



Fermions bounded by kinks of false vacuum models

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ABSTRACT

In this work we address the question of existence of fermion bound-states and zero modes in the background of kinks of models presenting a structure with a false and a true vacuum. These models are important for the description of cosmological and condensed-matter systems. The spectrum of fermion bound-states on the background of kinks of a class of asymmetrical scalar field potentials is analytically obtained and the general validity of the result is argued.

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1. Introduction

The study of zero modes in topological structures appeared first in a work by Caroli, Gennes and Matricon [1] and, later in a work by Jackiw and Rossi [2]. After those seminal works, a large amount of works relating zero modes and topological structures have appeared in the literature. In fact, the presence of zero modes in such structures have important impact over the study of fractional quantum numbers [3], and in cosmic strings [4]. Very recently, Chu and Vachaspati [5] addressed the problem of fermion zero modes in kink–antikink structures. They have shown that there are bound states on kink–antikink pairs whose energy vanishes exponentially fast with separation of the kink and antikink, in contrast with it was obtained in [6]. This was done by studying the analytical solutions for the solitons of the $\lambda\phi^4$ model [7,8]. After that, Brihaye and Delsate [9] considered the case of fermion modes in the background of lump-like structures. Usually, lump-like structures are unstable [9,10]. Notwithstanding, even non-stable structures can be physically relevant as in the case of the so-called spharelons appearing in the electroweak model [11], which can be important to explain the baryon asymmetry of the Universe [12]. Another interesting question is that of the confining of fermions in the brane world scenarios [13,14].

On the other hand, as observed by Coleman and Callan [19,20], asymmetrical potentials can be important to describe nucleation processes on statistical physics, crystallization of a supersaturated solution, the boiling of a superheated fluid and even in the case of the evolution of cosmological models. In this last application, one can suppose that when the Universe have been created it was far from any vacuum state. As it has expanded and cooled down it evolved first to a false vacuum instead of the true one. Thus, in such a scenario, when the time goes by, the Universe should finally be settled in the true vacuum state. Furthermore, those classes of models present some fluctuating solutions as recently observed [15–18], which can be responsible for a retarded decaying process when compared with the non-fluctuating configurations [18]. Considering the possibility that our Universe could present such a kind of underlying structure, it is important to study the possible existence of fermion bound-states and zero modes in the background of kinks of asymmetrical scalar field potentials. Here we will show through analytical calculations, that those kind of potentials can support fermion bound-states in the background of kink-like structures. However, it is not a simple task to get even the kinks for this type of asymmetrical field potential and, certainly even more difficult to verify the possible existence of fermion bound-states. Thus, in order to circumvent this problem, we will treat with some piecewise potentials which present a discontinuity in its derivative. Despite of this, their solutions present a continuous energy density. The advantage is that, in each region the model is exactly solvable. As far as we know, it was in [21] that a model

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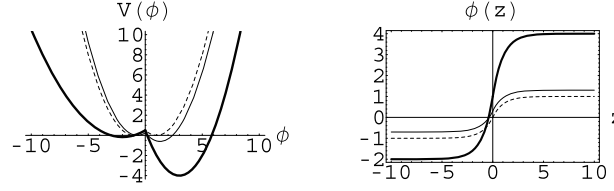


Fig. 1. Piecewise potentials and their respective kinks: The dashed line corresponds to the DQ model, the thin continuous line to the ADQ model for $\epsilon = 0.3$, and the thick line to the GADQ model with $V_1 = -1$, $V_2 = -8$, $\phi_1 = 2$, $\phi_2 = 3$ and $\lambda = 0.5$.

with those features was proposed for the first time. After that, the so-called Doubly Quadratic (DQ) model was introduced [22–25]. It is represented by the potential

$$V(\phi_{DQ}) = \frac{1}{2}\phi_{DQ}^2 - |\phi_{DQ}| + \frac{1}{2}. \quad (1)$$

More recently an asymmetrical version of this, the Asymmetrical Doubly Quadratic (ADQ) model represented by

$$V(\phi_{ADQ}) = \frac{1}{2}\phi_{ADQ}^2 - |\phi_{ADQ}| - \epsilon\phi_{ADQ} + \frac{1}{2}(\epsilon - 1)^2, \quad (2)$$

where $0 < \epsilon < 1$, was studied [24,18]. In this work, we will introduce a generalized version of the ADQ model, what we will call GADQ model. It has the advantage that, having the previous mentioned potentials as their limits, it can be used to study systems in which the curvature of the potential is different in each side of the discontinuity point. Moreover, the vacua of the model can be chosen to represent a kind of slow-roll potential, giving rise to inflaton fields which are important in cosmological inflationary scenarios. In fact, a similar model was used to study wet surfactant mixtures of oil and water [28]. The model is such that

$$V(\phi_{GADQ}) = \begin{cases} \lambda[(\phi_{GADQ} - \phi_2)^2 + V_2], & \phi \geq 0, \\ \lambda\left(\frac{\phi_2^2 + V_2}{\phi_1^2 + V_1}\right)[(\phi_{GADQ} + \phi_1)^2 + V_1], & \phi \leq 0, \end{cases} \quad (3)$$

where λ , ϕ_1 , ϕ_2 , V_1 and V_2 are constant parameters which obey the following restrictions

$$\phi_2 > 0, \quad \phi_1 > 0, \quad V_2 > -\phi_2^2 \quad \text{and} \quad V_1 > -\phi_1^2. \quad (4)$$

All the three potentials and their corresponding kinks are plotted in Fig. 1.

