Orthomodular Lattices Induced by the Concurrency Relation

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We apply to locally finite partially ordered sets a construction which associates a complete lattice to a given poset; the elements of the lattice are the closed subsets of a closure operator, defined starting from the *concurrency* relation. We show that, if the partially ordered set satisfies a property of local density, i.e.: N-density, then the associated lattice is also orthomodular. We then consider occurrence nets, introduced by C.A. Petri as models of concurrent computations, and define a family of subsets of the elements of an occurrence net; we call those subsets *causally closed* because they can be seen as subprocesses of the whole net which are, intuitively, closed with respect to the forward and backward local state changes. We show that, when the net is K-dense, the causally closed sets coincide with the closed sets induced by the closure operator defined starting from the concurrency relation. K-density is a property of partially ordered sets introduced by Petri, on the basis of former axiomatizations of special relativity theory.

1   Introduction

We consider models of concurrent behaviours based on partial orders, and in particular occurrence nets. Occurrence nets were introduced by C.A. Petri ([8]) as a model of non sequential processes which are physically realizable. They are a special kind of Petri nets, where occurrences of local states (also called conditions) and of events are partially ordered. The partial order is interpreted as a kind of causal dependence relation (for background on partial orders and occurrence nets, see [4]).

Any partially ordered set (or *poset* for short) induces a *concurrency* relation, defined as the complement of the partial order. This relation is symmetric but in general non transitive.

By applying known techniques of lattice theory, one can derive, from such a relation, a complete, orthocomplemented lattice of subsets of the underlying set.

In a recent paper ([2]) we showed that this technique, applied to an occurrence net, always gives an orthomodular lattice.

In the present paper, we consider N-density and K-density, properties defined by Petri, inspired by former axiomatizations of special relativity theory (see, for example, [6]). A partial order is K-dense if any *line* (namely, a maximal subset of pairwise ordered elements) intersects any *cut* (a maximal subset of pairwise incomparable elements). This corresponds to the intuitive idea that in a given global state (represented by a cut) any sequential subprocess is in some state, given by a point along the subprocess. N-density is a sort of local, and weaker, form of K-density.

We show that N-density, together with two local finiteness properties, of a poset is sufficient to produce an orthomodular lattice. This generalizes one of the main results of [2].

We then restrict attention to degree-finite and interval-finite occurrence nets. On these nets we introduce the notion of *causally closed set*, which corresponds to a set of elements of the net which identifies a sort of *causally closed* subprocess, i.e.: a subprocess uniquely constructible starting from a set of concurrent conditions. We show that closed sets, as defined in [2], are causally closed and prove that, in the case of K-dense occurrence nets, closed sets and causally closed sets coincide.
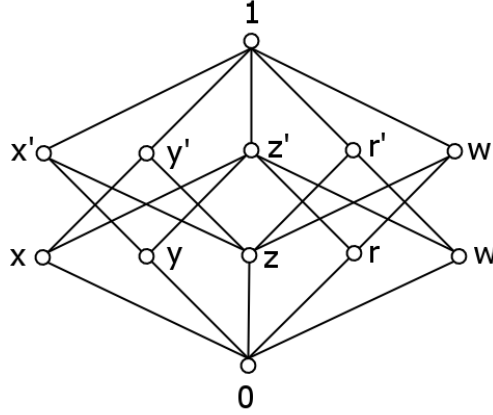
Figure 1: A finite orthomodular lattice.

The next section collects some definitions and results to be used later. In Section 3, we show that N-density and interval-finiteness suffice to grant the orthomodularity of the lattice of closed sets generated starting from the concurrency relation. Section 4 introduces the notion of causally closed set, and shows that in K-dense occurrence nets, closed sets and causally closed sets coincide.

Proofs are omitted, but can be found in [3].

2 Preliminary Definitions

2.1 Orthomodular Posets and Lattices

In this section we recall the basic definitions related to orthomodular posets and lattices.

