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*Review Article*

# Variational Methods for NLEV Approximation Near a Bifurcation Point

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We review some more and less recent results concerning bounds on nonlinear eigenvalues (NLEV) for gradient operators. In particular, we discuss the asymptotic behaviour of NLEV (as the norm of the eigenvector tends to zero) in bifurcation problems from the line of trivial solutions, considering perturbations of linear self-adjoint operators in a Hilbert space. The proofs are based on the Lusternik-Schnirelmann theory of critical points on one side and on the Lyapounov-Schmidt reduction to the relevant finite-dimensional kernel on the other side. The results are applied to some semilinear elliptic operators in bounded domains of $\mathbb{R}^N$. A section reviewing some general facts about eigenvalues of linear and nonlinear operators is included.

## 1. Introduction and Examples

The term "nonlinear eigenvalue" (NLEV) is a frequent shorthand for "eigenvalue of a nonlinear problem," see, for instance [1–3]. While for the estimation of eigenvalues of linear operators there is wealth of abstract and computational methods (see, e.g., Kato's [4] and Weinberger's [5] monographs), for NLEV, the question is relatively new and there is not much literature available. In this paper, we review some abstract methods which allow for the computation of upper and lower bounds of NLEV near a bifurcation point of the linearized problem. Moreover, as one of our aims is to stimulate further research on the subject, we spend some effort in presenting it in a sufficiently general context and emphasize the question of the *existence* of eigenvalues for a nonlinear operator. In fact, Section 2 is entirely devoted to this, and to a parallel consideration of similar facts for linear operators.

Thus, generally speaking, consider two nonlinear (= not necessarily linear) operators $A, B : E \to F$ ($E, F$ real Banach spaces) such that $A(0) = B(0) = 0$. If for some $\lambda \in \mathbb{R}$ the equation

$$A(u) = \lambda B(u) \tag{1.1}$$

has a solution $u \neq 0$, then we say that $\lambda$ is an *eigenvalue* of the pair $(A, B)$ and $u$ is an *eigenvector* corresponding to $\lambda$. This definition is a word-by-word copy of the standard one for pairs of *linear* operators, where most frequently one takes $E = F$ and $B(u) = u$, and of course it may be of very little significance in general. However, it goes back at least to Krasnosel'skii [6] the demonstration of the importance of this concept for operator equations such as (1.1), with a view in particular to nonlinear integral equations of Hammerstein or Urysohn type.

In this paper, we consider (1.1) under the following qualitative assumptions:

(A) (1.1) possesses infinitely many eigenvalues $\lambda_n$;

(B) (1.1) has a linear reference problem $A_0(u) = \lambda B_0(u)$ which also possesses infinitely many eigenvalues $\lambda_n^0$.

It is then natural to try to approximate or estimate $\lambda_n$ in terms of $\lambda_n^0$. In the sequel, we will take $F = E'$, the dual space of $E$, and assume that all operators involved are *continuous gradient* operators from $E$ to $E'$; of course, this is done in order to exploit the full strength of variational methods. We emphasize in particular the case in which $E$ is a Hilbert space, identified with its dual.

Next, we note that two main routes are available to guarantee (A) and (B). The first involves the Lusternik-Schnirelmann (LS) theory of critical points for even functionals on symmetric manifolds (when $A$ and $B$ are odd mappings). The model example is the $p$-Laplace equation, briefly recalled in Example 1.1, exhibiting infinitely many eigenvalues and having the ordinary Laplace equation ($p = 2$) as linear reference problem. From our point of view, a main advantage of LS theory is precisely that it grants—provided that the constraint manifold contains subsets of arbitrary genus and that the Palais-Smale condition is satisfied at all candidate critical levels—the existence of infinitely many distinct eigenvalue/eigenvector pairs of (1.1), see, for instance, Amann [7], Berger [8], Browder [9], Palais [10], and Rabinowitz [11].

The domain of applicability of LS theory embraces as a particular case of (1.1) NLEV problems of the form

$$(A(u) \equiv) A_0(u) + P(u) = \lambda u (\equiv \lambda B(u)), \tag{1.2}$$

where the operators act in a real Hilbert space $H$, $A_0$ is linear and self-adjoint, and $P$ is odd and viewed as a perturbation of $A_0$. Under appropriate compactness and positivity assumptions on $A_0$ and $P$, (A) and (B) will be satisfied. More general forms of (1.2)—such as $A_0(u) + P(u) = \lambda B(u)$ where $A_0, P$, and $B$ are operators of $E$ into its dual $E'$ and $A_0$ behaves as the $p$-Laplacian—have been considered by Chabrowski [12], see Example 1.4 in this section.

However, problems of the form (1.2) can be studied in our framework also when $P$ is not necessarily an odd mapping, but rather satisfies the local condition

$$P(u) = o(\|u\|) \quad \text{as } u \longrightarrow 0. \tag{1.3}$$

Indeed in this case, Bifurcation theory ensures (see, e.g., [11]) that each isolated eigenvalue $\lambda_0$ of finite multiplicity of $A_0$ is a bifurcation point for (1.2), which roughly speaking means that solutions $u \neq 0$ of the unperturbed problem $A_0 u = \lambda_0 u$ (i.e., eigenfunctions associated with $\lambda_0$) do survive for the perturbed problem (1.2) in a neighborhood of $u = 0$ and for $\lambda$ near $\lambda_0$. Therefore, the framework described above at the points (A) and (B) is *grosso modo* respected also in this case provided that $A_0$ has a countable discrete spectrum.

When applicable, LS theory yields existence of eigenfunctions of *any* norm (provided of course that the relevant operators be defined in the whole space), in contrast with Bifurcation theory which only yields (in this context) information near $u = 0$.

In the main part of this paper (Section 3), we focus our attention upon equations of the form (1.2), having in mind—with a view to the applications—a $P$ that is odd *and* satisfies (1.3). For such a $P$, both methods are applicable and can be tested to see which of them yields better quantitative information on the eigenvalues associated with small eigenvectors. More precisely, given an isolated eigenvalue $\lambda_0$ of finite multiplicity of $A_0$, the assumptions on $P$ guarantee bifurcation at $(\lambda_0, 0)$ from the line $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ of trivial solutions, and in particular ensure the existence for $R > 0$ sufficiently small of solutions $(\lambda_R, u_R)$ of (1.2) such that

$$\|u_R\| = R \quad \text{for each } R, \qquad \lambda_R \longrightarrow \lambda_0 \quad \text{as } R \longrightarrow 0, \tag{1.4}$$

that is, parameterized by the norm $R$ of the eigenvector $u_R$ and bifurcating from $(\lambda_0, 0)$. If we qualify the condition $P(u) = o(\|u\|)$ with the more specific requirement that, for some $q > 2$,

$$P(u) = O\left(\|u\|^{q-1}\right) \quad \text{as } u \longrightarrow 0, \tag{1.5}$$

then the information in (1.4) can be made more precise to yield estimates of the form (as $R \to 0$)

$$C_1 R^{q-2} + o\left(R^{q-2}\right) \leq \lambda_R - \lambda_0 \leq C_2 R^{q-2} + o\left(R^{q-2}\right). \tag{1.6}$$

We are interested in the evaluation of the constants $C_1$ and $C_2$. It turns out that these can be estimated in terms of $\lambda_0$ itself and other known constants related to $P$. We do this in two distinct ways, as indicated before.

(i) Using Lusternik-Schnirelmann's theory in order to estimate the difference $\lambda_R - \lambda_0$ through the LS "minimax" critical levels. This approach was first used by Berger [8, Chapter 6, Section 6.7A] and then pursued by the author (see [13], e.g.) and subsequently by Chabrowski [12].

(ii) Using the Lyapounov-Schmidt method to reduce (1.2) to an equation in the finite-dimensional space $N \equiv N(A_0 - \lambda_0 I)$, and then working carefully on the reduced equation in order to exploit the stronger condition (1.5). We have recently followed this approach in [14].

Our computations in Section 3 show that the second method is both technically and conceptually simpler, requires less on $P$ ($P$ need not be odd), and yields sharper results. We conclude Section 3 and the present work on applying these abstract results to a simple semilinear elliptic equation, see Example 1.3. Let us remark on passing that in the case of ordinary differential equations, detailed estimates for NLEV near a bifurcation point have been recently proved by Shibata [15]. The techniques employed by him are elementary and straightforward—direct integration and manipulation of the differential equation, series expansion, and so on—but very efficiently used. Some earlier result in this style can be found, for instance, in [16].

The remaining parts of this paper are organised as follows. We complete this introductory section presenting (as a matter of example) some boundary-value problems for nonlinear differential equations, depending on a real parameter $\lambda$ and admitting the zero solution for all values of $\lambda$, that can be cast in the form (1.1) with an appropriate choice of the function space $E$ and of the operators $A, B$.

Section 2 is intended to recall for the reader's convenience some basic facts from the calculus of variations and critical point theory. We first indicate the reduction of (1.1) to the search of critical points of the potential $a$ of $A$ on the manifold $V \equiv \{u \in E : b(u) = \text{const}\}$, $b$ the potential of $B$. Some details are spent to show that absolute minima or maxima correspond to the *first* eigenvalue—we do this for the elementary case of homogeneous operators such as the $p$-Laplacian—while minimax critical levels correspond to *higher order* eigenvalues, both for linear and nonlinear operators. In this circle of ideas, we recall a few elements of LS theory that are helpful to state and prove our subsequent results.

Let us finally mention that foundations and inspiration for the study, of NLEV problems are to be found in (among many others) Krasnosel'skii [6], Vainberg [17], Fučík et al. [18], Ambrosetti and Prodi [19], Nirenberg [20], Rabinowitz [11, 21, 22], Berger [8], Stackgold [23], and Mawhin [3].

*Example 1.1* (the $p$-Laplace equation). The most famous (and probably most important) example of a nonlinear problem exhibiting the features described in the points (A) and (B) above is provided by the $p$–Laplace equation ($p > 1$):

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \mu |u|^{p-2}u, \tag{1.7}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$, subject to the Dirichlet boundary conditions $u = 0$ on the boundary $\partial\Omega$ of $\Omega$. Fix $p > 1$, let $E$ be the Sobolev space $W_0^{1,p}(\Omega)$, equipped with the norm

$$\|v\|^p_{W_0^{1,p}} = \int_\Omega |\nabla v|^p, \tag{1.8}$$

and let $E' = W^{-1,p'}(\Omega)$ be the dual space of $E$. A (weak) solution of (1.7) is a function $u \in E$ such that

$$A_p(u) = \lambda B_p(u), \tag{1.9}$$

where $\lambda = \mu^{-1}$ $(\mu \neq 0)$ and $A_p, B_p : E \to E'$ are defined by duality via the equations

$$\langle B_p(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx, \qquad \langle A_p(u), v\rangle = \int_\Omega |u|^{p-2}uv\, dx, \tag{1.10}$$

where $u, v \in E$ and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $E$ and $E'$.

