaluated by applying a numerical integration technique, based on the Gauss method. In this way, Equation (12) becomes:

$$\mathbf{K}_e = \sum_{i=1}^{n^\circ layers} b_i \sum_{k=1}^{K} \sum_{j=1}^{J} \mathbf{B}(\xi_k,\eta_j)^T \mathbf{D}_i\, \mathbf{B}(\xi_k,\eta_j) \det[J(\xi_k)]\, \frac{y_i - y_{i-1}}{2}\, w_k w_j \tag{13}$$

where K and J respectively represent the number of Gauss points in x-direction and along the layer thickness (here assumed equal to two and one, the latter placed at mid-height of each layer), while w_k and w_j are the corresponding weights. Subsequently, the program evaluates the secant global stiffness matrix $\mathbf{K}$, by first assembling the stiffness matrices $\mathbf{K}_e$ of all the elements forming the mesh, and then applying the external constraint conditions by means of the Lagrange multipliers method. The determination of beam total displacements requires the construction of the total nodal force vector $\mathbf{F}$, which is also in this case evaluated by assembling the contribution due to each element $\mathbf{F}_e$. The latter is generally expressed as the sum of three terms, respectively related to body forces, surface forces and concentrated loads acting on the considered element, so having:

$$\mathbf{F}_e = \int_{V_e} (\mathbf{Y}(y)\mathbf{N}(x))^T \mathbf{f}\ dV + \int_{S_f} (\mathbf{Y}(y)\mathbf{N}(x))^T \mathbf{p}\ dS + \sum_{m=1}^{M} (\mathbf{Y}(y_m)\mathbf{N}(x_m))^T \mathbf{W} \tag{14}$$

where **f** and **p** are the body force and the surface force vectors eventually acting on the element, while **W** is a vector containing the concentrated forces, which are supposed to be applied in m points having coordinates (x_m, y_m).
By rewriting Equation (14) in the intrinsic coordinate system (ξ, η) and adopting the Gauss integration technique, the following expression, implemented in the program, can be obtained:

$$\mathbf{F}_e = \sum_{i=1}^{n^\circ layers} b_i \sum_{k=1}^{K} \sum_{j=1}^{J} (\mathbf{Y}(\eta_j)\mathbf{N}(\xi_k))^T \mathbf{f}\ \det[J]\ \frac{y_i - y_{i-1}}{2} w_k w_j + \sum_{h=1}^{H} b_h \sum_{k=1}^{K} (\mathbf{Y}(\eta_h)\mathbf{N}(\xi_k))^T p\ \det[J]\ w_k + \\ + \sum_{m=1}^{M} (\mathbf{Y}(\eta_m)\mathbf{N}(\xi_m))^T \mathbf{W} \tag{15}$$

where H represents the number of layers subjected to a distributed load (usually two, namely beam extrados and intrados), η_h and b_h are the coordinate and the width of beam cross-section in correspondence of the loaded side, while M represents the number of nodes where a concentrated load is applied.

At each loading step, the total nodal force vector **F** of the whole structure is properly updated and then the equilibrium system $\mathbf{F} = \mathbf{K\,S}$ is solved through an incremental-iterative procedure. Starting from the global displacements **S**, the program first calculates the strain and stress fields in the beam (Eqs. (8) and (9)) and then determines the unbalanced force vector $\mathbf{e_F} = \mathbf{F} - \boldsymbol{K}\,\boldsymbol{S}$, which is used to perform the convergence check $\|\mathbf{e_F}\| < \varepsilon_F \|\mathbf{F}\|$, ε_F being the admitted tolerance value. If this check is satisfied, convergence is achieved for the considered load increment; otherwise, the stiffness matrix is updated on the basis of the current strain field and the procedure is repeated until the condition is verified.