Definition 1 *An* orthocomplemented poset $\mathscr{P} = \langle P, \leq, 0, 1, (\,.\,)' \rangle$ *is a partially ordered set* $(P, \leq)$*, equipped with a minimum and a maximum element, respectively denoted by 0 and 1, and with a map* $(\,.\,)' : P \to P$*, such that the following conditions are verified (where* $\vee$ *and* $\wedge$ *denote, respectively, the least upper bound and the greatest lower bound with respect to* $\leq$*, when they exist):* $\forall x, y \in P$

(i) $(x')' = x$;

(ii) $x \leq y \Rightarrow y' \leq x'$;

(iii) $x \wedge x' = 0$ and $x \vee x' = 1$.

The map $(\,.\,)' : P \to P$ is called an *orthocomplementation* in P. In an orthocomplemented poset, $\wedge$ and $\vee$, when they exist, are not independent: in fact, the so-called De Morgan laws hold: $(x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$. In the following, we will sometimes use *meet* and *join* to denote, respectively, $\wedge$ and $\vee$. Meet and join can be extended to families of elements in the obvious way, denoted by $\bigwedge$ and $\bigvee$.

A poset $(P, \leq)$ is called *orthocomplete* when it is orthocomplemented and every countable subset of pairwise orthogonal elements of P has a least upper bound.

A lattice $\mathscr{L}$ is a poset in which for any pair of elements meet and join always exist. A lattice $\mathscr{L}$ is *complete* when the meet and the join of any subset of $\mathscr{L}$ always exist.

Definition 2 *[1] An* orthomodular poset $\mathscr{P} = \langle P, \leq, 0, 1, (.)' \rangle$ *is an orthocomplete poset which satisfies the condition:*

$$x \leq y \Rightarrow y = x \vee (y \wedge x')$$

which is usually referred to as the *orthomodular* law.

The orthomodular law is weaker than the distributive law. A lattice $\mathscr{L}$ is called *distributive* if and only if $\forall x, y, z \in \mathscr{L}$ the equalities $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ hold. Orthocomplemented distributive lattices are called Boolean algebras. Orthomodular posets and lattices can therefore be considered as a generalization of Boolean algebras and have been studied as algebraic models for quantum logic [10].

Any orthomodular lattice can be seen as a family of partially overlapping Boolean algebras. Figure 1 shows a finite orthomodular lattice.

2.2   Closure Operators

Definition 3 *Let X be a set and* $\mathbb{P}(X)$ *the powerset of X. A map* $\mathscr{C} : \mathbb{P}(X) \to \mathbb{P}(X)$ *is a* closure operator *on X if, for all* $A, B \subseteq X$,

(i) $A \subseteq \mathscr{C}(A)$,

(ii) $A \subseteq B \Rightarrow \mathscr{C}(A) \subseteq \mathscr{C}(B)$,

(iii) $\mathscr{C}(\mathscr{C}(A)) = \mathscr{C}(A)$.

A subset A of X is called *closed* with respect to $\mathscr{C}$ if $\mathscr{C}(A) = A$. If $\mathscr{C}$ is a closure operator on a set X, the family $\{A \subseteq X \mid \mathscr{C}(A) = A\}$ of closed subsets of X forms a complete lattice, when ordered by inclusion, in which

$$\bigwedge \{A_i : i \in I\} = \bigcap_{i \in I} A_i, \quad \bigvee \{A_i : i \in I\} = \mathscr{C}(\bigcup_{i \in I} A_i).$$

The proof of this statement can be found in [5].