Equation (1.7) possesses countably many eigenvalues $\mu_n(p)$ $(n \in \mathbb{N})$, which are values of the real function $\phi_p$ defined via

$$\phi_p(u) = \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \quad \left(u \in W_0^{1,p}(\Omega),\ u \neq 0\right), \tag{1.11}$$

and can be naturally arranged in an increasing sequence

$$\mu_1(p) \le \mu_2(p) \le \cdots \mu_n(p) \le \cdots, \qquad \lim_{n\to\infty} \mu_n(p) = +\infty. \tag{1.12}$$

This relies on the very special nature of (1.7), because $A_p$ and $B_p$ are

(i) *odd* ($F : E \longrightarrow E'$ is said to be odd if $F(-u) = -F(u)$ for $u \in E$);

(ii) *positively homogeneous* of the same degree $p - 1 > 0$ ($F$ positively homogeneous of degree $\alpha$ means that $F(tu) = t^\alpha F(u)$ for $t > 0$ and $u \in E$);

(iii) *gradient* ($F$ gradient means that $\langle F(u), v\rangle = f'(u)v$ for some functional $f$ on $E$).

The existence of the sequence $(\mu_n(p))$ then follows (using the compactness of the embedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$) by the Lusternik-Schnirelmann theory of critical points for even functionals on sphere-like manifolds (see the references cited in Section 1). The eigenvalues $\mu_n(p)$ have been studied in detail, and in particular as to their *asymptotic behaviour* García Azorero and Peral Alonso [24] and Friedlander [25] have proved the two-sided inequality

$$A|\Omega|n^{p/N} \le \mu_n(p) \le B|\Omega|n^{p/N}, \tag{1.13}$$

to hold for all sufficiently large $n$ and for suitable positive constants $A$ and $B$ depending only on $N$ and $p$; $|\Omega|$ stands for the (Lebesgue) $N$-dimensional volume of $\Omega$. This generalizes in part the classical result of Weyl [26] for the linear case (corresponding to $p = 2$ in (1.7)), that is, for the eigenvalues $\mu_n^0$ of the Dirichlet Laplacian $-\Delta u = \mu u$, $u \in W_0^{1,2}(\Omega)$:

$$\mu_n^0 \equiv \mu_n(2) = K|\Omega|n^{2/N} + o\left(n^{2/N}\right) \quad (n \longrightarrow \infty). \tag{1.14}$$

Evidently, this and similar questions would be of greater interest, should one be able to prove that the $\mu_n(p)$ are the *only* eigenvalues of (1.7); however, this is demonstrated only for $N = 1$, in which case they can be computed by explicit solution of (1.7). For this, as well as for a general discussion of the features of (1.7), its eigenvalues, and in particular the very special properties owned by the first eigenvalue $\mu_1(p)$ (corresponding to the minimum of the functional $\phi_p$ defined in (1.11)) and the associated eigenfunctions, we refer the reader to the beautiful Lecture Notes by Lindqvist [27]. For an interesting discussion on the existence of eigenvalues outside the LS sequence in problems related to (1.7), we recommend the very recent papers in [28, 29].

*Remark 1.2.* The existence of countably many eigenvalues for (1.7) has been recently proved by means of a completely different method than the LS theory, namely, the representation theory of compact linear operators in Banach spaces developed in [30]. Actually the eigenfunctions associated with these eigenvalues are defined in a weaker sense, and only upper bounds of the type in (1.13) are proved. As emphasized in [30], it is not clear what connection (if any) there is between the higher eigenvalues found by the two procedures, nor whether there are eigenvalues not found by either method.

*Example 1.3* (semilinear equations). As a second example, consider the semilinear elliptic eigenvalue problem

$$-\Delta u = \mu\big(u + f(x,u)\big), \qquad u \in H_0^1(\Omega) \equiv W_0^{1,2}(\Omega), \tag{1.15}$$

again in a bounded domain of $\Omega \subset \mathbb{R}^N$, where the nonlinearity is given by a real-valued function $f = f(x,s)$ defined on $\Omega \times \mathbb{R}$ and satisfying the following hypotheses:

(HF0) $f$ satisfies Carathéodory conditions (i.e., to say, $f$ is continuous in $s$ for a.e. $x \in \Omega$ and measurable in $x$ for all $s \in \mathbb{R}$);

(HF1) there exist a constant $a \geq 0$ and an exponent $q$ with $2 < q < 2N/(N-2) \equiv 2^*$ if $N > 2$, $2 < q < \infty$ if $N \leq 2$ such that

$$\left|f(x,s)\right| \leq a|s|^{q-1} \quad \text{for } x \in \Omega(\text{a.e.}),\ s \in \mathbb{R}. \tag{1.16}$$

Here we take the Hilbert space $H = H_0^1(\Omega)$ equipped with the scalar product

$$(u,v) = \int_\Omega \nabla u \nabla v \, dx, \tag{1.17}$$

and consider again weak solutions of (1.15), defined now as solutions of the equation in $H$

$$A_0 u + P(u) = \lambda u, \tag{1.18}$$

whereas before $\lambda = 1/\mu$ $(\mu \neq 0)$ while the operators $A_0, P : H \to H$ are defined (using the self-duality of $H$ based on (1.17)) by the equations

$$(A_0 u, v) = \int_\Omega uv \, dx, \qquad (P(u), v) = \int_\Omega f(x,u)v \, dx, \tag{1.19}$$

for $u, v \in H$ (note: we write here and henceforth $A_0$ for $A_2$). Then we see that also (1.15) can be cast in the form (1.1), with $B(u) = u$ and

$$A = A_0 + P. \tag{1.20}$$

Despite this formal similarity, the present Example is essentially different from Example 1.1. To see this, first note that the basic eigenvalue problem for the Dirichlet Laplacian, $-\Delta u = \mu u$, $u \in H_0^1(\Omega)$, takes (in our notations) the form

$$A_0(u) = \lambda u, \tag{1.21}$$

and involves—of course—only linear operators, $A_0$ and the identity map. These can be seen as a special type of 1-homogeneous operators; now while in the former example they are replaced with the $(p-1)$-homogeneous operators $A_p$ and $B_p$ defined in (1.10), here we deal with an *additive* perturbation, $A_0 + P$, of $A_0$. This new operator $A = A_0 + P$ is still a gradient, and will be odd if so is taken $f$ in its dependence upon the second variable, but plainly is *not* 1-homogeneous any longer (except when $f(x,s) = a(x)s$, $a \in L^\infty(\Omega)$, in which case of course we would be dealing with a *linear* perturbation of a linear problem: see for this [31] or [5]).

Nevertheless, the assumptions (HF0)-(HF1) and results from Bifurcation theory ensure all the same (as indicated before in Section 1, see also Section 3 for more details) that eigenvalues $\mu_R$ for (1.15) do exist, associated with eigenfunctions $u_R$ of small $H_0^1$ norm $R$, near each fixed eigenvalue $\mu_0 = \mu_k^0$ of the Dirichlet Laplacian; we put here and henceforth $\mu_k^0 = (\lambda_k^0)^{-1}$, with $\lambda_k^0$ the $k$th eigenvalue of (1.21).

Two main differences with the former situation must be noted at once.

(i) First, the loss of homogeneity causes that the eigenvalues "depend on the norm of the eigenfunction," unlike in Example 1.1 where it suffices to consider normalized eigenvectors. Indeed in general, if the operators $A$ and $B$ appearing in (1.1) are both homogeneous of the same degree, it is clear that if $u_0$ is an eigenvector corresponding say to the eigenvalue $\hat{\mu}$, then so does $tu_0$ for any $t > 0$.

(ii) Second, Bifurcation theory provides in the present "generic" situation only *local* results, that is, results holding in a neighborhood of $(\mu_0, 0) \in \mathbb{R} \times H_0^1(\Omega)$, and thus concerning eigenfunctions of small norm. "Generic" means that we here ignore the multiplicity of $\mu_0$: for in case we knew that this is an odd number, then *global* results would be available from the theory [32] to grant the existence of an unbounded "branch" (in $\mathbb{R} \times H_0^1(\Omega)$) of solution pairs $(\mu, u)$ bifurcating from $(\mu_0, 0)$.

Under the assumptions (HF0)-(HF1) we have shown in particular (see [14, 33, 34]) that

$$\mu_R = \mu_0 + O\left(R^{q-2}\right) \quad \text{as } R \longrightarrow 0, \tag{1.22}$$

(i.e., $|\mu_R - \mu_0| \le KR^{q-2}$ for some $K \ge 0$ and all sufficiently small $R > 0$), and more precisely that

$$CR^{q-2} + o\left(R^{q-2}\right) \le \mu_0 - \mu_R \le DR^{q-2} + o\left(R^{q-2}\right) \quad \text{as } R \longrightarrow 0, \tag{1.23}$$

for suitable constants $C, D$ related to $f$; here $o(R^{q-2})$ denotes as usual an unspecified function $h = h(R)$ such that $h(R)/R^{q-2} \to 0$ as $R \to 0$.

In Section 3, we explain how estimates like (1.23) follow independently both from LS theory (when $f$ is odd in $s$) and from Bifurcation theory and refine our previous results in the estimate of the constants $C$ and $D$.

*Example 1.4* (quasilinear equations). The results indicated in Examples 1.1 and 1.3 can be partly extended to the problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \mu\left(|u|^{p-2}u + f(x,u)\right) \quad u \in W_0^{1,p}(\Omega), \tag{1.24}$$

where $p > 1$ and $f = f(x, s)$ is dominated by $|s|^{q-1}$ for some $p < q < p^*$, with

$$p^* \equiv \frac{Np}{N-p} \quad \text{if } N > p, \qquad p^* \equiv \infty \quad \text{if } N \le p. \tag{1.25}$$

Equation (1.24) reduces to (1.7) if $f \equiv 0$ and to (1.15) if $p = 2$, and therefore formally provides a common framework for both equations. However, it must be noted—looking at the bifurcation approach indicated in the previous example—that the desired extension can only be partial, because $p \neq 2$ (1.24) is no longer a perturbation of a linear problem, but of the homogeneous problem (1.7). Bifurcation should thus be considered from the eigenvalues of the $p$-Laplace operator, but to my knowledge there is (in the general case) no abstract result about bifurcation from the eigenvalues of a homogeneous operator (let alone stand from those of a general nonlinear operator). A fundamental exception is that of the first eigenvalue of a homogeneous operator (see Theorem 2.4 and Remark 2.6 in Section 2) which possesses—under additional assumptions on the operator itself—remarkable properties such as the positivity of the associated eigenfunctions, see [35]. These properties have been extensively used (in [36, 37], e.g.) in order to prove *global* bifurcation results for (1.24) from the first eigenvalue of the $p$-Laplacian. Related results can be found in [38, 39].

