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ABSTRACT

A family of oligopolies that possess a unique equilibrium was identified in the second authors doctoral dissertation. For such a family, it is therein specified a class of functions-economically related to the price function of a Cournot oligopoly – that satisfy a particular type of quasi-concavity. The first part of the present article (i) conceptualizes that type of quasi-concavity by introducing the notion of demi-concavity, (ii) considers two possible variants and (iii) provides some calculus properties. The second part, by relying on the results on demi-concavity, proves a Cournot equilibrium uniqueness theorem which is new for the journal literature and subsumes various results thereof. A third part shows an example that illustrates the novelty of the result.

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1. Introduction

Quasi-concavity is an important property in optimization theory and is often conveniently employed in game theory and economics. In the present article we introduce a special type of quasi-concavity which we refer to as demi-concavity. As far as we know, demi-concavity has not been systematically analyzed in the literature. The only exception – or perhaps the most important exception by far – is [1], where the closely related notion of semi-convexity is investigated.

Even though it might not be evident at a first glance, the structure of the problem of the necessity and sufficiency of the conditions for the existence of a unique Cournot equilibrium propounded in [2] is intrinsically related to the definition of semi-convexity enunciated in [1] and hence – by virtue of the characterization Theorem 1.6.2 in [1] – to our definition of demi-concavity. A relatively simple proof of equilibrium semi-uniqueness (i.e. of the existence of at most one equilibrium) in Cournot oligopolies with convex cost functions had been developed in the mentioned doctoral thesis. The method of that proof makes a crucial – but tacit – use of the properties of demi-concave functions. The subsequent articles [3,4] showed that in presence of convex cost functions such a method can be conveniently applied in the proof of equilibrium semi-uniqueness in Cournot oligopolies when either the 'integrated price flexibility' function or the 'industry revenue' function is concave; however, none of the two articles gives evidence of the role of demi-concavity in that method. The main purpose of this work is to clarify such a role with an equilibrium semi-uniqueness result (our Theorem 3.3) which is new for the journal literature and subsumes various results thereof. As we shall explain in the body of this work, the mere problem of the existence of at least one equilibrium is not in fact a real issue because of standard equilibrium existence results like that in [5].

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The paper consists of three parts: Section 2 on the notion of demi-concavity, Section 3 on the new equilibrium semi-uniqueness result and Section 4 on the novelty of that result. In Section 2 we conceptualize quasi-concavity introducing the qualitative definition of a (continuous) demi-concave function and of its variants; as our main results on equilibrium semi-uniqueness are generally valid only under certain assumptions on the differentiability of the price functions, most of our examination of the properties of demi-concave functions restricts attention to the differentiable case. (The necessity of this condition is investigated in [2] and various examples of the literature show that such a condition cannot be dispensed with.) In Section 3 we fix the oligopolistic setting and we present the new equilibrium semi-uniqueness result: Theorem 3.3. In Section 4 we first seek a specific class of price functions that satisfy the conditions imposed in Theorem 3.3 and then, by means of a numerical example, we show the novelty of that theorem in regard to that class. Appendix 1, with its Lemmas A.1-A.3 recalls some results of convex analysis for price functions.

Our examination of demi-concavity is restricted to some properties that apply to the main results of this work on equilibrium semi-uniqueness in Cournot oligopolies. Nevertheless, we maintain that a more independent and systematic analysis of demi-concavity is called for. Indeed, we maintain that such a notion of quasi-concavity can be frequently used in economics because of the very basic 'counting' structure of many problems in which a (possibly parametrized) objective function controlled by an agent is a real continuous quasi-concave function f on a real interval that satisfies the equality $f = g - h$, where g is usually interpreted as an 'income' function and h as a cost function. As it is clear from Section 2.2 below, if we assume that costs are increasing and convex – which is a standard assumption in many economic applications and which embodies the idea of decreasing returns to scale – and income is continuous, one of the weakest assumptions we can impose to guarantee the quasi-concavity of f is, in some loose sense, the demi-concavity of g: the stronger assumption of concavity of g is unnecessary and the weaker assumption of quasi-concavity of g does not generally guarantee the desired result. With the present article we hope also to contribute to the development of a strand of literature on this type of quasi-concavity.

2. Demi-concavity

2.1. Notions

Henceforth the letter I will denote a real interval and we shall use Euler's notation for derivatives (and hence, for instance, Dg will denote the derivative of a differentiable real-valued function g defined on a proper real interval). The interval I will be tacitly assumed to be proper when it will be the domain of a differentiable function. As usual, $\mathrm{Int}(I)$ will denote the topological interior of I, $\inf\left(I\right)$ the infimum of I in the extended reals and $\sup\left(I\right)$ the supremum of I in the extended reals. When the interval I will be proper, the interval $I \setminus \left\{\inf\left(I\right)\right\}$ will be denoted by $I^{\oplus}$ and the interval $I \setminus \left\{\sup\left(I\right)\right\}$ by $I^{\ominus}$. By a **sequential partition** (into two parts) of I we mean an ordered pair $\left(I_1, I_2\right)$ of possibly empty real intervals $I_1 \subseteq I$ and $I_2 \subseteq I$ such that $I_1 \cup I_2 = I$ and $I_1 < I_2$ (i.e. such that $x < y$ for all $\left(x, y\right) \in I_1 \times I_2$).

Definition 1: Let $g : I \to \mathbb{R}$ be a continuous function.

(1) g is **demi-concave** if there exists a sequential partition (I_1, I_2) of I such that $g\restriction_{I_1}$ is concave and $g\restriction_{I_2}$ is decreasing.
(2) g is **semi-strictly demi-concave** if there exists a sequential partition (I_1, I_2) of I such that $g\restriction_{I_1}$ is strictly concave and $g\restriction_{I_2}$ is decreasing.
(3) g is **strongly demi-concave** if g is differentiable and there exists a sequential partition (I_1, I_2) of I such that $g\restriction_{I_1}$ is strictly concave and $Dg < 0$ on I_2.

Remark 1: In Definition 1 (1), (2) the preliminary continuity condition is simply inessential. Nonetheless, in the results of this Section 2 such a condition is always present – mostly for the sake of expositional convenience – and hence we have preliminarily assumed it in Definition 1.

2.2. Some general properties

Proposition 2.1 relates the three notions of demi-concavity introduced so far with that of quasi-concavity.

Proposition 2.1: *Let $g : I \to \mathbb{R}$ be a continuous function. The following sequence of implications is true: g is strongly demi-concave $\Rightarrow$ g is semi-strictly demi-concave $\Rightarrow$ g is demi-concave $\Rightarrow$ g is quasi-concave.*

Proof: Only the last implication may not be evident. Its proof is as follows. Fix a sequential partition (I_1, I_2) of I such that $g\restriction_{I_1}$ is concave and $g\restriction_{I_2}$ is decreasing. As $g\restriction_{I_1}$ is concave, $g\restriction_{I_1}$ is quasi-concave and hence there exists a sequential partition (I_{11}, I_{12}) of I_1 such that $g\restriction_{I_{11}}$ is increasing and $g\restriction_{I_{12}}$ is decreasing. Thus $(I_{11}, I_{12} \cup I_2)$ is a sequential partition of I and g is decreasing on $I_{12} \cup I_2$ by the continuity of g. As g is increasing on I_{11}, we can conclude that the upper level sets of g are convex, or equivalently that g is quasi-concave. □

Definition 1 implies the following simple important result.

Theorem 2.2: *Let $g : I \to \mathbb{R}$ be a continuous function and let $h : I \to \mathbb{R}$ be continuous, decreasing and concave. If g is demi-concave (semi-strictly-demi-concave), then $g + h$ is demi-concave (semi-strictly-demi-concave).*

2.3. Some calculus properties

Lemma 2.3: *Suppose $g : I \to \mathbb{R}$ is differentiable. Besides suppose*

$$x_1, x_2 \in I,\ x_1 < x_2 \text{ and } Dg\left(x_1\right) > 0 \Rightarrow Dg\left(x_1\right) (>) \geq Dg\left(x_2\right).$$

Put $H_> := \left\{x \in I \mid Dg\left(x\right) > 0\right\}$ and $H_\leq := \left\{x \in I \mid Dg\left(x\right) \leq 0\right\}$.