We now describe a well-known construction from binary relations to closure operators (see, for example, [5]). Let X be a set, and $\alpha \subseteq X \times X$ be a symmetric relation. Define an operator $(.)^{\perp}$ on the powerset of X: given $A \subseteq X$

$$A^{\perp} = \{x \in X \mid \forall y \in A : (x, y) \in \alpha\}.$$

By applying twice the operator $(.)^{\perp}$, we get a new operator $C(.) = (.)^{\perp\perp}$. The map C on the powerset of X is a closure operator on X. A subset A of X is called *closed* with respect to $(.)^{\perp\perp}$ if $A = A^{\perp\perp}$. The family $L(X)$ of all closed sets of X, ordered by set inclusion, is a complete lattice.

When α is also irreflexive, the operator $(.)^{\perp}$ applied to elements of $L(X)$ is an orthocomplementation; the structure $\mathscr{L}(X) = \langle L(X), \subseteq, \emptyset, X, (.)^{\perp} \rangle$ then forms an orthocomplemented complete lattice.

2.3   Occurrence Nets

Definition 4 *A* net *is a triple* $N = (B, E, F)$, *where B and E are countable sets,* $F \subseteq (B \times E) \cup (E \times B)$, *and*

(i) $B \cap E = \emptyset$

(ii) $\forall e \in E \;\; \exists x, y \in B : (x, e) \in F$ *and* $(e, y) \in F$.

The elements of B are called *local states* or *conditions*, the elements of E *local changes of state* or *events*, and F is called the *flow relation*. Note that we allow isolated conditions but not isolated events.

Local states correspond to properties which can be true or false in a given global state of the system; potential global states of the system modeled by N are subsets of local states.

For each $x \in B \cup E$, define $^\bullet x = \{y \in B \cup E \mid (y,x) \in F\}$, $x^\bullet = \{y \in B \cup E \mid (x,y) \in F\}$. For $e \in E$, an element $b \in B$ is a *precondition* of e if $b \in {}^\bullet e$; it is a *postcondition* of e if $b \in e^\bullet$.

Occurrences of events are in accord with the following *firing rule*: an event may occur if its preconditions are true and its postconditions are false; when the event occurs its preconditions become false, while its postconditions become true. In this way the occurrence of an event changes the current global state by only changing the *local* states directly connected to the event itself.

Occurrence nets are a special class of nets used to model non-sequential processes ([8], [4]) by recording the partial order of the validity of conditions and of the event occurrences in the evolution of the system behaviour.

Definition 5 *A net $N = (B,E,F)$ is an* occurrence net *iff*

(i) $\forall b \in B : |^\bullet b| \leq 1 \,\wedge\, |b^\bullet| \leq 1$ *and*

(ii) $\forall x,y \in B \cup E : (x,y) \in F^+ \Rightarrow (y,x) \notin F^+,$

where F^+ denotes the transitive closure of F.

Definition 5(i) means that an occurrence net does not contain non-deterministic choices, the idea being that all conflicts are resolved at the behavioural level. Definition 5(ii) means that an occurrence net contains no cycles, the idea being that all loops are unfolded at the behavioural level.

Because of Definition 5(ii), the structure $(X, \sqsubseteq)$ derived from an occurrence net N by putting $X = B \cup E$ and $\sqsubseteq = F^*$ (F^* denotes the reflexive and transitive closure of F) is a partially ordered set (shortly *poset*). We will use $\sqsubset$ to denote the associated strict partial order.

Given a partial order relation $\leq$ on a set P, we can derive the relations $li = \leq \cup \geq$, and $co = (P \times P) \setminus li$. We will be interested in such relations derived from $(X, \sqsubseteq)$. In such case, intuitively, $x\ li\ y$ means that x and y are connected by a causal relation, and $x\ co\ y$ means that x and y are causally independent. The relations li and co are symmetric and not transitive. Note that li is a reflexive relation, while co is irreflexive. Given an element $x \in X$ and a set $S \subseteq X$, we write $x\ co\ S$ if $\forall y \in S : x\ co\ y$. Moreover, given two sets $S_1 \subseteq X$ and $S_2 \subseteq X$, we write $S_1\ co\ S_2$ if $\forall x \in S_1, \forall y \in S_2 : x\ co\ y$. In the following we will use $x\,co\,y$ or $(x,y) \in co$ indifferently, and similarly for li.