This work is organized as follows: In the next section we compute with some detail the zero mode and other bound states for the fermion in the presence of the kink of the DQ model. The next two sections are devoted to those solutions respectively in the cases of the ADQ and GADQ models. Finally, in the Conclusions section we trace some comments about the consequences of the existence of those fermion bound-states.

2. Fermion bound states in the presence of DQ model kink

The Lagrangian density in 1 + 1 dimensions for a fermion field coupled with a scalar field which is going to be used in the work given in [5],

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) + i\bar{\psi}\gamma^\mu \partial_\mu \psi - g\phi\bar{\psi}\psi, \quad (5)$$

where $\mu = 0, 3$, $V(\phi)$ is the potential, given in terms of a real scalar field, defining the bosonic sector of the specific model under analysis, ψ is the two component spinor, g is the corresponding Yukawa coupling constant and γ are the Dirac matrices which in this work will be written as

$$\gamma^t = \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma^z = i\sigma^1 = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (6)$$

In the particular case of the DQ model, the equations of motion from which the kinks come from, and the fermion modes in its fixed scalar field background should obey, are respectively written as

$$\partial^\mu \partial_\mu \phi_{DQ} + \phi_{DQ} - \frac{\phi_{DQ}}{|\phi_{DQ}|} = 0, \quad (7)$$

$$(i\gamma^\mu \partial_\mu - g\phi_{DQ})\psi = 0. \quad (8)$$

Note that, in the above equations we are assuming that the backreaction due to the Yukawa coupling between the Dirac field and the scalar one can be neglected [2,5], in other words, the scalar field behaves like a classical background field [26]. In this situation, it is not difficult to obtain the static configuration of the ϕ_{DQ} , which is given by

$$\phi_{DQ}(z) = \text{sign}(z - z_0)(1 - e^{-|z - z_0|}), \quad (9)$$

where we have chosen an arbitrary point z_0 such that, at its left $\phi_{DQ}(z) < 0$ and at the right $\phi_{DQ}(z) > 0$. Thus, when $z \rightarrow \pm\infty$ one gets $\phi_{DQ}(z) \rightarrow \pm 1$. Moreover, it was granted that the field and its derivative are continuous at the transition point.

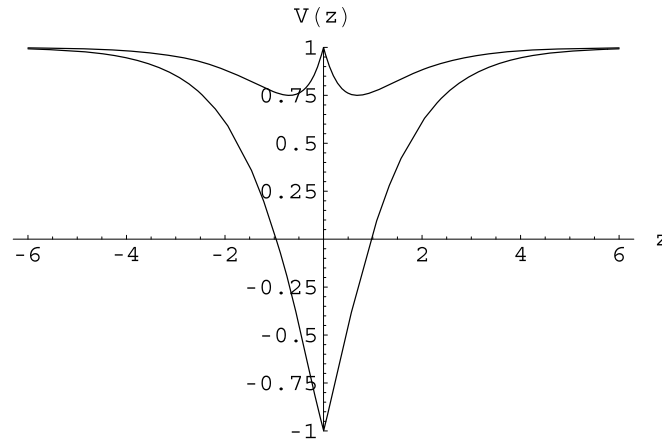


Fig. 2. Effective potentials of the DQ model with $g = 1$.

Now, in order to solve the Dirac equation (8), it is convenient to choose [5] that

$$\psi = \frac{e^{-iEt}}{\sqrt{2}} \begin{bmatrix} (\chi_+ - \chi_-) \\ (\chi_+ + \chi_-) \end{bmatrix}, \quad (10)$$

and the corresponding normalization constant is

$$\int_{-\infty}^{\infty} dz |\psi|^2 = \int_{-\infty}^{\infty} dz (|\chi_+|^2 + |\chi_-|^2) = 1. \quad (11)$$

In terms of the fields χ_{\pm} , the coupled equations for the components are

$$(\partial_z + g\phi_{DQ})\chi_+ = -E\chi_-, \quad (12)$$

$$(\partial_z - g\phi_{DQ})\chi_- = +E\chi_+. \quad (13)$$

They can be decoupled by, for instance, isolating the function χ_- in (12) and substituting it in (13), and vice-versa. After that procedure, one recovers a one-dimensional Schrödinger like equation. Those mentioned manipulations lead us to

$$-\partial_z^2 \chi_{\pm} + g(g\phi_{DQ}^2 \mp \partial_z \phi_{DQ})\chi_{\pm} = E^2 \chi_{\pm}, \quad (14)$$

where the effective potential $V_{\pm}(\phi_{DQ}) \equiv g(g\phi_{DQ}^2 \mp \partial_z \phi_{DQ})$ can ultimately be cast in the form

$$V_{\pm}(z) = g[g(e^{-2|z|} - 2e^{-|z|} + 1) \mp e^{-|z|}], \quad (15)$$

in the present case. For the sake of simplicity, we adopt $z_0 = 0$, which means simply that the center of the kink is at the origin of the coordinate system. It is important to remark that, in the interval $0 < g \leq 1/2$, the potential presents a barrier shape and, as a consequence, there is no room for the presence of bound states. Furthermore, in order to grant the existence of at least one bound state, one can apply a theorem introduced by Simon [27], where one subtracts g^2 from $V_{\pm}(z)$ and integrates over the entire real z -axis. The result of that integration shall be smaller than or equal to zero. In the case of DQ model, this establishes that for $g \geq 2/3$ the bound states are granted. The typical profile of the effective potential is represented in Fig. 2. In fact, it presents a double-well form when one deals with small coupling constants and it evolves for a simple well when g becomes large enough.

