Chapter 2
What Is or What Might Be the Legacy of Felix Klein?

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Abstract Felix Klein was an outstanding mathematician with an international reputation. He promoted many aspects of mathematics, e.g. practical applications and the relation between mathematics and natural sciences but also the theory of relativity, modern algebra, and didactics of mathematics. In this article about the *The Legacy of Felix Klein* we firstly refer to his ideas in university teaching of mathematics teacher students and the three books "Elementary Mathematics from a higher (advanced) standpoint" from the beginning of the last century. Secondly we refer to his interests in school mathematics and his influence to the "Merano Resolution" (1905) where he pleaded for basing mathematics education on the concept of function, an increased emphasis on analytic geometry and an introduction of calculus in secondary schools. And thirdly we especially discuss the meaning and the importance of Klein's ideas nowadays and in the future in an international, worldwide context.

Keywords Felix Klein · History · Legacy · University teaching

2.1 Felix Klein as a Sensitised Mathematician

When we talk about the legacy of Felix Klein, we are interested in the significance of Felix Klein's work for mathematics and especially mathematics education, for our current theory and practice, and above all for tomorrow's ideas concerning the teaching and learning of mathematics. We are interested in Felix Klein as a mathematician, as a mathematics teacher; but most of all, we are interested in his ideas on teaching and learning mathematics, the problems he saw at university and at secondary school level, and the solutions that he suggested for these problems. We are interested in these solutions because we recognize that we are nowadays confronted with similar or even the same problems as 100 years ago (Klein 1909–1916). Talking about Felix Klein's legacy means hoping to find answers to some of the problems we

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are struggling with today. Talking about the Felix Klein's legacy today means giving answers to—at least—three basic questions:

- Which situations and which problems at the end of the 19th and the beginning of the 20th century can be seen in analogy to present situations?
- How did Felix Klein react to these problems and which solutions did he suggest?
- What do we know nowadays about the effect of the answers and solutions provided by Felix Klein 100 years ago?

Analogies between the situation 100 years ago and today can immediately be seen if we think about the current discussions concerning the goals and contents of teacher education at university level and especially the problems of students with the transition from high school to college or university and the transition back to high school. The problems with these transitions are expressed in Felix Klein's most famous statement, the "double discontinuity" from the introduction to the "Elementary mathematics from a higher standpoint, Volume I" (1908):

> The young university student finds himself, at the outset, confronted with problems, which do not remember, in any particular, the things with which he had been concerned at school. Naturally he forgets all these things quickly and thoroughly. When, after finishing his course of study, he becomes a teacher, he suddenly finds himself expected to teach the traditional elementary mathematics according to school practice; and, since he will be scarcely able, unaided, to discern any connection between this task and his university mathematics, he will soon fell in with the time honoured way of teaching, and his university studies remain only a more or less pleasant memory which has no influence upon his teaching. (Klein 2016 [1908], Introduction, Volume 1, p. 1)

When we hear the lamentations of today's university professors about the decreasing abilities of freshmen, and when we note the negative views of young teachers about the effects of their mathematics studies, you can surely be in doubt whether there has been any change, or indeed any change at all in the last 100 years.

However, we also know that answers to problems in education—not limited to mathematics education—can only be offered taking full recognition of the current political, social and scientific situation. Answers are not and will never be general statements, they always have to be newly evaluated in an ongoing process of discussion between different social groups. What is or what might be the impact of Felix Klein's ideas on these current discussion processes?

Felix Klein's life shows that it always needs a sensitised person to analyse the environment and to think in a visionary manner. Felix Klein is an example of just such a person who recognized problems, thought about solutions, suggested changes, was driven by external requests and changed his mind based on personal experiences. In the following we try to highlight some characteristics of Felix Klein we see as the background of his way of thinking and the basis of the legacy of Felix Klein.