On the basis of the flow relation F and its transitive closure F^+, to each $x \in X$ we associate the elements in its *past* and in its *future*, denoted by:

$$F^-(x) = \{y \in X \mid y \sqsubset x\} \text{ and}$$
$$F^+(x) = \{z \in X \mid x \sqsubset z\}$$

respectively. By generalizing to subsets S of X, we denote the *past* and the *future* of S by

$$F^-(S) = \{x \in X \mid x \notin S, \exists y \in S : x \in F^-(y)\} \text{ and}$$
$$F^+(S) = \{x \in X \mid x \notin S, \exists y \in S : x \in F^+(y)\}.$$

From the definition, it follows that an element x belongs neither to its future nor to its past.

A *clique* of a binary relation is a set of pairwise related elements. From the *co* and *li* relations one can define *cuts* and *lines* of a poset $\mathscr{P} = (P, \leq)$ as maximal cliques of *co* and *li*, respectively:
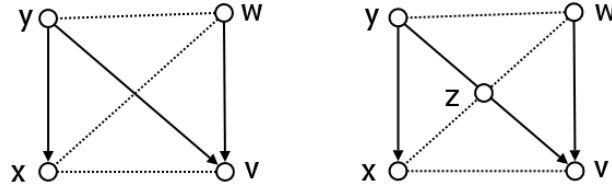
Figure 2: Illustration of N-density.

$$\text{Cuts}(\mathscr{P}) = \{c \subseteq P \mid c \text{ is a maximal clique of } co \cup id_P\};$$
$$\text{Lines}(\mathscr{P}) = \{\, l \subseteq P \mid l \text{ is a maximal clique of } li\}.$$

Given an occurrence net $N = (B, E, F)$, we will denote by $\text{Cuts}(N)$ and $\text{Lines}(N)$, respectively, the set of cuts and the set of lines of the poset associated to N. We will always assume the Axiom of Choice, so that any clique of co and of li can be extended to a maximal clique. In particular, we will denote cliques and maximal cliques of co which contain only conditions by B-cosets and B-cuts, respectively.

C.A. Petri formalized some properties which intuitively should hold for posets corresponding to non-sequential processes which are actually feasible [9], see also [4]. In particular, we will consider interval-finiteness, degree-finiteness, a sort of local density called N-density, and K-density.

Definition 6 $\mathscr{P} = (P, \leq)$ *is* interval-finite $\Leftrightarrow \forall x, y \in P : |[x, y]| < \infty$, *where* $[x, y] = \{z \in P \mid x \leq z \leq y\}$.

For $x, y \in P$, we write $x \lessdot y$ if $x < y$ and, for all $z \in P$, $x < z \leq y \Rightarrow z = y$. Let $^\bullet x = \{y \mid y \lessdot x\}$, and $x^\bullet = \{y \mid x \lessdot y\}$.

Definition 7 $\mathscr{P} = (P, \leq)$ *is* degree-finite $\Leftrightarrow \forall x \in P : |^\bullet x| < \infty$ *and* $|x^\bullet| < \infty$.

Definition 8 $\mathscr{P} = (P, \leq)$ *is* N-dense $\Leftrightarrow \forall\, x, y, v, w \in P$: $(y < v$ *and* $y < x$ *and* $w < v$ *and* $(y\, co\, w\, co\, x\, co\, v)) \Rightarrow \exists z \in P : (y < z < v$ *and* $(w\, co\, z\, co\, x))$.

For a graphical representation of N-density condition see Figure 2.

Proposition 1 *[4] Let* $(X, \sqsubseteq)$ *be the poset associated to an occurrence net* $N = (B, E, F)$, $X = (B \cup E)$. *Then* $(X, \sqsubseteq)$ *is N-dense.*

K-density is based on the idea of interpreting cuts as (global) states and lines as sequential subprocesses. K-density postulates that every occurrence of a subprocess must be in some state.