From the above analysis, one can see that in the range $0 < g < 2/3$, there are only zero mode states where $\chi_-(z) = 0$ and $E = 0$. In fact, as one can verify from its expression presented below, the only restriction over the value of the coupling constant is that $g > 0$, in order to grant the normalizability of the zero mode state. Thus, Eq. (12) becomes

$$\partial_z \chi_+ + g\phi \chi_+ = 0, \quad (16)$$

and the normalized solutions with $z_0 = 0$, are given by

$$\chi_+^{(0)}(z) = c \exp[-g(|z| + e^{-|z|})], \quad (17)$$

with $c \equiv \frac{(2g)^g}{\sqrt{2[\Gamma(2g) - \Gamma(2g, 2g)]}}$. Here, $\Gamma(z)$ stands for the gamma function and $\Gamma(a, z)$ is the incomplete gamma function. This lead us to the following spinor of the zero mode state,

$$\psi^{(0)}(z, t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_+^{(0)}(z) \\ \chi_+^{(0)}(z) \end{pmatrix}. \quad (18)$$

Now, in order to get more bound states, one should work with $g \geq 2/3$. In this case, the effective differential equation appears like

$$-\partial_z^2 \chi_{\pm} + V_{\pm} \chi_{\pm} = E^2 \chi_{\pm}, \tag{19}$$

and it was used that

$$\chi_{\mp} = \mp \frac{1}{E} \left[\frac{d\chi_{\pm}}{dz} \pm g\phi \chi_{\pm} \right], \tag{20}$$

in order to decouple the pair of first-order differential equations. Here, it is interesting to note that the potential $V_{\pm}(z)$ takes on a different functional form on either side of $z = 0$. Thus, we can write the pair of Eqs. (19) as

$$-\chi_{<}''(z) + V_{<}(z)\chi_{<}(z) = E^2 \chi_{<}(z), \quad z < 0, \tag{21}$$

$$-\chi_{>}''(z) + V_{>}(z)\chi_{>}(z) = E^2 \chi_{>}(z), \quad z > 0, \tag{22}$$

where the prime means derivative with respect to z , and $\chi(z) = \chi_{+}(z)$ or $\chi_{-}(z)$, $V(z) = V_{+}(z)$ or $V_{-}(z)$. At this point one can choose, without loss of completeness of solutions, one of the functions χ_{\pm} in order to generate the complete set of eigenstates. By choosing χ_{+} one can construct the spinor

$$\psi(z, t) = \frac{e^{-iEt}}{\sqrt{2}} \begin{bmatrix} \chi_{+}(z) + \chi_{-}(z) \\ \chi_{+}(z) - \chi_{-}(z) \end{bmatrix}. \tag{23}$$

The effective potential in both sides of the above differential equations can be written as

$$V_{<}(z) = Ae^{2z} + Be^z + A, \quad z \leq 0, \tag{24}$$

$$V_{>}(z) = Ae^{-2z} + Be^{-z} + A, \quad z \geq 0 \tag{25}$$

with $A \equiv g^2$, $B \equiv -2g^2 - g$. Then, we can rewrite Eqs. (21) and (22) as

$$\chi_{<}''(z) - (Ae^{2z} + Be^z + \omega^2)\chi_{<}(z) = 0, \quad z < 0, \tag{26}$$

$$\chi_{>}''(z) - (Ae^{-2z} + Be^{-z} + \omega^2)\chi_{>}(z) = 0, \quad z > 0 \tag{27}$$

with $\omega^2 \equiv A - E^2$.

Now, let us begin our search for a solution in the region where $z < 0$. It is interesting to note that this expression resembles the one of the Morse potential [30]. In order to get rid of the exponential terms in (26), it is usual to perform the transformation

$$y = Be^z. \tag{28}$$

Then, we get

$$y^2 \frac{d^2 \chi_{<}(y)}{dy^2} + y \frac{d\chi_{<}(y)}{dy} - \left(\frac{A}{B^2} y^2 + y + \omega^2 \right) \chi_{<}(y) = 0. \tag{29}$$

At this point we perform a function redefinition

$$\chi_{<}(y) = y^{\omega} e^{\alpha y} f_{<}(y) \tag{30}$$

where α is a positive definite constant. In this case, imposing that $\alpha \equiv -\sqrt{A}/B$, and recalling that $B \equiv -2g^2 - g$ and $g \geq 2/3$, we arrive at the differential equation

$$y \frac{d^2 f_{<}(y)}{dy^2} + [(2\omega + 1) - 2\sqrt{A}y/B] \frac{df_{<}(y)}{dy} - [(2\omega + 1)\sqrt{A}/B + 1] f_{<}(y) = 0. \tag{31}$$

Finally, by doing the variable scaling $y = B\tilde{y}/(2\sqrt{A})$, we get

$$\tilde{y} \frac{d^2 f_{<}(\tilde{y})}{d\tilde{y}^2} + (b - \tilde{y}) \frac{df_{<}(\tilde{y})}{d\tilde{y}} - af_{<}(\tilde{y}) = 0, \tag{32}$$

where we defined

$$b \equiv 2\omega + 1, \quad a \equiv \omega + \frac{1}{2} + \frac{B}{2\sqrt{A}}. \tag{33}$$