4. Model validation: comparison between numerical and experimental results

In order to check the effectiveness of the proposed computational method, numerical results are here compared with some well-known test results, relative to statically determinate and indeterminate beams (Leonhardt and Walther (1962) – series ET and GT, and Leonhardt et al. (1964) – series HH, respectively). These classic experimental programs, which were devoted to the investigation of beam behavior in presence of shear, represent indeed a good trial for the above described procedure, with a large amount of examined specimens and analyzed parameters, such as longitudinal and transversal reinforcement ratios, shear span to depth ratios, transversal cross-section shapes, failure modes. For sake of brevity, only the results relative to the isostatic beam GT1 and to the hyperstatic beam HH4 are reported in the following; however, further comparisons can be found in Michelini et al. (2006) and Michelini (2007), to which reference is made. Specimen GT1 was a 3.4 m long simply supported beam (with a net span equal to 3 m), characterized by a 300 mm wide and 350 mm high rectangular cross-section, and subjected to a distributed load p. The beam was reinforced with four 20 mm rebars placed in a double layer in the bottom part of the cross-section, and with two 8 mm rebars in the upper part. The stirrups, having a diameter equal to 6 mm, were characterized by a linearly decreasing spacing from the midspan to the support in the left part of the beam, and by a constant spacing in the right part. Further details on the geometric and mechanical characteristics of the examined specimen, as well as on test setup, can be found in Leonhardt and Walther (1962). Due to the unsymmetrical distribution of the stirrups, it is necessary to model the whole beam; however, in order to obtain satisfying results with a limited computational effort, a not much dense mesh is adopted along the span, with three-node elements, each of them characterized by 13 layers. Displacement filed is modelled with third order polynomial functions. Some comparisons between numerical and experimental curves are reported in Figure 2. In more detail, Figure 2a is relative to the global response of specimen GT1 in terms of total applied load P *vs*. midspan deflection v_M. The examined specimen was characterized by a flexural failure, with crushing of the upper concrete chord. As can be seen, the proposed computation method is able to correctly catch the experimental behavior both at serviceability and near failure, providing a realistic prediction of the ultimate load and of the collapse mode. Moreover, the model provides satisfactory results also in terms of local response, which represents a fundamental aspect in order to perform serviceability verifications. As an example, Figure 2b reports the sum of crack widths Σw measured between midspan and the support for increasing values of the total load P, but satisfying results can be also obtained in terms of maximum crack width evolution as a function of the external load, here omitted for brevity. Furthermore, Figures 2c-d show the stresses in the concrete struts of the web σ_c and those in the stirrups σ_s, still expressed as a function of the total load P. It can be observed that the model correctly predicts the stress field in the materials; in case of stirrups, the numerical prediction confirms that they are subjected to compressive stresses until the appearance of an inclined shear crack, which intersects them, as also results from experimental measurements.

As regards statically indeterminate beams, the attention is here focused on a two span continuous specimen named HH4, having a net span equal to 2.57 m, with a 250 mm wide and 320 mm high rectangular cross-section. The beam was subjected to two point loads (P/2) applied in the middle of each span. It was reinforced with five 14 mm rebars in the tension regions and two 14 mm rebars in the compression ones, while shear reinforcement was formed by ϕ8 stirrups, with a constant spacing (see Leonhardt et al. (1964) for further details). This specimen was selected since it was characterized by a critical value of the ratio between the acting moment and the shear force, equal to 2.60, and it failed in shear in correspondence of the lower collapse load registered for all the beams belonging to series HH. In order to find a proper balance between computational complexity and model reliability, the same modeling choices already described for specimen GT1 are adopted; however, taking advantage of the symmetry of the problem, only one half of the beam is analyzed in this case.