Clearly, in case the $f$ appearing in (1.24) be odd in its second variable, typically when $f$ is of the form

$$f(x, s) = a(x)|s|^{q-2}s, \quad a \in L^\infty(\Omega), \tag{1.26}$$

then one can resort again to LS theory, because the resulting abstract equation

$$A_p(u) + P(u) = \lambda B_p(u) \tag{1.27}$$

(with $A_p$ and $B_p$ as in (1.10) and $P$ defined via (1.19) and (1.26)) involves operators which are all odd, and one can prove in this way bifurcation from each eigenvalue $\mu_n(p)$ of (1.7). For the corresponding results, see Chabrowski [12]. To be precise, the problem dealt with by Chabrowski is slightly different as he considers the modified form of (1.24) in which $f$ sits on the left-hand side (i.e., it is added to the $p$-Laplacian) rather than on the right-hand side of the equation. Needless to say, this does not change the essence of our remark, nor the results for (1.24) would be much different from those in [12].

## 2. Existence of Eigenvalues for Gradient Operators

Consider (1.1) where $A, B : E \to E'$ ($E$ a real, infinite dimensional, reflexive Banach space) and suppose that $\langle B(u), u\rangle \neq 0$ for $u \neq 0$. If $\lambda$ is an eigenvalue of $(A, B)$ and $u$ is a corresponding eigenvector, then

$$\lambda = \frac{\langle A(u), u\rangle}{\langle B(u), u\rangle} \equiv R(u). \tag{2.1}$$

Thus, the eigenvalues of $(A, B)$—if any—must be searched among the values of the function $R$ defined on $E \setminus \{0\}$ by means of (2.1). $R$ is called the *Rayleigh quotient* relative to $(A, B)$, and its importance for pairs of linear operators is well established [5].

Well-known simple examples (just think of linear operators) show that without further assumptions, there may be no eigenvalues at all for $(A, B)$. On the other hand, we know that a real symmetric $n \times n$ matrix has at least one eigenvalue, and so does any self-adjoint linear operator in an infinite-dimensional real Hilbert space, provided it is compact. The nonlinear analogue of the class of self-adjoint operators is that of gradient operators, which are the natural candidates for the use of variational methods.

In their simplest and oldest form traced by the Calculus of Variations, variational methods consist in finding the minimum or the maximum value of a functional on a given set in order to find a solution of a problem in the set itself. Basically, if we wish to solve the equation

$$A(u) = 0, \quad u \in E, \tag{2.2}$$

and $A : E \to E'$ is a *gradient* operator, which means that

$$\langle A(u), v \rangle = a'(u)v \quad \forall u, v \in E, \tag{2.3}$$

for some differentiable functional $a : E \to \mathbb{R}$ [(the *potential* of $a$)], then we need to just find the *critical points* of $a$, that is, the points $u \in E$ where the derivative $a'(u)$ of $a$ vanishes. The images $a(u)$ of these points are by definition the *critical values* of $a$, and the simplest such are evidently the minimum and the maximum values of $a$ (provided of course that they are attained). However, from the standpoint of eigenvalue theory, the relevant equation is (1.1), whose solutions $u$ are—when also $B$ is a gradient—the critical points of $a$ constrained to $b(u) = \text{const}$, where $b$ is the potential of $B$. To be precise, normalize the potentials assuming that $a(0) = b(0) = 0$ and consider for $c \neq 0$ the "surface"

$$V_c \equiv \{u \in E : b(u) = c\}. \tag{2.4}$$

Then at a critical point $u \in V_c$ of the restriction of $a$ to $V_c$, we have

$$a'(u)v = \lambda b'(u)v \quad \forall v \in E, \tag{2.5}$$

for some Lagrange multiplier $\lambda$. This is the same as to write $A(u) = \lambda B(u)$, and thus yields an eigenvalue-eigenvector pair $(\lambda, u) \in \mathbb{R} \times V_c$ for (1.1); note that $0 \notin V_c$ if $c \neq 0$. Of course to derive (2.5) we need some regularity of $V_c$, and this is ensured (if $B$ is continuous) by the assumptions made upon $B$, which guarantee—since $b'(u)u = \langle B(u), u \rangle \neq 0$ for $u \neq 0$ and $0 \notin V_c$—that $V$ is indeed a $C^1$ submanifold of $E$ of codimension one [40].

Let us collect the above remarks stating formally the basic assumptions on $A, B$ and the basic fact on the existence of at least one eigenvalue for $A$, $B$.

(AB0) $A, B : E \to E'$ are continuous gradient operators with $\langle B(u), u \rangle \neq 0$ for $u \neq 0$.

**Theorem 2.1.** *Suppose that $A$, $B$ satisfy (AB0) and let $V_c$ be as in* (2.4). *Suppose, moreover, that the potential $a$ of $A$ is bounded above on $V_c$ and let $M \equiv \sup_{u\in V_c} a(u)$. If $M$ is attained at $u_0 \in V_c$, then there exists $\lambda_0 \in \mathbb{R}$ such that*

$$A(u_0) = \lambda_0 B(u_0). \tag{2.6}$$

*That is, $u_0$ is an eigenvector of the pair $(A, B)$ corresponding to the eigenvalue $\lambda_0$. A similar statement holds if $a$ is bounded below, provided that $m \equiv \inf_{u\in V} a(u)$ is attained.*

### *2.1. The First Eigenvalue (for Linear and Nonlinear Operators)*

Looking at the statement of Theorem 2.1, we remark that in general there may be more points/eigenvectors $u \in V_c$ (if any at all) where $M$ is attained, and consequently different corresponding eigenvalues (the values taken by the Rayleigh quotient (2.1) at such points). However, in a special case, $\lambda_0$ is uniquely determined by $M$ and plays the role of "first eigenvalue" of $(A, B)$: this is when $A$ and $B$ are positively homogeneous of the same degree. Recall that $A$ is said to be *positively homogeneous* of degree $\alpha > 0$ if $A(tu) = t^\alpha A(u)$ for $u \in E$ and $t > 0$. For such operators pairs, it is sufficient to consider a fixed level set (that is, to consider normalized eigenvectors), for instance,

$$V \equiv \{u \in E : b(u) = 1\}. \tag{2.7}$$

**Theorem 2.2.** *Let $A, B : E \longrightarrow E'$ satisfy (AB0) and let $a, b$ be their respective potentials. Suppose in addition that $A$, $B$ are positively homogeneous of the same degree. If $a$ is bounded above on $V$ and $M = \sup_{u\in V} a(u)$ is attained at $u_0 \in V$, then*

$$A(u_0) = MB(u_0). \tag{2.8}$$

*Moreover, $M$ is the largest eigenvalue of the pair $(A, B)$. Likewise, if $a$ is bounded below and $m = \inf_{u\in V} a(u)$ is attained, then $m$ is the smallest eigenvalue of the pair $(A, B)$.*

Let us give the direct easy proof of Theorem 2.2, that does not even need Lagrange multipliers. The homogeneity of $A$ and $B$ implies that

$$\sup_{u\in V} a(u) = \sup_{u\neq 0} \frac{a(u)}{b(u)} = \sup_{u\neq 0} \frac{\langle A(u), u\rangle}{\langle B(u), u\rangle}. \tag{2.9}$$

Indeed recall (see [7] or [8]) that a continuous gradient operator $A$ is related to its potential $a$ (normalized so that $a(0) = 0$) by the formula

$$a(u) = \int_0^1 \langle A(tu), u\rangle dt. \tag{2.10}$$

Thus, if $A$ is homogeneous of degree $\alpha$ we have

$$a(u) = \frac{\langle A(u), u\rangle}{\alpha + 1}, \tag{2.11}$$

and similarly for $b$; in particular, $a$ and $b$ are $(\alpha+1)$-homogeneous. Therefore, if for $u \neq 0$ we put $t(u) = (b(u))^{-1/(\alpha+1)}$, we have

$$\frac{a(u)}{b(u)} = a(t(u)u), \tag{2.12}$$

and as $b(t(u)u) = 1$ (i.e., $t(u)u \in V$), the first equality in (2.9), follows immediately, and so does the second by virtue of (2.11). By (2.9) and the definition of $M$ we have

$$a(u) - Mb(u) \leq 0 \quad \text{for any } u \in E. \tag{2.13}$$

Suppose now that $M$ is attained at $u_0 \in V$. Then $a(u_0) - Mb(u_0) = 0$. Thus, $u_0$ is a point of absolute maximum of the map $K \equiv a - Mb : E \to \mathbb{R}$ and therefore its derivative $K'(u_0) = a'(u_0) - Mb'(u_0)$ at $u_0$ vanishes, that is,

$$A(u_0) = MB(u_0). \tag{2.14}$$

This proves (2.8). To prove the final assertion, observe that by (2.9), $M$ is also the maximum value of the Rayleigh quotient, and therefore the largest eigenvalue of $(A, B)$ by the remark made above.

So the real question laid by Theorems 2.1 and 2.2 is how can we ensure that (i) $a$ is bounded and (ii) $a$ attains its maximum (or minimum) value on $V$? The first question would be settled by requiring in principle that $V$ is bounded and that $A$ (and therefore $a$) is bounded on bounded sets. However, to answer affirmatively (ii), we need anyway some *compactness*, and as $E$ has infinite dimension—which makes hard to hope that $V$ be compact—such property must be demanded to $a$ (or to $A$).

*Definition 2.3.* A functional $a : E \to \mathbb{R}$ is said to be *weakly sequentially continuous* (wsc for short) if $a(u_n) \to a(u)$ whenever $u_n \to u$ weakly in $E$, and *weakly sequentially lower semicontinuous* (wslsc) if

$$\liminf_{n\to\infty} a(u_n) \geq a(u), \tag{2.15}$$

whenever $u_n \to u$ weakly in $E$. Finally, $a$ is said to be *coercive* if $a(u_n) \to +\infty$ whenever $\|u_n\| \to +\infty$.

**Theorem 2.4.** *Let $A, B : E \to E'$ satisfy (AB0) and let $a$, $b$ be their respective potentials. Suppose that*

(i) *$a$ is wsc;*

(ii) *$b$ is coercive and wslsc.*

*Then $a$ is bounded on $V$. Suppose moreover that $A$ and $B$ are positively homogeneous of the same degree. If $M \equiv \sup_{u\in V} a(u) > 0$ (resp., $m \equiv \inf_{u\in V} a(u) < 0$), then it is attained and is the largest (resp., smallest) eigenvalue of $(A, B)$.*

*Proof.* Suppose by way of contradiction that $a$ is not bounded above on $V$, and let $(u_n) \subset V$ be such that $a(u_n) \to +\infty$. As $b$ is coercive, $(u_n)$ is bounded (in fact, $V$ itself is a bounded set) and therefore as $E$ is reflexive we can assume—passing if necessary to a subsequence—that $(u_n)$ converges weakly to some $u_0 \in E$. As $a$ is wsc, it follows that $a(u_n) \to a(u_0)$, contradicting the assumption that $a(u_n) \to +\infty$. Thus, $M$ is finite, and we can now let $(u_n) \subset V$ be a *maximizing sequence*, that is, a sequence such that $a(u_n) \to M$. As before, we can assume that $(u_n)$ converges weakly to some $u_0 \in E$, and the weak sequential continuity of $a$ now implies that $a(u_0) = M$.