Eq. (32) can be recognized as a confluent hypergeometric equation [29], also known as the Kummer equation [31], whose complete solution can be written in the form

$$f_{<}(\tilde{y}) = \tilde{C}^{(1)} M(a, b; \tilde{y}) + \tilde{C}^{(2)} (\tilde{y})^{1-b} M(1 + a - b, 2 - b; \tilde{y}), \tag{34}$$

with the definitions

$$\tilde{C}^{(1)} \equiv C^{(1)} + \frac{C^{(2)}\pi}{\sin(\pi b)} \frac{1}{\Gamma(1 + a - b)\Gamma(b)}, \quad \tilde{C}^{(2)} \equiv -\frac{C^{(2)}\pi}{\sin(\pi b)} \frac{1}{\Gamma(a)\Gamma(2 - b)}. \tag{35}$$

Then, coming back to the original variables, one can write

$$\chi_{<}(z)_{DQ} = N_1 \exp[\omega z - g e^z] M(a, b; 2g e^z) + W_1 \exp(-\omega z - g e^z) M(1 + a - b, 2 - b; 2g e^z), \quad (36)$$

where

$$N_1 \equiv \tilde{C}^{(1)}(B)^\omega, \quad W_1 \equiv \tilde{C}^{(2)}(B)^\omega (2g)^{1-b}. \quad (37)$$

At this point we note that due to the asymptotic behavior of the Kummer function $M(\alpha, \beta; 2g e^z) = 1$ when $z \rightarrow -\infty$, and the presence of $\exp(-\omega z)$ in the second term of the above expression, we conclude that, in order to grant a normalized solution, we shall choose $W_1 = 0$.

This leads us finally to the following set of solutions:

$$\chi_{<}(z)_{DQ} = N_1 \exp(\omega z - g e^z) M(a, b; 2g e^z). \quad (38)$$

The solution in the region where $z > 0$ can be obtained through a quite analogous procedure and its normalizable solution is written as

$$\chi_{>}(z)_{DQ} = N_2 \exp[-\omega z - g e^{-z}] M(a, b; 2g e^{-z}). \quad (39)$$

Note that it is still necessary to impose the continuity of the function and its first derivative at the origin. At this point it is important to remark that the original physical spinor ψ only needs to be continuous, its derivative is not required to be continuous due to the fact that the probability current density does not depend on its derivative (in contrast with what happens in the non-relativistic case). However, since that spinor must be continuous and one of the $\chi(z)$ spinor components depends both on the $\chi_{\pm}(z)$ (see Eq. (23)) and the corresponding derivative, both the $\chi(z)$ component and its derivative must be continuous simultaneously. As a consequence, one must impose the continuity of the function and its derivative at the origin

$$\chi_{<}(0)_{DQ} = \chi_{>}(0)_{DQ}, \quad \chi'_{<}(0)_{DQ} = \chi'_{>}(0)_{DQ}. \quad (40)$$

This leads us to

$$\chi_{<}(z)_{DQ} = N_1 \exp(\omega z - g e^z) M(a, b; 2g e^z), \quad z \leq 0, \quad (41)$$

$$\chi_{>}(z)_{DQ} = \pm N_1 \exp[-\omega z - g e^{-z}] M(a, b; 2g e^{-z}), \quad z \geq 0. \quad (42)$$

Therefore, it can be seen that the eigenvalues are respectively determined by

$$\chi'_{<}(0)_{DQ} = \chi'_{>}(0)_{DQ} = 0 \quad \text{or} \quad \chi_{<}(0)_{DQ} = \chi_{>}(0)_{DQ} = 0, \quad (43)$$

that is, we obtain the even parity ($N_1 = N_2$) energy eigenvalues by requiring the vanishing of the first derivative of the solution at $z = 0$, and the odd parity ($N_1 = -N_2$) ones by requiring the vanishing of the solution at $z = 0$. Thus, we have

$$\begin{aligned} M(a + 1, b; 2g) &= 0, \quad \text{for } N_1 = N_2, \\ M(a, b; 2g) &= 0, \quad \text{for } N_1 = -N_2, \end{aligned} \quad (44)$$

and this allows us to numerically obtain the energy levels of the DQ model. At this point it is necessary to remark that, since the effective Schrödinger-like equation is invariant under the transformation $z \rightarrow -z$, the corresponding solutions will present definite parity. The energy eigenvalues of the system come from the above conditions, which are done by finding the zeroes of the Kummer functions. In fact, one should still remember that, since the energy values for the both components of the fermion states must be the same, one must to use $a = \omega - g$ and $b = 2\omega + 1$. Finally, since the acceptable values for the bound-state energies cannot be below the minimum of the effective potential and also not above of their asymptotic values, one must restrict the search for eigenvalues to the interval where $g - \frac{1}{4} \leq E^2 \leq g^2$. In Fig. 3, we present a comparison of the exact values of the fermion bound-state energies of the DQ and $\lambda\phi^4$ models, and in Fig. 4 it can be seen the plot of the functions χ_{\pm} of the first two levels in the case where $g = 2$.