Definition 9 $\mathscr{P} = (P, \leq)$ *is* K-dense $\Leftrightarrow \forall c \in \text{Cuts}(\mathscr{P}), \forall l \in \text{Lines}(\mathscr{P}) : c \cap l \neq \emptyset$.

Obviously, in general $|c \cap l| \leq 1$. An occurrence net N is K-dense if its associated poset is K-dense.

3   Closed Sets Induced by the Concurrency Relation

In this section we apply the construction recalled at the end of Section 2.2 to the co relation in partially ordered sets, and study the resulting algebraic structure of closed sets.

Let $\mathscr{P} = (P, \leq)$ be a poset. We can define an operator on subsets of P, which corresponds to an orthocomplementation, since co is irreflexive, and by this operator we define closed sets.
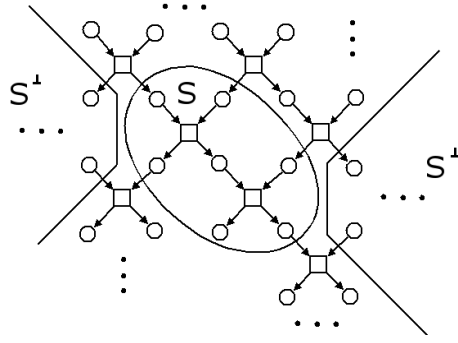
Figure 3: A closed set S and its orthocomplement $S^\perp$.

Definition 10 *Let $S \subseteq P$, then*

(i) $S^\perp = \{x \in P \mid \forall y \in S : x \ co \ y\}$ *is the* orthocomplement *of S;*

(ii)   *if $S = (S^\perp)^\perp$, then S is a* closed set *of P.*

The set $S^\perp$ contains the elements of P which are not in causal relation with any element of S. Obviously, $S \cap S^\perp = \emptyset$ for any $S \subseteq P$. In the following, we sometimes denote $(S^\perp)^\perp$ by $S^{\perp\perp}$. Note that: $\forall c \in Cuts(\mathscr{P})$, $c^\perp = \emptyset$ and $c^{\perp\perp} = P$.

Example 1 *Figure 3 shows a closed set S and its orthocomplement $S^\perp$ on the poset associated to an occurrence net.*

A closed set S and its orthocomplement $S^\perp$ share the past and the future.

Proposition 2 *[2] Let $S = S^{\perp\perp}$. Then $F^-(S) = F^-(S^\perp)$ and $F^+(S) = F^+(S^\perp)$.*

Now we study the algebraic structure induced by the closure operator defined above. We call $L(P)$ the collection of closed sets of $\mathscr{P} = (P, \leq)$. By the results on closure operators recalled in Section 2.2, we know that

$$\mathscr{L}(P) = \langle L(P), \subseteq, \emptyset, P, (\,.\,)^\perp \rangle$$

is an orthocomplemented complete lattice, in which the meet is just set intersection, while the join of a family of elements is given by set union followed by closure.

In general, the structure $\mathscr{L}(P)$ is not orthomodular and a fortiori non distributive, as can be seen in the following example.

Example 2 *Let us consider the poset $\mathscr{P} = (P, \leq)$ shown in the left side of Fig. 2 and the closed sets $\{w\}$ and $\{v, w\}$; $\{w\}^\perp = \{x, y\}$. The orthomodular law is not valid since $\{w\} \subset \{v, w\}, (\{x, y\} \wedge \{v, w\}) = \emptyset$ and hence $(\{w\} \vee \emptyset)$ is not equal to $\{v, w\}$.*

The poset considered in the previous example is not N-dense. It is natural to investigate if there is a relation between N-density of a poset P and the orthomodularity of the associated structure of closed sets $\mathscr{L}(P)$. It turns out that N-density of an interval-finite poset is a sufficient condition even if not necessary for orthomodularity.