It remains to prove—under the stated additional assumptions—that $u_0 \in V$. To do this, first observe that (as $b$ is wslsc)

$$1 = \liminf_{n\to\infty} b(u_n) \geq b(u_0). \tag{2.16}$$

We claim that $b(u_0) = 1$. Indeed suppose by way of contradiction that $b(u_0) < 1$, and let $t_0 > 0$ be such that $t_0 u_0 \in V$; such a $t_0$ is uniquely determined by the condition

$$b(t_0 u_0) = t_0^{\alpha+1} b(u_0) = 1, \tag{2.17}$$

which yields $t_0 = ((b(u_0))^{-1/(\alpha+1)}$ and shows that $t_0 > 1$. But then, as $M > 0$, we would have

$$a(t_0 u_0) = t_0^{\alpha+1} a(u_0) = t_0^{\alpha+1} M > M, \tag{2.18}$$

which contradicts the definition of $M$ and proves our claim. The proof that $m$ is attained if it is strictly negative is entirely similar. □

*Example 2.5* (the first eigenvalue of the $p$-Laplace operator). If $A = A_p$ and $B = B_p$ are defined as in Example 1.1, we have

$$a(u) = \frac{\int_\Omega |u|^p}{p}, \quad b(u) = \frac{\int_\Omega |\nabla u|^p}{p} \quad \left(u \in W_0^{1,p}(\Omega)\right), \tag{2.19}$$

for their respective potentials (see (2.11)), and therefore

$$\frac{a(u)}{b(u)} = \frac{\langle A_p(u), u\rangle}{\langle B_p(u), u\rangle} = \frac{\int_\Omega |u|^p}{\int_\Omega |\nabla u|^p} = \left(\phi_p(u)\right)^{-1} \quad (u \neq 0), \tag{2.20}$$

with $\phi_p$ as in (1.11). The compact embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ implies that $a$ is wsc (see the comments following Definition 2.7); moreover, looking at (1.8) we see that $b$ is coercive, while its weak sequential lower semicontinuity is granted as a property of the norm of any reflexive Banach space [41]. It follows by Theorem 2.4 that

$$\lambda_1(p) \equiv \sup \frac{\int_\Omega |u|^p}{\int_\Omega |\nabla u|^p} \tag{2.21}$$

is attained and is the largest eigenvalue of $A_p$, which is the same as to say that $\mu_1(p) \equiv (\lambda_1(p))^{-1}$ is the smallest eigenvalue of (1.7). This shows the existence and variational characterization of the first point in the spectral sequence (1.12).

*Remark 2.6.* Much more can be said about $\mu_1(p)$, in particular, $\mu_1(p)$ is *isolated* and *simple* (i.e., the corresponding eigenfunctions are multiple of each other), and moreover the eigenfunctions *do not change sign in* $\Omega$. These fundamental properties (proved, e.g., in [27]) are among others at the basis of the global bifurcation results for equations of the form (1.24) due to [36, 37]. For an abstract version of these properties of the first eigenvalue, see [35].

Let us now indicate very briefly some conditions on $A$, $B$ ensuring the properties required upon $a$, $b$ in Theorem 2.4.

*Definition 2.7.* A mapping $A : E \to F$ ($E$, $F$ Banach spaces) is said to be *strongly sequentially continuous* (strongly continuous for short) if it maps weakly convergent sequences of $E$ to strongly convergent sequences of $F$.

It can be proved (see, e.g., [7]) that if a gradient operator $A : E \to E'$ is strongly continuous, then its potential $a$ is wsc. Moreover, it is easy to see that a strongly continuous operator $A : E \to F$ is *compact*, which means by definition that $A$ maps bounded sets of $E$ onto relatively compact sets of $F$ (or equivalently, that any bounded sequence $(u_n)$ in $E$ contains a subsequence $(u_{n_k})$ such that $A(u_{n_k})$ converges in $F$). Moreover, when $A$ is a linear operator, then it is strongly continuous if and only if it is compact [42].

*Definition 2.8.* A mapping $A : E \to E'$ is said to be *strongly monotone* if

$$\langle A(u) - A(v), u - v\rangle \geq k\|u - v\|^2, \tag{2.22}$$

for some $k > 0$ and for all $u, v \in E$.

It can be proved (see, e.g., [9]) that if a gradient operator $A$ is strongly monotone, then its potential $a$ is coercive and wslsc.

With the help of Definitions 2.7 and 2.8, Theorem 2.4 can be easily restated using hypotheses which only involve the operators $A$ and $B$. Rather than doing this in general, we wish to give evidence to the special case that $E = E' = H$, a real Hilbert space (whose scalar product will be denoted $(\cdot,\cdot)$), and that $B(u) = u$. In fact, this is the situation that we will mainly consider from now on. Note that in this case, if $A$ is positively homogeneous of degree 1, we have by (2.9)

$$\sup_{b(u)=1} a(u) = \sup_{u\neq 0} \frac{(A(u), u)}{\|u\|^2} = \sup_{u\in S} (A(u), u), \tag{2.23}$$

where

$$S = \{u \in H : \|u\| = 1\}. \tag{2.24}$$

**Corollary 2.9.** *Let $H$ be a real, infinite-dimensional Hilbert space and let $A : H \to H$ be a strongly continuous gradient operator which is positively homogeneous of degree 1. Let*

$$M = \sup_{u\in S}(A(u), u), \qquad m = \inf_{u\in S}(A(u), u). \tag{2.25}$$

*Then $M, m$ are finite and moreover if $M > 0$ (resp., $m < 0$), it is attained and is the largest (resp., smallest) eigenvalue of $A$.*

*Remark 2.10.* The result just stated holds true under the weaker assumption that $A$ be compact, see [43, Theorem 1.2 and Remark 1.2], where also noncompact maps are considered. In this case, however, the condition $M > 0$ must be replaced by $M > \alpha(A)$, with $\alpha(A)$ the *measure of noncompactness* of $A$.

Among the 1-positively homogeneous operators, a distinguished subclass is formed by the *bounded linear* operators acting in $H$. Denoting such an operator with $T$, we first recall (see, e.g., [8]) that $T$ is a gradient if and only if it is *self-adjoint* (or *symmetric*), that is, $(Tu, v) = (u, Tv)$ for all $u, v \in H$. Next, a classical result of functional analysis (see, e.g., [42]) states that if a linear operator $T : H \to H$ is self-adjoint and compact, then it has at least one eigenvalue. The precise statement is as follows: put

$$\lambda_1^+(T) \equiv \sup_{u\in S}(Tu, u), \qquad \lambda_1^-(T) \equiv \inf_{u\in S}(Tu, u). \tag{2.26}$$

Then $\lambda_1^+(T) \geq 0$, and if $\lambda_1^+(T) > 0$ then it (is attained) is the largest eigenvalue of $T$. Similar statements—with reverse inequalities—hold for $\lambda_1^-(T)$. Evidently, these can be proven as particular cases of Corollary 2.9, except for the nonstrict inequalities, which are due to our assumptions that $H$ has infinite dimension and that $T$ is compact. Indeed, if for instance, we had $\lambda_1^+(T) < 0$, then the very definition (2.26) would imply that $|(Tu, u)| \geq \alpha\|u\|^2$ for some $\alpha > 0$ and all $u \in H$, whence it would follow (by the Schwarz' inequality) that $\|Tu\| \geq \alpha\|u\|$ for all $u \in H$, implying that $T$ has a bounded inverse $T^{-1}$ and therefore that $S = T^{-1}T(S)$ is compact, which is absurd. Finally, note that $\lambda_1^+(T) = \lambda_1^-(T) = 0$ can only happen if $(Tu, u) = 0$ for all $u \in H$, implying that $T \equiv 0$ [42]. The conclusion is that *any compact self-adjoint operator has at least one nonzero eigenvalue* provided that it is not identically zero.

### *2.2. Higher Order Eigenvalues (for Linear and Nonlinear Operators)*

Let us remain for a while in the class of bounded linear operators. For these, the use of variational methods in order to study the existence and location of *higher order* eigenvalues is entirely classical and well represented by the famous *minimax principle* for the eigenvalues of the Laplacian [31]. By the standpoint of operator theory (see e.g., [44] or [45], Chapter

XI, Theorem 1.2), this consists in characterizing the (positive, e.g.) eigenvalues of a compact self-adjoint operator $T$ in a Hilbert space $H$ as follows. For any integer $n \geq 0$ let

$$\mathcal{U}_n = \{V \subset H : V \text{ subspace of dimension } \leq n\}, \tag{2.27}$$

and for $n \geq 1$ set

$$c_n(T) = \inf_{V \in \mathcal{U}_{n-1}} \sup_{u \in S \cap V^{\perp}} (Tu, u), \tag{2.28}$$

where $S = \{u \in H : \|u\| = 1\}$ and $V^{\perp}$ is the subspace orthogonal to $V$. Then

$$\left( \sup_{u \in S}(Tu, u) = \right) c_1(T) \geq c_2(T) \geq \cdots \geq c_n(T) \geq \cdots \geq 0, \tag{2.29}$$

and if $c_n(T) > 0$, $T$ has $n$ eigenvalues above 0, precisely, $c_i(T) = \lambda_i^+(T)$ for $i = 1, \ldots, n$ where $(\lambda_i^+(T))$ denotes the (possibly finite) sequence of *all* such eigenvalues, arranged in decreasing order and counting multiplicities. There is also a "dual" formula for the positive eigenvalues:

$$\lambda_n(T) = \sup_{V \in \mathcal{V}_n} \inf_{u \in S \cap V} (Tu, u), \tag{2.30}$$

where

$$\mathcal{V}_n \equiv \{V \subset H : V \text{ subspace of dimension } \geq n\}. \tag{2.31}$$

The above formulae (2.28)–(2.30) may appear quite involved at first sight, but the principle on which they are based is simple enough. Suppose we have found the first eigenvalue $\lambda_1(T) \equiv \lambda_1^+(T)(> 0)$ as in (2.26). For simplicity we consider just positive eigenvalues and so we drop the superscript +. Now, iterate the procedure: let

(i) $v_1 \in S$ be such that $Tv_1 = \lambda_1(T)v_1$;

(ii) $V_2 \equiv \{u \in H : (u, v_1) = 0\} \equiv {v_1}^{\perp}$;

(iii) $\lambda_2(T) \equiv \sup_{u \in S \cap V_2}(Tu, u)$.

Then $\lambda_1(T) \geq \lambda_2(T) \geq 0$, and if $\lambda_2(T) > 0$ then it is attained and is an eigenvalue of $T$: indeed—due to the symmetry of $T$—the restriction $T_2$ of $T$ to $V_2$ is an operator *in* $V_2$, and so one can apply to $T_2$ the same argument used above for $T$ to prove the existence of $\lambda_1$. Moreover, in this case, if we let

(i) $v_2 \in S$ be such that $Tv_2 = \lambda_2 v_2$,

(ii) $Z_2 \equiv [v_1, v_2] \equiv \{\alpha v_1 + \beta v_2 : \alpha, \beta \in \mathbb{R}\}$,

then it is immediate to check that

$$\lambda_2(T) = \inf_{u \in S \cap Z_2} (Tu, u). \tag{2.32}$$

Collecting these facts, and using some linear algebra, it is not difficult to see that

$$\lambda_2(T) = \inf_{V \in \mathcal{U}_1} \sup_{u \in S \cap V^{\perp}} (Tu, u) = \sup_{V \in \mathcal{V}_2} \inf_{u \in S \cap V} (Tu, u), \tag{2.33}$$

where

$$\mathcal{V}_2 \equiv \{V \subset H : V \text{ subspace of dimension } \geq 2\}. \tag{2.34}$$

For a rigorous discussion and complete proofs of the above statements, we refer the reader to [44, 45] or [5], for instance.