In Fig. 3, one can observe that the number of allowed fermionic bound states of the DQ model is bigger than the one of the $\lambda\phi^4$ case. In part this can be explained because the two models have the same top value for the potential energy (g^2), but the bottom of DQ ($g - 1/4$) potential is lower than the one for the $\lambda\phi^4$ model (g). For instance, in the case where $g = 1$, beyond the zero-mode, the DQ model allows two bound states, respectively with energies $E_1 = 0.952837$ and $E_2 = 0.984584$, while the $\lambda\phi^4$ model permits only the existence of the zero-mode. As it should be expected from an inspection of the shape of the effective potentials, the higher excited states of the fermions become more and more close to each other, as a consequence of the fact that those energies are far from the bottom of the effective potential, where the difference of these models is much more evident. It is precisely this property that lead us to believe that one can deal with exact models like the DQ, in order to study expected features of smooth non-exactly solvable models.

3. The case of the ADQ model

Considering the potential defined in Eq. (2), the kink connecting the vacuum at $\phi_{ADQ} = -1 + \epsilon$ with the one at $\phi_{ADQ} = 1 + \epsilon$ (see Fig. 1), of the ADQ model [24,18], can be expressed as

$$\phi_{ADQ}(z) = \begin{cases} e^z - (1 - \epsilon), & z \leq 0, \\ (1 + \epsilon) - e^{-z}, & z \geq 0. \end{cases} \quad (45)$$

In this model the corresponding zero mode is written as

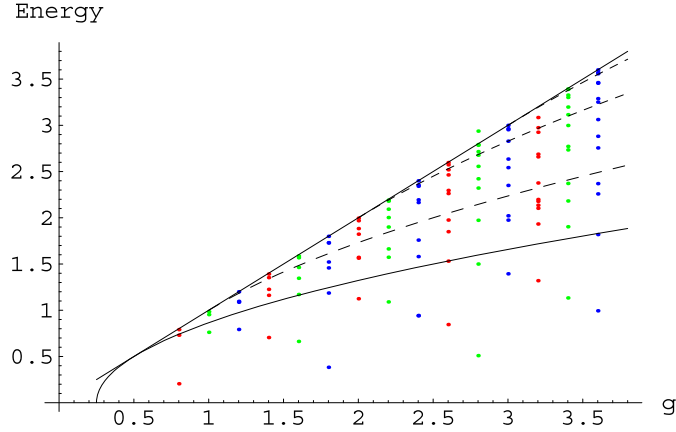


Fig. 3. Numerically calculated fermion bound-state energies of the DQ (dots) and $\lambda\phi^4$ (dashed lines) models. The valid energy values are the ones located between the two external solid lines.

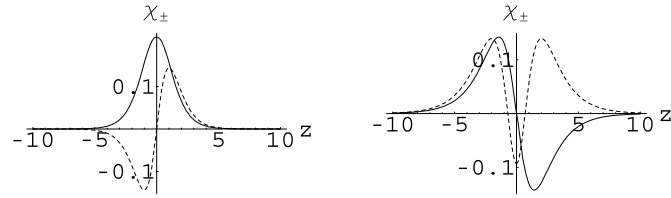


Fig. 4. χ functions of the first two eigenenergies of the DQ model with $g=2$. χ_+ is represented by the solid lines, and χ_- appears in the dashed ones.

$$\chi_+^{(0)}(z) = c_0 \exp\{-g[e^z - (1 - \epsilon)z]\}, \quad z \leq 0,$$

$$\chi_+^{(0)}(z) = c_0 \exp\{-g[e^{-z} + (1 + \epsilon)z]\}, \quad z \geq 0,$$

(46)

where

$$c_0 = \frac{(2g)^g}{\sqrt{(2g)^{-2g\epsilon} \Gamma[2g(1 + \epsilon), 0, 2g] - (2g)^{2g\epsilon} \Gamma[-2g(\epsilon - 1), 2g, 0]}},$$

(47)

with $\Gamma(a, z_0, z_1)$ being the generalized incomplete gamma function. Note that, as in the case of the DQ model, the only restriction over the coupling constant is that it must be positive definite.

Once more we use the representation where the spinor is given by (10). Thus, after similar calculations to that performed in the case of the DQ model, we get the following effective differential equations for the case of the ADQ model,

$$\chi_<''(z) - \{g^2 e^{2z} + [-2(1 - \epsilon)g^2 \mp g]e^z + g^2(1 - \epsilon)^2 - E^2\} \chi_<(z) = 0, \quad z < 0,$$

(48)

$$\chi_>''(z) - \{g^2 e^{-2z} + [-2(1 + \epsilon)g^2 \mp g]e^{-z} + g^2(1 + \epsilon)^2 - E^2\} \chi_>(z) = 0, \quad z > 0.$$

(49)

Once more, we choose $\chi_+(z)$ in order to generate the complete set of solutions.

Now applying again the Simon theorem, one can verify that the bound states are granted once more if $g \geq 2/3$. In this case one can arrive at the following normalizable solutions

$$\chi_<(z)_{ADQ} = N_1 \exp\left[z\sqrt{g^2(1 - \epsilon)^2 - E^2 - ge^z}\right] M(a_<, b_<; 2ge^z), \quad z < 0,$$

$$\chi_>(z)_{ADQ} = N_2 \exp\left\{-\left[z\sqrt{g^2(1 + \epsilon)^2 - E^2 + ge^{-z}}\right]\right\} M(a_>, b_>; 2ge^{-z}), \quad z > 0,$$

(50)

where

$$a_< = \omega_< - (1 - \epsilon)g, \quad b_< = 2\omega_< + 1, \quad \omega_< = \sqrt{g^2(1 - \epsilon)^2 - E^2},$$

$$a_> = \omega_> - (1 + \epsilon)g, \quad b_> = 2\omega_> + 1, \quad \omega_> = \sqrt{g^2(1 + \epsilon)^2 - E^2}.$$