2.2 Felix Klein Recognized Problems and Described Them in Detail

In 1872—at the age of 23—Felix Klein became professor at the University of Erlangen. In his inaugural address, the *Erlanger Antrittsrede* (see Rowe 1985)—which was not published during his lifetime and must not be confused with the "Erlangen programme"—he considered the dichotomy, the division, between humanistic and scientific education. He therefore felt there was a lack of widespread knowledge of mathematics in society. For Felix Klein mathematics had been a formal educational tool for training the mind and he claimed mathematics lessons at school were not "developing a proper feeling for mathematical operations or promoting a lively, intuitive grasp of geometry." (ibid., p. 139). Further, he voiced his view on mathematics education:

> We want the future teacher to stand *above* his subject, that he have a conception of the present state of knowledge in his field, and that he generally be capable of following its further development. (ibid., p. 128)

Felix Klein recognized problems concerning the acceptance of mathematics in society and deficits of school mathematics, but—at this age or state of his thinking—he did not have a detailed plan or strategy to solve these problems. But it was the beginning of a long standing lifelong involvement in mathematics education at the university and at secondary school level.

2.3 Felix Klein Thought About Solutions for Problems

Felix Klein wanted to improve secondary mathematics by improving the preparation of teachers.

> It is here that we, as university teachers of mathematics, have a wide, and hopefully rewarding, field for our activity. At stake is the task, precisely in the sense just mentioned, of raising the standards of mathematical education for later teaching candidates to a level that has not been seen for many years. If we educate better teachers, then mathematics instruction will improve by itself, as the old consigned form will be filled with a new, revitalized content! In recent years the situation has already improved in many respects, as the number of younger teachers. (Rowe 1985, p. 139)

"Better education" means—for Felix Klein—going beyond the contents of school level, but moreover, teachers should be aware of the present state of mathematics science.

> We want the future teacher to stand *above* his subject, that he has a conception of the present state of knowledge in his field, and that he generally be capable of following its further development. (ibid.)

Also nowadays, we—of course—support Felix Klein in his opinion on teachers standing above their subject, and we also agree and support him for wanting teacher

students to do “an independent research study” and asked for “mathematical exercises and seminars for student participants” (ibid.). In the meanwhile, bachelor or master thesis and seminars are compulsory for teacher students which means this is a possibility to integrate them into research studies, either in mathematics, mathematics education, pedagogy or psychology. But it was and still is an open question how this education influenced and influences mathematics teaching and learning at school. Moreover, the more general question can be asked of how the connection between school and university mathematics can be established.

2.4 Felix Klein Suggested Changes not Only in General, but also in a Specific Way

Criticizing mathematics teacher education, mathematics in school or the way mathematics is taught at school was and is quite popular. The present state of an education system is always a compromise and will never fulfil the widespread and sometimes contradictory interests of professors, teachers, students, parents, heads of schools, policymakers and economic people. But criticizing is only a first step; moreover, it is important to provide suggestions for changes or alternative ways of teaching. Felix Klein not only criticized education circumstances and thought about alternatives in a general way, he suggested changes in specific ways and presented very particular moves to new approaches.

In the following we give two examples for Klein’s ideas about changes in teacher education:

- In his *Antrittsrede* (inaugural address) 1880 at the university of Leipzig “Über die Beziehungen der neueren Mathematik zu den Anwendungen” (Concerning the connection between the newer mathematics and the applications—published first in 1895a), Felix Klein wanted to respond to the fragmentation of the science of mathematics by introducing *general* elementary as well as *specialization* lessons and—what was completely new at this time—a with his university colleagues concerted study plan for students.
- Nowadays, we have the suggested subdivision in the form of bachelor and master studies. But although these ideas might point in a direction of Felix Klein’s ideas, it cannot be assumed that he also had supported the reduction and bureaucratic regimentation of the bachelor studies especially.
- Teacher education at university should be restructured by introducing new lectures, seminars and student exercises especially. Felix Klein supported exercises at university because he saw the necessity of educating students to work individually and independently and he created working and reading rooms for students at the university. These suggestions are well-accepted nowadays and hit the spirit of the *Reform Pädagogik* at the beginning of the last century. Moreover, he emphasized the importance of individual *scientific homework for students*, nowadays called bachelor or master thesis (Klein 1895b).