(1) *The pair $\left(H_>, H_\leq\right)$ is a sequential partition of I.*
(2) *The function g is decreasing on $H_\leq$, strictly increasing on $H_>$ and (strictly) concave on $H_>$.*

Proof:

(1) As $H_> \cap H_\leq = \emptyset$ and $H_> \cup H_\leq = I$, we are done if we show that $\left(x_1, x_2\right) \in H_> \times H_\leq$ implies $x_1 \leq x_2$. By contradiction, suppose $x_1 \in H_>$, $x_2 \in H_\leq$ and $x_2 < x_1$. As Dg has the Darboux property, there exists $x_1^* \in]x_2, x_1[$ such that $Dg\left(x_1^*\right) = Dg\left(x_1\right)/2$. But then $Dg\left(x_1\right) > Dg\left(x_1^*\right) > 0$ and $x_1^* < x_1$, in contradiction with the implication in the statement of this lemma both in the case between parentheses and in that without.
(2) By part 1, $H_\leq$ and $H_>$ are intervals. Thus g is decreasing on $H_\leq$, strictly increasing on $H_>$ and the implication in the statement of this lemma entails the (strict) decreasingness of Dg on $H_>$ and hence the (strict) concavity of g on $H_>$. □

The statements after the double implications in Theorem 2.4 are meant to hold for every $x_1, x_2 \in I$ such that $x_1 < x_2$.

Theorem 2.4: *Suppose $g : I \to \mathbb{R}$ is differentiable.*

(1) *g is demi-concave $\Leftrightarrow$ $\left[Dg\left(x_2\right) > 0 \Rightarrow Dg\left(x_1\right) \geq Dg\left(x_2\right)\right]$.*
(2) *g is semi-strictly demi-concave $\Leftrightarrow$ $\left[Dg\left(x_2\right) > 0 \Rightarrow Dg\left(x_1\right) > Dg\left(x_2\right)\right]$.*
(3) *g is strongly demi-concave $\Leftrightarrow$ $\left[Dg\left(x_2\right) \geq 0 \Rightarrow Dg\left(x_1\right) > Dg\left(x_2\right)\right]$.*

Proof:

(1) '$\Rightarrow$'. Suppose g is demi-concave. Besides suppose $x_1, x_2 \in I$, $x_1 < x_2$ and $Dg\left(x_2\right) > 0$. Let $\left(I_1, I_2\right)$ be a sequential partition of I such that g is concave on I_1 and decreasing on I_2. If I_2 is not proper, then g is concave on I and hence $Dg\left(x_1\right) \geq Dg\left(x_2\right)$. Suppose I_2 is proper. Then Dg is non-positive on I_2. Thus $x_1, x_2 \in I_1$ and $Dg\left(x_1\right) \geq Dg\left(x_2\right)$ by the concavity of g on I_1.

'$\Leftarrow$'. Suppose the right-hand side of the double implication is true. Then, a fortiori, the part without parentheses of the implication in the statement of Lemma 2.3 is true and hence g is demi-concave by that lemma.

(2) '$\Rightarrow$'. Suppose g is semi-strictly demi-concave. Besides suppose $x_1, x_2 \in I$, $x_1 < x_2$ and $Dg\left(x_2\right) > 0$. Let $\left(I_1, I_2\right)$ be a sequential partition of I such that g is strictly concave on I_1 and decreasing on I_2. If I_2 is not proper, then g is strictly concave on I and hence $Dg\left(x_1\right) > Dg\left(x_2\right)$. Suppose I_2 is proper. Then Dg is non-positive on I_2. Thus $x_1, x_2 \in I_1$ and $Dg\left(x_1\right) > Dg\left(x_2\right)$ by the strict concavity of g on I_1.
'$\Leftarrow$'. Suppose the right-hand side of the double implication is true. Then, a fortiori, the part with parentheses of the implication in the statement of Lemma 2.3 is true and hence g is semi-strictly demi-concave by that lemma.

(3) '$\Rightarrow$'. Suppose g is strongly demi-concave. Besides suppose $x_1, x_2 \in I, x_1 < x_2$ and $Dg\left(x_2\right) \geq 0$. Let $\left(I_1, I_2\right)$ be a sequential partition of I such that g is strictly concave on I_1 and Dg is negative on I_2. Therefore $x_2 \in I_1$. Thus $x_1 \in I_1$ and $Dg\left(x_1\right) > Dg\left(x_2\right)$ by the strict concavity of g on I_1.
'$\Leftarrow$'. Suppose the right-hand side of the double implication is true. A fortiori, the right-hand side of the double implication of part 2 is true and by part 2 there exists a sequential partition $\left(I_1, I_2\right)$ of I such that g is strictly concave on I_1 and decreasing on I_2. If I_2 is not proper, then the differentiable function g is strictly concave (and hence strongly demi-concave) on I. Suppose I_2 is proper. Then Dg is non-positive on I_2. The function Dg is even negative on $I_2^{\oplus}$: if Dg vanished at some $x \in I_2^{\oplus}$, then the right-hand side of the double implication would entail the positivity of Dg on the non-empty subset $]\inf I_2, x[$ of $I_2^{\oplus}$ and this would contradict the non-positivity of Dg on $I_2^{\oplus}$. If $I_2 = I_2^{\oplus}$, then $\left(I_1, I_2\right)$ is a sequential partition characterizing the strong demi-concavity of g. If $I_2 \neq I_2^{\oplus}$, then $\inf I_2 \in I_2$ and $\left(I_1 \cup \{\inf I_2\}, I_2^{\oplus}\right)$ is a sequential partition characterizing the strong demi-concavity of g. □

Corollary 2.5 clarifies the relation between the implication used in the main results in [2] and the characterizing implication of Theorem 2.4 (2).

Corollary 2.5: *Suppose $g : I \to \mathbb{R}$ is a differentiable function. Then g is semi-strictly demi-concave if and only if*

$$x_1, x_2 \in I,\ x_1 < x_2 \text{ and } Dg\left(x_1\right) > 0 \ \Rightarrow\ Dg\left(x_1\right) > Dg\left(x_2\right).$$

Proof: The 'if' part follows from Lemma 2.3 with parentheses. The 'only if' part follows from Theorem 2.4 (2): the validity of the right-hand side of the double implication of Theorem 2.4 (2) implies, a fortiori, the validity of the implication in the statement of this corollary. □

We provide a final result used in a proof in Section 4.

Proposition 2.6: *A differentiable function $g : I \to \mathbb{R}$ is strongly demi-concave if and only if it is strongly demi-concave on $I^{\oplus}$.*

Proof: The 'only if' part is evident. Now we prove the 'if' part. Suppose g is strongly demi-concave on $I^{\oplus}$. We may suppose that $I \neq I^{\oplus}$. Let $a \in I$ be such that $I = I^{\oplus} \cup \{a\}$. Let (I_1, I_2) be a sequential partition of $I^{\oplus}$ such that $g\restriction_{I_1}$ is strictly concave and $Dg < 0$ on I_2. Now $(I_1 \cup \{a\}, I_2)$ is a sequential partition of I such that $g\restriction_{I_1}$ is strictly concave and $Dg < 0$ on I_2. Thus g is strongly demi-concave. □

Remark 2: Proposition 2.6 is false if one replaces $I^{\oplus}$ with $\mathrm{Int}(I)$. For instance, $\cos\restriction_{[0,\pi]}$ is strongly demi-concave on $]0, \pi[$ but not on $]0, \pi]$.

3. Cournot oligopolies

3.1. Setting

A **(homogeneous) Cournot oligopoly** is a game in strategic form with a player set $N := \{1, \ldots, n\}$ whose elements are called **firms**. We assume that each firm i has a strategy set X_i which is a (possible right-open and possibly unbounded) proper interval of $\mathbb{R}_+$ containing 0. The elements of X_i, that is the strategies, are also called **production levels** and those of the Minkowski-sum $Y := \sum_{l \in N} X_l$ **industry production levels**. Putting $\mathbf{X} := \mathsf{X}_{i=1}^n X_i$, each firm i's payoff function $u_i : \mathbf{X} \to \mathbb{R}$ is defined by

$$u_i(\mathbf{x}) := p(\sum_{l \in N} x_l)x_i - c_i(x_i)$$

and is called firm i's **profit function**. Henceforth $p : Y \to \mathbb{R}$ is called **price function** (also known as **inverse demand function**) and $c_i : X_i \to \mathbb{R}$ is called firm i's **(net) cost function**. A Nash equilibrium of a Cournot oligopoly is called a **(Cournot) equilibrium**. When either $X_i = [0, m_i]$ or $X_i = [0, m_i[$, firm i is said to have a **capacity constraint** m_i. When $X_i = [0, m_i]$ the capacity constraint will be called a **binding** capacity constraint. Needless to say, Y is a proper interval of $\mathbb{R}$ with $0 \in Y \subseteq \mathbb{R}_+$ and Y is compact if and only if $X_i = [0, m_i]$ for each player i; in such a case $Y = [0, \sum_{l \in N} m_l]$. The profit functions – as well as the set of equilibria – of these oligopolies do not depend on the value of p at 0. Thus only the **proper price function**

$$\tilde{p} := p \restriction Y^{\oplus}$$

matters (recall that $Y^{\oplus} = Y \setminus \{0\}$).