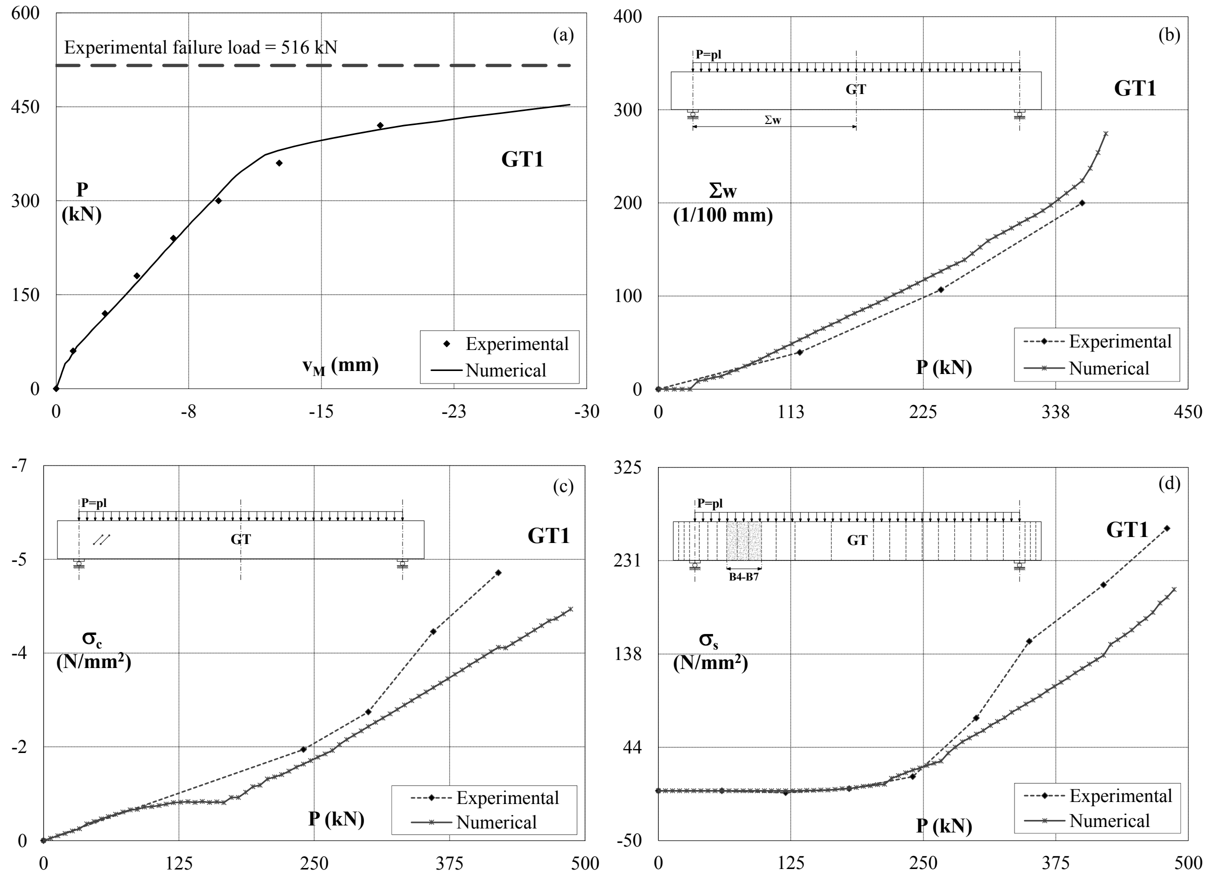

Fig. 2. Comparison between numerical and experimental (Leonhardt and Walther (1962)) results in terms of: (a) total applied load P *vs*. midspan deflection v_M, (b) sum of crack widths Σw *vs*. total applied load P, stresses in (c) concrete struts σ_c and in (d) stirrups σ_s *vs*. total applied load P.

A selection of the most interesting comparisons between numerical and experimental results is shown in Figure 3, which reports both the global response of the beam (in terms of total applied load *P vs.* deflection under the loading point v_M, Fig. 3a), and the evolution of some selected local variables for increasing values of the external load.