Theorem 1 *Let $\mathscr{P} = (P, \leq)$ be an N-dense, interval-finite poset. Then $\mathscr{L}(P)$ is an orthomodular lattice.*
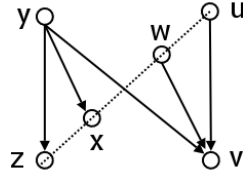
Figure 4: A non N-dense poset generating an orthomodular lattice.

The reverse implication is not true, as can be seen in the following example.

Example 3 *Figure 4 shows a poset $\mathscr{P} = (P, \leq)$ which is not N-dense. The family of closed sets $L(P)$ forms a Boolean, hence orthomodular, lattice, whose atoms are $\{z\}, \{x\}, \{w\}, \{u\}$.*

From Theorem 1 and Proposition 1 it follows that the family $L(N)$ of closed sets of the poset $(X, \sqsubseteq)$ associated to an interval-finite occurrence net $N = (B, E, F)$, with $X = (B \cup E)$, forms an orthomodular lattice $\mathscr{L}(N) = \langle L(N), \subseteq, \emptyset, X, (.) \perp \rangle$. This result constitutes a generalization and a different proof of a theorem in [2], where interval-finite and degree-finite occurrence nets have been considered and where closed sets have been called "causally closed sets".

4   Causally Closed Sets in Occurrence Nets

In this section we consider only interval-finite and degree-finite occurrence nets. We introduce particular subsets of elements of such an occurrence net $N = (B, E, F)$, which we call "causally closed sets", since they can be interpreted as subprocesses which can be uniquely obtained by a particular B-coset.

We show that, for K-dense occurrence nets, "causally closed sets" coincide with closed sets of the poset associated to the occurrence net, as defined in the previous section.
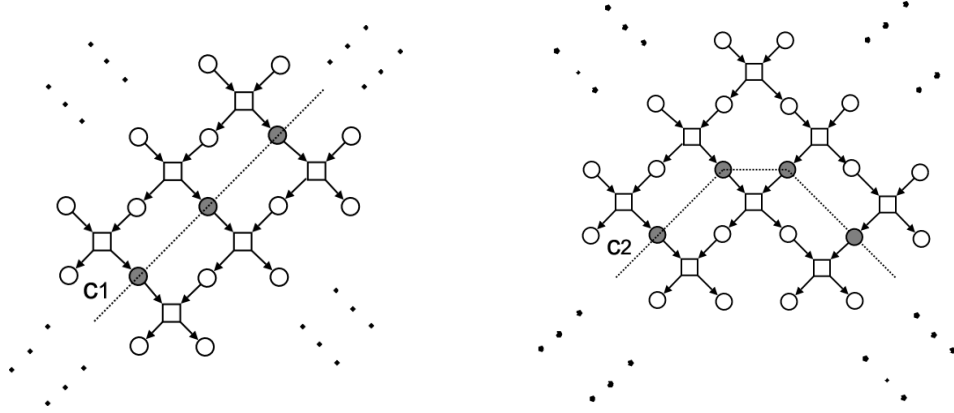
Definition 11 *Let $N = (B, E, F)$ be an occurrence net. $C \subseteq B \cup E$ is a* causally closed set *iff*

   (i) $\forall e \in E,\ ^\bullet e \subseteq C \Rightarrow e \in C,$

   (ii) $\forall e \in E,\ e^\bullet \subseteq C \Rightarrow e \in C,$

   (iii) $\forall e \in E,\ e \in C \Rightarrow {}^\bullet e \cup e^\bullet \subseteq C,$

   (iv) $\forall x, y \in C,\ x\ li\ y \Rightarrow [x, y] \subseteq C.$

A causally closed set is therefore a convex set which, intuitively, is closed with respect to the firing rule, i.e.: if it contains an event then it contains also all its preconditions and postconditions, and, moreover, if it contains all the preconditions or all the postconditions of an event then it contains the event itself.