Before proceeding we introduce some useful notation for a function $g : J \to \mathbb{R}$ whose domain is a proper real interval $J \subseteq \mathbb{R}_+$. For $k \in \mathbb{R}_+$, let

$$J_k := (J - \{k\}) \cap \mathbb{R}_+.$$

Clearly $J_0 = J$ and J_k is an interval. For each $k \in J^{\ominus}$ the interval J_k is proper. When $k \in \mathbb{R}_+$, we define the function $g^{(k)} : J_k \to \mathbb{R}$ by

$$g^{(k)}(x) := g(x + k).$$

We define the function $r_g : J \to \mathbb{R}$ by

$$r_g := g \cdot \mathrm{Id}$$

and hence $r_g(x) = g(x)x$. When g vanishes nowhere, we define $\eta_g : J \to \mathbb{R}$ by

$$\eta_g := Dg \frac{\mathrm{Id}}{g}$$

and hence $\eta_g(x) = Dg(x)\frac{x}{g(x)}$. When g is positive and continuous and given $q \in J$, we define the function $L_g : J \to \mathbb{R}$ by

$$L_g(y) := y \ln g(y) - \int_q^y \ln g(\xi)\, \mathrm{d}\xi. \tag{1}$$

This function depends on q, but several of its properties (such as its concavity, its strict concavity and its decreasingness) do not depend on q. Clearly, when g is differentiable, then

$$DL_g = \eta_g. \tag{2}$$

When g is interpreted as a price function, we refer to r_g as the **industry revenue** associated to g, to L_g as the **integrated price flexibility of** g **(with respect to** q**)** and to η_g as the **price flexibility** of

g. We say that g has a **concave** (resp. **strictly concave**, **decreasing**, **strictly decreasing**) **integrated price flexibility** – and write that L_g is concave (resp. strictly concave, decreasing, strictly decreasing) – if there exists q such that L_g is concave (resp. strictly concave, decreasing, strictly decreasing).

The functions $\tilde{p}^{(k)} : (Y^{\oplus})_k \to \mathbb{R}$ (where $k \in \mathbb{R}_+$) will play an important role in our analysis of oligopolies. It is worth to remark that $(Y^{\oplus})_k = (Y \setminus \{0\} - \{k\}) \cap \mathbb{R}_+$, that $0 \in (Y^{\oplus})_k$ when $k \in Y^{\oplus}$ and that $(Y^{\oplus})_k$ is proper when $k \in Y^{\ominus}$.

3.2. The equilibrium (semi-)uniqueness result

In this subsection we consider the equilibrium uniqueness problem for Cournot oligopolies with convex cost functions. Proposition 3.1 shows a preliminary equilibrium existence result. The proof of Proposition 3.1 tacitly relies on the following variant for oligopolies of Theorem 1 in [6][1]: *An oligopoly has an equilibrium if $\tilde{p}$ is decreasing and for each firm i: u_i is continuous; u_i is quasi-concave in its i-th argument; there exists $\overline{x}_i \in X_i$ such that $p(x_i)c_i(x_i) < c_i(0)$ for every $x_i \in X_i$ with $x_i > \overline{x}_i$.* Note that the last condition holds if each firm has a binding capacity constraint.

Proposition 3.1: *Consider a Cournot oligopoly where for each firm i there exists $\overline{x}_i \in X_i$ such that $r_p(x_i) - c_i(x_i) \leq -c_i(0)$ for every $x_i \in X_i$ with $x_i > \overline{x}_i$. Further suppose that each cost function c_i is continuous, increasing and convex, the industry revenue function r_p is continuous and $\tilde{p}$ is decreasing. Then a sufficient condition for the existence of an equilibrium is that for every $k \in Y$ the function $r_{p^{(k)}}$ is demi-concave.*

Proof: Fix a player i. The continuity of r_p entails that of $\tilde{p}$. This implies that u_i is continuous at all $\mathbf{x} \neq \mathbf{0}$ as c_i is continuous. Also, u_i is continuous at $\mathbf{0}$ as

$$|u_i(\mathbf{x}) - u_i(\mathbf{0})| = |r_p(\sum_l x_l)\frac{x_i}{\sum_l x_l} - c_i(x_i) + c_i(0)| \leq |r_p(\sum_l x_l) - r_p(0)| + |c_i(0) - c_i(x_i)|$$

for all $\mathbf{x} \neq \mathbf{0}$. Suppose $r_{p^{(k)}}$ is demi-concave for every $k \in Y$. Fix the production level x_l of each firm $l \neq i$ and consider u_i as a function of its i-th argument. Putting $k := \sum_{l \neq i} x_l$, this function equals $r_{p^{(k)}} \upharpoonright X_i - c_i$ and is demi-concave by Theorem 2.2. Thus, by Proposition 2.1, it is also quasi-concave. □

Thus, for Cournot oligopolies like those in the previous proposition, equilibrium existence is not a real issue. However, this proposition does not guarantee equilibrium semi-uniqueness. Theorem 3.3 below provides sufficient additional conditions for such an oligopoly to have at most one equilibrium. The proof of that theorem will make use of the following lemma.

Lemma 3.2: *Suppose $\tilde{p}$ is positive and differentiable.*

(1) *If $r_{\tilde{p}^{(k)}}$ is demi-concave for every $k \in \mathrm{Int}(Y)$, then $D\tilde{p} \leq 0$.*
(2) *If $r_{\tilde{p}^{(k)}}$ is semi-strictly demi-concave for every $k \in \mathrm{Int}(Y)$, then $D\tilde{p} < 0$.*

Proof:

(1) Suppose $r_{\tilde{p}^{(k)}}$ is demi-concave for every $k \in \mathrm{Int}(Y)$. By contradiction, suppose $D\tilde{p}\left(y\right) > 0$ for some $y \in Y^{\oplus}$. As $D\tilde{p}$ has the Darboux property we can assume w.l.o.g. that y belongs to $\mathrm{Int}(Y)$. Put $\upsilon := y/2$ and note that also υ belongs to $\mathrm{Int}(Y)$ and hence υ is positive. As $\tilde{p}\left(y\right) = \tilde{p}^{(\upsilon)}\left(\upsilon\right)$ and $D\tilde{p}\left(y\right) > 0$, we have

$$D\tilde{p}^{(\upsilon)}\left(\upsilon\right) > 0; \tag{3}$$

so $Dr_{\tilde{p}^{(\upsilon)}}\left(\upsilon\right) = D\tilde{p}^{(\upsilon)}\left(\upsilon\right) \cdot \upsilon + \tilde{p}^{(\upsilon)}\left(\upsilon\right) > 0$ by the positivity of $\tilde{p}^{(\upsilon)}$ at υ. Theorem 2.4 (1) and the previous inequality imply the decreasingness of $Dr_{\tilde{p}^{(\upsilon)}}$ on $]0, \upsilon]$; so $r_{\tilde{p}^{(\upsilon)}}$ is concave on $]0, \upsilon]$. But then, by Lemma A.2 (3), we get a contradiction with (3): to verify this claim identify the restrictions of the functions $\tilde{p}^{(\upsilon)}$ and $r_{\tilde{p}^{(\upsilon)}}$ to $]0, \upsilon]$ of this Lemma 3.2 with the functions $\tilde{p}$ and $r_{\tilde{p}}$ in the statement of Lemma A.2.