**Corollary 2.11.** *If $T : H \to H$ is compact, self-adjoint, and positive (i.e., such that $(Tu, u) > 0$ for $u \neq 0$), then it has infinitely many eigenvalues $\lambda_n^0$:*

$$\left(\sup_{u \in S}(Tu, u)\right) = \lambda_1^0 \geq \lambda_2^0 \geq \cdots \lambda_n^0 \geq \cdots > 0. \tag{2.35}$$

*Moreover, $\lambda_n^0 \to 0$ as $n \to \infty$.*

The last statement is easily proved as follows: suppose instead that $\lambda_n^0 \geq k > 0$ for all $n \in \mathbb{N}$. For each $n$, pick $u_n \in S$ with $Tu_n = \lambda_n^0 u_n$; we have $(u_n, u_m) = 0$ for $n \neq m$ because $T$ is self-adjoint. Then $T(u_n/\lambda_n^0) = u_n$, and the compactness of $T$ would now imply that $(u_n)$ contains a convergent subsequence, which is absurd since $\|u_n - u_m\|^2 = 2$ for all $n \neq m$.

We now finally come to the nonlinear version of the minimax principle, that is, the Lusternik-Schnirelmann (LS) theory of critical points for even functionals on the sphere [17]. There are various excellent accounts of the theory in much greater generality (see, for instance, Amann [7], Berger [8], Browder [9], Palais [10], and Rabinowitz [11, 21]), and so we need just to mention a few basic points of it, these will lead us in short to a simple but fundamental statement (Corollary 2.17), that is, a striking proper generalization of Corollary 2.11 and that will be used in Section 3.

For $R > 0$, let

$$S_R \equiv RS = \{u \in H : \|u\| = R\}. \tag{2.36}$$

If $K \subset S_R$ is *symmetric* (i.e., $u \in K \Rightarrow -u \in K$) then the *genus* of $K$, denoted $\gamma(K)$, is defined as

$$\gamma(K) = \inf\{n \in \mathbb{N} : \text{there exists a continuous odd mapping of } K \text{ into } \mathbb{R}^n \setminus \{0\}\}. \tag{2.37}$$

If $V$ is a subspace of $H$ with $\dim V = n$, then $\gamma(S_R \cap V) = n$. For $n \in \mathbb{N}$ put

$$K_n(R) = \{K \subset S_R : K \text{ compact and symmetric, } \gamma(K) \geq n\}. \tag{2.38}$$

In the search of critical points of a functional, the so-called *Palais-Smale condition* is of prime importance. For a continuous gradient operator $A : H \to H$ (with potential $a$), and for a given $R > 0$, put

$$D(x) \equiv A(x) - \frac{(A(x), x)}{R^2} x \quad (x \in H) \tag{2.39}$$

and call $D$ the *gradient of* $a$ *on* $S_R$. Essentially, for a given $x \in S_R$, $D(x)$ is the tangential component of $A(x)$, that is, the component of $A(x)$ on the tangent space to $S_R$ at $x$.

*Definition 2.12.* Let $A : H \to H$ be a continuous gradient operator and let $a$ be its potential. $a$ is said to satisfy the *Palais-Smale condition at* $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ for short) on $S_R$ if any sequence $(x_n) \subset S_R$ such that $a(x_n) \to c$ and $D(x_n) \to 0$ contains a convergent subsequence.

**Lemma 2.13.** *Let $A : H \to H$ be a strongly continuous gradient operator and let $a : H \to \mathbb{R}$ be its potential. Suppose that $a(x) \neq 0$ implies $A(x) \neq 0$. Then $a$ satisfies* $(\mathrm{PS})_c$ *on* $S_R$ *for each* $c \neq 0$.

*Proof.* It is enough to consider the case $R = 1$. So let $(x_n) \subset S$ be a sequence such that $a(x_n) \to c \neq 0$ and

$$D(x_n) = A(x_n) - (A(x_n), x_n)x_n \longrightarrow 0. \tag{2.40}$$

We can assume—passing if necessary to a subsequence—that $(x_n)$ converges weakly to some $x_0$. Therefore, $A(x_n) \to A(x_0)$ and moreover—as $a$ is wsc—$a(x_n) \to a(x_0)$ and similarly $(A(x_n), x_n) \to (A(x_0), x_0)$. Thus, $a(x_0) = c \neq 0$ and therefore $A(x_0) \neq 0$ by assumption. It follows from (2.40) that $(A(x_n), x_n)x_n \to -A(x_0) \neq 0$. This first shows that $(A(x_0), x_0) \neq 0$ (otherwise we would have $(A(x_n), x_n)x_n \to 0$) and then implies—since $(x_n) = (A(x_n), x_n)^{-1}(A(x_n), x_n)x_n$—that $(x_n)$ converges to $-(A(x_0), x_0)^{-1}A(x_0)$, of course. □

*Example 2.14.* Here are two simple but important cases in which the assumption mentioned in Lemma 2.13 is satisfied.

(i) $A$ is a *positive* (resp., *negative*) operator, that is, $(A(u), u) > 0$ (resp., $(A(u), u) < 0$) for $u \in H$, $u \neq 0$.

(ii) $A$ is a positively homogeneous operator.

Indeed if $A$ is, for instance, positive, then in particular $A(u) \neq 0$ for $u \neq 0$, and so the conclusion follows because $a(u) \neq 0$ implies that $u \neq 0$. While if $A$ is positively homogeneous of degree say $\alpha$, then $a(u) = \langle A(u), u \rangle / (\alpha + 1)$ and so the conclusion is immediate.

**Theorem 2.15.** *Suppose that $A : H \to H$ is an odd strongly continuous gradient operator, and let $a$ be its potential. Suppose that $a(x) \neq 0$ implies $A(x) \neq 0$. For $n \in \mathbb{N}$ and $R > 0$ put*

$$C_n(R) \equiv \sup_{K_n(R)} \inf_K a(u), \tag{2.41}$$

*where* $K_n(R)$ *is as in* (2.38)*. Then*

$$\left(\sup_{S_R} a(u)\right) = C_1(R) \geq \cdots C_n(R) \geq C_{n+1}(R) \geq \cdots \geq 0. \tag{2.42}$$

*Moreover,* $C_n(R) \to 0$ *as* $n \to \infty$*, and if* $C_k(R) > 0$ *for some* $k \in \mathbb{N}$*, then for* $1 \leq n \leq k, C_n(R)$ *is a critical value of* $a$ *on* $S_R$*. Thus, there exist* $\lambda_n(R) \in \mathbb{R},\ u_n(R) \in S_R$ $(1 \leq n \leq k)$ *such that*

$$C_n(R) = a(u_n(R)), \tag{2.43}$$

$$A(u_n(R)) = \lambda_n(R)u_n(R). \tag{2.44}$$

*Remark 2.16.* A similar assertion holds for the negative minimax levels of $a$,

$$\left(\inf_{S_R} a(u)\right) = D_1(R) \leq \cdots D_n(R) \leq D_{n+1}(R) \leq \cdots \leq 0. \tag{2.45}$$

*Indication of the Proof of Theorem 2.15*

The sequence $(C_n(R))$ is nondecreasing because for any $n \in \mathbb{N}$, we have $K_n(R) \supset K_{n+1}(R)$ as shown by (2.38). Also, $C_1(R) = \sup_{S_R} a(u)$ because $K_1(R)$ contains all sets of the form $\{x\} \cup \{-x\}$, $x \in S_R$ [11]. For the proof that $C_n(R) \to 0$ as $n \to \infty$ we refer to Zeidler [46]. Finally, if $C_k(R) > 0$, since by Lemma 2.4 we know that $a$ satisfies (PS) at the level $C_k(R)$, it follows by standard facts of critical point theory (see any of the cited references) that $C_k(R)$ is attained and a critical value of $a$ on $S_R$.

**Corollary 2.17.** *Let* $A : H \to H$ *be an odd strongly continuous gradient operator, and suppose moreover that* $A$ *is positive. Then the numbers* $C_n(R)$ *defined in* (2.41) *are all positive. Thus, for each* $R > 0$*, there exists an infinite sequence of "eigenpairs"* $(\lambda_n(R), u_n(R)) \in \mathbb{R} \times S_R$ *satisfying* (2.43)-(2.44)*.*

In conjunction with Corollary 2.17, the following result—for which we refer to [8]—will be used to carry out our estimates in Section 3.

**Proposition 2.18.** *Let* $A_0 : H \to H$ *satisfy the assumptions of Corollary 2.17. Suppose moreover that* $A_0$ *is linear (and therefore is a linear compact self-adjoint positive operator in* $H$*). Then*

$$C_n^0(R) \equiv \sup_{K_n(R)} \inf_K a_0(u) = \frac{1}{2}\lambda_n^0 R^2, \tag{2.46}$$

*where* $a_0(u) = (1/2)(A_0(u), u)$ *and* $(\lambda_n^0)$ *is the decreasing sequence of the eigenvalues of* $A_0$*, as in Corollary 2.11.*

## 3. Nonlinear Gradient Perturbation of a Self-Adjoint Operator

In this section we restrict our attention to equations of the form

$$A(u) \equiv A_0(u) + P(u) = \lambda u, \tag{3.1}$$

in a real Hilbert space $H$, where,

(i) $A_0$ is a (linear) bounded self-adjoint operator in $H$;

(ii) $P$ is a continuous gradient operator in $H$.

We suppose moreover that

$$P(u) = o(\|u\|) \quad \text{as } u \longrightarrow 0. \tag{3.2}$$

Note that—due to the continuity condition on $P$—this is the same as to assume that $P(0) = 0$ and that $P$ is is Fréchet differentiable at 0 with

$$P'(0) = 0. \tag{3.3}$$

*Remark 3.1.* We are assuming for convenience that $P$ is defined on the whole of $H$, but it will be clear from the sequel that *our conclusions hold true when P is merely defined in a neighborhood of* 0. The only modification would occur in the first statement of Theorem 3.2, where the words "for each $R > 0$" should be replaced by "for each $R > 0$ sufficiently small."