From the continuity of the function at the origin ($\chi_<(0)_{ADQ} = \chi_<(0)_{ADQ}$) we get

$$N_1 = \frac{M(a_>, b_>; 2g)}{M(a_<, b_<; 2g)} N_2.$$

(51)

Furthermore, the continuity of the first derivative at $z=0$ ($\chi_<'(0)_{ADQ} = \chi_<'(0)_{ADQ}$) implies that

$$N_1 = -\left[\frac{\epsilon g M(a_>, b_>; 2g) + a_> M(a_> + 1, b_>; 2g)}{-\epsilon g M(a_<, b_<; 2g) + a_< M(a_< + 1, b_<; 2g)}\right] N_2.$$

(52)

Thus, one can arrive at the condition which determines the allowed energies. In this case it is obtained by substituting the relation (51) in (52), which lead us to

$$a_{<}M(a_{<} + 1, b_{<}; 2g)M(a_{>}, b_{>}; 2g) + a_{>}M(a_{>} + 1, b_{>}; 2g)M(a_{<}, b_{<}; 2g) = 0. \quad (53)$$

Observing the effective potential, one can conclude that the energies of the fermion bound-states shall be looked for in the interval $g(1 - \epsilon) - 1/4 \leq E^2 \leq g^2(1 - \epsilon)^2$. As it should be, the results of the DQ model are recovered by taking $\epsilon = 0$.

It is interesting to note that (53) in the limit where $\epsilon = 0$, is written as

$$M(a, b; 2g)M(a + 1, b; 2g) = 0, \quad (54)$$

from which (44) is recovered. Moreover, in that limit, Eqs. (51) and (52) become respectively: $N_1 = N_2$ and $N_1 = -N_2$.

The corresponding spinor ψ for the ADQ model can be straightforwardly obtained. Here we will not present those spinors explicitly for the sake of economy of space.

4. Fermion bounded by the GADQ kinks

As mentioned in the Introduction section, the GADQ model (Eq. (3)) compared with the previous two models presents, despite the fact that those models are limiting cases of it, the advantage of allowing to study systems in which the curvature of the potential is different in each side of the discontinuity point, as well as the possibility of regulating the distance of each minimum of the potential in relation to the point of discontinuity.

It is not difficult to show that the kink like solution (see Fig. 1), connecting the two vacua of the model, is given by

$$\phi_{\text{GADQ}} = -\phi_1 + \delta e^{az}, \quad z \leq 0, \quad (55)$$

$$\phi_{\text{GADQ}} = \phi_2 - \Delta e^{-bz}, \quad z \geq 0, \quad (56)$$

where we defined

$$\delta \equiv \left[\frac{(\phi_2 + \phi_1)\sqrt{\phi_1^2 + V_1}}{\sqrt{(\phi_1^2 + V_1) + \sqrt{(\phi_2^2 + V_2)}}} \right], \quad \Delta \equiv \left[\frac{(\phi_2 + \phi_1)\sqrt{\phi_2^2 + V_2}}{\sqrt{(\phi_1^2 + V_1) + \sqrt{(\phi_2^2 + V_2)}}} \right],$$

$$a \equiv \sqrt{2\lambda(\phi_2^2 + V_2)/(\phi_1^2 + V_1)} \quad \text{and} \quad b \equiv \sqrt{2\lambda}.$$

Usually, the kink like solutions are stable, at least when they are related to a topological charge as happens with the solutions coming from the first-order differential equations in the BPS [32] approach. In this regard, it is important to remark that in this work, and also in the cases considered in [18] and [24], we will in general be working with non-topological configurations, where the solutions come directly from the second-order differential equations, so that the stability is not granted from topological charges. Thus, the verification of the stability through a direct calculation becomes mandatory. The linear stability is checked by performing a small perturbation around the static exact configuration. So, we consider that

$$\phi(z, t) = \phi_{\text{class}}(z) + \eta(z, t). \quad (57)$$

Now, performing a Taylor expansion for $V(\phi)$ around the classical static solution $\phi_{\text{class}}(z)$, and keeping terms up to the second order in the perturbation $\eta(z, t)$ in the Lagrangian density, one gets after straightforward calculations that

$$-\frac{d^2 n(z)}{dz^2} + V_{st}(z)n(z) = E_{st}n(z), \quad \frac{d^2 T(t)}{dt^2} + E_{st}T(t) = 0, \quad (58)$$

where we made the usual separation of variables where $\eta(z, t) = n(z)T(t)$, and $V_{st} = d^2V(\phi)/d\phi^2|_{\phi=\phi_{\text{class}}}$. For the time-dependent equation (58) one obtains the solution

$$T(t) = ce^{t\sqrt{-E}} + de^{-t\sqrt{-E}}, \quad (59)$$

with c and d being arbitrary integration constants. It is simple to conclude that, in order to keep the solution finite and non-vanishing forever, one must to impose that $E \geq 0$, so keeping the kink stable. Otherwise the perturbation will destroy it when the time goes by.