(2) Suppose $r_{\tilde{p}^{(k)}}$ is semi-strictly demi-concave for every $k \in \mathrm{Int}(Y)$. By contradiction, suppose $D\tilde{p}\left(y\right) \geq 0$ for some $y \in Y^{\oplus}$. Part 1 ensures that $D\tilde{p}\left(y\right) \leq 0$. Thus $D\tilde{p}\left(y\right) = 0$ and $Dr_{\tilde{p}}\left(y\right) = \tilde{p}\left(y\right) > 0$. As $r_{\tilde{p}}$ is semi-strictly demi-concave, Theorem 2.4 (2) implies that $r_{\tilde{p}}$ is strictly concave on $]0, y]$. Lemma A.2 (3) applies and guarantees that $D\tilde{p} < 0$ on $]0, y]$: a contradiction. □

Henceforth $\mathbb{N}^*$ will denote the set of positive integers. When $\tilde{p}$ is differentiable, we fix an antiderivative P_k of $\tilde{p}^{(k)}$ for every $k \in Y^{\ominus}$ and we put

$$R^{k,s} := r_{\tilde{p}^{(k)}} + \left(s-1\right) P_k \tag{4}$$

for every $k \in Y^{\ominus}$ and $s \in \mathbb{N}^*$.

Theorem 3.3: *Consider a Cournot oligopoly and suppose each cost function c_i is convex. Suppose the proper price function $\tilde{p}$ is positive and differentiable. Besides suppose at least one of the following conditions holds.*

(Ia) *Every $R^{k,s}$ is strictly concave.*
(Ib) *Every $R^{k,s}$ is strongly demi-concave and every c_i is increasing.*
(IIa) *Every $R^{k,s}$ is concave and every c_i is strictly convex.*
(IIb) *Every $R^{k,s}$ is demi-concave and every c_i is strictly convex and increasing.*
(III) *Every $R^{k,s}$ is semi-strictly demi-concave and every c_i is strictly increasing.*

Then there exists at most one equilibrium.

***Proof*:** For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $M \subseteq N$ we write $x_M := \sum_{l \in M} x_l$.

Suppose $\mathbf{a}, \mathbf{b}$ are distinct equilibria and w.l.o.g. suppose

$$a_N \leq b_N.$$

Let $J := \{l \in N \mid a_l < b_l\}$ and $s := \#J$. Note that $\mathbf{b} \neq \mathbf{0}$, $b_N > 0$, $1 \leq s \leq n$, $a_J \leq a_N$, $b_J \leq b_N$, $a_J < b_J, b_J - a_J \geq b_N - a_N$ and $b_N - b_J \in Y^{\ominus}$.

As $r_{\tilde{p}^{(k)}} = R^{k,1}$, Lemma 3.2 implies

$$D\tilde{p} \begin{cases} \leq 0, \\ < 0 \text{ in cases Ia, Ib and III.} \end{cases} \tag{5}$$

We next prove by contradiction that

$$a_N > 0.$$

Suppose $a_N = 0$. Then $\mathbf{a} = 0$. As $J \neq \emptyset$, we can fix $j \in J$. So $b_j > a_j = 0$. As $\tilde{p}$ is decreasing, $\overline{p}(0) := \lim_{y \downarrow 0} \tilde{p}(y)$ exists in $\mathbb{R} \cup \{+\infty\}$ As c_j is convex, $D^+c_j(0) \neq +\infty$. This implies that the right partial derivative with respect to the j-th variable of u_i at $\mathbf{0}$ exists in $\mathbb{R} \cup \{+\infty\}$ and equals $\overline{p}(0) - D^+c_j(0)$. As $\mathbf{a}, \mathbf{b}$ are equilibria, first order conditions imply that

$$Dp(b_N)b_j + p(b_N) - D^-c_j(b_j) \geq 0 \geq \overline{p}(0) - D^+c_j(0).$$

But $Dp(b_N)b_j + p(b_N) - D^-c_j(b_j) \leq p(b_N) - D^-c_j(b_j) < \overline{p}(0) - D^+c_j(0)$, in contradiction with the previous centered inequalities. (The strict inequality holds as either c_j is strictly convex or $\tilde{p}$ is strictly decreasing.)

As $\mathbf{a}$ and $\mathbf{b}$ are equilibria and $a_N \neq 0$, we have that $D_i^+u_i(\mathbf{a}) \leq 0 \leq D_i^-u_i(\mathbf{b})$ for all $i \in J$. That is

$$Dp(a_N)a_i + p(a_N) - D^+c_i(a_i) \leq 0 \leq Dp(b_N)b_i + p(b_N) - D^-c_i(b_i) \tag{6}$$

for all $i \in J$. This implies $Dp(b_N)b_J + sp(b_N) \geq \sum_{l \in J} D^- c_l(b_l)$. As cost functions are convex and increasing in case Ib, strictly convex and strictly increasing in case IIb and convex and strictly increasing in case III, we obtain

$$Dp(b_N)b_J + sp(b_N) \begin{cases} \geq 0 \text{ in case Ib,} \\ > 0 \text{ in cases IIb and III.} \end{cases} \tag{7}$$

Next we prove that

$$Dp(a_N)a_J + sp(a_N) \begin{cases} > Dp(b_N)b_J + sp(b_N) \text{ in cases Ia, Ib and III,} \\ \geq Dp(b_N)b_J + sp(b_N) \text{ in cases IIa and IIb.} \end{cases} \tag{8}$$

Clearly, the previous inequalities are immediately implied by (5) when $a_N = b_N$ because $a_J < b_J$. Hence suppose $a_N < b_N$. As for each strategy profile $\mathbf{x}$ and subset J of N such that $x_N > 0$ and $x_N - x_J \in Y^{\ominus}$ the equality $DR^{x_N - x_J, s}(x_J) = Dp(x_N)x_J + sp(x_N)$ is true, (7) gives

$$Dp(b_N)b_J + sp(b_N) = DR^{b_N - b_J, s}(b_J) \begin{cases} \geq 0 \text{ in case Ib,} \\ > 0 \text{ in cases IIb and III.} \end{cases}$$

As $a_N > 0$, we see that $a_N - (b_N - b_J)$ belongs to the domain $(Y^{\oplus})_{b_N - b_J}$ of $R^{b_N - b_J, s}$. As $a_N - (b_N - b_J) < b_J$, from Theorem 2.4 we obtain that

$$DR^{b_N - b_J, s}(a_N - (b_N - b_J)) \begin{cases} > DR^{b_N - b_J, s}(b_J) \text{ in cases Ia, Ib and III,} \\ \geq DR^{b_N - b_J, s}(b_J) \text{ in case IIa and IIb.} \end{cases}$$

As $Dp(a_N) \leq 0$ and $a_N - (b_N - b_J) \geq a_J$, we have

$$DR^{b_N - b_J, s}(a_N - (b_N - b_J)) = Dp(a_N)(a_N - (b_N - b_J)) + sp(a_N) \leq Dp(a_N)a_J + sp(a_N)$$

and hence (8) is true.

As cost functions are convex and in cases IIa and IIb even strictly convex,

$$-\sum_{l \in J} D^+ c_l(a_l) \begin{cases} \geq -\sum_{l \in J} D^- c_l(b_l) \text{ in cases Ia, Ib and III,} \\ > -\sum_{l \in J} D^- c_l(b_l) \text{ in cases IIa and IIb} \end{cases}$$

and hence, by (8), we have that

$$Dp(a_N)a_J + sp(a_N) - \sum_{l \in J} D^+ c_l(a_l) > Dp(b_N)b_J + sp(b_N) - \sum_{l \in J} D^- c_l(b_l)$$

or equivalently that $\sum_{i \in J} (Dp(a_N)a_i + p(a_N) - D^+ c_i(a_i)) > \sum_{i \in J} (Dp(b_N)b_i + p(b_N) - D^- c_i(b_i))$, which is in contradiction with the inequalities in (6). □

Remark 3: In Theorem 3.3 no monotonicity property of the price function is mentioned. However, Lemma 3.2 shows that in all five cases a monotonicity condition is implicitly imposed on the proper price function.

Remark 4: In Theorem 3.3 the price function p may be discontinuous at 0. This possibility is important not only for games of Cournot competition but also for rent-seeking games – see for instance [7,8] – which are structurally analogous.