Example 4 *The set S in Figure 3 is causally closed. The set of grayed elements on the left side of Figure 5 is causally closed. On the contrary, the set of grayed elements on the right side of Figure 5 is not causally closed because, for example, it contains the preconditions of an event but it does not contain the event itself.*

We call $CC(N)$ the family of causally closed sets of N. It is easy to prove that $CC(N)$ is closed by intersection, $\emptyset \in CC(N)$, $B \cup E \in CC(N)$. Hence the family $CC(N)$ forms a complete lattice, where meet is given by set intersection. In general this lattice is not orthocomplemented.

Figure 5: Examples illustrating that in general $\phi(A) \neq A^{\perp\perp}$.

Definition 12 *Let $N = (B, E, F)$ be an occurrence net, $X = B \cup E$, and $\mathbb{P}(X)$ be the powerset of X. Define $\phi : \mathbb{P}(X) \to \mathbb{P}(X)$ as follows: $\forall A \subseteq X$, $\phi(A) = \bigcap\{C_i \mid C_i \in CC(N) \text{ and } A \subseteq C_i\}$.*

Note that ϕ is a closure operator, $A \subseteq \phi(A)$ and the frontier of $\phi(A)$ is a subset of conditions, where the frontier of a subset A of X is the set $\{x \in A \mid \exists y \in X \setminus A \text{ such that: } xFy \text{ or } yFx\}$.

Closed sets, as defined in Section 3, are causally closed sets, and this directly follows from a characterization of closed sets given in [2]. However, in general a causally closed set is not a closed set, in fact the following example shows two cases in which $\phi(A) \neq A^{\perp\perp}$.

Example 5 *Figure 5 shows a non K-dense occurrence net. Remember that $\forall c \in Cuts(N), c^\perp = \emptyset$ and $c^{\perp\perp} = X$. Consider now the B-cut c_1 formed by the gray conditions on the left side of the figure; $\phi(c_1) = c_1$. For the B-cut c_2 on the right side of the figure, $c_2 \subset \phi(c_2) \subset X$.*

When A is a B-coset of $N = (B, E, F)$, $\phi(A)$ can be inductively constructed, as shown in the following.

Definition 13 *Given $A_i \subseteq B \cup E$, define $A_{i+1} = A_i \cup \{Int(e) \mid e \in E, {}^\bullet e \subseteq A_i \vee e^\bullet \subseteq A_i\}$, where for $e \in E$, $Int(e) = \{e\} \cup {}^\bullet e \cup e^\bullet$.*

Note that $A_i \subseteq A_{i+1}$ for every i.

Proposition 3 *Let A be a B-coset of N. Then $\bigcup_{i \in \mathbb{N}} A_i = \phi(A)$, where $A_0 = A$.*

The inductive construction of a causally closed set from a B-coset is shown in Figure 6. This motivates the name of *causally closed* sets and suggests an interpretation of these as *non sequential subprocesses*, which are *causally closed* in the sense that are uniquely constructed starting by a B-coset of the whole process. The construction, in fact, simulates the forward and backward run of the system starting from a B-coset: it adds all the events such that either all their preconditions or all their postconditions belong to the starting B-coset, and then it proceeds by adding to the set all the post/preconditions of the added events, and so on until no other event may be added.

Now we show that for K-dense occurrence nets, $CC(N)$ coincides with the collection $L(N)$ of closed sets of N generated by the *co* relation.

Let $N = (B, E, F)$ be a K-dense, interval-finite and degree-finite occurrence net.