For the GADQ model, we can write the potential as

$$V(\phi) = \lambda \left\{ [(\phi - \phi_2)^2 + V_2]S(\phi) + \left(\frac{\phi_2^2 + V_2}{\phi_1^2 + V_1} \right) [(\phi + \phi_1)^2 + V_1]S(-\phi) \right\}, \quad (60)$$

where $S(\phi)$ is the Heaviside function. Thus, the stability potential is given by

$$V_{st} = \lambda \left\{ 2S[\phi(z)] + 2\left(\frac{\phi_2^2 + V_2}{\phi_1^2 + V_1} \right) S[-\phi(z)] + \left[-2\phi_2 - 4\phi_1 \left(\frac{\phi_2^2 + V_2}{\phi_1^2 + V_1} \right) + \frac{2\phi_2^2\phi_1}{\phi_1^2 + V_1} + \frac{2V_2\phi_1}{\phi_1^2 + V_1} \right] \delta[\phi(z)] \right\}, \quad (61)$$

and the corresponding stability equation is written as

$$-\frac{d^2n(z)}{dz^2} + \lambda \left\{ 2S[\phi(z)] + 2\left(\frac{\phi_2^2 + V_2}{\phi_1^2 + V_1}\right)S[-\phi(z)] + \left[-2\phi_2 - 4\phi_1\left(\frac{\phi_2^2 + V_2}{\phi_1^2 + V_1}\right) + \frac{2\phi_2^2\phi_1}{\phi_1^2 + V_1} + \frac{2V_2\phi_1}{\phi_1^2 + V_1} \right] \delta[\phi(z)] \right\} n(z) = E_{st}n(z). \quad (62)$$

Solving the above equation, we obtain

$$E_{st} = \frac{2\lambda\{4c_0 - [1 + c_0 + (\phi_2V_1 + \phi_1V_2)^2/\phi_2^2]\}}{4[2 + 2c_0 + (\phi_2V_1 + \phi_1V_2)^2/\phi_2^2]}, \quad (63)$$

with $c_0 = 2\lambda(\phi_2^2 + V_2)/(\phi_1^2 + V_1)$. By using this result we obtain $E_{st} = -10.0304$ to $\lambda = 1/2$, $\phi_2 = 3$, $\phi_1 = 2$, $V_2 = -8$ and $V_1 = -1$, so indicating that this is an unstable solution. On the other hand, for the case that $\lambda = 1/2$, $\phi_2 = 1$, $\phi_1 = 1$, $V_2 = 0$ and $V_1 = 0$, we concluded that the ground-state solution of the stability equation have $E_{st} = 0$, so indicating that this is a stable configuration.

The zero mode solution for the GADQ is straightforwardly obtained as

$$\begin{aligned} \chi_+^{(0)}(z) &= c_0 \exp\left(g\phi_1 z - \frac{g\delta}{a} e^{az}\right), \quad z \leq 0, \\ \chi_+^{(0)}(z) &= c_0 \exp\left[g\left(\frac{\Delta}{b} - \frac{\delta}{a}\right)\right] \exp\left(-g\phi_2 z - \frac{g\Delta}{b} e^{-bz}\right), \quad z \geq 0, \end{aligned} \quad (64)$$

with $c_0 = \frac{1}{\sqrt{I_1 + I_2 \exp[g(\frac{\Delta}{b} - \frac{\delta}{a})]}}$, and

$$\begin{aligned} I_1 &= \frac{1}{a} (4)^{-g\phi_1/a} \left(\frac{g\delta}{a}\right)^{-2g\phi_1/a} [\Gamma(2g\phi_1/a) - \Gamma(2g\phi_1/a, 2g\delta/a)], \\ I_2 &= \frac{1}{b} (4)^{-g\phi_2/b} \left(\frac{g\Delta}{b}\right)^{-2g\phi_2/b} [\Gamma(2g\phi_2/b) - \Gamma(2g\phi_2/b, 2g\Delta/b)], \end{aligned} \quad (65)$$

which is also normalizable if $g > 0$ since, by construction a and b and, according to (4), ϕ_1 and ϕ_2 are all positive definite constants.

Now, the use of the Simon theorem lead us to following the restriction over the parameters which is necessary in order to grant the existence of at least one bound state. In this case one shall have

$$g \geq \frac{4ab\gamma}{b[4\gamma^2 - (\phi_1 - 2\gamma)^2] + a[4\gamma^2 - (\phi_2 - 2\gamma)^2]}, \quad (66)$$

with $\gamma \equiv (\phi_2 + \phi_1)\sqrt{\phi_1^2 + V_1}/[\sqrt{\phi_1^2 + V_1} + \sqrt{\phi_2^2 + V_2}]$. It is compatible with the corresponding restriction of the DQ and ADQ models in their respective limits.

In this case, after straightforward calculations, one can arrive at the following effective differential equations for the fermion bounded in this kink like configuration,

$$-\partial_z^2 \chi_{\pm} + (\mu^< e^{2az} + \xi_{\pm}^< e^{az} + \nu^<) \chi_{\pm} = E^2 \chi_{\pm}, \quad z < 0, \quad (67)$$

$$-\partial_z^2 \chi_{\pm} + (\mu^> e^{-2bz} + \xi_{\pm}^> e^{-bz} + \nu^>) \chi_{\pm} = E^2 \chi_{\pm}, \quad z > 0. \quad (68)$$

Again, we can simplify the problem by solving only Eqs. (21) and (22), where we defined

$$\begin{aligned} V_<(z) &\equiv (\mu^< e^{2az} + \xi^< e^{az} + \nu^<), & V_>(z) &\equiv \mu^> e^{-2bz} + \xi^> e^{-bz} + \nu^>, \\ \xi^< &\equiv -g\delta(2g\phi_1 + a), & \xi^> &\equiv -g\Delta(2g\phi_2 + b), & \nu^< &\equiv \phi_1^2 g^2, & \nu^> &\equiv \phi_2^2 g^2, \\ \mu^< &\equiv g^2 \delta^2, & \mu^> &\equiv g^2 \Delta^2. \end{aligned}$$