4. Examples

4.1. Concave industry revenue and concave integrated price flexibility

In Section 4.2 we shall prove that Theorems 4.1 and 4.2 below are consequences of Theorem 3.3, the main result of this work. Such a consequentiality is worth to be remarked in that Theorems 4.1 and 4.2 *essentially* subsume some results of the literature like Theorem 3 in [3] and Theorem 2 in [4] (which, in turn, respectively, improve upon some results in [9] for log-concave price functions and in [10] for price functions with an associated concave industry revenue). We have written *essentially* because some structural conditions on price functions (i.e. the possible non-positivity of the price function in Theorem 2 in [4] and the possible non-differentiability of the price function at the maximum of its domain in the case of binding capacity constraints in Theorem 2 in [4] and in Theorem 3 in [3]) are ruled out by the assumptions of Theorem 3.3. We remark, however, that at the cost of a more technical presentation even such minor generalisations could have been handled without difficulty.

Theorem 4.1: *Consider a Cournot oligopoly. Suppose each cost function c_i is convex, the proper price function $\tilde{p}$ is positive, differentiable and decreasing and its associated industry revenue function $r_{\tilde{p}}$ is concave. Any of the following three additional conditions is sufficient for the existence of at most one equilibrium.*

(I) *$r_{\tilde{p}}$ is strictly concave.*
(II) *Each c_i is strictly convex.*
(III) *Each c_i is strictly increasing and $r_{\tilde{p}}$ is decreasing.*

Theorem 4.2: *Consider a Cournot oligopoly. Suppose each cost function c_i is convex, the proper price function $\tilde{p}$ is positive, differentiable and decreasing and its integrated price flexibility $L_{\tilde{p}}$ is concave. If each cost function is increasing, then any of the following three additional conditions is sufficient for the existence of at most one equilibrium.*

(I) *$L_{\tilde{p}}$ is strictly concave.*
(II) *Each c_i is strictly convex.*
(III) *Each c_i is strictly increasing and $L_{\tilde{p}}$ is strictly decreasing.*

Note the similarity of the previous two theorems. In order to obtain such a similarity we have reformulated the above-mentioned results in [3,4] in terms of the proper price function $\tilde{p}$.

4.2. Proofs of Theorems 4.1 and 4.2

We shall prove Theorem 4.1 and Theorem 4.2 showing that they follow, respectively, from Theorem 3.3 (Ia), (IIa), (III) and Theorem 3.3 (Ib), (IIb), (III).

Proposition 4.3: *Suppose the proper price function $\tilde{p}$ is positive and differentiable and $r_{\tilde{p}}$ is concave.*

(1) *If $r_{\tilde{p}}$ is strictly concave, then each function $R^{k,s}$ is strictly concave.*
(2) *Each function $R^{k,s}$ is concave.*
(3) *If $r_{\tilde{p}}$ is decreasing, then each function $R^{k,s}$ is semi-strictly demi-concave.*

Proof: We preliminarily remark some facts. For all parts of Proposition 4.3, Lemma A.2 guarantees the decreasingness of $\tilde{p}$; thus every antiderivative P_k is concave and hence every derivative DP_k is decreasing. Consequently, every $(s-1)\,P_k$ is concave and every $(s-1)\,DP_k$ is decreasing. Note that each $R^{k,s}$ is differentiable and recall that

$$R^{k,s} = r_{\tilde{p}^{(k)}} + (s-1)\,P_k.$$

(1) Suppose $r_{\tilde{p}}$ is strictly concave. Lemma A.3 (2) guarantees the strict concavity of each $r_{\tilde{p}^{(k)}}$. Therefore each $R^{k,s}$ is strictly concave as it is the sum of the strictly concave function $r_{\tilde{p}^{(k)}}$ and of the concave function $(s-1)\,P_k$.

(2) Lemma A.3 (2) guarantees the concavity of each $r_{\tilde{p}^{(k)}}$. Thus each $R^{k,s}$ is concave as it is the sum of the concave functions $r_{\tilde{p}^{(k)}}$ and $(s-1)\,P_k$.
(3) Suppose $r_{\tilde{p}}$ is decreasing. We distinguish two cases.

Case $s+k=1$. Then $s=1$ and $k=0$ and hence $R^{k,s}=r_{\tilde{p}}$. Thus $R^{k,s}$ is decreasing (and hence semi-strictly demi-concave) as $R^{k,s}=r_{\tilde{p}}$ and $r_{\tilde{p}}$ is decreasing.

Case $s+k>1$. Then either $s>1$ or $k>0$. Lemma A.2 (4) guarantees the strict decreasingness of each $p^{(k)}$; so each $(s-1)\,p^{(k)}$ is decreasing and even strictly decreasing when $s>1$. Lemma A.3 (2) guarantees the decreasingness of each derivative $Dr_{\tilde{p}^{(k)}}$; Lemmas A.2 (4) and A.3 (3) guarantee the strict decreasingness of each derivative $Dr_{\tilde{p}^{(k)}}$ when $k>0$. Hence every $R^{k,s}$ is strictly concave (and hence semi-strictly demi-concave) as $DR^{k,s}=Dr_{\tilde{p}^{(k)}}+(s-1)\,p^{(k)}$ is strictly decreasing. □

Lemma 4.4: *Let $K\subseteq\mathbb{R}_{++}$ be a proper interval. Suppose $F_1: K\to\mathbb{R}$ is a differentiable function such that $\mathrm{Id}\cdot DF_1\,(\leq)\leq[<]\,1$. Pick $\lambda\in\mathbb{R}$ and suppose the implication*

$$x_2DF_1(x_2)+\lambda\,(\geq)\geq[>]0\ \Rightarrow\ x_1DF_1(x_1)+\lambda\,(>)\geq[\geq]x_2DF_1(x_2)+\lambda$$

is true for every $x_1,x_2\in K$ such that $x_1<x_2$. Fix an antiderivative F_2 of the avarage function e^{F_1}/Id of e^{F_1}. Then $e^{F_1}+\lambda F_2$ is (strongly demi-concave) demi-concave [semi-strictly demi-concave].

***Proof*:** Fix $x_1,x_2\in K$ such that $x_1<x_2$. By Theorem 2.4, we are done if we prove

$$D(e^{F_1}+\lambda F_2)(x_2)(\geq)\geq[>]0\ \Rightarrow D(e^{F_1}+\lambda F_2)(x_2)(>)\geq[>]D(e^{F_1}+\lambda F_2)(x_1). \tag{9}$$

Its proof is as follows. Denote e^{F_1}/Id by $\overline{e^{F_1}}$. As $De^{F_1}=e^{F_1}DF_1$ and $DF_2=\overline{e^{F_1}}$, we have to prove the validity of the implication

$$\begin{aligned}&\overline{e^{F_1}}(x_2)(x_2DF_1(x_2)+\lambda)\,(\geq)\,\geq[>]\,0\\ &\quad\Rightarrow\quad \overline{e^{F_1}}(x_1)(x_1DF_1(x_1)+\lambda)\,(>)\,\geq[>]\,\overline{e^{F_1}}(x_2)(x_2DF_1(x_2)+\lambda).\end{aligned}$$

Suppose $\overline{e^{F_1}}(x_2)(x_2DF_1(x_2)+\lambda)\,(\geq)\,\geq[>]\,0$. Then $x_2DF_1(x_2)+\lambda(\geq)\,\geq[>]\,0$ and

$$x_1DF_1(x_1)+\lambda(>)\,\geq[\geq]\,x_2DF_1(x_2)+\lambda\,(\geq)\,\geq[>]\,0. \tag{10}$$

As $\mathrm{Id}\cdot DF_1\,(\leq)\,\leq\,[<]\,1$, it follows that $D\overline{e^{F_1}}=e^{F_1}\frac{\mathrm{Id}\cdot DF_1-1}{\mathrm{Id}^2}=(\leq)\,\leq[<]\,0$ and therefore the function $\overline{e^{F_1}}$ is (decreasing) decreasing [strictly decreasing]. Thus

$$\overline{e^{F_1}}(x_1)(\geq)\,\geq[>]\,\overline{e^{F_1}}(x_2)>0. \tag{11}$$

Inequality (9) follows from inequalities (10) and (11). □

Proposition 4.5: *Suppose the proper price function $\tilde{p}$ is positive, differentiable and decreasing and $L_{\tilde{p}}$ is concave.*

(1) *If $L_{\tilde{p}}$ is strictly concave, then each function $R^{k,s}$ is strongly demi-concave.*
(2) *Each function $R^{k,s}$ is demi-concave.*
(3) *If $D\tilde{p}<0$, then each function $R^{k,s}$ is semi-strictly demi-concave.*

***Proof*:** We use parentheses for part 1 and brackets for part 3. As $D\tilde{p}\,(\leq)\,\leq\,[<]\,0$, we have $D\tilde{p}^{(k)}\,(\leq)\leq\,[<]\,0$ and hence $\eta_{\tilde{p}^{(k)}}+1\,(\leq)\leq\,[<]\,1$ on $(Y^{\oplus})_k\setminus\{0\}$. We apply Lemma 4.4 defining

$F_1 : (Y^{\oplus})_k \setminus \{0\} \to \mathbb{R}$ by $F_1(x) := \ln(\tilde{p}^{(k)}(x)) + \ln(x)$. The following equalities

$$\mathrm{Id} \cdot DF_1 = \eta_{\tilde{p}^{(k)}} + 1, \quad e^{F_1} = r_{\tilde{p}^{(k)}}, \quad \overline{e^{F_1}}\left(= \frac{e^{F_1}}{\mathrm{Id}}\right) = \tilde{p}^{(k)}$$

are true on $(Y^{\oplus})_k \setminus \{0\}$. Thus can fix F_2 as the restriction of P_k to $(Y^{\oplus})_k \setminus \{0\}$.