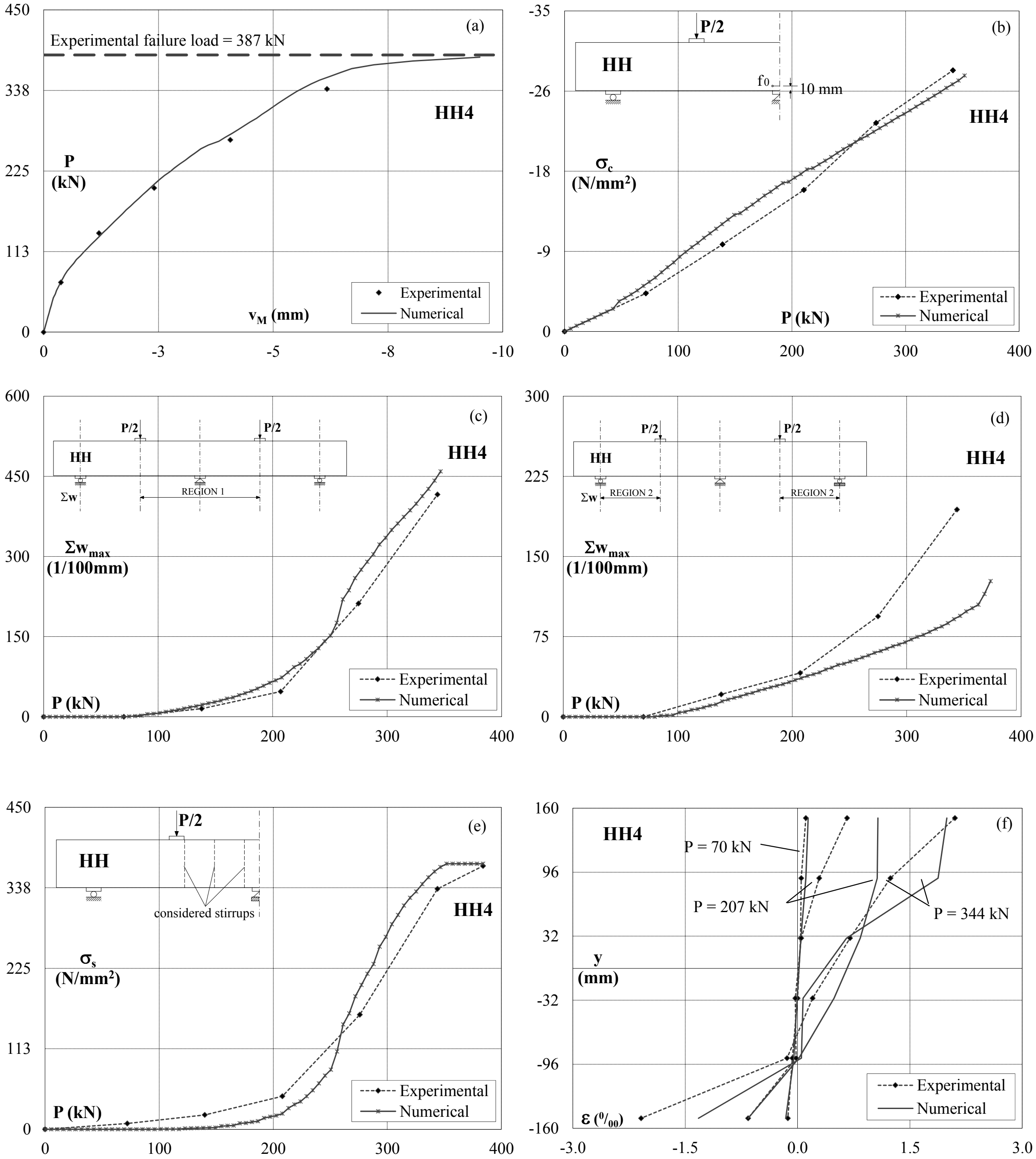

Fig. 3. Comparison between numerical and experimental (Leonhardt et al. (1964)) results in terms of: (a) total applied load *P vs.* deflection under the loading point v_M; (b) stress in concrete struts σ_c *vs.* total applied load *P*; sum of crack widths Σw *vs.* total applied load *P* in the (c) central and (d) lateral part of the beam; (e) stress in stirrups σ_s *vs.* total applied load *P*; (f) cross-section deformation for predefined values of the applied load.

As an example, Figure 3b reports the compressive stress in concrete σ_c in correspondence of the central support, while Figures 3c-d are relative to the sum of crack widths Σw registered at mid-height of the specimen, respectively in the central part of the beam (region 1), and in the lateral part, between the loading point and the support (region 2). Numerical and experimental stresses in the stirrups σ_s for an increasing applied load P are compared in Figure 3e; the plotted values represent the maximum stress among those registered in the stirrups placed between the applied load and the central support (as indicated in the upper part of the Figure) for each loading increment, according to Leonhardt et al. (1964). Finally, it can be observed that the adoption of a refined kinematic model allows a satisfactory prediction of beam behavior also in correspondence of discontinuity regions, as highlighted by the comparison plotted in Figure 3f, showing the normal strains ε along beam height in correspondence of the central support for predefined values of the external load P.

5. Conclusions

A computational method for the nonlinear analysis of statically determinate and indeterminate RC beams is here presented. The method is based on the development of a layered beam element with refined kinematic assumptions (the displacement field along beam axis and height is modelled through polynomial functions), able to account for material nonlinearity through the implementation of PARC smeared crack constitutive model. The capability and reliability of the procedure are verified through comparisons with significant well-known experimental data from literature. The reported examples prove that the model allows an accurate prediction of failure loads and modes, as well as deflections, crack widths and stresses in the materials for both isostatic and continuous RC beams. Consequently, it can represent a useful tool to perform refined nonlinear analysis for design purposes with a limited computational effort, requiring only the geometric characteristics of the element and the constitutive properties of materials. Moreover, thanks to its specific features, the model is able to represent also local failures due to high stress concentrations, for example near point loads or constraints, which could be hardly dealt with by using classic design methods based on sectional analysis.

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