As $P(0) = 0$, (3.1) possesses the *trivial solutions* $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$. Recall that a point $\lambda_0 \in \mathbb{R}$ is said to be a *bifurcation point* for (3.1) if any neighborhood of $(\lambda_0, 0)$ in $\mathbb{R} \times H$ contains nontrivial solutions (i.e., pairs $(\lambda, u)$ with $u \neq 0$) of (3.1). A basic result in this matter states that if $P$ satisfies (3.2), and if moreover $A_0$ is *compact* and $P$ is *strongly continuous* (so that $A = A_0 + P$ is strongly continuous), then each nonzero eigenvalue of $A_0 = A'(0)$ is a bifurcation point for (3.1), and in particular for any $R > 0$ sufficiently small, there exists a solution $(\lambda_R, u_R)$ such that

$$\|u_R\| = R \quad \text{for each } R, \qquad \lambda_R \longrightarrow \lambda_0 \quad \text{as } R \longrightarrow 0. \tag{3.4}$$

Essentially, this goes back to Krasnosel'skii [6, Theorem 6.2.2], who used a minimax argument of Lusternik-Schnirelmann type considering deformations of a certain class of compact, noncontractible subsets of the sphere $S_R$. Subsequently, the compactness (resp., strong continuity) conditions on $A_0$ (resp., on $P$) were removed and replaced by the assumption that $P$ should be of class $C^1$ near $u = 0$, by Böhme [47] and Marino [48], who strengthened the conclusions showing that in this case bifurcation takes place from every isolated eigenvalue of finite multiplicity of $A_0$ and moreover that for $R > 0$ sufficiently small, there exist (at least) *two* distinct solutions $(\lambda_R^i, u_R^i)$ satisfying (3.4) for $i = 1, 2$; "distinct" means here in particular that $u_R^1 \neq u_R^2$. Proofs of this result can be found also in Rabinowitz [11, Theorem 11.4] or in Stuart [49, Theorem 7.2], for example. Moreover, when $P$ is also odd, then the proper critical point theory of Lusternik and Schnirelmann for even functionals (briefly recalled in Section 2) can be further exploited to show that if $n$ is the multiplicity of $\lambda_0$, then for each $R > 0$ there are at least $2n$ distinct solutions $(\lambda_R^k, \pm u_R^k)$, $k = 1 \dots n$, which satisfy (3.4) for each $k$; see, for instance [11, Corollary 11.30]. Each of these sets of assumptions thus guarantees the existence of one or more families

$$\mathcal{F} = \{(\lambda_R, u_R) \mid 0 < R < R_0\}, \tag{3.5}$$

of solutions of (3.1) satisfying (3.4), that is, parameterized by the norm $R$ of the eigenvector $u_R$ for $R$ in an interval $]0, R_0[$ and bifurcating from $(\lambda_0, 0)$. In such situation, it is natural to study the *rate of convergence* of the eigenvalues $\lambda_R$ to $\lambda_0$ as $R \to 0$, and in order to perform such quantitative analysis we do strengthen and make more precise the condition (3.2) on $P$. Indeed throughout this section we consider a $P$ that satisfies the following basic growth assumption near $u = 0$:

$$P(u) = O\left(\|u\|^{q-1}\right) \quad \text{as } u \to 0 \text{ for some } q > 2, \tag{3.6}$$

that is, we suppose that there exist ($q > 2$ and) positive constants $M$ and $R_0$ such that

$$\|P(u)\| \leq M\|u\|^{q-1}, \tag{3.7}$$

for all $u \in H$ with $\|u\| \leq R_0$.

We suppose moreover that there exist constants $R_1 > 0$, $0 \leq k \leq K$ and $\beta, \gamma \in [0, \alpha]$, $\alpha \equiv q/2 > 1$ such that for all $u \in H$ with $\|u\| \leq R_1$,

$$k(A_0(u), u)^{\beta}\left(\|u\|^2\right)^{\alpha-\beta} \leq (P(u), u) \leq K(A_0(u), u)^{\gamma}\left(\|u\|^2\right)^{\alpha-\gamma}. \tag{3.8}$$

Note that as $A_0$ is a bounded linear operator, we have $\|A_0(u)\| \leq C\|u\|$ for some $C \geq 0$ and for all $u \in H$, which implies that $|(A_0(u), u)| \leq C\|u\|^2$ for all $u$. Inserting this in (3.8) thus yields

$$|P(u), u| \leq C_1\|u\|^{2\alpha} = C_1\|u\|^q, \tag{3.9}$$

for some $C_1 \geq 0$. On the other hand, (3.7) also implies—via the Cauchy-Schwarz inequality—a similar bound on $(P(u), u)$. Thus, we see that (3.8) is compatible with (3.7), and is essentially a more specific form of it carrying a sign condition on $P$. In our final Example 3.4, we will see that (3.8) is satisfied by the operator associated with simple power nonlinearities often considered in perturbed eigenvalue problems for the Laplacian. Before this, in the present section we develop eigenvalue estimates that follow by (3.7) and (3.8) in the general Hilbert space context.

### *3.1. NLEV Estimates via LS Theory*

In our first approach, we exploit LS theory in the simple form described in Section 2. We will therefore assume, in addition to the hypotheses already made in this section upon $A_0$ and $P$, that

(i) $P$ is odd;

(ii) $A_0$ is compact and $P$ is strongly continuous;

(iii) $A_0$ is positive and $P$ is nonnegative (i.e., $(P(u), u) \geq 0$ for all $u \in H$).

**Theorem 3.2.** (A) *Let $H$ be a real Hilbert space and suppose that*

(i) *$A_0$ is a linear, compact, self-adjoint, and positive operator in $H$;*

(ii) *$P$ is an odd, strongly continuous, gradient, and nonnegative operator in $H$.*

*Then for each fixed $R > 0$,* (3.1) *has an infinite sequence $(\lambda_n(R), u_n(R))$ of eigenvalue-eigenvector pairs with $\|u_n(R)\| = R$.*

(B) *Suppose in addition that $P$ satisfies* (3.2)*. Then for each $n \in \mathbb{N}$, $\lambda_n(R) \to \lambda_n^0$ as $R \to 0$, where $\lambda_n^0$ is the $n$th eigenvalue of $A_0$. Thus, each $\lambda_n^0$ is a bifurcation point for* (3.1)*.*

(C) *Suppose in addition that $P$ satisfies* (3.6)*. Then*

$$\lambda_n(R) = \lambda_n^0 + O\left(R^{q-2}\right) \quad \text{as } R \longrightarrow 0. \tag{3.10}$$

(D) *Finally, if in addition $P$ satisfies* (3.8)*, then as $R \to 0$ one has*

$$-K\left(\lambda_n^0\right)^{\gamma} R^{q-2} + o\left(R^{q-2}\right) \leq \lambda_n(R) - \lambda_n^0 \leq K\left(\frac{1}{\alpha} + 1\right)\left(\lambda_n^0\right)^{\gamma} R^{q-2} + o\left(R^{q-2}\right). \tag{3.11}$$

*Proof.* The conditions in (A) guarantee that $A \equiv A_0 + P$ satisfies the assumptions of Corollary 2.17. Therefore, for each $R > 0$, there exist an infinite sequence $C_n(R)$ of critical values and a corresponding sequence $(\lambda_n(R), u_n(R))$ of eigenvalue-eigenvector pairs satisfying (2.41)–(2.44). We will make use of these formulae to derive our estimates. The statement (B) is essentially due to Berger, see [8, Chapter 6, Section 6.7A]. As the third statement has been essentially proved elsewhere (see, e.g. [33]), it remains only to prove (D).

Let $a$, $a_0$ and $p$ be the potentials of $A$, $A_0$, and $P$, respectively. We have from (2.10)

$$a = a_0 + p, \qquad a_0(u) = \frac{1}{2}(A_0(u), u), \qquad p(u) = \int_0^1 \langle P(tu), u\rangle\, dt. \tag{3.12}$$

Also let $R_1 > 0$ be such that (3.8) holds for $\|u\| \leq R_1$. In the derivation of the estimates below, we assume without further mention that $\|u\| \leq R_1$.

*Step 1.* It follows from (3.8) that

$$k_1 a_0(u)^{\beta}\left(\|u\|^2\right)^{\alpha-\beta} \leq p(u) \leq K_1 a_0(u)^{\gamma}\left(\|u\|^2\right)^{\alpha-\gamma}, \tag{3.13}$$

where

$$k_1 := H\frac{2^{\beta-1}}{\alpha}, \qquad K_1 := \frac{2^{\gamma-1}}{\alpha}. \tag{3.14}$$

The definition (2.41) of $C_n(R)$ then shows, using (3.13) and (2.46), that

$$C_n^0(R) + k_1 C_n^0(R)^{\beta} R^{2(\alpha-\beta)} \le C_n(R) \le C_n^0(R) + K_1 C_n^0(R)^{\gamma} R^{2(\alpha-\gamma)}. \tag{3.15}$$

*Step 2*. Equation (2.44) implies in particular that

$$(A(u_n(R)), u_n(R)) = \lambda_n(R) R^2. \tag{3.16}$$

Whence—using (2.43) and (3.12)—we get

$$\begin{aligned} C_n(R) - \frac{1}{2}\lambda_n(R)R^2 &= a(u_n(R)) - \frac{1}{2}(A(u_n(R)), u_n(R)) \\ &= p(u_n(R)) - \frac{1}{2}(P(u_n(R)), u_n(R)). \end{aligned} \tag{3.17}$$

It also follows from (3.8) that

$$k_2 a_0(u)^{\beta}\left(\|u\|^2\right)^{\alpha-\beta} \le \frac{1}{2}(P(u), u) \le K_2 a_0(u)^{\gamma}\left(\|u\|^2\right)^{\alpha-\gamma}, \tag{3.18}$$

where

$$k_2 := k2^{\beta-1}, \qquad K_2 := K2^{\gamma-1}. \tag{3.19}$$

We see from (3.13) and (3.18) that both $p(u)$ and $(1/2)(P(u), u)$ vary in the interval of endpoints $k_1 a_0(u)^{\beta}(\|u\|^2)^{\alpha-\beta}$ and $K_2 a_0(u)^{\gamma}(\|u\|^2)^{\alpha-\gamma}$: indeed (as $\alpha > 1$) $\min\{k_1, k_2\} = k_1$, $\max\{K_1, K_2\} = K_2$. Therefore, writing for simplicity $u$ for $u_n(R)$, we have

$$\begin{aligned} \left|C_n(R) - \frac{1}{2}\lambda_n(R)R^2\right| &\le K_2 a_0(u)^{\gamma}\left(\|u\|^2\right)^{\alpha-\gamma} - k_1 a_0(u)^{\beta}\left(\|u\|^2\right)^{\alpha-\beta} \\ &\le K_2 a_0(u)^{\gamma}\left(\|u\|^2\right)^{\alpha-\gamma} \quad (\text{since } A_0 \ge 0) \\ &\le K_2 a(u)^{\gamma}\left(\|u\|^2\right)^{\alpha-\gamma} \quad (\text{since } P \ge 0) \\ &= K_2 C_n(R)^{\gamma} R^{2(\alpha-\gamma)} \quad (\text{by } (2.43)). \end{aligned} \tag{3.20}$$