Case $k = 0$. Note that $(Y^{\oplus})_0 \setminus \{0\} = Y^{\oplus}$ and recall that $DL_{\tilde{p}} = \eta_{\tilde{p}}$. As $\eta_{\tilde{p}} : Y^{\oplus} \to \mathbb{R}$ is (strictly decreasing) decreasing [decreasing], also $\mathrm{Id} \cdot DF_1 + s - 1 : Y^{\oplus} \to \mathbb{R}$ is (strictly decreasing) decreasing [decreasing]. Lemma 4.4 guarantees that $e^{F_1} + (s-1)F_2$ is (strongly demi-concave) demi-concave [semi-strictly demi-concave]. Thus, as desired, also the function $R^{0,s} = \mathrm{Id} \cdot \tilde{p}^{(0)} + (s-1)P_0 : (Y^{\oplus})_0 \to \mathbb{R}$ has this property.

Case $k \in \mathrm{Int}(Y)$. As $\eta_{\tilde{p}}$ is (strictly decreasing) decreasing [decreasing], Lemma A.3 (1) guarantees that also $\eta_{\tilde{p}^{(k)}} + 1 : (Y^{\oplus})_k \to \mathbb{R}$ is (strictly decreasing) decreasing [decreasing]. Therefore also $\mathrm{Id} \cdot DF_1 + s - 1 : (Y^{\oplus})_k \setminus \{0\} \to \mathbb{R}$ is (strictly decreasing) decreasing [decreasing]. Lemma 4.4 guarantees that $e^{F_1} + (s-1)F_2$ is (strongly demi-concave) demi-concave [semi-strictly demi-concave]. Thus the function $R^{k,s} = r_{\tilde{p}^{(k)}} + (s-1)P_k$ is (strongly demi-concave) demi-concave [semi-strictly demi-concave] on $(Y^{\oplus})_k \setminus \{0\}$. By Proposition 2.6, also $R^{k,s}$ has this property. □

Proof of Theorem 4.1: By Lemma A.2 (3), (4) we have that $D\tilde{p} < 0$ in cases I and III. Proposition 4.3 guarantees that every $R^{k,s}$ is strictly concave in case I, concave in case II and semi-strictly demi-concave in case III. Now apply Theorem 3.3 (Ia) in case I, Theorem 3.3 (IIa) in case II and Theorem 3.3 (III) in case III. □

Proof of Theorem 4.2: By Lemma A.2 (1), (2), $D\tilde{p} < 0$ in cases I and III. Proposition 4.5 guarantees that every $R^{k,s}$ is strongly demi-concave in case I, demi-concave in case II and semi-strictly demi-concave in case III. Now apply Theorem 3.3 (Ib) in case I, Theorem 3.3(IIb) in case II and Theorem 3.3 (III) in case III. □

4.3. A new class

Proposition 4.6 illustrates some of the novelties of Theorem 3.3.

Proposition 4.6: *Consider price functions $p^{\bullet}$ and p° with domain $\mathbb{R}_+$. Assume that $p^{\bullet}$ and p° are positive, continuously differentiable and that their derivatives are negative. Additionally assume that there is a point in $\mathbb{R}_{++}$, say y, such that*

$$p^{\bullet}(y) = p^{\circ}(y) \text{ and } Dp^{\bullet}(y) = Dp^{\circ}(y).$$

Let p be the price function with domain $\mathbb{R}_+$ such that

$$p\restriction[0,y] = p^{\bullet}\restriction[0,y] \text{ and } p\restriction[y,+\infty[= p^{\circ}\restriction[y,+\infty[.$$

Then the function p is positive and continuously differentiable and its derivative is negative. Assume that $\tilde{p}^{\bullet}$ has either a strictly concave integrated price flexibility $L_{\tilde{p}^{\bullet}}$ or a strictly concave associated revenue function $r_{\tilde{p}^{\bullet}}$; additionally assume that also $\tilde{p}^{\circ}$ has either a strictly concave integrated price flexibility $L_{\tilde{p}^{\circ}}$ or a strictly concave associated revenue function $r_{\tilde{p}^{\circ}}$. Then

1. *the function $R^{k,s}$ associated to p, defined like in (4) above, is strongly demi-concave for every pair $(k,s) \in \mathbb{R}_+ \times \mathbb{N}^*$;*
2. *however, the function p can possess neither a concave integrated price flexibility $L_{\tilde{p}}$ nor a concave associated revenue function $r_{\tilde{p}}$.*

Proof: The fact that p is positive and continuously differentiable and that its derivative is negative is immediate. Now we shall prove (1) and (2).

(1) Choose an arbitrary pair (k, s) in $\mathbb{R}_+ \times \mathbb{N}^*$. We denote by $R_\bullet^{k,s}$ and $R_\circ^{k,s}$ the two functions – defined like in (4) above – respectively associated to $p^\bullet$ and p° that satisfy the equalities $R_\bullet^{k,s}(0) = R^{k,s}(0)$ and $R_\circ^{k,s}(y) = R^{k,s}(y)$: clearly, given $R^{k,s}$, such two functions uniquely exist. (Recall that $R_\bullet^{k,s}$ and $R_\circ^{k,s}$ are unique up to a real constant when $s > 1$.)
Case $k \geq y$. Propositions 4.5 (1) and 4.3 (1) imply that $R_\circ^{k,s}$ is strongly demi-concave. As $k \geq y$, we have that $R^{k,s} = R_\circ^{k,s}$. Thus $R^{k,s}$ is strongly demi-concave.
Case $k < y$. Propositions 4.5 (1) and 4.3 (1) imply that $R_\bullet^{k,s}$ and $R_\circ^{k,s}$ are strongly demi-concave. Clearly, also $R_\bullet^{k,s} \upharpoonright [0, y-k]$ and $R_\circ^{k,s} \upharpoonright [y-k, +\infty[$ are strongly demi-concave. As $R^{k,s} \upharpoonright [0, y-k] = R_\bullet^{k,s} \upharpoonright [0, y-k]$ and $R^{k,s} \upharpoonright [y-k, +\infty[= R_\circ^{k,s} \upharpoonright [y-k, +\infty[$, we have that even $R^{k,s} \upharpoonright [0, y-k]$ and $R^{k,s} \upharpoonright [y-k, +\infty[$ are strongly demi-concave. Now we further distinguish two subcases.
Subcase $DR^{k,s}(y-k) < 0$. As $R^{k,s} \upharpoonright [y-k, +\infty[$ is strongly demi-concave, $DR^{k,s} \upharpoonright [y-k, +\infty[$ (and hence $DR^{k,s}$) is negative on $[y-k, +\infty[$ by an immediate logical consequence of Theorem 2.4 (3). As $R^{k,s} \upharpoonright [0, y-k]$ is strongly demi-concave, there exists an interval $C \subseteq [0, y-k]$ (with minimum 0 when non-empty) such that $R^{k,s} \upharpoonright [0, y-k]$ (and hence $R^{k,s}$) is strictly concave on C and has negative derivative on $[0, y-k] \setminus C$. Thus, $R^{k,s}$ is strictly concave on the interval C (with minimum 0 when non-empty) and has negative derivative on $\mathbb{R}_+ \setminus C$. We can conclude that the differentiable function $R^{k,s}$ is strongly demi-concave.
Subcase $DR^{k,s}(y-k) \geq 0$. As $R^{k,s} \upharpoonright [0, y-k]$ is strongly demi-concave, $R^{k,s} \upharpoonright [0, y-k]$ (and hence $R^{k,s}$) is strictly concave on $[0, y-k]$. As $R^{k,s} \upharpoonright [y-k, +\infty[$ is strongly demi-concave, there exists an interval $C \subseteq [y-k, +\infty[$ (with minimum $y-k$ when non-empty) such that $R^{k,s} \upharpoonright [y-k, +\infty[$ (and hence $R^{k,s}$) is strictly concave on C and has negative derivative on $[y-k, +\infty[\setminus C$. Thus $R^{k,s}$ is strictly concave on the interval $[0, y-k] \cup C$ and has negative derivative on $\mathbb{R}_+ \setminus ([0, y-k] \cup C)$. We can conclude that the differentiable function $R^{k,s}$ is strongly demi-concave.