2.5 Felix Klein Asked for Change Not Only on the Organizational Level, but He also Suggested Changes in the Way Mathematics Should Be Taught at University

The present discussion about the adequate way of teaching and learning mathematics at the university asks on the one side for the contents, the changes, and refreshment of the current contents, on the other side it asks for new methods of learning and teaching. While there is a common agreement on the importance of the traditional lectures "calculus" and "linear algebra", there are open questions about the necessity of "bridging-the-gap-lectures" and additional tutoring classes for freshmen or basic lectures in set theory, number theory, logics or computational mathematics. Concerning new teaching methods there are a lot of suggestions like integrating digital technologies, fostering self-reliance of students and introducing new concepts like the "inverted classroom", "learning by teaching", "research-based learning", "e-learning" or "blended learning" in university teaching.

For Felix Klein, the abstract character of mathematics was a big problem in teaching mathematics: "It is the great abstractness we have to combat"[1] (1895a, p. 538). He asked for more visualization or—in German—"Anschauung" in university lectures, but also in the whole learning process. "Anschauung" was of great importance not only for research but also for teaching. He saw "Anschauung" as a basis for a strict logical formal way of thinking. In this context, he had a wide view on "Anschauung"[2]:

- Working with graphs in the frame of functional thinking was part of "Anschauung".
- Felix Klein created collections of *geometrical models* at the universities of Erlangen, Munich, Leipzig and he completed the already existing collection in Göttingen (see also the article of Halverscheid and Labs in this book). He always emphasized the interrelationship between the representation of mathematical objects as models and in their symbolic form.
- Felix Klein always saw the connectivity between pure and applied mathematics. He pleaded for an education in applied mathematics and he even recommended a few semesters of study at a technical university for teacher students.[3]
- Moreover, Felix Klein saw the value of "new technologies" for universities, but also for high school teaching. In the first volume of "Elementary Mathematics from a higher Standpoint" (2016a [1908]) he recommended the calculation machine (Figs. 2.1 and 2.2), a tool which went into mass production towards the end of the 19th century and was widely used in industry and natural sciences: "Above all, every teacher of mathematics should be familiar with it." (2016a, p. 24). At these times, however, it was too expensive and too unwieldy to be actually used in class rooms. But he also expressed his wish or vision "that the calculating machine, in

[1] "Es ist ihre (*die der Mathematik, author*) große Abstraktheit, die wir bekämpfen müssen".

[2] For some information and more details, see the chapter "Intuitive Thinking and Visualisation" in this book.

[3] For an overview of the role of pure and applied mathematics in Germany, see Schubring (1989).

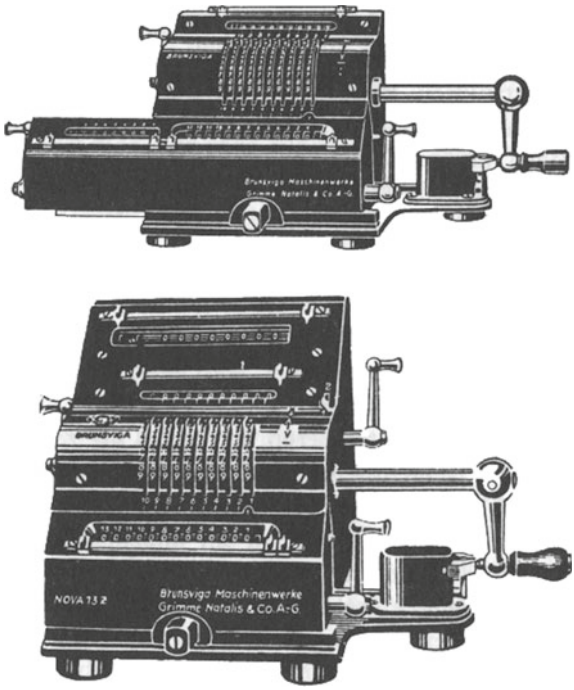

Fig. 2.1 Pictures in Klein (1908)

Fig. 2.2 A calculating machine from the beginning of the 20th century

view of its great importance, may become known in wider circles than is now the case." (ibid.).