*Step 3*. Using the right-hand side of (3.15), we get

$$\begin{aligned} K_2 C_n(R)^{\gamma} R^{2(\alpha-\gamma)} &\le K_2 \left\{ C_n^0(R) + K_1 C_n^0(R)^{\gamma} R^{2(\alpha-\gamma)} \right\}^{\gamma} R^{2(\alpha-\gamma)} \\ &= K_2 C_n^0(R)^{\gamma} \left\{ 1 + K_1 C_n^0(R)^{\gamma-1} R^{2(\alpha-\gamma)} \right\}^{\gamma} R^{2(\alpha-\gamma)} \\ &= K_2 \left( \frac{\lambda_n^0}{2} \right)^{\gamma} R^{2\alpha} \left\{ 1 + K_1 \left( \frac{\lambda_n^0}{2} \right)^{\gamma-1} R^{\epsilon} \right\}^{\gamma}, \end{aligned} \tag{3.21}$$

where we have replaced $C_n^0(R)$ with its value $(1/2)\lambda_n^0 R^2$—see (2.46)—and have put $\epsilon = 2(\gamma - 1) + 2(\alpha - \gamma) = 2(\alpha - 1) > 0$ as $\alpha > 1$. Thus, as $R \to 0$,

$$\left\{ 1 + K_1 \left( \frac{\lambda_n^0}{2} \right)^{\gamma-1} R^{\epsilon} \right\}^{\gamma} = 1 + O(R^{\epsilon}) = 1 + o(1), \tag{3.22}$$

so that

$$K_2 C_n(R)^{\gamma} R^{2(\alpha-\gamma)} \le K_2 \left( \frac{\lambda_n^0}{2} \right)^{\gamma} R^{2\alpha} (1 + o(1)). \tag{3.23}$$

Therefore, by (3.20), we end up this step with the estimate

$$\left| C_n(R) - \frac{1}{2} \lambda_n(R) R^2 \right| \le K_2 \left( \frac{\lambda_n^0}{2} \right)^{\gamma} R^{2\alpha} (1 + o(1)). \tag{3.24}$$

*Step 4* (upper bound). Using again the right hand side of (3.15) in (3.24), and then using again (2.46) we obtain

$$\begin{aligned} \frac{1}{2} \lambda_n(R) R^2 &\le C_n(R) + K_2 \left( \frac{\lambda_n^0}{2} \right)^{\gamma} R^{2\alpha} (1 + o(1)) \\ &\le C_n^0(R) + K_1 C_n^0(R)^{\gamma} R^{2(\alpha-\gamma)} + K_2 \left( \frac{\lambda_n^0}{2} \right)^{\gamma} R^{2\alpha} (1 + o(1)) \\ &= \frac{1}{2} \lambda_n^0 R^2 + K_1 \left( \frac{\lambda_n^0}{2} \right)^{\gamma} R^{q} + K_2 \left( \frac{\lambda_n^0}{2} \right)^{\gamma} R^{q} (1 + o(1)) \\ &= \frac{1}{2} \lambda_n^0 R^2 + Z \left( \frac{\lambda_n^0}{2} \right)^{\gamma} R^{q} + o(R^{q}), \end{aligned} \tag{3.25}$$

where

$$Z = K_1 + K_2 = K\frac{2^{\gamma-1}}{\alpha} + K2^{\gamma-1} = K2^{\gamma-1}\left(\frac{1}{\alpha}+1\right). \tag{3.26}$$

We conclude that as $R \to 0$

$$\lambda_n(R) \leq \lambda_n^0 + K\left(\frac{1}{\alpha}+1\right)\left(\lambda_n^0\right)^{\gamma} R^{q-2} + o\left(R^{q-2}\right). \tag{3.27}$$

*Step 5* (lower bound). Using now the left hand side of (3.15) in (3.24), and then using as before (2.46) we get

$$\begin{aligned}
\frac{1}{2}\lambda_n(R)R^2 &\geq C_n(R) - K_2\left(\frac{\lambda_n^0}{2}\right)^{\gamma} R^{2\alpha}(1+o(1)) \\
&\geq C_n^0(R) + k_1 C_n^0(R)^{\beta} R^{2(\alpha-\beta)} - K_2\left(\frac{\lambda_n^0}{2}\right)^{\gamma} R^{2\alpha}(1+o(1)) \\
&= \frac{1}{2}\lambda_n^0 R^2 + \left\{k_1\left(\frac{\lambda_n^0}{2}\right)^{\beta} - K_2\left(\frac{\lambda_n^0}{2}\right)^{\gamma}\right\} R^q + o(R^q) \\
&\geq \frac{1}{2}\lambda_n^0 R^2 - K_2\left(\frac{\lambda_n^0}{2}\right)^{\gamma} R^q + o(R^q).
\end{aligned} \tag{3.28}$$

We conclude that, as $R \to 0$,

$$\lambda_n(R) \geq \lambda_n^0 - K\left(\lambda_n^0\right)^{\gamma} R^{q-2} + o\left(R^{q-2}\right), \tag{3.29}$$

and this, together with (3.27), ends the proof of (3.11). □

### 3.2. NLEV Estimates via Bifurcation Theory

As already remarked, LS theory has a true global character from the standpoint of NLEV, in that for *any* fixed $R > 0$ it allows for the *"simultaneous consideration of an infinite number of eigenvalues"* $\lambda_n(R)$, if we can use the same words of Kato [4] for a situation involving nonlinear operators—though strictly parallel to that of compact self-adjoint operators, as shown by Corollaries 2.11 and 2.17.

In contrast, Bifurcation theory—at least in the way used here and based on the classical Lyapounov-Schmidt method, see for instance [23]—is (i) *local* (it yields information for $R$ small) and (ii) built starting from a *fixed* isolated eigenvalue of finite multiplicity of $A_0$: given such an eigenvalue $\lambda_0$, one reduces (via the Implicit Function Theorem) the original equation to an equation in the finite-dimensional kernel $N(\lambda_0) \equiv N(A_0 - \lambda_0 I)$. The use of he Implicit Function Theorem demands $C^1$ regularity on the operators involved, but dispenses from the assumptions made before of (oddness, positivity and) *compactness*.

These differences between Theorems 3.2 and 3.3 are stressed by the change of notation ($\lambda_0$ rather than $\lambda_n^0$) also in the formulae (3.11) and (3.31) for our estimates. On the other hand, the obvious relation existing between the two statements is that each nonzero eigenvalue of a compact operator is isolated and of finite multiplicity.

**Theorem 3.3.** (A) *Let $A_0$ be a bounded self-adjoint linear operator in a real Hilbert space $H$ and let $\lambda_0$ be an isolated eigenvalue of finite multiplicity of $A_0$. Consider* (3.1)*, where $P$ is a $C^1$ gradient map defined in a neighborhood of* 0 *in $H$ and satisfying* (3.6)*. Then $\lambda_0$ is a bifurcation point for* (3.1)*, and moreover if $\mathcal{F} = \{(\lambda_R, u_R) : 0 < R < R_0\}$ is any family of nontrivial solutions of* (3.1) *satisfying* (3.4)*, then the eigenvalues $\lambda_R$ satisfy the estimate*

$$\lambda_R = \lambda_0 + O\left(R^{q-2}\right) \quad \text{as } R \longrightarrow 0. \tag{3.30}$$

(B) *If, in addition, $P$ satisfies the condition* (3.8)*, then as $R \to 0$ one has*

$$k(\lambda_0)^{\beta} R^{q-2} + o\left(R^{q-2}\right) \leq \lambda_R - \lambda_0 \leq K(\lambda_0)^{\gamma} R^{q-2} + o\left(R^{q-2}\right). \tag{3.31}$$

*Proof.* Theorem 3.3 is merely a variant of Theorem 1.1 in [14]. We report here the main points of the proof of the latter—that makes systematic use of the condition (3.6)—and the improvements deriving by the use of the additional assumption (3.8).

Let $N = N(\lambda_0) \equiv N(A_0 - \lambda_0 I)$ be the eigenspace associated with $\lambda_0$, and let $W$ be the range of $A_0 - \lambda_0 I$. Then by our assumptions on $A_0$ and $\lambda_0$, $H$ is the orthogonal sum

$$H = N \oplus W. \tag{3.32}$$

Set $L = A_0 - \lambda_0 I$, $\delta = \lambda - \lambda_0$ and write (1.18) as

$$Lu + P(u) = \delta u. \tag{3.33}$$

Let $\Pi_1$, $\Pi_2 = I - \Pi_1$ be the orthogonal projections onto $N$ and $W$, respectively; then writing $u = \Pi_1 u + \Pi_2 u \equiv v + w$ according to (3.32) and applying in turn $\Pi_1$, $\Pi_2$ to both members of (3.33), the latter is turned to the system

$$\begin{gathered} \Pi_1 P(v + w) = \delta v, \\ Lw + \Pi_2 P(v + w) = \delta w. \end{gathered} \tag{3.34}$$

By the self-adjointness of $A_0$, we have $Lw \in W$ for any $w \in W$ and therefore $(Lu, u) = (Lw, w)$ for any $u = v + w \in H$. Now let $\mathcal{F} = \{(\lambda_0 + \delta_R, u_R) : 0 < R < R_0\}$ be as in the statement of Theorem 3.3. Then from (3.33),

$$(Lu_R, u_R) + (P(u_R), u_R) = \delta_R R^2, \tag{3.35}$$

for $0 < R < R_0$, and writing $u_R = v_R + w_R$ this yields

$$(Lw_R, w_R) + (P(u_R), u_R) = \delta_R R^2 \quad (0 < R < R_0). \tag{3.36}$$

Under assumption (1.5), the term $(P(u_R), u_R)$ in (3.36) is evidently $O(R^q)$. What matters is to estimate the first term $(Lw_R, w_R)$; we claim that the same assumption (1.5) also yields

$$(Lw_R, w_R) = o(R^q) \quad \text{as } R \longrightarrow 0. \tag{3.37}$$

Then (3.36) will immediately imply that $\delta_R = O(R^{q-2})$—which is (3.30)—and will thus prove the first assertion of Theorem 3.3. To prove our claim, we let $(\delta, u)$ be any solution of (3.33) and write $u = v + w,\ v \in N,\ w \in W$; then $(\delta, v, w)$ satisfies the system (3.34). The second of these equations is $Lw - \delta w = -\Pi_2 P(v + w)$ or, putting $H_\delta = -((L - \delta I)|_W)^{-1}$,

$$w = H_\delta \Pi_2 P(v + w). \tag{3.38}$$

As $P$ is $C^1$ near $u = 0$ and $P'(0) = 0$, a standard application of the Implicit Function Theorem guarantees the existence of neighborhoods $\mathcal{U}$ of $(0, 0)$ in $\mathbb{R} \times N$ and $\mathcal{W}$ of 0 in $W$ such that, for each fixed $(\delta, v)$ in $\mathcal{U}$, there exists a unique solution $w = w(\delta, v) \in \mathcal{W}$ of (3.38). Moreover, $w$ depends on a $C^1$ fashion upon $\delta$ and $v$ and

$$\|w(\delta, v)\| = o(\|v\|) \quad \text{as } v \longrightarrow 0,\ v \in N, \tag{3.39}$$

uniformly with respect to $\delta$ for $\delta$ in bounded intervals of $\mathbb{R}$. Our point is that using again the supplementary assumption (1.5), (3.39) can be improved (see [14]) to

$$\|w(\delta, v)\| = O\left(\|v\|^{q-1}\right) \quad \text{as } v \longrightarrow 0,\ v \in N, \tag{3.40}$$

uniformly for $\delta$ near 0.