(2) See the next example. $\square$

Example 4.7: Pick an arbitrary γ in $[0, 1]$ and consider the two price functions $p^\bullet : \mathbb{R}_+ \to \mathbb{R}$ and $p^\circ : \mathbb{R}_+ \to \mathbb{R}$ respectively defined by

$$p^\bullet(x) = \begin{cases} \frac{1}{1+x} + \frac{4}{5} - \frac{9\gamma}{1250}(x-4)^2 & \text{if } x \leq 4 \\ \frac{1}{1+x} + \frac{4}{5} & \text{if } x \geq 4 \end{cases} \quad \text{and} \quad p^\circ(x) = e^{\frac{4-x}{25}}.$$

Both $p^\bullet$ and p° are positive, are continuously differentiable on $\mathbb{R}_+$ (and so, in particular, $Dr_{\tilde{p}^\bullet}$ is continuous) and have negative derivatives. It can be readily checked that

$$p^\bullet(4) = p^\circ(4) = 1 \text{ and } Dp^\bullet(4) = Dp^\circ(4) = -\frac{1}{25}.$$

Note that $r_{\tilde{p}^\bullet}$ is strictly concave because the continuous function $Dr_{\tilde{p}^\bullet}$ is strictly decreasing as[2]

$$D^2 r_{\tilde{p}^\bullet}(x) = \begin{cases} \gamma \frac{72-27x}{625} - \frac{2}{(x+1)^3} < 0 & \text{if } x < 4 \\ -\frac{2}{(x+1)^3} < 0 & \text{if } x > 4. \end{cases}$$

Note that $L_{\tilde{p}^\circ}$ has a strictly concave integrated price flexibility because

$$D\eta_{\tilde{p}^\circ}(x) = -\frac{1}{25} < 0 \text{ for all } x \in \mathbb{R}_{++}.$$

Consider now the function $p : \mathbb{R}_+ \to \mathbb{R}_{++}$ defined by

$$p(x) = \begin{cases} p^\bullet(x) = \frac{1}{1+x} + \frac{4}{5} - \frac{9\gamma}{1250}(x-4)^2 & \text{if } x \le 4 \\ p^\circ(x) = e^{\frac{4-x}{25}} & \text{if } x \ge 4. \end{cases}$$

By Proposition 4.6 (1), the continuously differentiable decreasing function p has an associated function $R^{k,s}$ – defined like in (4) above – which is (continuously differentiable and) strongly demi-concave for every (k,s) in $\mathbb{R}_+ \times \mathbb{N}^*$. However, p has neither a concave integrated price flexibility nor an associated concave revenue function because

$$D^2 L_{\tilde{p}}(2) = D\eta_{\tilde{p}}(2) = D\eta_{\tilde{p}^\bullet}(2) = \frac{546875 + 54\gamma\left(6875 - 972\gamma\right)}{9\left(54\gamma - 2125\right)^2} > 0$$

for all $\gamma \in [0,1]$ and

$$D^2 r_{\tilde{p}}(54) = D^2 r_{\tilde{p}^\circ}(54) = \frac{4}{625}e^{-2} > 0.$$

Finally, it is worth to note that – by the definition of $R^{k,s}$ and by Proposition 2.6 – the strong demi-concavity of $R^{k,s}$ for every (k,s) in $\mathbb{R}_+ \times \mathbb{N}^*$ implies the strong demi-concavity of $r_{p^{(k)}}$ for every $k \in \mathbb{R}_+$.

Theorem 4.2 in [11] does not imply the equilibrium uniqueness results of the present article as the assumptions of that theorem entail that a positive price function is twice continuously differentiable at all positive points of its domain. However, even if we restrict our attention to twice continuously differentiable price functions, Theorem 4.2 in [11] does not imply the equilibrium uniqueness results of this article.[3] The following Remark clarifies.

Remark 5: Let $\gamma \in [0,1]$ and consider an oligopoly with $N = \{1,2\}$ where $X_1 = X_2 = [0,1000]$, $c_1(x_1) = x_1$ and $c_2(x_2) = x_2$ and p is defined by

$$p(x) = \begin{cases} \frac{1}{1+x} + \frac{4}{5} - \frac{9\gamma}{1250}(x-4)^2 & \text{if } x \le 4 \\ e^{(4-x)/25} & \text{if } x \ge 4. \end{cases}$$

As it is clear from Example 4.7, the continuously differentiable decreasing function p has associated functions

$$R^{k,s} \text{ and } r_{p^{(k)}}$$

that are strongly demi-concave for, respectively, every (k,s) in $\mathbb{R}_+ \times \mathbb{N}^*$ and every $k \in \mathbb{R}_+$. So, by our Theorem 3.3 and our Proposition 3.1, there exists a unique Cournot equilibrium for this oligopoly. Theorem 4.2 in [11] does not guarantee the existence of a unique equilibrium when $\gamma \in [0,1[$ since p is not twice differentiable. However, also when $\gamma = 1$ and the price function

$$p(x) = \begin{cases} \frac{1}{1+x} + \frac{4}{5} - \frac{9}{1250}(x-4)^2 & \text{if } x \le 4 \\ e^{\frac{4-x}{25}} & \text{if } x \ge 4 \end{cases}$$

is twice continuously differentiable we have that Theorem 4.2 in [11] does not guarantee the existence of a unique equilibrium for this oligopoly because that theorem can apply only if – using the notation in [11] – there exists a solution (α,β) in $\mathbb{R} \times \mathbb{R}$ to the system[4]

$$\begin{cases} 0 \le \alpha + \beta \\ 0 \ge \Delta^p_{\alpha,\beta}(2) = \frac{428\,738}{31\,640\,625}\alpha + \frac{958\,873}{10\,546\,875}\beta + \frac{865\,637}{31\,640\,625} \\ 0 \ge \Delta^p_{\alpha,\beta}(120) = \frac{1}{125}e^{-\frac{232}{25}}\left(24\alpha + 5\beta - 5\right), \end{cases}$$

which indeed has no solution in $\mathbb{R} \times \mathbb{R}$. (Note that the equality

$$\Delta^{p}_{\alpha,\beta}\left(x\right) = \left(\alpha - 1\right) \cdot x \cdot Dp\left(x\right) \cdot Dp\left(x\right) + x \cdot p\left(x\right) \cdot D^2 p\left(x\right) + \left(1 - \beta\right) \cdot p\left(x\right) \cdot Dp\left(x\right)$$

is defined by (2) in [11] and in particular note that – because of the linearity of cost functions and the negativity of Dp – the first inequality of the system is a consequence of inequality (7) assumed in Theorem 4.2 in [11] and that – by Lemma 1.3 in [11] – the last two inequalities of the system follow from the assumption of Theorem 4.2 in [11] that p is $\left(\alpha, \beta\right)$ -biconcave at all positive points of its domain – and hence in particular at 2 and at 120 – for some fixed pair $\left(\alpha, \beta\right)$).

Notes

1. Such a Theorem 1 is in turn a variant of the Nikaido–Isoda theorem [5].
2. To check that $D^2 r_{\tilde{p}^\bullet}$ is negative on $[0, 4[$ notice that $72 - 27x \leq 0$ if $x \in]8/3, 4[$ and that $D^2 r_{\tilde{p}^\bullet}$ is majorized on $[0, 8/3]$ by the strictly concave real-valued map on $[0, 8/3]$ defined at x by $\left(72 - 27x\right)/625 - 2\left(x + 1\right)^{-3}$ (that is negative at its unique maximizer $5\sqrt[4]{2}/\sqrt{3} - 1$).
3. Analogously, Theorem 4.2 in [11] does not imply the equilibrium uniqueness results in [2].
4. In fact the conditions are actually stronger.