Theorem 2 *Let $Y \subseteq B \cup E$. Then $Y \in CC(N) \iff Y \in L(N)$*

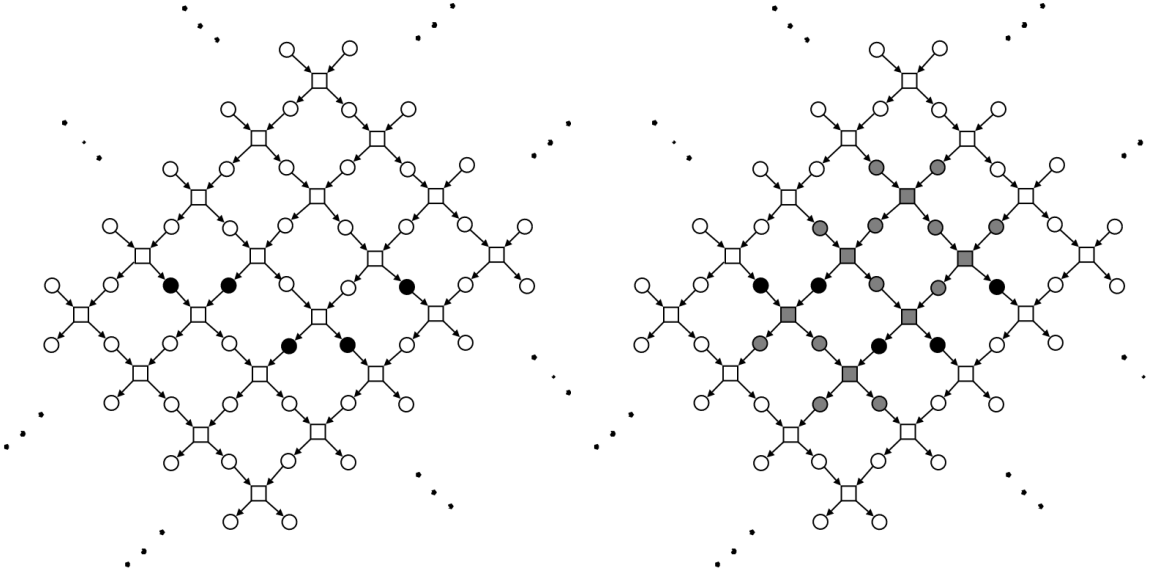The theorem is an immediate consequence of the following propositions.

Figure 6: Inductive construction of a causally closed set.

Proposition 4 *Let A be a B-coset of N. Then $\phi(A) = A^{\perp\perp}$.*

Proposition 5 *Let $Y \in CC(N)$, and $A \subseteq Y$ be a B-cut of the subposet induced by Y. Then $\phi(A) = Y$.*

Therefore, in the case of K-dense, interval-finite and degree-finite occurrence nets, the families of causally closed sets and of closed sets coincide. In particular, any closed set can be uniquely obtained starting from a B-coset by applying to it the inductive construction as in Definition 13.

5   Conclusion

The main contribution of this paper is twofold. On one hand, we have applied to locally finite partially ordered sets a construction which associates a complete lattice to a given poset; the elements of the lattice are certain subsets of the poset, precisely the closed subsets of a closure operator, defined starting from the *co* relation. We have shown that, if the partially ordered set satisfies a property of local density, i.e.: N-density, then the associated lattice is also orthomodular. Orthomodular posets are studied in the frame of quantum logic (see, for example, [7]). This suggests to interpret closed sets as propositions in a logical language.

On the other hand, we have focused attention on occurrence nets as models of concurrent computations, and defined a family of subsets of the elements of an occurrence net; we call those subsets *causally closed* because they can be seen as subprocesses of the whole net which are, intuitively, closed with respect to the (forward and backward) firing rule of the net. We have shown that, when the net is K-dense, the causally closed sets coincide with the closed sets induced by the closure operator defined starting from the *co* relation.

Starting from these first results, we intend to pursue the investigation of lattices of subprocesses in different directions. On one hand, we will study further properties of such lattices, and their relations with domain theory. On the other hand, we will extend the construction to cyclic Petri nets, in which a sensible concurrency relation can be defined even in the absence of a global partial order.

Acknowledgments

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