Repeating the same procedure used in the DQ and ADQ cases, one can arrive after a lengthy but straightforward computation at the normalizable solution

$$\begin{aligned} \chi_<(z)_{GADQ} &= N_1 \exp(k^< az - g e^{az}/a) M(\Lambda_{\pm}^<, 2\omega^< + 1; 2g e^{az}/a), \quad z < 0, \\ \chi_>(z)_{GADQ} &= N_2 \exp(-k^> bz - g e^{-bz}/b) M(\Lambda_{\pm}^>, 2\omega^> + 1; 2g e^{-bz}/b), \quad z > 0, \end{aligned} \quad (69)$$

where it was defined that $(k^<)^2 \equiv (\omega^</a)^2$, $(k^>)^2 \equiv (\omega^>/b)^2$, $\Lambda^< \equiv k^< + 1/2 + \xi^</(2g\phi_1 a)$ and $\Lambda^> \equiv k^> + 1/2 + \xi^>/(2g\phi_2 b)$. The continuity condition of the solution at the origin is such that one must to impose that

$$N_1 = \frac{e^{-g/b} M(\Lambda^>, 2\omega^> + 1; 2g/b)}{e^{-g/a} M(\Lambda^<, 2\omega^< + 1; 2g/a)} N_2. \quad (70)$$

Again we must impose the condition that $\chi'_{<}(0)_{GADQ} = \chi'_{>}(0)_{GADQ}$, thus we find

$$N_1 = -\frac{e^{-g/b} [(k^> b - g - b\Lambda^>) M(\Lambda^>, 2\omega^> + 1; 2g/b) + b\Lambda^> M(\Lambda^> + 1, 2\omega^> + 1; 2g/b)]}{e^{-g/a} [(k^< a - g - a\Lambda^<) M(\Lambda^<, 2\omega^< + 1; 2g/a) + a\Lambda^< M(\Lambda^< + 1, 2\omega^< + 1; 2g/a)]} N_2. \quad (71)$$

Once more we will not present the explicit expression of ψ which can be straightforwardly obtained. Finally using the relation (70) in (71) or vice-versa, one can verify that the following condition which must be respected by the self-energies of the GADQ fermionic bound states,

$$\begin{aligned} & M(\Lambda^<, 2\omega^< + 1; 2g/a) [(-k^>b - g - \Lambda^>b)M(\Lambda^>, 2\omega^> + 1; 2g/b) + \Lambda^>bM(\Lambda^> + 1, 2\omega^> + 1; 2g/b)] \\ & + M(\Lambda^>, 2\omega^> + 1; 2g/b) [(-k^<a - g - \Lambda^<a)M(\Lambda^<, 2\omega^< + 1; 2g/a) \\ & + \Lambda^<aM(\Lambda^< + 1, 2\omega^< + 1; 2g/a)] = 0. \end{aligned} \quad (72)$$

Note that $\omega^<$ and $\omega^>$ are defined as $\omega^< \equiv \sqrt{(g\phi_1)^2 - E^2}$ and $\omega^> \equiv \sqrt{(g\phi_2)^2 - E^2}$, where the energy must be the same for the two components of the spinor. Analyzing the numerically obtained energy levels of the GADQ for a fixed value of the coupling constant and a range of the asymmetry parameter ϕ_2 , it can be seen that the number of energy levels diminishes when the field vacua becomes far from each other.

Since the GADQ model allows the controlling of the points where the vacua are, the effective potential is correspondingly affected. So, the global minimum of the effective potential can be at right or at left of the discontinuity in the derivative. Considering the case where $V_{eff-}^< \leq V_{eff-}^>$, the energy eigenvalues will be in the interval

$$v^< - \xi_-^< \left(1 + \frac{\xi_-^<}{2v^<}\right) \leq E^2 \leq v^<, \quad (73)$$

where $\xi_-^< \equiv -g\delta(2g\phi_1 - a)$. As in the previous example, the fermion configurations of the GADQ model can recover those of the ADQ and DQ, simply by adjusting conveniently the set of parameters which define the potential profile. Thus, we have shown that a model with a vacuum structure which presents a true and a false vacuum can support fermion zero modes and bound states and that the case of a usual symmetric and degenerate potential is a particular case of this more general one.

5. Conclusions

In this work we have shown that models with a false vacuum, which are important in a number of physically important situations, can support not only fermion zero modes but also higher excited fermion bound-states. This has been done through a direct and exact construction of the fermion configurations in the presence of kink like bosonic configurations backgrounds. From inspection of the results coming from the DQ model in comparison with the $\lambda\phi^4$ one, one can be convinced that under certain conditions those models presenting derivative discontinuities may represent approximate solutions for more soft potentials. This is important because, as far as we know, there are no exact solutions for an asymmetric version of the $\lambda\phi^4$ model, but we can construct exact solutions in the case of the GADQ model. The existence of fermion bound-states in the background of asymmetric kinks can have important consequences for cosmological and condensed matter systems, specially in the case of models where the Universe presents a non-trivial evolutionary behavior. In particular, considering the temporal evolution of the kink configuration, one could expect the appearance of unbounded chiral fermions, coming from the bounded ones.

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