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Appendix 1. Some results of convex analysis for price functions

Various results of this Appendix can be found in [3] and in [4]: they are presented in this Appendix in order for this article to be more self-contained.

Lemma A.1: *Suppose $\tilde{p}$ is non-negative and $r_{\tilde{p}}$ is (strictly) concave. Then r_p is (strictly) concave.*

***Proof*:** The function r_p is lower semi-continuous at 0 as $r_p \geq 0 = r_p\left(0\right)$. Thus r_p is lower semi-continuous and $r_{\tilde{p}}$ is (strictly) concave. The lower semi-continuous function r_p is (strictly) concave as $r_{\tilde{p}}$ is (strictly) concave. □

Lemma A.2: *Suppose $\tilde{p}$ is positive and differentiable.*

(1) *If $L_{\tilde{p}}$ is strictly concave and $\tilde{p}$ is decreasing, then $D\tilde{p} < 0$.*
(2) *If $L_{\tilde{p}}$ is concave and strictly decreasing, then $D\tilde{p} < 0$.*
(3) *If $r_{\tilde{p}}$ is (strictly) concave, then $D\tilde{p}\ (<) \leq 0$.*
(4) *If $r_{\tilde{p}}$ is concave and decreasing, then $D\tilde{p} < 0$.*

Proof:

(1) Suppose $L_{\tilde{p}}$ is strictly concave and $\tilde{p}$ is decreasing. Clearly $\eta_{\tilde{p}} \leq 0$ as $D\tilde{p} \leq 0$ by the decreasingness of the positive function $\tilde{p}$. If $D\tilde{p}\left(y\right) = 0$ then $\eta_{\tilde{p}}\left(y\right) = 0$ and $y > 0$. But this is impossible as the strict concavity of $L_{\tilde{p}}$ implies the strict decreasingness of $\eta_{\tilde{p}}$ and hence that $\eta_{\tilde{p}}\left(y/2\right) > 0$.
(2) Suppose $L_{\tilde{p}}$ is concave and $\tilde{p}$ is strictly decreasing. The derivative $\eta_{\tilde{p}}$ of the strictly decreasing concave function $L_{\tilde{p}}$ defined on the left open interval $Y^{\oplus}$ must be negative. Thus $D\tilde{p} < 0$ as $\tilde{p} > 0$.
(3) Suppose $r_{\tilde{p}}$ is (strictly) concave. By Lemma A.1 also r_p is (strictly) concave. Thus, for each $y \in Y^{\oplus}$, we have that $Dr_p\left(y\right) \leq (<) \left(r_p\left(y\right) - r_p\left(0\right)\right) / \left(y - 0\right)$ and hence that $Dp\left(y\right) \leq (<) 0$ as the equalities $Dr_p\left(y\right) = Dp\left(y\right)y + p\left(y\right)$ and $\left(r_p\left(y\right) - r_p\left(0\right)\right) / \left(y - 0\right) = p\left(y\right)$ hold true.
(4) Suppose $r_{\tilde{p}}$ is concave and decreasing. Hence $Dr_{\tilde{p}} \leq 0$. Thus $D\tilde{p}\left(y\right)y + \tilde{p}\left(y\right) \leq 0$ for all $y \in Y^{\oplus}$ and hence $D\tilde{p} < 0$ by the positivity of $\tilde{p}$. □

Lemma A.3: *Consider a price function p and pick $k \in \mathrm{Int}(Y)$.*

(1) *Suppose $\tilde{p}$ is positive and differentiable. If $L_{\tilde{p}}$ is (strictly) concave and decreasing, then $L_{\tilde{p}^{(k)}}$ is (strictly) concave.*
(2) *If $r_{\tilde{p}}$ is (strictly) concave, then $r_{\tilde{p}^{(k)}}$ is (strictly) concave.*
(3) *If $r_{\tilde{p}}$ is concave and $\tilde{p}$ is strictly decreasing, then $r_{\tilde{p}^{(k)}}$ is strictly concave.*

Proof:

(1) Suppose $L_{\tilde{p}}$ is (strictly) concave and decreasing. Then $\eta_{\tilde{p}}$ is (strictly) decreasing. Besides $\eta_{\tilde{p}} \leq 0$ as $D\tilde{p} \leq 0$ and $\tilde{p}$ is positive. We are done if we prove that $\eta_{\tilde{p}^{(k)}}$ is (strictly) decreasing. The proof of such fact is as follows. Let $x_1, x_2 \in (Y^{\oplus})_k$ with $x_1 < x_2$. We know that $\eta_{\tilde{p}} \leq 0$ and hence $\eta_{\tilde{p}} < 0$ when $\eta_{\tilde{p}}$ is strictly decreasing because the domain of $\eta_{\tilde{p}}$ is left open. Thus $0\ (>) \geq \eta_{\tilde{p}}(x_1 + k)(>) \geq \eta_{\tilde{p}}(x_2 + k)$. As $k > 0$, we have $\frac{x_1}{x_1+k} < \frac{x_2}{x_2+k}$. We obtain, as desired, $\eta_{\tilde{p}^{(k)}}(x_1) = \eta_{\tilde{p}}(x_1 + k)\frac{x_1}{x_1+k} \geq \eta_{\tilde{p}}(x_2 + k)\frac{x_1}{x_1+k}(>) \geq \eta_{\tilde{p}}(x_2 + k)\frac{x_2}{x_2+k} = \eta_{\tilde{p}^{(k)}}(x_2)$.
(2) We prove, by contradiction, only the case of strict concavity of $r_{\tilde{p}}$. The proof of the other case is analogous. Suppose $r_{\tilde{p}}$ is strictly concave and $r_{\tilde{p}^{(k)}}$ is not strictly concave. Then there exist $x, y \in (Y^{\oplus})_k$ with $x < y$ and $t \in\ ]0, 1[$ such that the inequality $r_{\tilde{p}^{(k)}}(z) \leq tr_{\tilde{p}^{(k)}}(x) + (1 - t)r_{\tilde{p}^{(k)}}(y)$ holds true for $z = tx + (1 - t)y$. By Lemma A.2 (3) the function $\tilde{p}$ is strictly decreasing; thus also $\tilde{p}^{(k)}$ is strictly decreasing. As $z + k \geq 0$, the inequality

$$r_{\tilde{p}^{(k)}}(z)(z + k) \leq (tr_{\tilde{p}^{(k)}}(x) + (1 - t)r_{\tilde{p}^{(k)}}(y))(z + k)$$

is true. As $x + k, y + k \in Y^{\oplus}$ and $t \in\ ]0, 1[$, also $z + k = t(x + k) + (1 - t)(y + k) \in Y^{\oplus}$. As $r_{\tilde{p}}$ is strictly concave and $z > 0$, the strict inequality

$$r_{\tilde{p}}(z + k)z > (tr_{\tilde{p}}(x + k) + (1 - t)r_{\tilde{p}}(y + k))z$$

holds true. The two previous centered inequalities together with the equality $r_{\tilde{p}}(z+k)z = r_{\tilde{p}^{(k)}}(z)(z+k)$ imply

$$(tr_{\tilde{p}^{(k)}}(x) + (1 - t)r_{\tilde{p}^{(k)}}(y))(z + k) > (tr_{\tilde{p}}(x + k) + (1 - t)r_{\tilde{p}}(y + k))z$$

and hence, by the definition of $r_{\tilde{p}}$ and $r_{\tilde{p}^{(k)}}$, we obtain

$$tx(z + k)\tilde{p}(x + k) + (1 - t)y(z + k)\tilde{p}(y + k) > t(x + k)z\tilde{p}(x + k) + (1 - t)(y + k)z\tilde{p}(y + k).$$

Therefore $kt\left(1 - t\right)\tilde{p}\left(y + k\right)\left(y - x\right) > kt\left(1 - t\right)\tilde{p}\left(x + k\right)\left(y - x\right)$ as $y - z = t\left(y - x\right)$ and $z - x = \left(1 - t\right)\left(y - x\right)$. As $k > 0$, we can conclude that

$$\tilde{p}\left(y + k\right) > \tilde{p}\left(x + k\right).$$

But the previous inequality contradicts the strict decreasingness of $\tilde{p}^{(k)}$.
(3) Analogous to the proof of part 2. □