Now to prove the claim (3.37), first observe that $L|_W : W \to W$ is a bounded linear operator, so that $\|Lw\| \le C\|w\|$ for some $C > 0$ and for all $w \in W$. Thus, $|(Lw, w)| \le C\|w\|^2$, and it follows by (3.40) that

$$(Lw(\delta, v), w(\delta, v)) = O\left(\|v\|^{2(q-1)}\right) \quad \text{as } v \longrightarrow 0. \tag{3.41}$$

Returning to the solutions $(\lambda_0 + \delta_R, u_R) \in \mathcal{F}$, and writing as above $u_R = v_R + w_R$, we can suppose—diminishing $R_0$ if necessary—that $(\delta_R, v_R, w_R) \in \mathcal{U} \times \mathcal{W}$ for all $R : 0 < R < R_0$. This implies by uniqueness that $w_R = w(\delta_R, v_R)$ for all $R : 0 < R < R_0$. The estimate (3.40) thus yields in particular that $\|w_R\| = O(\|v_R\|^{q-1})$ as $R \to 0$ and in turn (since $\|v_R\| \le \|u_R\| = R$), (3.41)) yields that $(Aw_R, w_R) = O(R^{2(q-1)})$ as $R \to 0$. Since $2(q - 1) > q$ (because $q > 2$), this implies (3.37).

In order to improve the rudimentary estimate (3.30), one has to look more closely at the term $(P(u_R), u_R)$ in (3.36). Indeed as shown in [14], under the stated assumptions on $P$ we also have

$$(P(u_R), u_R) = (P(v_R), v_R) + o(R^q) \quad \text{as } R \longrightarrow 0. \tag{3.42}$$

Using (3.37) and (3.42) in (3.36), we have therefore

$$\delta_R R^2 = (P(v_R), v_R) + o(R^q) \quad \text{as } R \longrightarrow 0. \tag{3.43}$$

To conclude the proof of Theorem 3.3, we introduce as in [14] constants $k_{\lambda_0}$ and $K_{\lambda_0}$ via the formulae

$$k_{\lambda_0} \equiv \inf_{0<\|v\|<R_0, v\in N} \frac{(P(v), v)}{\|v\|^q}, \qquad K_{\lambda_0} \equiv \sup_{0<\|v\|<R_0, v\in N} \frac{(P(v), v)}{\|v\|^q}. \tag{3.44}$$

These yield the inequalities

$$k_{\lambda_0}\|v\|^q \leq (P(v), v) \leq K_{\lambda_0}\|v\|^q \quad (v \in N,\ \|v\| < R_0). \tag{3.45}$$

We know that as $v \rightarrow 0$, $w(\delta, v) = o(\|v\|)$ and so $\|v + w(\delta, v)\| = \|v\| + o(\|v\|)$. It follows that as $R \rightarrow 0$, $\|v_R\| = R + o(R)$ for the solutions $(\lambda_0 + \delta_R, v_R + w_R) \in \mathcal{F}$; using this in (3.45), we conclude that

$$k_{\lambda_0} R^q + o(R^q) \leq \langle P(v_R), v_R\rangle \leq K_{\lambda_0} R^q + o(R^q) \quad (R \longrightarrow 0). \tag{3.46}$$

Replacing this in (3.43), we obtain the inequalities

$$k_{\lambda_0} R^{q-2} + o\left(R^{q-2}\right) \leq \lambda_R - \lambda_0 \leq K_{\lambda_0} R^{q-2} + o\left(R^{q-2}\right). \tag{3.47}$$

Note that these have been derived using merely the assumption (3.6), which implies that $|(P(u), u)| \leq M\|u\|^q$ for some constant $M$ in a neighborhood of $u = 0$ and thus guarantees that $k_{\lambda_0}$, $K_{\lambda_0}$ are finite. Suppose now that $A_0$ satisfies the additional assumption (3.8), that we report here for the reader's convenience:

$$k(A_0(u), u)^{\beta}\left(\|u\|^2\right)^{\alpha-\beta} \leq (P(u), u) \leq K(A_0(u), u)^{\gamma}\left(\|u\|^2\right)^{\alpha-\gamma}. \tag{3.48}$$

As $A_0 v = \lambda_0 v$ for $v \in N \equiv N(A_0 - \lambda_0 I)$, we have $(A_0(v), v) = \lambda_0\|v\|^2$ for such $v$ and therefore (3.48) yields

$$(P(v), v) \leq K(\lambda_0)^{\gamma}\|v\|^{2\gamma}\left(\|v\|^2\right)^{\alpha-\gamma} = K(\lambda_0)^{\gamma}\|v\|^q, \quad v \in N. \tag{3.49}$$

A similar inequality, based on the left-hand side of (3.48), provides a lower bound to $(P(v),v)$ for $v \in N$. It follows by the definitions (3.44) of $k_{\lambda_0},\ K_{\lambda_0}$ that

$$k_{\lambda_0} \geq k(\lambda_0)^{\beta}, \qquad K_{\lambda_0} \leq K(\lambda_0)^{\gamma}. \tag{3.50}$$

Using these in (3.47) yields the desired inequalities (3.31). □

*Example 3.4.* Let us now reconsider Example 1.3, and take in particular the basic example of a nonlinearity satisfying (HF0) and (HF1), namely,

$$f(x,s) = |s|^{q-2}s \quad (2 < q < 2^*). \tag{3.51}$$

In this case, we see from (1.19) that

$$(P(u),u) = \int_{\Omega} |u|^q \, dx = \|u\|_q^q. \tag{3.52}$$

The following inequality for functions of $H_0^1(\Omega)$ permits to estimate $(P(u),u)$.

**Proposition 3.5.** *Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ $(N > 2)$, let $2^*$ be defined by $1/2^* = 1/2 - 1/N$, and let $q$ be such that $2 \leq q \leq 2^*$. Then*

$$C\|u\|_2^q \leq \|u\|_q^q \leq D\|u\|_2^{q-(q-2)N/2}\|\nabla u\|_2^{(q-2)N/2}, \tag{3.53}$$

*for all $u \in H_0^1(\Omega)$, with*

$$C = |\Omega|^{-(q-2)/2}, \qquad D = S(2,N)^{(q-2)N/2}. \tag{3.54}$$

*Here $|\Omega|$ stands for the (Lebesgue) measure of $\Omega$ in $\mathbb{R}^N$ and $S(2,N)$ for the best constant of the Sobolev embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$:*

$$S(2,N) = \sup_{u \in W_0^{1,2}(\Omega)} \frac{\|u\|_{2^*}}{\|\nabla u\|_2}. \tag{3.55}$$

*Proof.* The proof of the left-hand side of (3.53) is very simple and amounts to verify the inequality

$$\int_{\Omega} |v|^q dx \geq |\Omega|^{-(q-2)/2}\left(\int_{\Omega} v^2 dx\right)^{q/2}, \tag{3.56}$$

which holds true for any $q \geq 2$ and for any measurable function $v$ on $\Omega$. To see this, first observe that (3.56) is trivial if $q = 2$. While if $q > 2$, then $q/2 > 1$, and so by Hölder's inequality,

$$\int_{\Omega} v^2 dx \leq \left(\int_{\Omega} |v|^q dx\right)^{2/q}\left(\int_{\Omega} dx\right)^{(q-2)/q} = |\Omega|^{(q-2)/q}\left(\int_{\Omega} |v|^q dx\right)^{2/q}. \tag{3.57}$$

It follows that

$$\left(\int_\Omega v^2 dx\right)^{q/2} \leq |\Omega|^{(q-2)/2}\left(\int_\Omega |v|^q dx\right), \tag{3.58}$$

which gives (3.56).

The proof of the right-hand side of (3.53) requires more work and is based on an interpolation inequality which makes use of Hölder's and Sobolev's inequality (see [41], e.g.). A detailed proof can be found in [34]. □

Consider the operators $A_0$ and $P$ in $H \equiv H_0^1(\Omega)$ defined as in (1.19). If we put

$$\beta = \alpha = \frac{q}{2}, \qquad \gamma = \frac{q}{2} - (q-2)\frac{N}{4}, \tag{3.59}$$

then (3.53) can be written as

$$C(A_0(u), u)^\beta \leq \langle P(u), u\rangle \leq D(A_0(u), u)^\gamma \left(\|u\|^2\right)^{\alpha-\gamma}, \tag{3.60}$$

and shows that $P$ satisfies (3.8) with $k = C$, $K = D$ and $\alpha$, $\beta$, $\gamma$ as shown in (3.59). It is straightforward to check (see, e.g., [21] or [11]) that $A_0$ and $P$ satisfy the remaining assumptions of Theorem 3.3. Therefore, we can use the inequality (3.31), that in the present case takes the form

$$C(\lambda_0)^{q/2}R^{q-2} + o\left(R^{q-2}\right) \leq \lambda_R - \lambda_0 \leq D(\lambda_0)^{q/2-(q-2)N/4}R^{q-2} + o\left(R^{q-2}\right). \tag{3.61}$$

Putting $\mu_R = (\lambda_R)^{-1}$, we then have a corresponding family $\{(\mu_R, u_R)\}$ of solutions of the original problem (1.15) such that, as $R \to 0$,

$$\mu_0\mu_R\left[C{\mu_0}^{-q/2}R^{q-2} + o\left(R^{q-2}\right)\right] \leq \mu_0 - \mu_R \leq \mu_0\mu_R\left[D{\mu_0}^{-q/2+\epsilon}R^{q-2} + o\left(R^{q-2}\right)\right], \tag{3.62}$$

where

$$\epsilon \equiv (q-2)\frac{N}{4} > 0. \tag{3.63}$$

Since $\mu_R = \mu_0 + o(1)$ anyway, this yields in turn

$$C\mu_0^2{\mu_0}^{-q/2}R^{q-2} + o\left(R^{q-2}\right) \leq \mu_0 - \mu_R \leq D\mu_0^2{\mu_0}^{-q/2+\epsilon}R^{q-2} + o\left(R^{q-2}\right), \tag{3.64}$$

as $R \to 0$, or putting $\alpha = 2 - q/2 = (4-q)/2$,

$$C{\mu_0}^{\alpha}R^{q-2} + o\left(R^{q-2}\right) \leq \mu_0 - \mu_R \leq D{\mu_0}^{\alpha+\epsilon}R^{q-2} + o\left(R^{q-2}\right). \tag{3.65}$$

We remark that (3.65) can be used for actual computation, in view of the expressions (3.54) of $C$ and $D$: indeed $S(2, N)$ is explicitly known for any $N$ [50] and can be found, for instance, in [51, page 151]. Some work on numerical computation of NLEV for equations of the form (1.15) can be found, for instance, in [52].

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