The mechanical calculating machine is an example of a tool that enhances human skills by performing mechanical calculations quickly. But it is also a visualizer for

arithmetic calculation methods during multiplication, division or square root extraction.

2.6 Felix Klein Was—Like Many of Us—(also) Driven by External Requests, but When He Was Involved in an Activity, He Was Extensively Committed

Until 1900, Felix Klein criticized mathematics instruction at secondary school level, he gave some constructive proposals for changes, but he did not have or give an overarching strategy for new approaches.[4] In 1900, he was asked by the Prussian ministry to compile an expert report for changes in high school mathematics. Now he thought more deeply about mathematics classrooms and he suggested analytic geometry, descriptive geometry and calculus as new subjects for high school mathematics. With this external request, Felix Klein began his commitment to high school mathematics, leading to the Merano reform in 1905 and finally to the international involvement of Felix Klein as the first president of ICMI in 1908.

It is characteristic of a competent, committed and assertive person who is convinced of the correct goals which are recognised as important that he or she thinks globally about achieving these goals. Felix Klein wanted not only to change the curriculum in schools and teaching education at the university level, he also asked for a special in-service teacher training. At this point, he could build on his experience because he had previously organized courses for teachers during their holidays (see Tobies 2000). And—like always—Felix Klein saw the interrelationship of his activities: On the one hand, teacher training is professional development for the teacher, but on the other hand, he saw these courses as a possibility to give university teachers feedback on the effect of their teaching education.

Nowadays, "scaling up", or the transfer of research results to schools and classrooms, is an important aspect in educational research (e.g. Wylie 2008). To make this transfer constructive, a close cooperation of teachers, teacher educators, professors from universities, administration people and policy makers is necessary. Felix Klein's commitment in mathematics education at high school (Gymnasium) level is an example of the effect of the cooperation of different institutions in the education process.

[4] In 1898, Felix Klein presented his ideas about future structural changes of the high school system (Klein 1900) in public for the first time. See also Mattheis (2000).

2.7 Felix Klein Permanently Critically Considered and Reconsidered His Own Ideas

In his 1923 published memoirs ("Lebenserinnerungen"), Felix Klein mentioned that he already presented a "detailed programme" of his "planned teaching activities" in his inaugural address at Erlangen (Erlanger Antrittsrede). If you read the text of the Antrittsrede and especially the "summary of the Antrittsrede in fifteen points" (Rowe 1985, p. 125), you only recognize fragments of this programme.

Compared to his ideas in his inaugural address in 1872, Felix Klein later on—based on his experience at the Technical University Munich[5]—emphasised much more the meaning of applications in mathematics education, and he also changed his mind concerning teaching of mathematics at school and university. David E. Rowe summarizes these changes of mind:

> The 'Erlanger Antrittsrede' of 1872, presented herein, gives a clear expression of Klein's views on mathematics education at the very beginning of his career. While previous writers, including Klein himself, have stressed the continuity between the Antrittsrede and his later views on mathematics education, the following commentary presents an analysis of the text together with external evidence supporting exactly the opposite conclusion. (1985, p. 123)

Originally, Felix Klein saw the teaching of mathematics at the university not in relation to special lectures; later on he emphasized the importance of lectures like "Elementary mathematics from a higher standpoint". Initially, he was very cautious about new contents or subjects at secondary school, later on—especially in the *Meraner Lehrplan*—he emphasized the meaning of calculus as "the coronation of functional thinking".

These changes of mind should be seen very positively. It shows Felix Klein as a person, who continuously reflected his own ideas.

2.8 Final Remark

The "Legacy of Felix Klein" can only be understood and evaluated if you value the competent, committed and assertive person who reflected throughout his professional life and with his background as a mathematics scientist upon mathematics teaching and learning. We are convinced that Felix Klein is, in his attitude, belief and strength, an example for all people nowadays who are interested in improving mathematics education at university and high school. Many of Felix Klein's ideas can be reinterpreted in the context of the current situation, and give some hints and advice for dealing with problems in teacher education and teaching mathematics in secondary schools today.

[5] Technische Hochschule München.

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